Information Theory. 6th Chapter Problems

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Table: Continuous Random Variables

Probability Distribution	Density $f(x)$	Characteristics m	σ^2	$h(X)$
Uniform	$\frac{1}{b-a},\quad x \in [a,b]$ $0,\quad x < a, x > b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\log(b-a)$
Exponential	$\lambda e^{-\lambda x},\quad x \geq 0$ $0,\quad x < 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\log \frac{e}{\lambda}$
Laplace	$\frac{\lambda}{2} e^{-\lambda\|x-m\|}$	m	$\frac{1}{\lambda^2}$	$\log \frac{2e}{\lambda}$
Normal (Gaussian)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$	m	σ^2	$\log \sqrt{2\pi e \sigma^2}$
Generalized Gaussian	$\frac{\alpha\eta(\alpha,\sigma)}{2\Gamma(1/\alpha)} e^{-\eta(\alpha,\sigma)\|x-m\|^{\alpha}}$ $\eta(\alpha,\sigma) = \frac{1}{\sigma}\sqrt{\frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)}}$	m	σ^2	$\log \frac{2\Gamma(1/\alpha)e^{1/\alpha}}{\alpha\eta(\alpha,\sigma)}$

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Problems

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2. Prove that Generalized Gaussian distribution with $\alpha = 1$, $\alpha 2$, $\alpha \to \infty$ is equivalent to Laplace, Gaussian and Unifoen respectively.
3. Evaluate on sone examples formulas from table (1) for differential entropy.
4. At what parameters of a distribution the differential entropy is 0? Without doing precise calculations, draw on the same plot densities of distributions when differential entropy is equal to 0 and 1.

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2. $$\begin{aligned} h(a+X) &= h(X); \\ h(aX) &= h(X) + \log|a|. \end{aligned}$$
3. For independent X and Y
$$\begin{aligned} h(XY) &= h(X) + h(Y|X), \\ h(X|Y) &\leq h(X), \end{aligned}$$

Properties of Differential Entropy:

4 $\forall \quad f_1(x)$ и $f_2(x)$

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5 If Random variable X is defined on segment of length a, then $\forall f(x)$ holds

$$h(X) \leq \log a,$$

Equality is achieved in case if Uniform distribution.

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7 For Random Variables with dispersion σ^2 $\forall f(x)$

$$h(X) \leq \log\sqrt{2\pi e\sigma^2},$$

Equality is achieved in case of Gaussian distribution.

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7. Write joint distribution of two Gaussian Random variables X, Y. Count Differential entropy of them and mutual information $I(X;Y)$. Draw a plot of dependence between mutual information and correlation coefficient of X и Y.

- *hint to task 7.* According to gaussian distribution density formula

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2} \det K^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{m})K^{-1}(\boldsymbol{x}-\boldsymbol{m})^T}, \qquad (1)$$

We need only correlation matrix which is defined by dispersions

$$\sigma_x^2 = \mathbf{M}[(x - m_x)^2]; \quad \sigma_y^2 = \mathbf{M}[(y - m_y)^2].$$

and by correlation matrix

$$\rho = \frac{\mathbf{M}[(x - m_x)(y - m_y)]}{\sigma_x \sigma_y},$$

Answer:

$$I(X;Y) = -\frac{1}{2}\log(1 - \rho^2).$$

8. Consider Random Variable $Y = \{y\}$, which is a sum of independent Gaussian Random Variables $X = \{x\}$ and $Z = \{z\}$, i.e $y = x + z$. As example, y can be considered as channel output when transmitting message x with addition of noise z. Find mutual information $I(X; Y)$.

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Hint: Sum of Gaussian variables is a Gaussian variable. Thus, this problem is a generalization of the previous one. We need to find correlation coefficient of X and Y.
Answer:

$$\rho = \frac{\sigma_x}{\sqrt{\sigma_x^2 + \sigma_z^2}},$$

$$I(X:Y) = \frac{1}{2}\log\left(1 + \frac{\sigma_x^2}{\sigma_z^2}.\right)$$

Note, that with increasing of noise dispersion, the mutual information tends to zero.

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Hint. Correllation matrix satisfies

$$K_n = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho^{n-1} \\ \rho & 1 & \cdots & \rho^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{bmatrix} \qquad (2)$$

Reduce matrix by Gauss method to the lower triangular form. Multiply each row, starting from second to ρ.

10 Prove the properties of correlation function and spectral power density of a discrete time stationary random process.

$$K(n) = K(-n); \tag{3}$$

$$K(0) \geq |K(n)|; \quad n > 0; \tag{4}$$

$$S(\omega) = S(-\omega); \tag{5}$$

$$S(\omega) - \text{ real function } \omega. \tag{6}$$

$$\int_{-\pi}^{\pi} S(\omega) d\omega = K(0) = \sigma^2. \tag{7}$$

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Hint. In order to prove (4), consider knowingly non-negative value $\mathbf{M}[(x_t - x_{t+n})^2]$.

11 Consider representative sample (for example, 1000 readings) of a real signal. it can be line of an image, audio fragment, fragment of speech, etc. Build the dependence between differential entropy of autoregressive process and it's order.

Hint to task 11. Use MATLAB. For reading audio file use function `wavread.m`, for reading images use `imread.wav`. After that use `corrcoeff.m` to calculate correlation coefficients. Use `levinson.m` for getting parameters of autoregressive filter of required order. Otherwise, parameters of autoregressive filter can be found by `lpc.m`. Substitute parameters into formula (8)

$$S(\omega) = \frac{\sigma^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-ik\omega}\right|^2}, \tag{8}$$

and integrate with "rectangle method". Check your results using formula (9)

$$h_\infty(X) = \frac{1}{2}\log(2\pi e) + \lim_{n\to\infty} \frac{\log\det K_n}{n}. \tag{9}$$