Reduced basis stochastic Galerkin methods for partial differential equations with random inputs

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Abstract

We present a reduced basis stochastic Galerkin method for partial differential equations with random inputs. In this method, the reduced basis methodology is integrated into the stochastic Galerkin method, such that the cost of solvers for the Galerkin system is significantly reduced. To reduce the main cost of matrix-vector manipulation involved in our reduced basis stochastic Galerkin approach, the secant method is applied to identify the number of reduced basis functions. We present a general mathematical framework of the methodology, validate its accuracy and demonstrate its efficiency with numerical experiments.

Keywords: PDEs with random data, reduced basis, generalized polynomial chaos, stochastic Galerkin method.

1. Introduction

In the past decades, there has been a rapid development in efficient numerical methods for solving partial differential equations (PDEs) with random inputs. This explosion in interest has been driven by the need of conducting uncertainty quantification for modeling realistic problems such as diffusion problems and acoustic scattering problems. The uncertainty sources for these problems typically arise from lack of knowledge or measurement of realistic model parameters such as permeability coefficients and refraction coefficients.

It is of interest to compute the mean and variance of the solutions of PDEs with random inputs. To this end, a lot of efforts are made. One of the simplest ways to deal with this problem is the Monte Carlo method (MCM) and its variants [1, 2]. In MCM, large numbers of realizations of the random inputs are generated based on their probability density functions. For each realization, the associated deterministic problem can be solved by available numerical schemes. Then the mean and variance of the stochastic solution can be approached by the statistical information of those solutions of deterministic problems. Although MCM is easy to implement, the solution statistics converge relatively slowly and typically numerous realizations are required. To improve the efficiency of sampling based methods, the stochastic collocation method (SCM) based on sparse grids is developed in [3] (and see [4] for a detailed review). After that, the reduced basis collocation

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method (RBCM) is proposed in [5], which significantly reduces the computational costs of colloca-
tion methods without loss of accuracy. In addition, reduced basis methods are actively developed
for solving PDEs with random inputs [6, 7, 8, 9], and especially, efficient preconditioning techniques
for reduced basis methods are developed in [10].

Another kind of efficient approaches for PDEs with random inputs is the stochastic Galerkin
method (SGM) [11, 12, 13]. In stochastic Galerkin methods, the stochastic solution is represented by
finite terms of trial basis expansions. Then the expansion coefficients can be determined by solving a
linear system resulted from the Galerkin projection. The trial functions mainly include polynomial
chaos (PC) [11], generalized polynomial chaos (gPC) [13, 12], piecewise polynomial bases [14],
multi-element generalized polynomial chaos (ME-gPC) [15, 16], and the dynamically bi-orthogonal
polynomials [17, 18, 19, 20]. In this work, we mainly focus on the gPC approach for the stochastic
solution. Since the stochastic Galerkin method usually results in a large coupled linear system, how
to design efficient solvers is a very important and challenging problem. Various iterative solvers
such as the mean-based preconditioning iterative method [21], the reduced basis solver based on
low-rank approximation [22] and the preconditioned low-rank projection methods [23] are vigorously
studied. In this work, we develop a reduced basis stochastic Galerkin method (RBSGM), which
integrate the reduced basis methodology into the stochastic Galerkin method, such that the cost
of solvers for the Galerkin system is significantly reduced. To reduce main cost of our RBSGM,
which includes matrix-vector manipulation in the iterative procedure, the secant method is applied
to identify the number of reduced basis functions. While the reduced basis method for physical
approximation in stochastic Galerkin is studied in [24], we give a systematic procedure to generate
the basis functions and to significantly reduce the number of matrix-vector multiplications in the
iterative procedure.

The outline of this work is as follows. In Section 2, we describe the PDE formulation considered
in the paper, introduce the stochastic Galerkin method, and discuss the iterative method for the
stochastic Galerkin system. In section 3, our reduced basis stochastic Galerkin method is presented.
Section 4 gives numerical results, and Section 5 concludes the paper.

2. Problem setting and stochastic Galerkin method

This section describes the considered PDEs with random inputs in this study, introduce the
stochastic Galerkin method, and discuss the iterative method for the stochastic Galerkin system.

2.1. Problem setting

Let $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_m]^{\mathrm{T}}$ be an m-variable random vector. The image of ξ_i is denoted by Γ_i and
the probability density function of ξ_i by $\rho_i(\xi_i)$. If we further assume that the components of $\boldsymbol{\xi}$, i.e.,
$\xi_1, \ldots, \xi_m$ are mutually independent, then the image of $\boldsymbol{\xi}$ is given by $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$ and the
probability density function of $\boldsymbol{\xi}$ by $\rho(\boldsymbol{\xi}) = \prod_{i=1}^{m} \rho_i(\xi_i)$.

In this work, we consider the following partial differential equations (PDEs) with random in-
puts, which is widely used in modeling steady state diffusion problems [13] and acoustic scattering
problems [25, 26]

$$
\begin{cases}
-\nabla \cdot [a(\boldsymbol{x}, \boldsymbol{\xi})\nabla u(\boldsymbol{x}, \boldsymbol{\xi})] - \kappa^2(\boldsymbol{x}, \boldsymbol{\xi})u(\boldsymbol{x}, \boldsymbol{\xi}) = f(\boldsymbol{x}) & \forall (\boldsymbol{x}, \boldsymbol{\xi}) \in D \times \Gamma, \\
\mathfrak{b}(\boldsymbol{x}, \boldsymbol{\xi}, u(\boldsymbol{x}, \boldsymbol{\xi})) = g(\boldsymbol{x}) & \forall (\boldsymbol{x}, \boldsymbol{\xi}) \in \partial D \times \Gamma,
\end{cases}
\tag{1}
$$

where $\mathfrak{b}$ is a boundary operator, f is the source function and g specifies the boundary condition. Both $a(\boldsymbol{x}, \boldsymbol{\xi})$ and $\kappa(\boldsymbol{x}, \boldsymbol{\xi})$ are assumed to have the following form:

$$a(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=0}^{m} a_0(\boldsymbol{x})\xi_i, \ \kappa(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=0}^{m} \kappa_i(\boldsymbol{x})\xi_i, \ \xi_0 = 1.$$

Note that we denote 1 as ξ_0 for simplify. The problem (1) is a diffusion equation when $\kappa(\boldsymbol{x}, \boldsymbol{\xi}) \equiv 0$ and a Helmholtz equation when $a(\boldsymbol{x}, \boldsymbol{\xi}) \equiv 1$.

To simplify the presentation, we assume the problem (1) satisfies homogeneous Dirichlet boundary conditions. It is straightforward to generalize the approach to non-homogeneous boundary conditions.

