Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

The Quadratic Assignment Problem

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THE QUADRATIC ASSIGNMENT PROBLEM*†

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This paper presents a formulation of the quadratic assignment problem, of which the Koopmans-Beckmann formulation is a special case. Various applications for the formulation are discussed. The equivalence of the problem to a linear assignment problem with certain additional constraints is demonstrated. A method for calculating a lower bound on the cost function is presented, and this forms the basis for an algorithm to determine optimal solutions. Further generalizations to cubic, quartic, N-adic problems are considered.

1. Introduction

The *linear assignment problem* is a special case of the wellknown transportation problem, and can be stated as follows: Given an $n \times n$ cost matrix $C = \| x_{ij} \|$, determine an $n \times n$ solution matrix $X = \| x_{ij} \|$ so as to

$$\text{(1)} \qquad \text{minimize} \quad \sum_{ij} c_{ij} x_{ij}$$

$$\text{(2)} \qquad \text{subject to} \quad \sum_{j} x_{ij} = 1 \qquad (i = 1, 2, \cdots, n),$$

$$\text{(3)} \qquad \sum_{i} x_{ij} = 1 \qquad (j = 1, 2, \cdots, n),$$

$$\text{(4)} \qquad \text{and} \quad x_{ij} = 0 \text{ or } 1 \qquad (1, j = 1, 2, \cdots, n).$$

There are several efficient methods for solving this problem, e.g., the method of Ford and Fulkerson [2]. The fact that the solution must be discrete, i.e., a permutation matrix, causes no difficulty. Any fractional solution, i.e., a doubly stochastic matrix, is a convex combination of permutation matrices, each having the same cost.

The *quadratic assignment problem* is defined as follows: Given n^4 cost coefficients c_{ijpq} $(i, j, p, q = 1, 2, \cdots, n)$, determine an $n \times x$ solution matrix $X = \| x_{ij} \|$ so as to

$$\text{(5)} \qquad \text{minimize} \quad \sum_{ij} \sum_{pq} c_{ijpq} x_{ij} x_{pq}$$

subject to constraints (2), (3), and (4). In this case, the discrete nature of the problem does cause difficulty. The only exact methods of solution known to the author are those presented in this paper.

The next section describes the original Koopmans-Beckmann version of the

* Received August 1962.

† This paper constitutes, with minor revisions, one chapter of the author's Ph.D. thesis in applied mathematics at Harvard University. The writing of the thesis was supported in part by a contractural arrangement with the Bell Telephone Laboratories. The author wishes to acknowledge the advice and encouragement of Dr. Willard Eastman during the writing of the thesis, and that of Dr. Kenneth Iverson during the early stages of investigation.

quadratic assignment problem. The sections that follow describe the formulation of a problem in sequencing, the transformation to an equivalent integer linear program and a method for computing minimal solutions. A final section discusses extensions to cubic, quartic, $\cdots$ n-adic assignment problems.

For notational convenience, the *dot product* of two matrices will sometimes be referred to. If A and B are compatible,

$$A \cdot B = \sum_{ij} a_{ij} b_{ij} .$$

2. The Koopmans-Beckmann Formulation

The first published statement of the quadratic assignment problem was by Koopmans and Beckmann [6], in the context of an analysis of the location of economic activity. Their formulation of the problem, which is somewhat less general than that given above, is as follows.

Let it be required to assign n plants to n locations in such a way that the total cost of interplant transportation is minimized. Two $n \times n$ matrices, $D = \| d_{jq} \|$ and $T = \| t_{ip} \|$ are given, where

d_{jq} = cost of transporting one unit of commodity from location j to location q.

t_{ip} = number of units of commodity to be transported from plant i to plant p.

