Correspondence

The Weight Spectra of Some Short Low-Rate Convolutional Codes

JEAN CONAN

Abstract—**This paper reports the results of a computer analysis of the distance properties of some of the best known rate 1/2, 1/3, and 1/4 codes to constraint length 14. The data include the truncated weight distributions of the codewords belonging to the incorrect subset which specifies the performance of the Viterbi algorithm as well as the minimum asymptotic growth rate of the weights of unmerged codewords which has been conjectured to be related to the length of error events produced by Viterbi decoders.**

I. Introduction and Notation

It is now well established that convolutional codes are among the most powerful means known to combat random errors occuring during the transmission of digital information [1], [2]. However, it is quite impractical to evaluate exactly the performance of such systems even for the simplest of codes. Some upper bounds known to be asymptotically tight at high signal-to-noise ratios have been developed in [1] and recently refined in [3], [4]. These bounds make use of weight enumerating functions of the codewords whose first terms are important in computing the probabilities of occurrence of the most significant error events. This paper reports results of a computer analysis of these enumerators for some of the best rate 1/2, 1/3, and 1/4 convolutional codes as compiled by Odenwalder [5] and Larsen [6]. Furthermore, Hemmati and Costello [7] demonstrated the role played by the minimum asymptotic growth rate of the weights in the set of unmerged codewords. Since we believe this parameter to be significant, its value has also been computed for those codes.

Following Forney [8], we define a binary $(n, 1)$ convolutional code of rate $R = 1/n$ and memory order m as the output of some feedforward one-input/n-output linear sequential machine of minimal state space cardinality 2^m. Given the n binary generating sequences or generators of the encoders, $g_j \triangleq (g_0{}^j, g_1{}^j, \cdots, g_m{}^j)$, $j = 1, \cdots, n$, the jth output can be computed from the input x_u as

$$y_t{}^j = \sum_{l=0}^{m} x_{t-l} g_l{}^j, \qquad j = 1, \cdots, n. \tag{1}$$

The state of the convolutional encoder at time t is defined as the m-tuple

$$s_t \triangleq (x_{t-1}, x_{t-2}, \cdots, x_{t-m}) \tag{2}$$

and clearly uniquely specifies, together with the input at time t, the corresponding n output symbols. A concise description of a code is contained in the associated Good–DeBruijn graph Γ. Such a directed graph $\Gamma \triangleq (S, U)$ with S the state space can be constructed by defining U as the set of arcs linking the elements of S after a one-step transition. Each element $u \in U$, hereafter referred to as a branch, corresponds to the transition between some pair of states (s_1, s_2) and will be assigned, as a label, the pair (x/y) where x and y represent, respectively, the input symbol and output n-tuple corresponding to the state transition $s_1 \rightarrow s_2$. If $w_H(z)$ represents the Hamming weight of the binary vector z, the symbolic label of the transition (s_1, s_2) will be denoted as $(W^{w_H(y)}, N^{w_H(x)})$. The Good–DeBruijn graph of Fig. 1 illustrates the concepts for a (2.1) convolutional code described by the minimal encoder with generators $g_1 = 1111_2 = 17_8, g_2 = 1101_2 = 15_8$.

II. Weight Enumerators of Convolutional Codes

A key role in the evaluation of the error performance is played by the weight distribution of the codewords belonging to the incorrect subset, also called the weight spectrum. Such a distribution is formed from the paths of Γ stemming from the null state s_0 on a nonzero output branch and reconverging for the first time to s_0 at some later time.