2.2. Variational formulation

To introduce the variational form of (1), some notations are required. We first consider the Hilbert space

$$L^2(D) := \left\{ v : D \to \mathbb{C} \ \middle| \ \|v\|_{L^2} < \infty \right\},$$

of square integrable functions equipped with the inner product

$$\langle u, v \rangle_{L^2} = \int_D v^* u \, \mathrm{d}\boldsymbol{x},$$

and the norm

$$\|v\|_{L^2} := \sqrt{\langle v, v \rangle_{L^2}}.$$

Moreover, let

$$H_0^1(D) := \left\{ v \in H^1(D) \,|\, v = 0 \text{ on } \partial D_D \right\},$$

where $H^1(D)$ is the Sobolev space

$$H^1(D) := \left\{ v \in L^2(D) \,, \, \partial v / \partial x_i \in L^2(D), i = 1, \ldots, d \right\}.$$

Next, we define the Hilbert space

$$L_\rho^2(\Gamma) := \left\{ v(\boldsymbol{\xi}) : \Gamma \to \mathbb{R} \ \middle| \ \|v\|_{L_\rho^2}^2 < \infty \right\},$$

which is equipped with the inner product

$$\langle u, v \rangle_{L_\rho^2} = \int_\Gamma v^*(\boldsymbol{\xi}) u(\boldsymbol{\xi}) \rho(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi},$$

and the norm

$$\|v\|_{L_\rho^2} := \sqrt{\langle v, v \rangle_{L_\rho^2}}.$$

Following [14], the tensor product space of $L^2(D)$ and $L_\rho^2(\Gamma)$ is defined as

$$L^2(D) \otimes L_\rho^2(\Gamma) := \left\{ w(\boldsymbol{x}, \boldsymbol{\xi}) \middle| w(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{n} u_i(\boldsymbol{x}) v_i(\boldsymbol{\xi}), u_i(\boldsymbol{x}) \in L^2(D), v_i(\boldsymbol{\xi}) \in L_\rho^2(\Gamma), n \in \mathbb{N}^+ \right\},$$

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which is equipped with the inner product

$$\langle w_1, w_2 \rangle_{L^2 \otimes L^2_\rho} = \int_\Gamma \int_D \rho(\boldsymbol{\xi}) w_2^* w_1 \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{\xi}.$$

Then, the variational form of (1) with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2 \otimes L^2_\rho}$ can be written as:

$$\int_\Gamma \int_D \rho(\boldsymbol{\xi}) \left[a \nabla u \cdot \nabla w^* - \kappa^2 u w^* \right] \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\xi} = \int_\Gamma \int_D \rho(\boldsymbol{\xi}) f w^* \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\xi}, \; \forall w \in L^2(D) \otimes L^2_\rho(\Gamma). \quad (2)$$

2.3. Discretization

To obtain the discrete version of (2), we need to introduce a finite dimensional subspace of $L^2(D) \otimes L^2_\rho(\Gamma)$ and find an approximation lies in it. Given subspace of the physical and the stochastic spaces

$$V_h = \mathrm{span} \{v_s(\boldsymbol{x})\}_{s=1}^{N_h} \subset H_0^1(D), \; S_p = \mathrm{span} \{\Phi_j(\boldsymbol{\xi})\}_{j=1}^{N_p} \subset L^2_\rho(\Gamma),$$

where h is the mesh size of the physical space and p is the order of generalized polynomial chaos (gPC) of the stochastic space, the finite dimensional subspace of $L^2(D) \otimes L^2_\rho(\Gamma)$ can be defined as

$$W_{hp} := V_h \otimes S_p = \mathrm{span} \left\{ v_s(\boldsymbol{x}) \Phi_j(\boldsymbol{\xi}) \mid s = 1, \ldots, N_h \text{ and } j = 1, \ldots, N_p \right\}.$$

In this work, the basis functions $\{v_s(\boldsymbol{x})\}_{s=1}^{N_h}$ are taken to be the standard trial (test) functions of finite element approximations. Meanwhile, $\{\Phi_j(\boldsymbol{\xi})\}_{j=1}^{N_p}$ are taken to be the generalized polynomial chaos (gPC) basis, which include the orthonormal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2_\rho}$, i.e.,

$$\langle \Phi_j(\boldsymbol{\xi}), \Phi_k(\boldsymbol{\xi}) \rangle_{L^2_\rho} = \int_\Gamma \rho(\boldsymbol{\xi}) \Phi_j(\boldsymbol{\xi}) \Phi_k^*(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} = \delta_{jk},$$

where δ_{jk} is the Kronecker delta function. As usual, $\{\Phi_j(\boldsymbol{\xi})\}_{j=1}^{N_p}$ consists of orthonormal polynomials with total degrees up to p, and p is referred as the gPC order. Then the dimension of S_p is given by $N_p = (m + p)!/(m!p!)$. For more details about gPC, the readers can consult Refs. [4, 12].

Suppose that $u_{hp}(\boldsymbol{x}, \boldsymbol{\xi})$ is an approximation of $u(\boldsymbol{x}, \boldsymbol{\xi})$ lies in W_{hp}, i.e.,

$$u_{hp}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{j=1}^{N_p} \sum_{s=1}^{N_h} u_{js} v_s(\boldsymbol{x}) \Phi_j(\boldsymbol{\xi}). \quad (3)$$

By restricting the test functions in weak form (2) to W_{hp}, we have

$$\int_\Gamma \int_D \rho(\boldsymbol{\xi}) \left[a \nabla u_{hp} \cdot \nabla w^* - \kappa^2 u_{hp} w^* \right] \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\xi} = \int_\Gamma \int_D \rho(\boldsymbol{\xi}) f w^* \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\xi}, \; \forall w \in W_{hp}.$$

This gives rise to a linear system

$$\boldsymbol{A} \boldsymbol{u} = \boldsymbol{b}, \quad (4)$$

where $\boldsymbol{u}^{\mathrm{T}} = [\boldsymbol{u}_1^{\mathrm{T}}, \ldots, \boldsymbol{u}_{N_p}^{\mathrm{T}}]$, $\boldsymbol{u}_j \in R^{N_h \times 1}$, $\boldsymbol{u}_j(s) = u_{js}$ and

$$\boldsymbol{A} = \sum_{i=0}^m \boldsymbol{G}_{i0} \otimes \boldsymbol{A}_i - \sum_{i=0}^m \sum_{j=0}^m \boldsymbol{G}_{ij} \otimes \boldsymbol{B}_{ij}, \; \boldsymbol{b} = \boldsymbol{h} \otimes \boldsymbol{f}. \quad (5)$$

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In (5), $\boldsymbol{G}_{ij} \in R^{N_p \times N_p}$ and $\boldsymbol{h} \in R^{N_p \times 1}$ only depending on the basis functions in the stochastic space, and thus are named as the stochastic Galerkin matrices (vector). They are given by

$$\boldsymbol{G}_{ij}(l,n) = \int_\Gamma \xi_i \Phi_l(\boldsymbol{\xi}) \xi_j \Phi_n^*(\boldsymbol{\xi}) \mathrm{d}\rho(\boldsymbol{\xi}), \ \boldsymbol{h}(l) = \int_\Gamma \Phi_l(\boldsymbol{\xi}) \mathrm{d}\rho(\boldsymbol{\xi}), \ \xi_0 = 1,$$

where $i,j \in \{0,1,\ldots,m\}$ and $l,n \in \{1,2,\ldots,N_p\}$.

On the other hand, the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_{ij}$ and vector $\boldsymbol{f}$ are given by

$$\boldsymbol{A}_i(s,t) = \int_D a_i(\boldsymbol{x}) \nabla v_s(\boldsymbol{x}) \cdot \nabla v_t^*(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \ \boldsymbol{B}_{ij}(s,t) = \int_D \kappa_i \kappa_j v_s v_t^* \mathrm{d}\boldsymbol{\xi}, \ \boldsymbol{f}(s) = \int_D f v_s^* \mathrm{d}\boldsymbol{x} \qquad (6)$$

where $i,j \in \{0,1,\ldots,m\}$ and $s,t \in \{1,2,\ldots,N_h\}$. It is clear that $\boldsymbol{A}_i$, $\boldsymbol{B}_{ij}$ and $\boldsymbol{f}$ only depending on the basis functions in the physical space and the coefficients $a_i(\boldsymbol{x})$, $\kappa_i(\boldsymbol{x})$.