Each assignment of plants to locations is represented by an $n \times n$ permutation matrix $X = \| x_{ij} \|$ where

$$x_{ij} = 1 \text{ if plant } i \text{ is assigned to location } j,$$
$$= 0 \text{ otherwise.}$$

The Koopmans-Beckmann problem is to minimize the dot product of D and T with respect to a symmetric permutation of the rows and columns of one of the matrices. That is, to

$$\text{minimize} \quad T \cdot (XDX^t). \tag{6}$$

It is seen that (6) is a special case of the quadratic form (5) with $c_{ijpq} = t_{ip} d_{jq}$. The coefficient c_{ijpq} represents "the cost of transportation from plant i at location j to plant p at location q." Additional linear costs of the form $B \cdot X$ can be incorporated into the quadratic form (5) by setting $c_{ijij} = t_{ii} d_{jj} + b_{ij}$.

Although not so generalized by Koopmans and Beckmann, a multicommodity problem can also be stated. Let $D^{(1)}, T^{(1)}; D^{(2)}, T^{(2)}; \cdots ; D^{(m)}, T^{(m)}$ be m pairs of matrices for m different commodities. Then it may be required to

$$\text{minimize} \quad \sum_k T^{(k)} \cdot (XD^{(k)} X^t),$$

which is equivalent to a quadratic form with

$$c_{ijpq} = \sum_k t_{ip}^{(k)} d_{jq}^{(k)} .$$

3. The Candidates' Problem

It is noteworthy that the traveling-salesman problem is a special case of the Koopmans-Beckmann formulation, in which D is a distance matrix and T is a cyclic permutation matrix of the form

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is therefore not surprising that a variety of problems involving permutations and sequences can be formulated as quadratic assignment problems. For example, consider the following "candidates' problem":

The Presidential and Vice-Presidential candidates of one of the major parties are planning their campaign itinerary. They decide that there are $2n$ cities that should be subjected to a full day's visit by one or the other of them during a particular n-day period. It does not matter which candidate visits any given city, or on what day within the n-day period he visits it. Each night during the period the two candidates will confer for a fixed length of time by long distance telephone. The day before his tour begins, the Presidential candidate is scheduled to be in city P_0, and the day after in P_{n+1}. Similar engagements for the Vice-Presidential candidate are V_0 and V_{n+1}.

The problem is to plan the tours for the two candidates in such a way that the total of the transportation costs and the telephone costs is minimized. Travel costs for the Presidential candidate and his entourage are p cents per airline mile, and for the Vice-Presidential candidate v cents. Telephone charges are t cents per airline mile.

Let a $2n \times 2n$ matrix D represent the distance between the $2n$ cities. Also define a $2n \times 2n$ matrix T of the form

$$T = \left|\begin{array}{ccccc:ccccc} 0 & p & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

The three nonzero quadrants are related to the Presidential candidate's tour, the telephone conferences, and the Vice-Presidential candidate's tour, respectively.

The solution is represented by a $2n \times 2n$ permutation matrix $X = \| x_{ij} \|$, whose elements are defined as follows:

For $i \leq n$:

$$x_{ij} = 1 \text{ if the Presidential candidate visits city } j \text{ on the } i^{\text{th}} \text{ day,}$$
$$= 0 \text{ otherwise.}$$

For $i \geqq n + 1$:

$x_{ij} = 1$ if the Vice-Presidential candidate visits city j on the $(i - n)^{\text{th}}$ day,
$\quad = 0$ otherwise.

The quantity

$$T \cdot (XDX^t)$$

represents all costs except the travel costs to and from the cities P_0, P_{n+1}, V_0 and V_{n+1}. For this purpose, define a $2n \times 2n$ matrix B, where

$$b_{1j} = p \text{ times the distance from city } P_0 \text{ to city } j,$$

$$b_{n+1,j} = v \text{ times the distance from city } V_0 \text{ to city } j,$$

$$b_{nj} = p \text{ times the distance from city } j \text{ to city } P_{n+1},$$

$$b_{2n,j} = v \text{ times the distance from city } j \text{ to city } V_{n+1},$$

and the remaining b_{ij} are zero. Then the objective of the problem is to minimize

$$T \cdot (XDX^t) + B \cdot X,$$

which is a single commodity Koopmans-Beckmann problem with additional linear costs.