The spectrum is conveniently described by the pair of functions

$$\sum_{l=d_f}^{+\infty} a_l W^l, \quad \sum_{l=d_f}^{+\infty} b_l W^l$$

where

$$\begin{cases} a_l \triangleq \text{number of finite length codewords of Hamming weight } l \\ b_l \triangleq \text{number of input 1's in all the finite length codewords of Hamming weight } l. \end{cases} \tag{3}$$

These function are related to the function $T(W, N)$ through the relations

$$\sum_{l=d_f}^{+\infty} a_l W^l = T(W, N)\Big|_{N=1}, \quad \sum_{l=d_f}^{+\infty} b_l W^l = \frac{dT(W, N)}{dN}\Big|_{N=1} \tag{4}$$

where formal evaluation of $T(W, N)$ can be carried out, in principle, by introducing, similarly to the approach of Hankamer [9], the one-step symbolic transition matrix F of Γ whose elements $F_{ij}, i, j = 1, 2, \cdots, 2^m$, are defined as

$$\begin{cases} F_{ij} \triangleq \text{symbolic label associated with the arc of } \Gamma \text{ linking the states } i \text{ and } j \text{ with } i \text{ and } j \text{ the binary } m\text{-tuples whose decimal values are } i-1 \text{ and } j-1, \text{ respectively,} \\ F_{ij} = 0 \qquad \text{if } i = j = 1,^1 \\ F_{ij} = 0 \qquad \text{otherwise} \end{cases} \tag{5}$$

[1] Through this condition the loop around the null state of Γ is removed.

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The author is with the Department of Electrical Engineering, Ecole Polytechnique de Montréal, Montreal, P. Q., Canada H3C 3A7.

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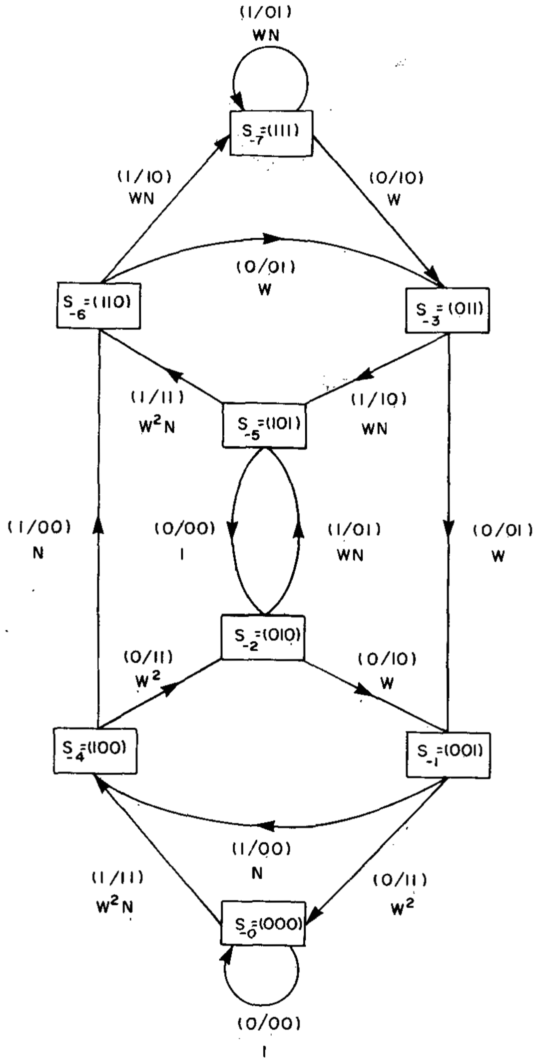

Fig. 1. Good-DeBruijn graph for the (2,1) convolutional code associated with the encoder specified by the generators $g_1 = 17_8$, $g_2 = 15_8$.

and completing the calculation (with I the unit square matrix)

$$T(W,N) = 1 - \frac{1}{[I - F]}. \tag{6}$$

To illustrate the procedure, the F matrix corresponding to the code of Fig. 1 is

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & W^2N & 0 & 0 & 0 \\ W^2 & 0 & 0 & 0 & N & 0 & 0 & 0 \\ 0 & W & 0 & 0 & 0 & WN & 0 & 0 \\ 0 & W & 0 & 0 & 0 & WN & 0 & 0 \\ 0 & 0 & W^2 & 0 & 0 & 0 & N & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W^2N & 0 \\ 0 & 0 & 0 & W & 0 & 0 & 0 & WN \\ 0 & 0 & 0 & W & 0 & 0 & 0 & WN \end{bmatrix} \tag{7}$$