Once the unknowns are solved through (4), the approximation $u_{hp}(\boldsymbol{x},\boldsymbol{\xi})$ can be easily reconstructed by (3). Since $\{\Phi_j(\boldsymbol{\xi})\}_{j=1}^{N_p}$ is orthonormal, the mean and the variance of $u_{hp}(\boldsymbol{x},\boldsymbol{\xi})$ are given by

$$\mathbb{E}[u_{hp}] = \sum_{s=1}^{N_h} u_{1s} v_s(\boldsymbol{x}), \ \mathbb{V}[u_{hp}] = \sum_{j=2}^{N_p} \sum_{s=1}^{N_h} (u_{js} v_s(\boldsymbol{x}))^2.$$

2.4. Iterative methods

The coefficient matrix $\boldsymbol{A}$ in (4) is usually a block-wise sparse matrix [27, 28]. Note that the size of matrix $\boldsymbol{A}$ in (4) is $N_h N_p$, where N_h is the DOF used in the spatial discretization and N_p is the number of basis used in the stochastic space. For high accuracy approximation, N_h and N_p are very large, and thus a large sparse linear system is need to be solved. In this section, we discuss the iterative methods for this kind of linear systems. In particular, we focus on Krylov subspace methods [29, 30], which based on the projection of the linear system (4) into a consecutively constructed Krylov subspace

$$\mathcal{K}_n(\boldsymbol{A}, \boldsymbol{r}^{(0)}) = \mathrm{span}\{\boldsymbol{r}^{(0)}, \boldsymbol{A}\boldsymbol{r}^{(0)}, \boldsymbol{A}^2 \boldsymbol{r}^{(0)}, \ldots, \boldsymbol{A}^{n-1} \boldsymbol{r}^{(0)}\},$$

where

$$\boldsymbol{r}^{(0)} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{u}^{(0)}$$

and $\boldsymbol{u}^{(0)}$ is an initial approximation.

In Krylov subspace methods, we usually need to compute the matrix-vector products of the form $\boldsymbol{A}\boldsymbol{x}$ and/or $\boldsymbol{A}^\mathrm{T} \boldsymbol{x}$. Since the coefficient matrix has a Kronecker products structure, the matrix-vector products can be done efficiently without assembling the coefficient matrix explicitly. To show this, we introduce the `vec` operation [29]. If $\boldsymbol{X} \in \mathbb{R}^{m \times n}$, then `vec`$(\boldsymbol{X})$ is an mn-by-1 vector obtained by "stacking" the columns of $\boldsymbol{X}$, i.e.,

$$\mathtt{vec}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{X}(:,1) \\ \vdots \\ \boldsymbol{X}(:,n) \end{bmatrix},$$

where $\boldsymbol{X}(:,j)$, $j = 1,\ldots,n$ is the j-th column of $\boldsymbol{X}$. Then the matrix-vector product in the iterative methods can be rewritten as

$$\boldsymbol{A}\boldsymbol{x} = \left(\sum_{i=0}^m \boldsymbol{G}_{i0} \otimes \boldsymbol{A}_i - \sum_{i=0}^m \sum_{j=0}^m \boldsymbol{G}_{ij} \otimes \boldsymbol{B}_{ij} \right) \boldsymbol{x} = \mathtt{vec} \left(\sum_{i=0}^m \boldsymbol{A}_i \boldsymbol{X} \boldsymbol{G}_{i0}^\mathrm{T} - \sum_{i=0}^m \sum_{j=0}^m \boldsymbol{B}_{ij} \boldsymbol{X} \boldsymbol{G}_{ij}^\mathrm{T} \right), \quad (7)$$

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where $\texttt{vec}(\boldsymbol{X}) = \boldsymbol{x}$ and $\boldsymbol{X} \in \mathbb{R}^{N_h \times N_p}$.

Preconditioning is another key ingredient for the success of iterative methods in solving the linear system (4). Instead of solving the original problem (4), we solve the preconditioned problem

$$\boldsymbol{A}\boldsymbol{P}^{-1}\boldsymbol{y} = \boldsymbol{b}, \text{ with } \boldsymbol{u} = \boldsymbol{P}^{-1}\boldsymbol{y},$$

where the nonsingular matrix $\boldsymbol{P}$ is called a preconditioner. A lot of preconditioners that exploit the structure of the linear system are developed (see, for example [21, 31, 32]). In particular, the mean based preconditioner is popular due to its non-intrusive nature. Denote the mean value of $\boldsymbol{\xi}$ by $\boldsymbol{\xi}^{(0)}$, and the identity matrix by $\boldsymbol{I}$. The mean based preconditioner is constructed based on the deterministic version of problem (1) associated with the realization $\boldsymbol{\xi}^{(0)}$. Let $\boldsymbol{A}_{\boldsymbol{\xi}^{(0)}}$ denotes the discrete version of the differential operator associated with $\boldsymbol{\xi}^{(0)}$ in (1), then the mean based preconditioner is given by $\boldsymbol{P} = \boldsymbol{I} \otimes \boldsymbol{A}_{\boldsymbol{\xi}^{(0)}}$. In this work, the mean based preconditioner is used in the iterative solvers to improve the efficiency.

The main criterion for selecting an iterative method is the knowledge of the coefficient matrix properties. For example, if $\boldsymbol{A}$ is symmetric positive definite, the conjugate gradient (CG) method is the best choice; otherwise, we should use generalized minimal residual (GMRES) method or other iterative methods. For more about this topic, the readers can consult Refs. [33, 34].

3. The reduced basis stochastic Galerkin method

In Section 2, we introduced the framework of the stochastic Galerkin method. As we have mentioned, the stochastic Galerkin method will lead to a linear system with coefficient matrix represented as Kronecker products. The number of unknowns is $N_h \times N_p$, where N_h is the DOF in the physical space, and N_p is the DOF in the physical space.

In practical implementation, N_h is large if the spatial mesh is sufficiently fine so that the error in the physical space is acceptable. On the other hand, since $N_p = (m + p)!/(m!p!)$, the DOF of the stochastic space is also large if the dimension of random variables is high or high accuracy is needed in the stochastic space. This will lead to an unacceptable high cost for solving the $N_h^2 \times N_p^2$ linear system (4). In this section, we show that the costs can be reduced significantly by integrate the reduced basis methodology into the iterative method.

3.1. Reduced basis method

The variational formulation of the deterministic version of (1) associated with a given $\boldsymbol{\xi}$ is given by

$$\int_D \left[a\nabla u \cdot \nabla v^* - k^2 u v^* \right] \mathrm{d}\boldsymbol{x} = \int_D f v^* \mathrm{d}\boldsymbol{x}, \ \forall v \in H_0^1(D).$$

Let $V_h = \text{span}\{v_i(\boldsymbol{x})\}_{i=1}^{N_h}$ be a spatial finite element approximation space (e.g., piecewise linear or quadratic polynomial space) of dimension N_h. A finite element method seeks an approximation $u_h(\boldsymbol{x}, \boldsymbol{\xi}) \in V_h$ such that

$$\int_D \left[a\nabla u_h \cdot \nabla v^* - k^2 u_h v^* \right] \mathrm{d}\boldsymbol{x} = \int_D f v^* \mathrm{d}\boldsymbol{x}, \ \forall v \in V_h. \tag{8}$$

Suppose that

$$u_h(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{s=1}^{N_h} u_s(\boldsymbol{\xi}) v_s(\boldsymbol{x}),$$

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then we have the following linear system

$$\boldsymbol{A_\xi u_\xi} = \boldsymbol{f}, \tag{9}$$

where

$$\boldsymbol{A_\xi} = \sum_{i=0}^{m} \boldsymbol{A}_i \xi_i - \sum_{i=0}^{m} \sum_{j=0}^{m} \boldsymbol{B}_{ij} \xi_i \xi_j. \tag{10}$$

Note that we define $\xi_0 = 1$ for convince, and $\boldsymbol{A}_i$, $\boldsymbol{B}_{ij}$, $i,j \in \{0, \ldots, m\}$ are given by (6).