4. Other Problems

Among the other problems that can be formulated are the minimization of "latency" in magnetic drum computers [8] (for which a multicommodity Koopmans-Beckmann problem is an excellent model), the placement of electronic assemblies so as to reduce total wire length [3, 5, 9], and various problems in the synthesis of sequential switching circuits.

5. An Equivalent Linear Program

A quadratic assignment problem with n^2 variables x_{ij} can be linearized by defining n^4 variables y_{ijpq}, where, effectively,

$$y_{ijpq} = x_{ij} x_{pq}.$$

Consider the following integer linear program in the $n^4 + n^2$ variables y_{ijpq} and x_{ij}:

$$\text{(7)} \qquad \text{minimize} \quad \sum\nolimits_{ijpq} c_{ijpq} y_{ijpq}$$

$$\text{(8)} \qquad \text{subject to} \quad \sum\nolimits_j x_{ij} = 1 \qquad (i = 1, 2, \cdots, n),$$

$$\text{(9)} \qquad \sum\nolimits_i = 1 \qquad (j = 1, 2, \cdots, n),$$

$$\text{(10)} \qquad \sum\nolimits_{ijpq} y_{ijpq} = n^2$$

$$\text{(11)} \qquad x_{ij} + x_{pq} - 2y_{ijpq} \geqq 0 \quad (i, j, p, q = 1, 2, \cdots, n),$$

and

$$\text{(12)} \qquad x_{ij} = 0 \text{ or } 1 \qquad (i, j = 1, 2, \cdots, n),$$

$$\text{(13)} \qquad y_{ijpq} = 0 \text{ or } 1 \qquad (i, j, p, q = 1, 2, \cdots, n).$$

Let the quadratic assignment problem defined by conditions (2) through (5) be designated problem Q, and the integer linear programming problem defined by conditions (7) through (13) be designated problem L. The following theorem asserts the equivalence of Q and L for any given set of cost coefficients.

Theorem: The feasible solutions of problems Q and L can be placed in one-to-one correspondence with equal values of the cost functions. A feasible solution $X^{(Q)}$ of Q corresponds to a feasible solution $(X^{(L)}, Y)$ of L if and only if $X^{(Q)} = X^{(L)}$.

Proof: It is sufficient to show that the constraints of problem L are such that for any given permutation matrix $X^{(L)}$, Y is determined uniquely by the relation

$$y_{ijpq} = x_{ij}x_{pq} .$$

Since all of the variables are restricted to the values 0 and 1, this relation is equivalent to

$$y_{ijpq} = 1 \Leftrightarrow x_{ij} = x_{pq} = 1.$$

It follows immediately from (11) that

$$y_{ijpq} = 1 \Rightarrow x_{ij} = x_{pq} = 1.$$

In order to prove the converse, let $x_{ij_i} = 1$ and $x_{ij} = 0$ for $j \neq j_i$, and similarly let $x_{pq_p} = 1$ and $x_{pq} = 0$ for $q \neq q_p$. Then, from (11), $y_{ijpq} = 0$ unless $j = j_i$ and $q = q_p$. It follows that

$$\sum_{jp} y_{ijpq} \leqq 1$$

and this inequality is strict unless $y_{ij_ipq_p} = 1$. Now summing over all i and p, we have

$$\sum_{ip} (\sum_{jq} y_{ijpq}) < n^2$$

unless $y_{ijpq} = 1$ whenever $x_{ij} = x_{pq} = 1$. Q.E.D.

6. Cost Bounds

The quadratic assignment problem is a finite problem. In principle, each of the $n!$ feasible solutions could be evaluated to find the minimum. Alternatively, existing integer programming techniques could be applied to the equivalent linear problem L. However, neither of these alternatives is attractive. This section is devoted to a discussion of lower bounds on the quadratic cost function, as a basis for computational methods that are better suited to the special nature of the problem.

