Numerical Algorithms - Assignment 1

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December 2020

Introduction

In this assignment we were asked to perform the estimation of parameter p_m for the logistic growth model

$$p_m = \frac{p_m}{1 + (p_m/p_0 - 1)\exp(-Kp_m t)} \tag{1}$$

with different numerical methods, where $p_0 = 100$, $p_{60} = 25000$, $K = 90$, with $p_T = p(T)$. In what follows we will present our results and discuss the value obtained in relationship with the theoretical results for each method.

In order to carry out the assignment we have used the programming language Python. To make sure that there are no compatibility problem, I have put the file `numalg.yaml` in the submission folder which contains the conda environment that I have used for the computation. In the spirit of the assignment, we have implemented from scratch all methods and functions that were required as intended.

1 Newton-Raphson

For this iterative scheme we have used $x_0 = 30000$ as initial point. The Newton-Raphson scheme results are reported in Table 1 and the relative errors are in 1. Even though convergence is attained after $k = 6$ steps, we expected a much smaller d_2: since the order of convergence of the Newton-Raphson method is 2, we expected to have a d_2 for $k = 6$ closer to $|f''(\alpha)/2f'(\alpha)| = |f''(p_m^{(6)})/2f'(p_m^{(6)})| \approx 7.4977\text{e}{-06}$. The value of d_2 at the last iteration is reported in the 6th row of Table 1. Likewise, we can see on the first panel of Figure 1 that the error ratio between iterations decreases faster by using $p = 1$ and $p = 1.618$ rather than $p = 2$, and they stabilise around a lower value at iteration $k = 6$. It is evident that d_p performs well with respect to the other two methods for all p from the plots on Figure 2.

2 Fixed Secant Scheme

For this iterative scheme we have used $x_0 = 30000$ and $x_1 = 70000$ as initial points. Unlike the variable secant scheme, the coefficient

$$q = \left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \right)^{-1} \tag{2}$$

is computed only once, with $k = 0$, and used in all subsequent iterations. Clearly, given that it takes $k = 9$ iterations before halting, it is the less effective of the three methods. Like the variable secant scheme, for $p = 2$ we have that the final d_k explodes, whereas for $p = 1$ and $p = 1.618$ remains under control. We can see on the second and third panels of Figure 2 that d_p is anomalous with respect to the other two methods for $p = 1.618$ and $p = 2$, where its error sharply increases. This behaviour is expected: since the coefficient q is never updated and the shape of the function may be arbitrary there is no guarantee on the error convergence.

3 Variable Secant Scheme

For this iterative scheme we have used the same initial points as in the fixed secant scheme (See Section 2). Even though its order of convergence is slower than that of the Newton-Raphson scheme, on this instance of the problem the variable secant scheme reach convergence after just $k = 5$ steps. As it can be seen on the third panel of Figure 1, the value of d_p increases after iteration $k = 4$. Moreover, given that the order of convergence is $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, the expected $d_{1.618}$ for $k = 5$ is expected to be close to $|f''(\alpha)/2f'(\alpha)| = |f''(p_m^{(6)})/1.618 f'(p_m^{(6)})| \approx 1.0966\text{e}{-05}$, and instead we have obtained a much higher value (see Table 1).

Conclusions

Each iterative scheme has yielded unexpected results. Even though we have tried different starting points for all three methods, the number of iterations and the various d_p do not change significantly. The results are not in line with the theory. Either the function provided by the exercise is peculiar, or the implementation of the methods by the author of this report is wrong (although each estimated p_m is effectively a zero for the original function given the initial values for $p_0, p_6 0$ and K).