For the reduced basis method (RBM), we first find an r dimension reduced space $Q_r \subset V_h$, where $r \ll N_h$, and seek a reduced solution $u_h^{(r)}(\boldsymbol{x}, \boldsymbol{\xi}) \in Q_r$ such that

$$\int_D \left[a \nabla u_h^{(r)} \cdot \nabla v^* - k^2 u_h^{(r)} v^* \right] \mathrm{d}\boldsymbol{x} = \int_D f v^* \mathrm{d}\boldsymbol{x}, \ \forall v \in Q_r. \tag{11}$$

Typically, the reduced problem (11) is much smaller than that of the full problem (8). And thus the linear system is likely to be solved more efficiently. Since in the stochastic collocation method, we only need to solve the deterministic problem associated with each realization, using RBM of course can improve the efficiency. Moreover, note that the discretization of the physical space is independent of the discretization of the stochastic space, RBM can also be applied to the stochastic Galerkin method for the improvement of efficiency. In the following, we give the details of the construction of the reduced space Q_r.

Assume that a training set $\boldsymbol{\Xi} \subset \Gamma$ is given such that $\{u(\boldsymbol{x}, \boldsymbol{\xi}) : \boldsymbol{\xi} \in \boldsymbol{\Xi}\}$ is accurately approximated by $\{u_h(\boldsymbol{x}, \boldsymbol{\xi}) : \boldsymbol{\xi} \in \boldsymbol{\Xi}\}$, where $u_h(\boldsymbol{x}, \boldsymbol{\xi})$ is referred as a snapshot. Then the reduced space Q_r is constructed as a span of snapshots, i.e.,

$$Q_r = \mathrm{span}\{u_h(\boldsymbol{x}, \boldsymbol{\xi}^{(1)}), \ldots, u_h(\boldsymbol{x}, \boldsymbol{\xi}^{(r)})\},$$

where $\{\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(r)}\}$ are chosen in the training set $\boldsymbol{\Xi}$ by greedy algorithm.

Let $\{q_1(\boldsymbol{x}), \ldots, q_r(\boldsymbol{x})\}$ be the basis of Q_r, and $\boldsymbol{q}_i(\boldsymbol{x})$ be the vector of coefficient values associated with q_i, i.e.,

$$q_i(\boldsymbol{x}) = \boldsymbol{q_i}(1) v_1 + \cdots + \boldsymbol{q_i}(N_h) v_{N_h}, \ i = 1, 2, \ldots, r. \tag{12}$$

Then the linear system for the reduced problem associated with $\boldsymbol{\xi}$ is given by

$$\boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{A_\xi} \boldsymbol{Q}_r \boldsymbol{u}_{\boldsymbol{\xi}}^{(r)} = \boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{f}, \tag{13}$$

where $\boldsymbol{A_\xi}$, $\boldsymbol{f}$ are given by (10), $\boldsymbol{u}_{\boldsymbol{\xi}}^{(r)}$ is the coefficient vector of the basis $\{q_1(\boldsymbol{x}), \ldots, q_r(\boldsymbol{x})\}$ and

$$\boldsymbol{Q}_r = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r].$$

By (10), (13) can be rewritten as

$$\left(\sum_{i=0}^{m} \boldsymbol{A}_i^{(r)} \xi_i - \sum_{i=0}^{m} \sum_{j=0}^{m} \boldsymbol{B}_{ij}^{(r)} \xi_i \xi_j \right) \boldsymbol{u}_{\boldsymbol{\xi}}^{(r)} = \boldsymbol{f}^{(r)}.$$

where

$$\boldsymbol{A}_i^{(r)} = \boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{A}_i \boldsymbol{Q}_r, \ \boldsymbol{B}_{ij}^{(r)} = \boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{B}_{ij} \boldsymbol{Q}_r, \ \boldsymbol{f}^{(r)} = \boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{f}.$$

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Once the reduced matrices $\boldsymbol{A}_i^{(r)}$, $\boldsymbol{B}_{ij}^{(r)}$ and the reduced vector $\boldsymbol{f}^{(r)}$ are computed, the linear system for each $\boldsymbol{\xi} \in \Gamma$ can be assembled with cost $O(r^2)$. Note that $\boldsymbol{u}_{\boldsymbol{\xi}}^{(r)}$ is the coefficient vector of the basis $\{q_1(\boldsymbol{x}), \ldots, q_r(\boldsymbol{x})\}$, and the coefficient vector with respect to the basis $\{v_1(\boldsymbol{x}), \ldots, v_{N_h}(\boldsymbol{x})\}$ is given by $\boldsymbol{Q}_r \boldsymbol{u}_{\boldsymbol{\xi}}^{(r)}$.

Typically, the reduced space is constructed via greedy algorithm based on an indicator. In this study, the indicator is denoted $\Delta_n(\boldsymbol{\xi})$ for the n-dimension RBM space, and the greedy for constructing $\boldsymbol{Q}_r$ is given by Algorithm 1.

Algorithm 1 Greedy algorithm for construction of reduced basis space

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Input: a set of candidate parameters $\boldsymbol{\Xi}$, the RBM dimension r;
Randomly select $\boldsymbol{\xi}^{(1)} \in \boldsymbol{\Xi}$ as the first sample, and set $n = 1$;
Compute $\boldsymbol{u}_{\boldsymbol{\xi}^{(1)}}$ by solving (9) and set $\boldsymbol{Q}_1 = \boldsymbol{u}_{\boldsymbol{\xi}^{(1)}} / \|\boldsymbol{u}_{\boldsymbol{\xi}^{(1)}}\|_2$;
while $n < r$ **do**
    for each $\boldsymbol{\xi} \in \boldsymbol{\Xi}$ **do**
        Compute the indicator $\Delta_n(\boldsymbol{\xi})$;
    end for
    $\boldsymbol{\xi}^{(n+1)} = \arg \max_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \Delta_n(\boldsymbol{\xi})$;
    Compute the snapshots $\boldsymbol{u}_{\boldsymbol{\xi}^{(n+1)}}$ by solving (9);
    Compute $\boldsymbol{q}$, the orthogonal complement of $\boldsymbol{u}_{\boldsymbol{\xi}^{(n+1)}}$ with respect to $\boldsymbol{Q}_n$;
    Augment the reduced basis matrix $\boldsymbol{Q}_{n+1} = [\boldsymbol{Q}_n, \boldsymbol{q}]$ and set $n = n + 1$.
end while

The most commonly used indicator is the residual-based a posteriori error estimator, the readers can consult Refs. [7] for details. However, since only the most important samples and their corresponding solutions are needed in this work, we would like to apply a residual free indicator to generate the reduced basis space. We review the residual free indicator following the presentation in [8].

Recall that the reduced solution $u_{hp}^{(r)}(\boldsymbol{x}, \boldsymbol{\xi}) \in \text{span}\{u_h(\boldsymbol{x}, \boldsymbol{\xi}^{(1)}), \ldots, u_h(\boldsymbol{x}, \boldsymbol{\xi}^{(r)})\}$, thus we have

$$u_{hp}^{(r)}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{r} u(\boldsymbol{x}, \boldsymbol{\xi}^{(i)}) l_i(\boldsymbol{\xi}).$$

The expansion coefficients $l_i(\boldsymbol{\xi})$ are functions of $\boldsymbol{\xi}$, and play the role of basis functions. Moreover, they satisfy

$$l_i(\boldsymbol{\xi}^{(j)}) = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta. This means that the coefficient functions $l_j(\boldsymbol{\xi})$ are cardinal Lagrange interpolants associated to the space of functions defined by their span. We define the residual free indicator by

$$\Delta_n(\boldsymbol{\xi}) = \sum_{i=1}^{n} |l_i(\boldsymbol{\xi})|.$$

The above function is actually the norm of an interpolation operator and referred as Lebesgue function in interpolation theory. It was shown that the Lebesgue function matches the behavior of the residual based a posteriori error estimator, and therefore very useful in selecting the snapshots in RBM. For more details about the residual free indicator, the readers can consult Refs. [8].