One way to calculate a lower bound on the cost function is to replace (4) by

$$x_{ij} \geqq 0 \qquad (i, j, = 1, 2, \cdots, n)$$

and to apply quadratic programming techniques to the resulting quadratic program. Since these techniques can be applied only to symmetric and positive semidefinite quadratic forms, the cost coefficients should be revised as follows.

For symmetry, replace c_{ijpq} by c'_{ijpq}, where

$$c'_{ijpq} = \tfrac{1}{2}(c_{ijpq} + c_{pqij}).$$

For positive semidefiniteness, replace all "diagonal" coefficients c_{ijij} by c'_{ijij}, where

$$c'_{ijij} = c_{ijij} + k \tag{14}$$

and k is some suitably large positive constant.

The effect of (14) is to add a positive constant nk to the cost function of the original quadratic assignment problem. However, for relatively small values of k, this constant may exceed the value of the minimum for the quadratic program, making the resulting bound negative. This is a very unsatisfactory result, particularly if all cost coefficients are nonnegative to begin with. Therefore, another type of bound will be developed.

Consider the equivalent integer program L described in the previous section. It is possible to arrange the n^4 variables y_{ijpq} into an $n^2 \times n^2$ matrix Y in such a way that for any feasible solution (X, Y), Y is a Kronecker second power of the $n \times n$ permutation matrix X. That is,

$$Y = X \times X = \begin{bmatrix} x_{11}X & x_{12}X & \cdots & x_{1n}X \\ x_{21}X & x_{22}X & \cdots & x_{2n}X \\ \vdots & & & \vdots \\ x_{n1}X & x_{n2}X & \cdots & x_{nn}X \end{bmatrix} \tag{15}$$

For example, if $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then

$$Y = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

In fact, an $n \times n$ quadratic assignment problem is exactly equivalent to an $n^2 \times n^2$ linear assignment problem, with the additional constraint:

(16) the $n^2 \times n^2$ solution matrix must be the Kronecker second power of an $n \times n$ permutation matrix.

There is an analogy with the traveling salesman problem. The n-city traveling salesman problem is exactly equivalent to an $n \times n$ linear assignment problem, with the additional constraint [1]:

(17) the $n \times n$ solution matrix must be cyclic, i.e., represent a cyclic permutation of integers 1 to n.

The question in both cases is how to impose the additional constraint on the linear assignment problem. The measures that can be taken directly are limited in effect. In the case of the traveling salesman problem, subcycles of length one can be excluded by prohibiting nonzero elements on the main diagonal of the solution matrix. If the problem is symmetric, other measures can be imposed to exclude subcycles of length two [1].

In the case of the quadratic assignment problem, similar measures are possible. Suppose Y is partitioned into n^2 *minors* of n^2 elements each, as suggested by the form of the Kronecker product (15). If the matrix Y is to satisfy condition (16) each minor must contain either n 1's or no 1's at all. Each minor that contains n 1's must itself be a permutation matrix, but with one degree of freedom removed. This is due to the fact that the diagonal element y_{ijij} of minor (i, j) must be 1, if the minor contains any 1's at all. Such an element corresponds to a linear component of the cost function.