while relation (6) yields after some tedious manipulations

$$T(W,N) = \frac{W^6N(N + W - W^2N)}{(1 - 2WN - W^3N)}$$

As a consequence

$$\sum_{l=d_f}^{+\infty} a_l W^l = \frac{W^6(1 + W - W^2)}{(1 - 2W - W^3)}$$
$$= W^6 + 3W^7 + 5W^8 + 11W^9 + 25W^{10} + 55W^{11} + 121W^{12} + 267W^{13} + \cdots$$

$$\sum_{l=d_f}^{+\infty} b_l W^l = \frac{W^6(2 - W - 2W^2 + W^2 + W^5)}{(1 - 2W - W^3)^2}$$
$$= 2W^6 + 7W^7 + 18W^8 + 49W^9 + 130W^{10} + 333W^{11} + 836W^{12} + \cdots.$$

It should be clear that even for the simplest codes, the determination of the weight spectrum from the transfer func-

TABLE I

WEIGHT SPECTRA AND MINIMUM ASYMPTOTIC RATE OF GROWTH OF THE WEIGHTS IN THE INCORRECT SUBSET FOR THE BEST (2,1) CONVOLUTIONAL CODES UP TO CONSTRAINT LENGTH 14

m	generators (octal)	d_f	$\dfrac{(a_{d_f+\ell})}{[b_{d_f+\ell}]}$, $\ell = 0, 1, \ldots, 17$	d_0
2	* 5,7	5	(1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072) [1, 4, 12, 32, 80, 192, 448, 1024, 2304, 5120, 11264, 24576, 53248, 114688, 245760, 524288, 1114112, 2359296]	1/2
3	* 15,17	6	(1, 3, 5, 11, 25, 55, 121, 267, 589, 1299, 2865, 6319, 13937, 30739, 67797, 149531, 329801, 727399) [2, 7, 18, 49, 130, 333, 836, 2069, 5060, 12255, 29444, 70267, 166726, 393635, 925334, 2166925, 5057286, 11767305]	1/2
4	* 23,35	7	(2, 3, 4, 16, 37, 68, 176, 432, 925, 2156, 5153, 11696, 26868, 62885, 145085, 334024, 774966, 1793363) [4, 12, 20, 72, 225, 500, 1324, 3680, 8967, 22270, 57403, 142234, 348830, 867106, 2134239, 5205290, 12724352, 31022962]	4/11
5	* 53,75	8	(1, 8, 7, 12, 48, 95, 281, 605, 1272, 3334, 7615, 18131, 43197, 99210, 237248, 559238, 1312675, 3108350) [2, 36, 32, 62, 332, 701, 2342, 5503, 12506, 36234, 88576, 225685, 574994, 1400192, 3554210, 8845154, 21841106, 54350946]	8/23