3.2. The reduced basis Galerkin method

Suppose Q_r is a reduced basis space generated by the RBM described in Section 3, we can find a reduced basis approximate solution $u_{hp}^{(r)}(\boldsymbol{x}, \boldsymbol{\xi}) \in Q_r \otimes S_p$ such that

$$\int_\Gamma \int_D \rho(\boldsymbol{\xi}) \left[a \nabla u_{hp}^{(r)} \cdot \nabla w^* - \kappa^2 u_{hp}^{(r)} w^* \right] \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\xi} = \int_\Gamma \int_D \rho(\boldsymbol{\xi}) f w^* \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\xi}, \ \forall w \in Q_r \otimes S_p.$$

Assume

$$u_{hp}^{(r)}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{j=1}^{N_p} \sum_{s=1}^{r} u_{js}^{(r)} q_s(\boldsymbol{x}) \Phi_j(\boldsymbol{\xi})$$

and note (12), the coefficients of $u_{hp}^{(r)}(\boldsymbol{x}, \boldsymbol{\xi})$, i.e., $u_{js}^{(r)} = \boldsymbol{U}^{(r)}(s, j)$, where

$$\boldsymbol{u}^{(r)} = \mathrm{vec}(\boldsymbol{U}^{(r)}), \ \boldsymbol{U}^{(r)} \in \mathbb{R}^{r \times N_p},$$

can be solved by

$$\left(\sum_{i=0}^{m} \boldsymbol{G}_{i0} \otimes \boldsymbol{A}_i^{(r)} - \sum_{i=0}^{m} \sum_{j=0}^{m} \boldsymbol{G}_{ij} \otimes \boldsymbol{B}_{ij}^{(r)} \right) \boldsymbol{u}^{(r)} = \boldsymbol{f}^{(r)}, \tag{14}$$

where

$$\boldsymbol{A}_i^{(r)} = \boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{A}_i \boldsymbol{Q}_r, \ \boldsymbol{B}_{ij}^{(r)} = \boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{B}_{ij} \boldsymbol{Q}_r, \ \boldsymbol{f}^{(r)} = \boldsymbol{Q}_r^{\mathrm{T}} \boldsymbol{f}.$$

Since $Q_r \subset V$, the reduced basis approximate solution $u_{hp}^{(r)}(\boldsymbol{x}, \boldsymbol{\xi})$ can be expanded as

$$u_{hp}^{(r)}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{j=1}^{N_p} \sum_{s=1}^{N_h} u_{js} v_s(\boldsymbol{x}) \Phi_j(\boldsymbol{\xi}),$$

where the coefficients are given by

$$u_{js} = \boldsymbol{U}(s, j), \ \boldsymbol{U} = \boldsymbol{Q}_r \boldsymbol{U}^{(r)}. \tag{15}$$

Note that the equation (14) has a same form, but much smaller size compared with (4), thus it can be solved much more efficient.

Now, let us consider the residual, i.e., $\|\boldsymbol{b} - \boldsymbol{A} \boldsymbol{u}^{(r)}\|_2$. By (5), (7) and (15), we have

$$\|\boldsymbol{b} - \boldsymbol{A} \boldsymbol{u}^{(r)}\|_2 = \left\| \boldsymbol{b} - \mathrm{vec} \left(\sum_{i=0}^{m} (\boldsymbol{A}_i \boldsymbol{Q}_r)(\boldsymbol{U}^{(r)} \boldsymbol{G}_{i0}^{\mathrm{T}}) + \sum_{i=0}^{m} \sum_{j=0}^{m} (\boldsymbol{B}_{ij} \boldsymbol{Q}_r)(\boldsymbol{U}^{(r)} \boldsymbol{G}_{ij}^{\mathrm{T}}) \right) \right\|_2 \tag{16}$$

Note that $\boldsymbol{A}_i$, $\boldsymbol{B}_{ij}$ and $\boldsymbol{G}_{ij}$ are typically sparse matrices, $\boldsymbol{Q}_r \in R^{N_h \times r}$, $\boldsymbol{U}^{(r)} \in R^{r \times N_p}$, the cost for (16) is given by

$$O\left((m+1)(m+2)[N_h r + N_p r + N_p N_h r]\right) \approx O(m^2 N_h N_p r).$$

Based on the above descriptions, we propose a reduced basis stochastic Galerkin method, which is referred as RBSGM in the following. Since we do not know the number of reduced basis r are needed, the reduced basis space is constructed adaptively in the RBSGM. The relative residual

$$\mathtt{res} = \|\boldsymbol{b} - \boldsymbol{A} \boldsymbol{u}^{(r)}\|_2 / \|\boldsymbol{b}\|_2 \tag{17}$$

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can be regarded as a function of the number of reduced basis r. In order to find r such that

$$\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{u}^{(r)}\|_2/\|\boldsymbol{b}\|_2 \leq \texttt{TOL}, \tag{18}$$

where TOL is the desired tolerance, we define a function $h(r)$ as the logarithmic (base 10) scale of the relative residual res, i.e.,

$$h(r) = \lg(\texttt{res}), \ r = 1, 2, \ldots,$$

Obviously, (18) is equivalent to

$$h(r) \leq \lg(\texttt{TOL}). \tag{19}$$

In order to reduce the times of residual computation (i.e., times of (16) need to be computed), the r satisfying (19) should be found in as few steps as possible. In this work, we use the secant method to find r satisfying (19). We first compute $h(r_1)$ and $h(r_2)$, and the new r is given by

$$r = r_1 + \frac{r_2 - r_1}{h(r_2) - h(r_1)} \left[\lg(\texttt{TOL}) - h(r_1)\right]. \tag{20}$$

The reduced basis stochastic Galerkin method is summarized in Algorithm 2.

Algorithm 2 The reduced basis stochastic Galerkin method

Input: The matrices $\boldsymbol{G}_{ij}$, $\boldsymbol{A}_i$, $\boldsymbol{B}_{ij}$ and vectors $\boldsymbol{h}$, $\boldsymbol{f}$, a set of candidate parameters $\boldsymbol{\Xi}$, the RBM dimension ns in each stage, the maximum RBM dimension nmax in total, the tolerance of the iterative method TOL;

Randomly select $\boldsymbol{\xi}^{(1)} \in \boldsymbol{\Xi}$ as the first sample, set $n = 1$, and set $\boldsymbol{Q}_1 = \boldsymbol{u}_{\boldsymbol{\xi}^{(1)}}/\|\boldsymbol{u}_{\boldsymbol{\xi}^{(1)}}\|_2$;
Solve (14), compute the relative residual res via (16)–(17), and set $\texttt{st} = 1$, $r_1 = 1$;
while $n < \texttt{nmax}$ & $\texttt{res} > \texttt{TOL}$ **do**
   Set $k = 0$;
   while $n < \texttt{nmax}$ & $k < \texttt{st} \cdot \texttt{ns}$ **do**
     for each $\boldsymbol{\xi} \in \boldsymbol{\Xi}$ **do**
       Compute the indicator $\Delta_n(\boldsymbol{\xi})$;
     end for
     $\boldsymbol{\xi}^{(n+1)} = \arg\max_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \Delta_n(\boldsymbol{\xi})$;
     Compute the snapshots $\boldsymbol{u}_{\boldsymbol{\xi}^{(n+1)}}$ by solving (9);
     Compute $\boldsymbol{q}$, the orthogonal complement of $\boldsymbol{u}_{\boldsymbol{\xi}^{(n+1)}}$ with respect to $\boldsymbol{Q}_n$;
     Augment the reduced basis matrices