It is possible to impose the restrictions described above in the following way. For fixed i, j, define

$$C^{(ij)} = \| c_{ijpq} \|.$$

For each minor $C^{(ij)}$ solve an $(n - 1) \times (n - 1)$ linear assignment problem, or equivalently, an $n \times n$ linear assignment problem subject to the condition $x_{ij} = 1$. Denote the solution $X^{(ij)}$, and its cost

$$f_{ij} = C^{(ij)} \cdot X^{(ij)}.$$

Then solve an $n \times n$ linear assignment problem for the matrix $F = \| f_{ij} \|$. Denote the solution to this last problem $Z = \| z_{ij} \|$. Then if the $n^2 \times n^2$ permutation matrix

$$Y = \begin{bmatrix} z_{11} X^{(11)} & z_{12} X^{(12)} & \cdots & z_{1n} X^{(1n)} \\ z_{21} X^{(21)} & z_{22} X^{(22)} & \cdots & z_{2n} X^{(2n)} \\ \vdots & & & \vdots \\ z_{n1} X^{(n1)} & z_{n2} X^{(n2)} & \cdots & z_{nn} X^{(nn)} \end{bmatrix}$$

satisfies condition (16), a minimal solution to the quadratic assignment problem has been found. Condition (16) is satisfied if and only if

$$\frac{1}{n} \sum_{ij} z_{ij} X^{(ij)} \tag{18}$$

is itself a permutation matrix.

In any case, a lower bound for the quadratic assignment problem is given by

$$\begin{aligned} C \cdot Y &= F \cdot Z \\ &= \textstyle\sum_{ij} z_{ij} C^{(ij)} \cdot X^{(ij)}. \end{aligned}$$

In the special case of a single-commodity Koopmans-Beckmann problem, a significant shortcut is possible.[1] In this case, each minor $C^{(ij)}$ is actually a Cartesian product of two n-vectors T^i and D^j (row i of T and row j of D). Finding a solution to the linear assignment problem for such a minor is equivalent to finding a permutation of the components of one of the vectors that minimizes the scalar product of T^i and D^j. It is not difficult to prove that the dot product of two vectors is minimized if and only if the k^{th} largest component of one vector is paired with the k^{th} smallest component of the other, for $k = 1, 2, \cdots, n$. Thus, the solution of the linear assignment problems for the n^2 minors is a trivial matter.

Example 1: Suppose it is desired to find a lower bound for the single commodity Koopmans-Beckmann problem, for which

$$D = \begin{bmatrix} 0 & 5 & 0 & 5 & 0 & 5 & 4 \\ 5 & 0 & 9 & 7 & 3 & 8 & 6 \\ 0 & 9 & 0 & 9 & 4 & 4 & 4 \\ 5 & 7 & 9 & 0 & 1 & 1 & 9 \\ 0 & 3 & 4 & 1 & 0 & 5 & 5 \\ 5 & 8 & 4 & 1 & 5 & 0 & 4 \\ 4 & 6 & 4 & 9 & 5 & 4 & 0 \end{bmatrix} \tag{19}$$

$$T = \begin{bmatrix} 0 & 0 & 6 & 1 & 1 & 8 & 4 \\ 0 & 0 & 1 & 0 & 3 & 1 & 3 \\ 6 & 1 & 0 & 8 & 8 & 4 & 2 \\ 1 & 0 & 8 & 0 & 7 & 6 & 4 \\ 1 & 3 & 8 & 7 & 0 & 0 & 6 \\ 8 & 1 & 4 & 6 & 0 & 0 & 9 \\ 4 & 3 & 2 & 4 & 6 & 9 & 0 \end{bmatrix} \tag{20}$$

In addition, let the problem have linear costs, represented by

$$B = \begin{bmatrix} 51 & 27 & 14 & 9 & 0 & 18 & 0 \\ 0 & 1 & 22 & 17 & 0 & 41 & 13 \\ 2 & 0 & 13 & 22 & 2 & 12 & 27 \\ 38 & 11 & 0 & 0 & 22 & 13 & 14 \\ 62 & 56 & 0 & 67 & 1 & 0 & 5 \\ 61 & 0 & 3 & 14 & 9 & 1 & 67 \\ 41 & 12 & 23 & 0 & 18 & 41 & 0 \end{bmatrix} \tag{21}$$

As noted in Section 2,

$$\begin{aligned} c_{ijpq} &= t_{iq}d_{jq} + b_{ij} \quad \text{if } i = p \text{ and } j = q, \\ &= t_{ip}d_{jq} \quad \text{otherwise.} \end{aligned}$$

Thus minor $C^{(ij)}$ is obtained by multiplying together elements the rows T^i and

[1] The independent work of Gilmore [6] on the single-commodity Koopmans-Beckmann problem is acknowledged. For this special case of the quadratic assignment problem, Gilmore's algorithm and the author's appear to be essentially the same.

