Information Theory. 5th Chapter Slides

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Noiseless coding problem statement

- $X = \{0, 1\}. Y = X$
- Discrete channel with noise.
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Table: Example 1

Message	Codeword	Decisive area
0	000	{000, 001, 010, 100}
1	111	{011, 101, 110, 111}

Noiseless coding problem statement

Table: Example 2

Message	Codeword	Decisive area
00	00000	{00000,00001,00010,00100, 01000,10000,11000,10001}
01	10110	{10110,10111,10100,10010, 11110,00110,01110,00111}
10	01011	{01011,01010,01001,01111, 00011,11011,10011,11010}
11	11101	{11101,11100,11111,11001, 10101,01101,00101,01100}

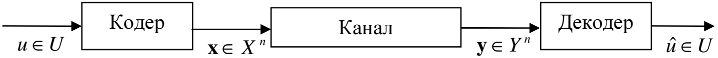
- 

Figure: Communication system Scheme

Noiseless coding problem statement

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- *Code of channel* over X is arbitrary set of sequences $A = \{\vec{x}_m\}$, $m = 1, ..., M$, $A \in X^n$.
- These sequences are *codewords*.
- Their length n is *code length*.
- Number of sequences M is *code cardinality*. R, defined as:

$$R = \frac{\log M}{n} \tag{1}$$

is called *code rate* (bits per symbol).
- Event when $\hat{u} \neq u$ is *decoding error*.
- And it's probability is *error probability*

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- Reminder: $\vec{x}_i^n = (x_i, ..., x_n)$. Channel is called *stationary*, if $\forall j, n$ and $\forall \vec{x}_{j+1}^{j+n} \in X^n$, $\vec{y}_{j+1}^{j+n} \in Y^n$ conditional probabilities $p(\vec{y}_{j+1}^{j+n}|\vec{x}_{j+1}^{j+n})$ are defined by sequence characters and do not depend from index j.

- Channel is called *memoryless*, if $\forall j, n$ and $\forall \vec{x}_{j+1}^{j+n} \in X^n$, $\vec{y}_{j+1}^{j+n} \in Y^n$

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- Stationary channel without memory is called discrete stationary channel.

To describe a Discrete Stationary Channel it's enough to define conditional probabilities $\{p(y|x), x \in X, y \in Y\}$. Let $X = \{0, ..., K-1\}$, $Y = \{0, ..., L-1\}$. Let $p_{ij} = p(y = j|x = i\}$, $i \in X$, $j \in Y$. Describe transition probabilities of channel p_{ij} in a *transition probability matrix*:

$$\begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,L-1} \\ p_{10} & p_{11} & \cdots & p_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K-1,0} & p_{K-1,1} & \cdots & p_{K-1,L-1} \end{bmatrix}.$$

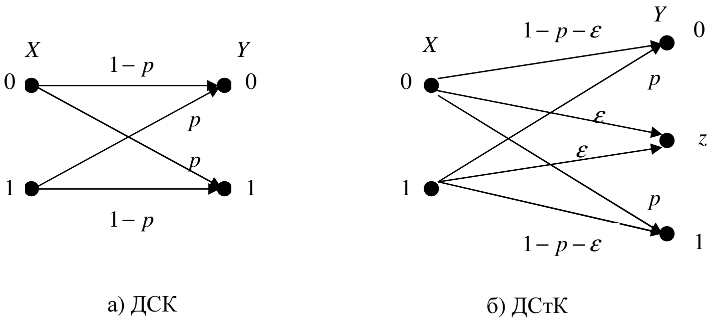

Figure: Discrete stationary channels examples

- Binary Symmetric Channel (BSC).
  $X = Y = \{0, 1\}$, $p_{10} = p_{01} = p$,
  $p_{00} = p_{11} = 1 - p$. Transition probability matrix:

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- Binary Symmetric Channel with Erasure (BSCE).
  $$P = \begin{bmatrix} 1-p-\varepsilon & \varepsilon & p \\ p & \varepsilon & 1-p-\varepsilon \end{bmatrix}.$$
  $X = 0, 1$, $Y = 0, 1, z$, where z is a special erasure symbol.

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- Mutual information:

$$I(x; y) = I(x) - I(x|y). \tag{2}$$

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$$I(X;Y) = \mathbf{M}\left[I(x;y)\right].$$

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- Dependence between average mutual information and joint probability distribution:

$$I(X;Y) = \sum_{x \in X} \sum_{y \in Y} p(x,y) \log \frac{p(y|x)}{p(y)}. \qquad (3)$$

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4. Nonnegativity: $I(X;Y) \geq 0$.
5. Identity $I(X;Y) = 0$ holds iff ensembles X and Y are independent.

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10. Mutual information $I(X;Y)$ is a convex $\cup$ function of conditional probabilities $p(y|x)$.

Conditional average mutual information.

- Consider $XYZ = \{(x, y, z), p(x, y, z)\}$. Fix $z \in Z$ and consider conditional probability distribution: $p(x, y|z) = \frac{p(x,y,z)}{p(z)}$.
- Average mutual information between X and Y: $I(X; Y|z) = \sum_{x \in X} \sum_{y \in Y} p(x, y|z) \log \frac{p(y|x,z)}{p(y|z)}$.

Conditional average mutual information.

- Conditional average mutual information between X and Y:
  $I(X;Y|Z) = \mathbf{M}\left[I(X;Y|z)\right] =$
  $\sum_{x \in X} \sum_{y \in Y} \sum_{z \in Z} p(x,y,z) \log \frac{p(y|x,z)}{p(y|z)}$
- Additional properties:
  $I(X;Y|Z) = H(Y|Z) - H(Y|XZ)$.
  $I(X;YZ) = I(X;Y) + I(X;Z|Y)$
  $I(X;YZ) = I(X;Z) + I(X;Y|Z)$

Conditional average mutual information.

A special case of information processing system, which has 3 probability ensembles:

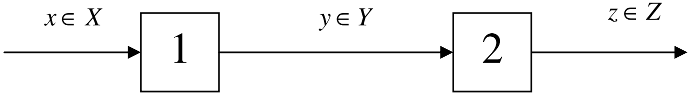

Figure: Information processing system

Conditional average mutual information.

Theorem

Let X, Y, Z be probability ensembles, which are formed by the information processing system at the previous slide. Then holds:

$$I(X;Y) \geq I(X;Z), \tag{4}$$

$$I(Y;Z) \geq I(X;Z). \tag{5}$$

proof. Use properties of conditional average mutual information:

$$
\begin{aligned}
I(X;YZ) &= I(X;Y) + I(X;Z|Y), \qquad (6)\\
I(X;YZ) &= I(X;Z) + I(X;Y|Z). \qquad (7)
\end{aligned}
$$

X and Z are independent. If Y is known, $I(X;Z|Y) = 0$. By equating the right sides of (6) and (7), we get

$$I(X;Y) = I(X;Z) + I(X;Y|Z).$$

Since the second term is non-negative, we obtain the inequality (4). Similarly we can prove (5).

Convexity of average mutual information