$$\boldsymbol{Q}_{n+1} = [\boldsymbol{Q}_n, \boldsymbol{q}], \quad \boldsymbol{Q}_{n+1}^{\mathrm{T}}\boldsymbol{f} = [\boldsymbol{Q}_n^{\mathrm{T}}\boldsymbol{f}; \boldsymbol{q}^{\mathrm{T}}\boldsymbol{f}],$$
$$\boldsymbol{A}_i\boldsymbol{Q}_{n+1} = [\boldsymbol{A}_i\boldsymbol{Q}_n, \boldsymbol{A}_i\boldsymbol{q}], \quad \boldsymbol{B}_{ij}\boldsymbol{Q}_{n+1} = [\boldsymbol{B}_{ij}\boldsymbol{Q}_n, \boldsymbol{B}_{ij}\boldsymbol{q}],$$
$$\boldsymbol{Q}_{n+1}^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{Q}_{n+1} = \begin{bmatrix} \boldsymbol{Q}_n^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{Q}_n & \boldsymbol{Q}_n^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{q} \\ \boldsymbol{q}^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{Q}_n & \boldsymbol{q}^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{q} \end{bmatrix}, \quad \boldsymbol{Q}_{n+1}^{\mathrm{T}}\boldsymbol{B}_{ij}\boldsymbol{Q}_{n+1} = \begin{bmatrix} \boldsymbol{Q}_n^{\mathrm{T}}\boldsymbol{B}_{ij}\boldsymbol{Q}_n & \boldsymbol{Q}_n^{\mathrm{T}}\boldsymbol{B}_{ij}\boldsymbol{q} \\ \boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_{ij}\boldsymbol{Q}_n & \boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_{ij}\boldsymbol{q} \end{bmatrix};$$

     Set $n = n + 1$, $k = k + 1$;
   end while
   Solve (14), and compute the relative residual res via (16)–(17);
   Set $r_2 = n$, predict r via (20), set $r_1 = r_2$, and set $\texttt{st} = \text{floor}\,((r - r_2)/\texttt{ns}) + 1$.
end while
return $\boldsymbol{u} = \texttt{vec}\left(\boldsymbol{Q}_n\boldsymbol{U}^{(n)}\right)$.

---

4. Numerical study

In this section, we consider two problems, one is a diffusion problem, and the other is a Helmholtz problem. In both problems, the approximation obtained by SGM with gPC order $p = 6$ is acting as the reference solution. In test problem 1, we use the standard finite element method to generate the matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_{ij}$, where $i, j \in \{0, 1, \ldots, m\}$. However, in test problem 2, the codes associated with [35] is applied for convenient. All results in this study are obtained in MATLAB R2015b on a desktop with 2.90GHz Intel Core i7-10700 CPU.

4.1. Test problem 1

In this test problem, we consider the diffusion equation:

$$\begin{cases} -\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{\xi})\nabla u) = 1, & \forall (\boldsymbol{x}, \boldsymbol{\xi}) \in D \times \Gamma \\ u = 0, & \forall (\boldsymbol{x}, \boldsymbol{\xi}) \in \partial D \times \Gamma \end{cases},$$

where $D = [-1, 1]^2$ and

$$a(\boldsymbol{x}, \boldsymbol{\xi}) = \mu + \sigma \sum_{i=1}^{m} \sqrt{\lambda_i} a_i(\boldsymbol{x}) \xi_i$$

is a truncated KL expansion of random field with mean function $\mu = 0.2$, standard deviation $\sigma = 0.1$ and covariance function

$$\mathrm{cov}(\boldsymbol{x}, \boldsymbol{y}) = \sigma^2 \exp\left(-|x_1 - y_1| - |x_2 - y_2|\right).$$

The random variables ξ_i are chosen to be identically independent distributed uniform random variables on $[-1, 1]$.

Table 1: CPU time for different m and N_h with $\texttt{ns} = 15$ and $p = 5$.

N_h \ m	$m = 5$	$m = 7$	$m = 10$	TOL
33^2	4.67 (0.42)	12.57 (1.70)	40.89 (9.05)	10^{-4}
65^2	5.97 (2.41)	15.39 (10.74)	52.79 (56.38)	10^{-4}
129^2	11.63 (19.14)	31.12 (93.41)	76.18 (417.95)	10^{-4}
33^2	13.11 (0.58)	28.55 (2.52)	75.73 (12.40)	10^{-5}
65^2	15.72 (3.30)	37.31 (13.61)	98.76 (77.58)	10^{-5}
129^2	25.49 (26.82)	63.03 (120.27)	162.88 (580.99)	10^{-5}

In Table 1, we give the CPU time of RBSGM for different m and N_h, the gPC order is 5, the number of reduced basis in each stage is 15, and the number of the candidate parameters is 500. We also give the CPU time of SGM for the same tolerance in the brackets. From the table, it is clear that the RBSGM is more efficient than SGM when the dimension of the stochastic space is high or when the grid in the physical space is sufficiently fine. All the stochastic Galerkin equation is solved by the preconditioned conjugate gradients method (PCG) with mean based preconditioner.

In Table 2, we give the CPU time for generating the reduced basis space and their percentages of the total CPU time are given in the brackets. It is clear that, in RBSGM, the most CPU time is
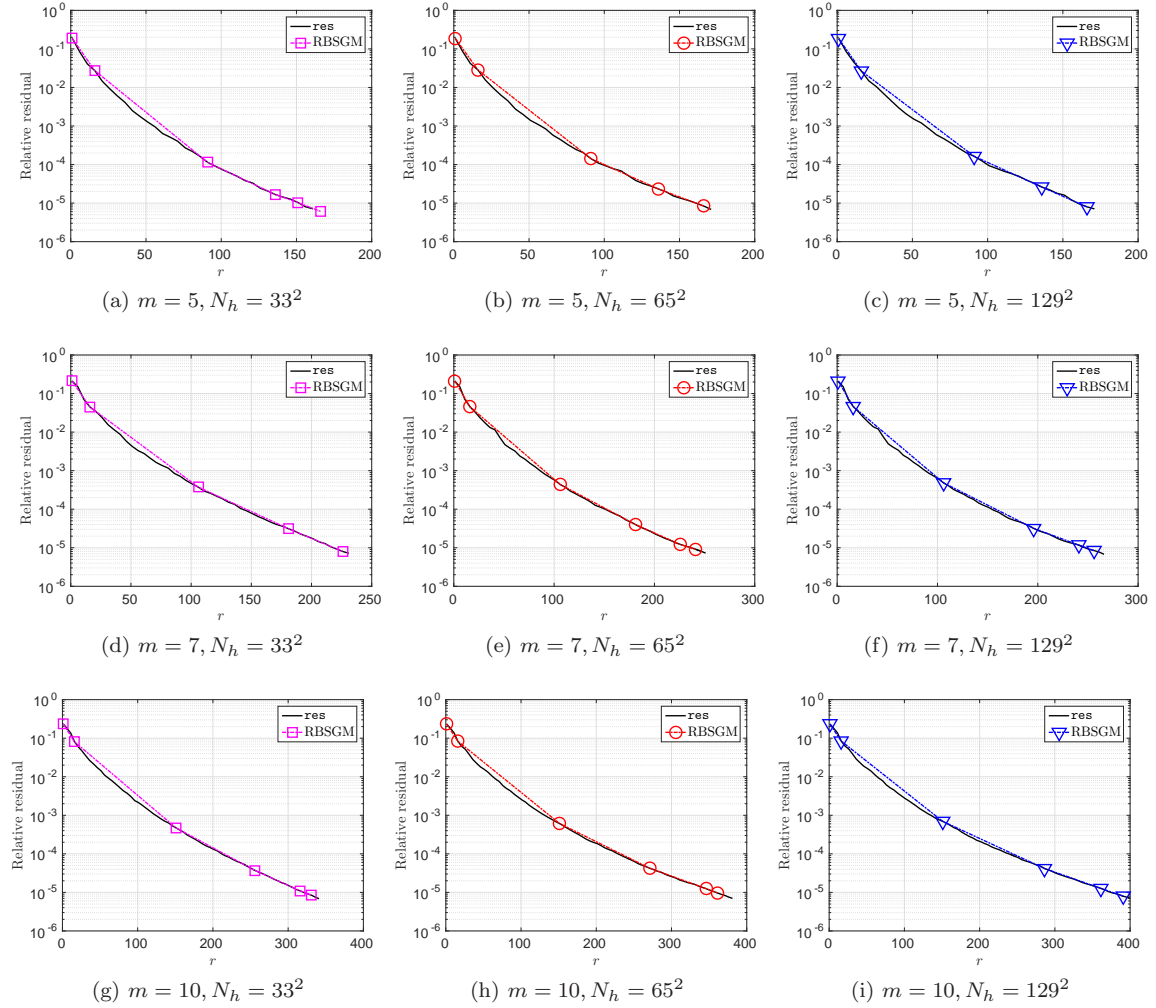
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(a) $m = 5, N_h = 33^2$