D^j and adding b_{ij} to the "diagonal" coefficient. For example,

$$(22)\qquad C^{(42)} = \begin{bmatrix} 5 & 0 & 9 & 7 & 3 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 0 & 72 & 56 & 24 & 64 & 48 \\ 0 & 11 & 0 & 0 & 0 & 0 & 0 \\ 35 & 0 & 63 & 49 & 21 & 56 & 42 \\ 30 & 0 & 54 & 42 & 18 & 48 & 36 \\ 20 & 0 & 36 & 28 & 12 & 32 & 24 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \\ 8 \\ 0 \\ 7 \\ 6 \\ 4 \end{bmatrix} = T^4,$$

$$D^2 = (\,5 \quad 0 \quad 9 \quad 7 \quad 3 \quad 8 \quad 6),$$

$$b_{42} = 11.$$

The diagonal coefficient, c_{4242}, is encircled.

The first step is to solve an $(n-1) \times (n-1)$ linear assignment problem for each of the n^2 minors. Thus, for $C^{(42)}$ there is a linear assignment problem with the cost matrix

$$\begin{bmatrix} 5 & 9 & 7 & 3 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 72 & 56 & 24 & 64 & 48 \\ 35 & 63 & 49 & 21 & 56 & 42 \\ 30 & 54 & 42 & 18 & 48 & 36 \\ 20 & 36 & 28 & 12 & 32 & 24 \end{bmatrix},$$

which is obtained from (22) by deleting the 4th row and 2nd column. As noted previously, this particular problem can be solved by permuting the elements of T^4 (with its 4th element removed) and D^2 (with its 2nd element removed) in such a way that their scalar product is minimized. Pairing the k^{th} largest element of one with the k^{th} smallest of the other gives:

$$131 = (8)(3) + (7)(5) + (6)(6) + (4)(7) + (1)(8) + (0)(9),$$

which is equivalent to the solution

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$X^{(42)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\begin{aligned} f_{42} &= C^{(42)} \cdot X^{(42)} \\ &= 131 + c_{4242} \\ &= 142. \end{aligned}$$

The complete F matrix is

$$F = \begin{bmatrix} 77 & 120 & 67 & 59 & 27 & 76 & 83 \\ 9 & 38 & 42 & 35 & 10 & 65 & 46 \\ 61 & 153 & 112 & 123 & 59 & 114 & 156 \\ 87 & 142 & 68 & 82 & 68 & 98 & 124 \\ 106 & 180 & 73 & 143 & 43 & 80 & 110 \\ 110 & 139 & 84 & 98 & 56 & 91 & 185 \\ 102 & 136 & 124 & 108 & 77 & 141 & 132 \end{bmatrix}.$$

The next step is to solve the $n \times n$ linear assignment problem for which F is the cost matrix. A minimal solution Z is

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

with a cost of 492.

Condition (16) is not satisfied by

$$Y = \| z_{ij} X^{(ij)} \|,$$

since

$$\frac{1}{n} \sum_{ij} z_{ij} X^{(ij)} = \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 & 3 \\ 1 & 3 & 1 & 1 & 0 & 0 & 1 \\ 4 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 6 & 0 \\ 1 & 1 & 0 & 3 & 1 & 0 & 1 \end{bmatrix}$$

is not a permutation matrix. Hence a feasible solution has not been found. However, 492 is a valid lower bound on the cost function. It will be shown in Example 2 that the actual minimum is 559.