6	* 133, 171	10	(11, 0, 38, 0, 193, 0, 1331, 0, 7275, 0, 40406, 0, 234969, 0, 1337714, 0, 7594819, 0) [36, 0, 211, 0, 1404, 0, 11633, 0, 77433, 0, 502690, 0, 3322763, 0, 21292910, 0, 134365911, 0]	4/13
7	* 247, 371	10	(1, 6, 12, 26, 52, 132, 317, 730, 1823, 4446, 10739, 25358, 60773, 146396, 350399, 842174, 2021290, 4853474) [2, 22, 60, 148, 340, 1008, 2642, 6748, 18312, 48478, 126364, 320062, 821350, 2102864, 5335734, 13549068, 34254388, 86441848]	5/18
8	* 561, 753	12	(11, 0, 50, 0, 286, 0, 1630, 0, 9639, 0, 55152, 0, 320782, 0, 1859184, 0, 10777264, 0) [33, 0, 281, 0, 2179, 0, 15035, 0, 105166, 0, 692330, 0, 4580007, 0, 29692894, 0, 190453145, 0]	8/27
9	1167, 1545	12	(2, 8, 15, 35, 68, 170, 458, 1084, 2574, 6177, 14939, 36200, 86856, 208847, 504561, 1217706, 2933502, 7066863) [14, 26, 74, 257, 496, 1378, 4122, 10832, 27988, 72209, 186920, 483102, 1234736, 3149395, 8033048, 20419644, 51688436, 130527021]	1/4
10	2335, 3661	14	(21, 0, 74, 0, 454, 0, 2687, 0, 15629, 0, 90518, 0, 526556, 0, 3067758, 0, 17845415, 0) [94, 0, 463, 0, 3783, 0, 26711, 0, 181571, 0, 1207474, 0, 7919894, 0, 51390913, 0, 329342619, 0]	2/15
11	4335, 5723	15	(16, 31, 44, 129, 309, 697, 1713, 4175, 10158, 24508, 58600, 141960, 343347, 826478, 1996843, 4820534, 11619637, 28039590) [76, 180, 374, 1142, 2783, 6836, 18709, 49242, 128178, 329408, 836478, 2151230, 5497355, 13931276, 35357451, 89485786, 225656685, 568414202]	14/53
12	10533, 17661	16	(33, 0, 111, 0, 779, 0, 4128, 0, 24173, 0, 142500, 0, 828402, 0, 4829478, 0, 28122349, 0) [152, 0, 971, 0, 6933, 0, 45436, 0, 303435, 0, 2036131, 0, 13256560, 0, 85514159, 0, 546034284, 0]	8/33
13	21675, 27123	16	(4, 17, 35, 76, 193, 454, 1047, 2624, 6138, 14944, 36179, 86640, 210568, 508233, 1225765, 2960696, 7146740, 17245991) [22, 99, 218, 608, 1724, 4404, 11108, 30438, 75942, 196714, 507232, 1289364, 3311290, 8425785, 21377872, 54168142, 136847122, 344912207]	27/103