(b) $m = 5, N_h = 65^2$

(c) $m = 5, N_h = 129^2$

(d) $m = 7, N_h = 33^2$

(e) $m = 7, N_h = 65^2$

(f) $m = 7, N_h = 129^2$

(g) $m = 10, N_h = 33^2$

(h) $m = 10, N_h = 65^2$

(i) $m = 10, N_h = 129^2$

Figure 1: Relative residual with respect to the number of reduced basis r.

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Table 2: CPU time for generating the reduced basis space and their percentage of the total CPU time.

N_h \ m	$m = 5$	$m = 7$	$m = 10$	TOL
33^2	4.53 (97)	11.74 (93)	30.07 (74)	10^{-4}
65^2	5.83 (98)	14.25 (93)	39.23 (74)	10^{-4}
129^2	11.19 (96)	28.70 (92)	60.51 (79)	10^{-4}
33^2	12.82 (98)	27.18 (95)	57.79 (76)	10^{-5}
65^2	15.46 (98)	35.10 (94)	76.20 (77)	10^{-5}
129^2	24.94 (98)	59.07 (94)	129.78 (80)	10^{-5}

cost by the computation of the reduced basis space, and thus the main bottleneck of the proposed method is the efficiency of generating the reduced basis space.

In Figure 1, we show the relative residual with respect to the number of reduced basis and the computed relative residual in the RBSGM. All the full linear systems (4) and the reduced linear systems (14) are solved by the preconditioned conjugate gradients method (PCG) with mean based preconditioner. From the figure, we can see that the number of reduced basis r is much smaller than the DOF of finite element method. Moreover, we can see that r depends on TOL but almost independent of N_h. In the RBSGM, the relative residual only need to be computed 5 or 6 times for TOL $= 10^{-5}$.

To access the accuracy of reduced basis Galerkin approach, we consider the relative errors of mean and variance function, which are defined through

$$\text{err}_\text{m} = \frac{\|\mathbb{E}[u_{hp}^{(r)}] - \mathbb{E}[u_\text{ref}]\|_{L^2}}{\|\mathbb{E}[u_\text{ref}]\|_{L^2}}, \ \text{err}_\text{V} = \frac{\|\mathbb{V}[u_{hp}^{(r)}] - \mathbb{V}[u_\text{ref}]\|_{L^2}}{\|\mathbb{V}[u_\text{ref}]\|_{L^2}}, \tag{21}$$

where u_ref is the reference solution. In the same way, we access the errors of the SGM through (21) with $u_{hp}^{(r)}$ replaced by u_{hp}. In this test problem, we use the approximation $u_{hp}(\boldsymbol{x}, \boldsymbol{\xi})$ obtained by the stochastic Galerkin method with gPC order $p = 6$ as the reference solution. The reference solution is solved by the preconditioned conjugate gradients method (PCG) with mean based preconditioner and tolerance 10^{-7}.

Table 3: Relative errors of mean and variance function.

	N_h \ m	$m = 5$	$m = 7$	$m = 10$	TOL
	33^2	4.4e-07 (7.8e-07)	1.5e-06 (7.3e-06)	4.0e-06 (2.3e-06)	10^{-4}
err_m	65^2	4.4e-07 (7.7e-07)	1.5e-06 (1.5e-06)	4.0e-06 (2.3e-06)	10^{-4}
	129^2	4.4e-07 (7.7e-07)	1.5e-06 (1.5e-06)	4.0e-06 (2.3e-06)	10^{-4}
	33^2	4.4e-05 (4.9e-05)	1.4e-04 (1.4e-04)	3.1e-04 (3.1e-04)	10^{-4}
err_V	65^2	4.4e-05 (4.8e-05)	1.4e-04 (1.5e-04)	3.1e-04 (3.1e-04)	10^{-4}
	129^2	4.4e-05 (4.8e-05)	1.4e-04 (1.5e-04)	3.1e-04 (3.1e-04)	10^{-4}

Table 3 shows the relative errors of mean and variance function for RBSGM. In the brackets, we

also give the relative errors of SGM for comparison. It can be seen from the table that the relative errors of mean and variance function for RBSGM are similar as that of the SGM.

4.2. Test problem 2

In this test problem, we consider a stochastic Helmholtz equation

$$\nabla^2 u + \kappa^2(\boldsymbol{x}, \boldsymbol{\xi})u = f(\boldsymbol{x}), \ \forall (\boldsymbol{x}, \boldsymbol{\xi}) \in D \times \Gamma,$$

with Sommerfeld radiation boundary condition. Let the domain of interest be $D = [0,1]^2$ and

$$\kappa(\boldsymbol{x}, \boldsymbol{\xi}) = \mu + \sigma \sum_{i=1}^{m} \sqrt{\lambda_i} \kappa_i(\boldsymbol{x}) \xi_i$$

is a truncated KL expansion of random field with mean function $\mu = 4 \cdot (2\pi)$, standard deviation $\sigma = 0.1\mu$, and covariance function

$$\text{cov}(\boldsymbol{x}, \boldsymbol{y}) = \sigma^2 \exp\left(-|x_1 - y_1|/4 - |x_2 - y_2|/4\right).$$

The random variables ξ_i are chosen to be identically independent distributed uniform random variables on $[-1, 1]$. The Gaussian point source at the center of the domain is served as the source term, i.e.,

$$f(\boldsymbol{x}) = e^{-(8\cdot4)^2((x_1-0.5)^2+(x_2-0.5)^2)}$$

In this example, we use the perfectly matched layers (PML) to simulate the Sommerfeld condition [36], and apply the codes associated with [35] to generate the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_{ij}$ and the snapshots in algorithm 1. Moreover, we use the approximation $u_{hp}(\boldsymbol{x}, \boldsymbol{\xi})$ obtained by the stochastic Galerkin method with gPC order $p = 6$ as the reference solution. The reference solution is solved by the bi-conjugate gradient stabilized method (Bi-CGSTAB) with mean based preconditioner and tolerance 10^{-7}.

Table 4: CPU time for different m and N_h with ns $= 10$ and $p = 5$.

N_h \ m	$m = 5$	$m = 7$	$m = 10$	TOL
33^2	4.94 (0.94)	14.84 (4.43)	56.48 (24.51)	10^{-4}
65^2	6.36 (6.01)	39.34 (25.88)	158.76 (189.03)	10^{-4}
129^2	17.97 (40.99)	69.45 (213.13)	417.23 (1123.50)	10^{-4}
33^2	11.24 (1.11)	43.15 (5.25)	150.57 (29.17)	10^{-5}
65^2	17.56 (6.96)	87.36 (29.82)	337.68 (246.95)	10^{-5}
129^2	38.19 (55.98)	162.84 (281.06)	695.23 (1468.82)	10^{-5}

In Table 4, we give the CPU time of RBSGM for different m and N_h, the gPC order is 5, the number of reduced basis in each stage is 10, and the number of the candidate parameters is 400. We also give the CPU time of SGM for the same tolerance in the brackets. From the table, it is clear that the RBSGM is more efficient than SGM when the dimension of the stochastic space is high or when the grid in the physical space is sufficiently fine.