7. Method of Solution

The feasible solutions to the quadratic assignment problem can be partitioned into subsets. For given i, j, subproblems can be defined for the alternatives $x_{ij} = 0$

and $x_{ij} = 1$. It is also convenient to partition a problem into more than two subproblems. For instance, an $n \times n$ problem can be broken down into n problems, each of dimension $(n-1) \times (n-1)$, corresponding to $x_{i1} = 1$, $x_{i2} = 1, \cdots, x_{in} = 1$.

For each subproblem that is created as a result of partitioning, a lower bound can be calculated. If a feasible solution of still lower cost is known, it follows that no solution to the subproblem corresponds to a minimal solution for the original problem. Thus, the lower bound described in the previous section can be used to good effect to eliminate subsets of solutions from consideration. Partitioning is terminated when there is no subproblem remaining with a bound that is lower than the cost of the best known feasible solution[2].

The precise manner in which partitioning should be carried out is a topic for further investigation. It is probable that the type of partitioning performed at each step should be controlled by the matrix (18).

Regardless of the type of partitioning performed, it is reasonable to expect that the lower bounds calculated for the resulting subproblems will be somewhat sharper than the bound for the original problem. This is partly due to a reduction in degrees of freedom. For an $n \times n$ problem there are $n!\,[(n-1)!]^n$ permutation matrices Y upon which to base a bound, and only $n!$ of these represent feasible solutions. When the dimension of the problem is reduced to $(n-1) \times (n-1)$, the ratio between $(n-1)!\,[(n-2)!]^{n-1}$ and $(n-1)!$ is very much smaller than the ratio between $n!\,[(n-1)!]^n$ and $n!$.

Another reason that lower bounds tend to improve with partitioning is that quadratic costs are converted into linear costs. Consider an $n \times n$ problem with no linear costs, i.e., $c_{ijij} = 0$ for all i, j. Suppose n-way partitioning is performed by setting $x_{n1} = 1$, $x_{n2} = 1, \cdots, x_{nn} = 1$. Let $C[k]$ denote the cost matrix for the k^{th} of the $(n-1) \times (n-1)$ problems so obtained. This problem involves linear costs represented by the coefficients

$$c_{ijij}[k] = c_{ijnk} + c_{nkij} \quad (i = 1, 2, \cdots, n-1; j = 1, 2, \cdots, k-1, k+1, \cdots, n),$$

and quadratic costs represented by the coefficients

$$c_{ijpq}[k] = c_{ijpq} \quad (i, p = 1, 2, \cdots, n-1; j, q = 1, 2, \cdots, k-1, k+1, \cdots, n).$$

Next, suppose $(n-1)$-way partitioning is performed by setting $x_{n-1,1} = 1$, $x_{n-1,2} = 1, \cdots, x_{n-1,k+1} = 1, \cdots, x_{n-1,n} = 1$. Let $C[kl]$ denote the cost matrix for the l^{th} $(n-2) \times (n-2)$ problem so obtained. The cost function for the problem involves a constant,

$$c_{nk(n-1)l} + c_{(n-1)lnk},$$

[2] Similar methods of partitioning have been proposed by Eastman [1] for the solution of the traveling salesman problem and by Land and Doig [7] for the solution of integer linear programming problems.

It should be pointed out that partitioning methods can easily be modified to produce, with less computation, solutions that differ from the optimum by no more than a certain specified amount.

linear costs represented by the coefficients

$$c_{ijij}[kl] = c_{ijnk} + c_{nkij} + c_{ij(n-1)l} + c_{(n-1)lij},$$

and quadratic costs represented by the coefficients

$$c_{ijpq}[kl] = c_{ijpq}.$$

After p partitioning operations, the cost function of the resulting $(n-p) \times (n-p)$ problems involves a constant term that is the sum of $p^2 - p$ nonzero coefficients, linear costs represented by coefficients that are the sum of $2p$ of the original quadratic coefficients, and quadratic costs that are essentially unchanged (except in dimension) from the original problem.