*Partial weight spectra for these codes were originally found by Odenwalder.

TABLE II

WEIGHT SPECTRA AND MINIMUM ASYMPTOTIC RATE OF GROWTH OF THE WEIGHTS IN THE INCORRECT SUBSET FOR THE BEST (3,1) CONVOLUTIONAL CODES UP TO CONSTRAINT LENGTH 14

m	generators (octal)	d_f	$\dfrac{(a_{d_f+\ell})}{[b_{d_f+\ell}]}$, $\ell = 0, 1, \ldots, 17$	d_0
2	* 5,7,7	8	(2, 0, 5, 0, 13, 0, 34, 0, 89, 0, 233, 0, 610, 0, 1597, 0, 4181, 0) [3, 0, 15, 0, 58, 0, 201, 0, 655, 0, 2052, 0, 6255, 0, 18687, 0, 54974, 0]	2/3
3	* 13, 15, 17	10	(3, 0, 2, 0, 15, 0, 24, 0, 87, 0, 188, 0, 557, 0, 1354, 0, 3713, 0) [6, 0, 6, 0, 58, 0, 118, 0, 507, 0, 1284, 0, 4323, 0, 11846, 0, 36009, 0]	4/5
4	* 25, 33, 37	12	(5, 0, 3, 0, 13, 0, 62, 0, 108, 0, 328, 0, 1051, 0, 2544, 0, 7197, 0) [12, 0, 12, 0, 56, 0, 320, 0, 693, 0, 2324, 0, 8380, 0, 23009, 0, 71016, 0]	2/3
5	* 47, 53, 75	13	(1, 3, 6, 4, 5, 12, 14, 33, 66, 106, 179, 317, 513, 766, 1297, 2251, 3964, 6721) [1, 8, 26, 20, 19, 62, 86, 204, 420, 710, 1345, 2606, 4343, 6790, 12305, 22356, 41090, 72820]	8/13
6	+ 133, 145, 175	15	(3, 5, 5, 6, 11, 15, 25, 54, 92, 164, 274, 450, 758, 1290, 2142, 3567, 6089, 10403) [11, 16, 19, 28, 55, 96, 169, 338, 636, 1276, 2172, 3628, 6580, 12048, 20820, 36358, 65009, 115368]	2/3
7	* 225, 331, 367	16	(1, 0, 8, 0, 24, 0, 51, 0, 133, 0, 405, 0, 1129, 0, 3532, 0, 9754, 0) [1, 0, 24, 0, 113, 0, 287, 0, 898, 0, 3020, 0, 9436, 0, 32644, 0, 98472, 0]	24/37
8	557, 663, 711	18	(5, 0, 7, 0, 36, 0, 85, 0, 204, 0, 636, 0, 1927, 0, 5416, 0, 15769, 0) [11, 0, 32, 0, 195, 0, 564, 0, 1473, 0, 5129, 0, 17434, 0, 54092, 0, 171117, 0]	10/17
9	1117, 1365, 1633	20	(8, 0, 18, 0, 41, 0, 132, 0, 395, 0, 981, 0, 2991, 0, 8843, 0, 25590, 0) [29, 0, 91, 0, 246, 0, 954, 0, 3138, 0, 8775, 0, 29185, 0, 94164, 0, 295578, 0]	5/9
10	2353, 2671, 3175	22	(14, 0, 18, 0, 59, 0, 160, 0, 463, 0, 1458, 0, 3971, 0, 115780, 34023, 0) [53, 0, 92, 0, 347, 0, 1104, 0, 3644, 0, 12692, 0, 38407, 0, 122297, 0, 389889, 0]	14/29
11	4767, 5723, 6265	24	(21, 0, 9, 0, 103, 0, 202, 0, 615, 0, 1811, 0, 5234, 0, 15358, 0, 43782, 0) [80, 0, 58, 0, 607, 0, 1563, 0, 5008, 0, 16474, 0, 52106, 0, 166791, 0, 515426, 0]	22/39
12	10533, 10675, 17661	24	(10, 0, 14, 0, 46, 0, 121, 0, 372, 0, 1055, 0, 3129, 0, 8848, 0, 26336, 0) [27, 0, 74, 0, 228, 0, 794, 0, 2757, 0, 8531, 0, 28250, 0, 88579, 0, 286193, 0]	20/37
13	21645, 35661, 37133	26	(12, 0, 32, 0, 54, 0, 167, 0, 506, 0, 1552, 0, 4404, 0, 12456, 0, 36522, 0) [41, 0, 165, 0, 319, 0, 1156, 0, 3937, 0, 13208, 0, 42284, 0, 129918, 0, 413986, 0]	4/7

*Partial weight spectra for these codes were originally found by Odenwalder.

+Overlooked by Odenwalder.

TABLE III
WEIGHT SPECTRA AND MINIMUM ASYMPTOTIC RATE OF GROWTH OF THE WEIGHTS IN THE INCORRECT SUBSET FOR THE BEST (4,1) CONVOLUTION CODES UP TO CONSTRAINT LENGTH 14

