What is homogeneous coordinate and homogeneous matrix?

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Cartisian coordinate

Students learn to present a point with Cartisian coodinate since their primary school. It is a very useful tool because once a base coordinate frame is setup, any point in the space can be presented with just a pair of number with respect to base coordinate frame:

$$(x_0, y_0)$$

Rotation matrix

I have a point p presented with Cartisian coordinate (x_a, y_a) with respect to frame F_a, if I rotate this point by axis Z of F_a with degree $-\theta$, and then we have the new point p', the question is what's the Cartisian coordinate value of p'_a, calculate (x'_a, y'_a), p_a means the coordinate value of point p with respect to frame F_a

Actually, rotating the point p by axis Z of Fa with $-\theta$ is equal to rotating the frame Fa by axis Z with θ, and then question can be transfered to what is the point's coordinate in the new frame F_b, aka p_b?

It's easy to prove the following equation from the following figure:

$$p'_a = p_b = \begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \cdot \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$

For convenience, we present rotation matrix with

$$R_z(-\theta) = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

or another more widely used notation:

$$R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Homogeneous matrix

Similarly, a translation can also be written in multiplication form: if $p_{ab} = p_b - p_a$, then

$$\begin{bmatrix} p_{b_x} \\ p_{b_y} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_{ab_x} \\ 0 & 1 & p_{ab_y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_{a_x} \\ p_{a_y} \\ 1 \end{bmatrix}$$

What's more interesting is we can combine the rotation and translation in one homogeneous matrix, if p_c is the coordinate of point p_a after a rotation by axis Z and a following translation p_{ca}:

then

$$p_c = R_z(\theta) \cdot p_a + p_{ca}$$

$$\begin{bmatrix} p_{c_x} \\ p_{c_y} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_{ca_x} \\ 0 & 1 & p_{ca_y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_{a_x} \\ p_{a_y} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p_{c_x} \\ p_{c_y} \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & p_{ca_x} \\ sin\theta & cos\theta & p_{ca_y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_{a_x} \\ p_{a_y} \\ 1 \end{bmatrix}$$

We call

$$\begin{bmatrix} p_{c_x} \\ p_{c_y} \\ 1 \end{bmatrix}$$

as homogeneous coordinate, which is different from Cartisian coordinate, and call

$$\begin{bmatrix} cos\theta & -sin\theta & p_{ca_x} \\ sin\theta & cos\theta & p_{ca_y} \\ 0 & 0 & 1 \end{bmatrix}$$

as homogeneous matrix, which contains a rotation part and a translation part.