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Table 5: CPU time for generating the reduced basis space and their percentage of the total CPU time.

N_h \ m	$m = 5$	$m = 7$	$m = 10$	TOL
33^2	4.49 (91)	11.75 (79)	28.97 (51)	10^{-4}
65^2	5.47 (86)	26.00 (66)	65.58 (41)	10^{-4}
129^2	12.89 (72)	37.86 (55)	122.01 (29)	10^{-4}
33^2	10.51 (94)	36.82 (85)	84.77 (56)	10^{-5}
65^2	16.00 (91)	67.04 (77)	158.35 (47)	10^{-5}
129^2	30.91 (81)	115.52 (71)	311.68 (45)	10^{-5}

In Table 5, we give the CPU time for generating the reduced basis space and their percentages of the total CPU time are given in the brackets. In this problem, since the non-zero matrices in (5) are $(m + 1)^2 + 1$, which is much bigger than that of test problem 1, the percentages of cost for generating the reduced basis space decrease, but still the main part of the proposed method.

In Figure 2, we show the relative residual with respect to the number of reduced basis and the computed relative residual in the RBSGM. All the full linear systems (4) and the reduced linear systems (14) are solved by the bi-conjugate gradient stabilized method (Bi-CGSTAB) with mean based preconditioner. From the figure, we can see that the number of reduced basis r is much smaller than the DOF of finite element method. Moreover, we can see that r depends on TOL but almost independent of N_h. In the RBSGM, the relative residual only need to be computed 5 or 6 times for TOL $= 10^{-5}$.

Table 6: Relative errors of mean and variance function.

	N_h \ m	$m = 5$	$m = 7$	$m = 10$	TOL
	33^2	4.1e-06 (2.5e-06)	7.7e-06 (3.0e-06)	8.0e-06 (3.1e-06)	10^{-4}
err_m	65^2	1.2e-05 (4.9e-06)	5.1e-06 (5.2e-06)	6.5e-06 (5.4e-06)	10^{-4}
	129^2	7.2e-06 (1.3e-05)	9.2e-06 (1.3e-05)	6.2e-06 (1.4e-05)	10^{-4}
	33^2	2.0e-05 (4.0e-05)	3.3e-05 (4.4e-05)	3.1e-05 (4.6e-05)	10^{-4}
err_V	65^2	4.6e-05 (1.3e-05)	1.8e-05 (1.4e-05)	2.0e-05 (1.6e-05)	10^{-4}
	129^2	2.1e-05 (2.2e-05)	2.3e-05 (2.1e-05)	1.6e-05 (2.3e-05)	10^{-4}

Table 6 shows the relative errors of mean and variance function for RBSGM. In the brackets, we also give the relative errors of SGM for comparison. It can be seen from the table that the relative errors of mean and variance function for RBSGM are similar as that of the SGM.

5. Conclusions

In this work, we develop a reduced basis stochastic Galerkin method for partial differential equations with random inputs. Compared with the standard stochastic Galerkin method, where the physical space is discretized by grid based approaches, our proposed method can very efficiently
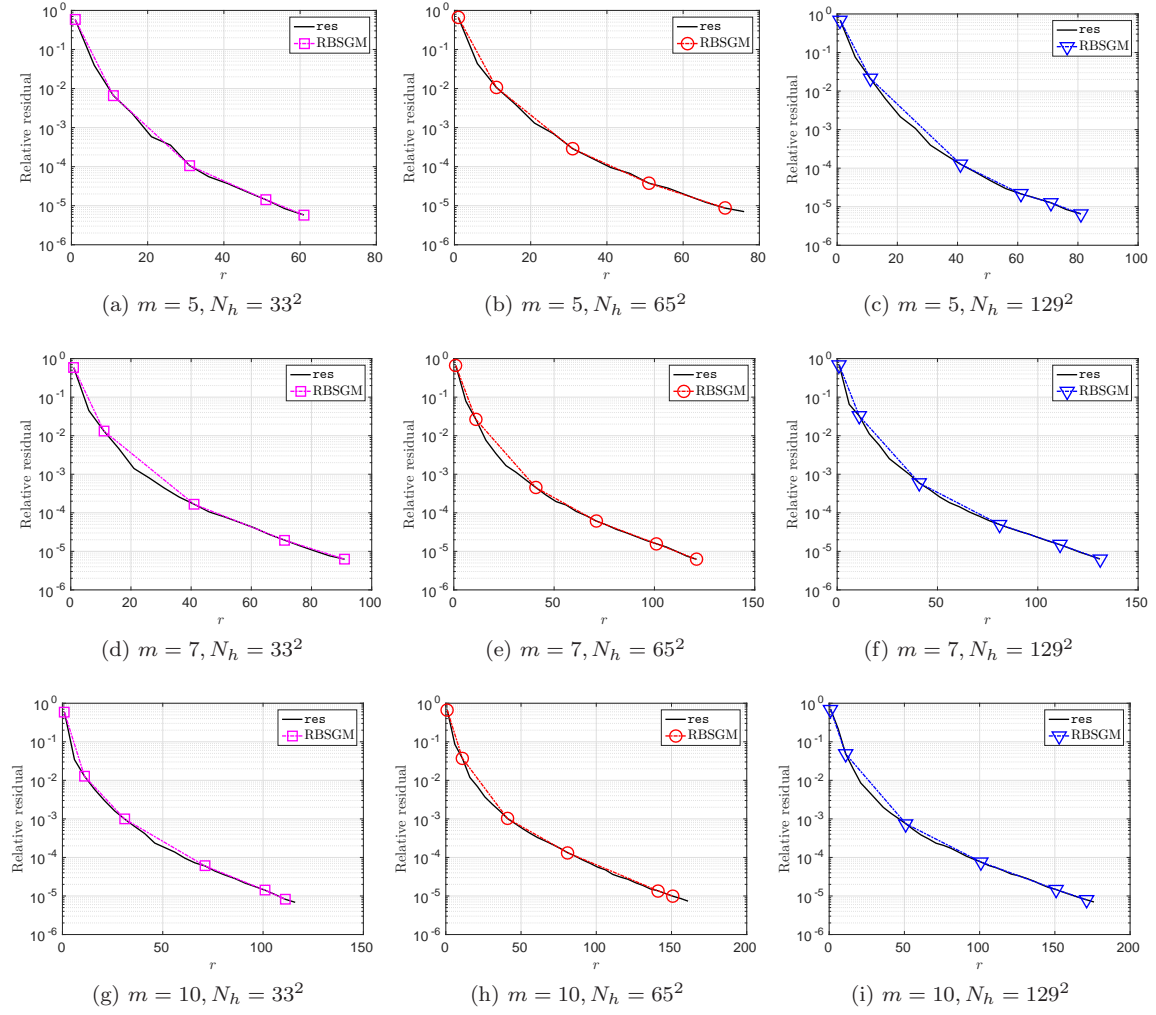
(a) $m = 5, N_h = 33^2$

(b) $m = 5, N_h = 65^2$

(c) $m = 5, N_h = 129^2$

(d) $m = 7, N_h = 33^2$

(e) $m = 7, N_h = 65^2$

(f) $m = 7, N_h = 129^2$

(g) $m = 10, N_h = 33^2$

(h) $m = 10, N_h = 65^2$

(i) $m = 10, N_h = 129^2$

Figure 2: Relative residual with respect to the number of reduced basis r.

compute the Galerkin solution, especially when the physical degrees of freedom are large. While we only consider how to efficiently identify reduced bases for the physical approximation in this work, we will focus on how to construct effective bases for the stochastic approximation in our future work.

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