Example 2: Consider the single-commodity Koopmans-Beckmann problem defined by (19), (20), (21) and for which a lower bound of 492 was calculated in Example 1. A minimal solution to this problem is found by performing n-way partitions as shown in Figure 1. An initial partition creates seven 6×6 subproblems corresponding to $x_{11} = 1$, $x_{21} = 1$, $\cdots$, $x_{71} = 1$, which have cost bounds of 583, 647, 529, 568, 570, 538, and 570, respectively. Four additional partitioning operations are required as indicated in the figure. The following is a tabulation of upper and lower bounds on the cost function after each successive partition. An upper bound is provided by a known feasible solution.

	Lower Bound	Upper Bound
Initial	492	$+\infty$
1	529	$+\infty$
2	532	$+\infty$
3	537	$+\infty$
4	538	559
5	559	559

The unique minimal solution to this problem is

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

with a cost of 559.

8. Cubic, Quartic, $\cdots$, N-adic Assignment Problems

It is not unreasonable to consider possible extensions to cubic, quartic, $\cdots$, even n-adic assignment problems. For example, the cubic assignment problem can be defined as follows: Given n^6 cost coefficients c_{ijpqrs} $(i, j, p, q, r, s =$

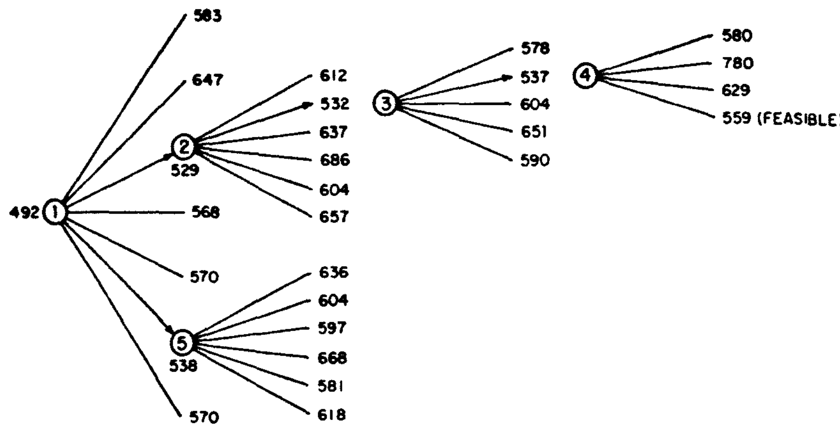

FIG. 1. Solution of example

$1, 2, \cdots, n)$, determine an $n \times n$ solution matrix $X = \| x_{ij} \|$ so as to

$$\text{minimize} \quad \sum_{ij} \sum_{pq} \sum_{rs} c_{ijpqrs} x_{ij} x_{pq} x_{rs}$$

subject to constraints (2), (3), (4).

The cubic problem is equivalent to an $n^3 \times n^3$ linear assignment problem with the additional constraint that the $n^3 \times n^3$ solution matrix must be the Kronecker third power of an $n \times n$ permutation matrix. The method of calculating lower bounds, the test for feasible solutions (18) and the partitioning procedure are easily generalized from the quadratic case.

It should not be overlooked that a more straightforward enumerative approach may be effective for certain higher-order problems. Suppose an $n \times n$ assignment problem is defined, for which it is necessary to minimize $f(X) + g(X)$, where $f(X)$ is a low-order polynomial function and $g(X)$ is any well-defined function that is "less important" than $f(X)$. Suppose λ is a lower bound on $g(X)$ and μ is the value of some known feasible solution. One approach is to find the set S of solutions, where

$$S = \{X \mid f(X) + \lambda < \mu\}.$$

The set S can be enumerated in a straightforward way by means of partitioning, and $g(X)$ can be evaluated for each $X \in S$ in order to find the minimum.

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