m	generators (octal)	d_f	$(a_{d_f+l}$ --- , $l = 0, 1, \ldots, 17)$ $[b_{d_f+l}$ --- $]$	d_0
2	5,7,7,7	10	(1,1,1,3,2,5,7,8,16,19,30,46,61,98,137,201,303,429) [2,1,4,9,8,25,32,52,100,131,240,366,554,930,1368,2187,3398,5141]	2/3
3	13,15,15,17	13	(2,1,0,3,1,4,8,4,15,16,18,45,40,73,119,122,244,313) [4,2,0,10,3,16,34,18,77,84,106,280,256,514,865,934,1988,2620]	1
4	25,27,33,37	16	(4,0,2,0,4,0,15,0,30,0,54,0,115,0,252,0,511,0) [8,0,7,0,17,0,60,0,140,0,301,0,707,0,1675,0,3739,0]	6/5
5	53,67,71,75	18	(3,0,5,0,6,0,12,0,23,0,67,0,157,0,283,0,610,0) [6,0,17,0,24,0,60,0,118,0,367,0,991,0,1980,0,4716,0]	10/9
6	135,135,147,163	20	(10,0,0,0,19,0,0,0,117,0,0,0,711,0,0,0,3084,0) [37,0,0,0,94,0,0,0,768,0,0,0,5558,0,0,0,28349,0]	4/5
7	235,275,313,357	22	(1,4,3,2,3,3,11,14,13,24,39,60,72,100,168,254,414,535) [2,10,10,8,10,11,54,64,68,140,218,382,478,660,1174,1846,3100,4139]	1
8	363,535,733,745	24	(2,0,6,0,10,0,18,0,37,0,95,0,179,0,358,0,810,0) [4,0,22,0,38,0,103,0,237,0,587,0,1251,0,2765,0,6666,0]	16/17
9	1117,1365,1633,1653	27	(4,4,4,8,5,14,22,17,33,46,75,112,168,248,317,501,703,1022) [10,12,18,44,31,72,120,108,221,320,545,786,1284,2054,2587,4272,6407,9376]	1
10	2327,2353,2671,3175	29	(5,6,4,6,7,7,10,22,33,46,85,118,162,243,341,487,690,1053) [13,24,18,22,35,34,56,108,187,292,531,784,1158,1828,2631,3896,5792,9048]	49/51
11	4767,5723,6265,7455	32	(14,0,10,0,14,0,47,0,105,0,180,0,452,0,973,0,1988,0) [49,0,40,0,82,0,267,0,640,0,1247,0,3362,0,8000,0,17453,0]	26/27
12	11145,12477,15573,16727	33	(5,5,3,9,7,8,22,23,28,53,79,116,165,224,346,537,809,1112) [19,16,15,46,29,48,124,140,174,336,555,830,1219,1764,2826,4626,7123,10022]	6/7
13	21113,23175,35527,35537	36	(19,0,16,0,30,0,83,0,153,0,333,0,736,0,1614,0,3298,0) [74,0,80,0,177,0,493,0,1098,0,2519,0,5872,0,13878,0,30678,0]	22/25

tion (6) is impractical. An alternate approach is to evaluate a truncated version $\Sigma_{l=d_f}^{T} a_l W^l$, $\Sigma_{l=d_f}^{T} b_l W^l$ for some truncation depth T. The advantage of this method is that the truncated version is suitable for the evaluation of refined bounds on the error events and bit error probabilities as suggested by Van De Meeberg [3] and Post [4].

A computer program has been written to determinate the truncated spectrum of any $(n, 1)$ convolutional code. The procedure operates as a forward sweep on Γ (forward dynamic program) starting at the null state and analyzing all paths of a given weight remerging to the null state at some later time.

The minimum weight $d_0(n)$ of the paths in the incorrect subset which diverge over n branches is governed by the minimum asymptotic growth rate of the weights in the subset of unmerged codewords d_0 formally defined as $d_0 \triangleq \lim (d_0(n)/n \rightarrow \infty n)$ [17]. The value of d_0 corresponds to the minimum ratio of Hamming weight over length taken over the set of all circuits of Γ excluding the loop around the null state. It follows that the minimum weight among the unmerged codewords increases periodically with depth as n grows. The corresponding ratio of the periodic increase in weight over the period constitutes a measure of d_0 which has been incorporated in our algorithm. The results obtained by this computational procedure are collected in Tables I–III. The free distance of the code, d_f, appears as the third entry in each table.

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Note on a Class of Codes Introduced by Séguin, Allard, and Bhargava

BERNARD COURTEAU AND JEAN GOULET

Abstract—**In this correspondence, we apply a theorem in finite geometry to obtain an improvement of the codes recently introduced by Séguin, Allard, and Bhargava to provide error protection for data in the ASCII format. Our construction yields a fast decoding algorithm.**

I. INTRODUCTION

In a recent paper published in this TRANSACTIONS [1], Séguin, Allard, and Bhargava introduced a class of codes

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The authors are with the Département de Mathématiques et d'Informatique, Université de Sherbrooke, Sherbrooke, P.Q., Canada J1K 2R1.

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