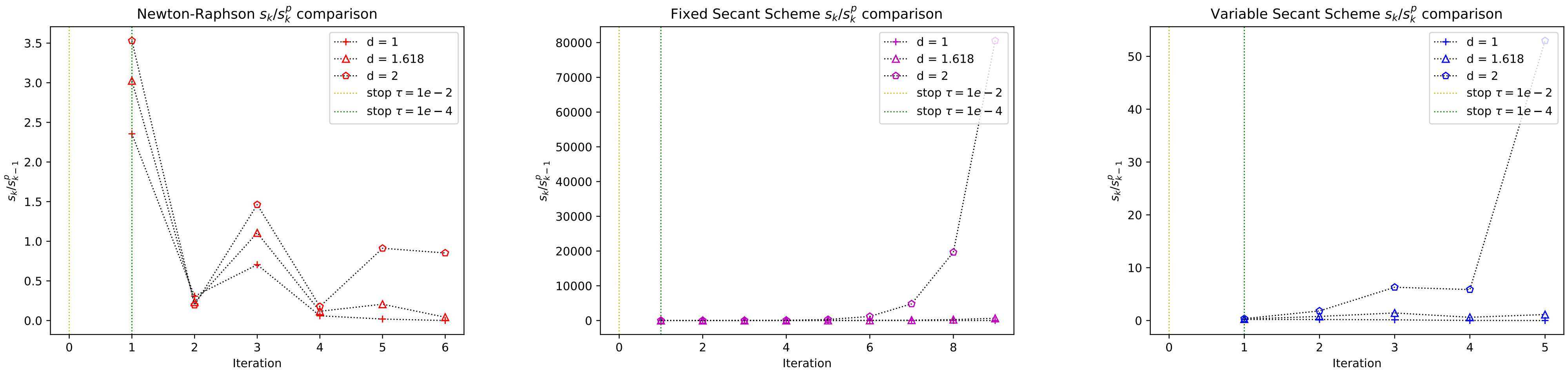

Figure 1: A comparison between d_p for each iterative method. Each plot ends with the last iteration, that is, when the estimated value of p_m is within the tolerance of τ = 1e−06.

	Newton-Raphson				Fixed Secant Scheme				Variable Secant Scheme			
k	p_m	d_1	$d_{1.618}$	d_2	p_m	d_1	$d_{1.618}$	d_2	p_m	d_1	$d_{1.618}$	d_2
0	3.0000e+04	–	–	–	3.0000e+04	–	–	–	3.0000e+04	–	–	–
1	9.013e+04	2.3554e+00	3.0248e+00	3.5305e+00	7.000e+04	2.4169e-01	3.5009e-01	4.4020e-01	7.000e+04	1.9464e-01	2.8195e-01	3.5452e-01
2	3.505e+04	3.0661e-01	2.3189e-01	1.9512e-01	4.519e+04	1.0580e-01	3.6862e-01	7.9735e-01	4.519e+04	1.9357e-01	7.7094e-01	1.8113e+00
3	6.764e+04	7.0451e-01	1.1063e+00	1.4622e+00	5.210e+04	2.5163e-01	3.5131e+00	1.7922e+01	5.060e+04	1.3021e-01	1.4307e+00	6.2945e+00
4	5.050e+04	5.9901e-02	1.1680e-01	1.7647e-01	5.138e+04	2.4179e-01	7.9197e+00	6.8443e+01	5.167e+04	1.5788e-02	6.1146e-01	5.8611e+00
5	5.155e+04	1.8534e-02	2.0583e-01	9.1154e-01	5.156e+04	2.4440e-01	1.9248e+01	2.8611e+02	5.153e+04	2.2526e-03	1.1329e+00	5.2970e+01
6	5.153e+04	3.2127e-04	4.1957e-02	8.5254e-01	5.152e+04	2.4377e-01	4.5861e+01	1.1677e+03	–	–	–	–
7	–	–	–	–	5.153e+04	2.4393e-01	1.0979e+02	4.7931e+03	–	–	–	–
8	–	–	–	–	5.153e+04	2.4389e-01	2.6252e+02	1.9647e+04	–	–	–	–
9	–	–	–	–	5.153e+04	2.4390e-01	6.2790e+02	8.0559e+04	–	–	–	–

Table 1: Iterations for each method used. For each section, the yellow row represents the stopping iteration for τ = 1e−02 and the green row represents the stopping iteration for τ = 1e−04. The last row which contains values is the one with stopping for τ = 1e−04. In this table the symbol d_p denotes the ratio s_k/s_{k-1}^p.

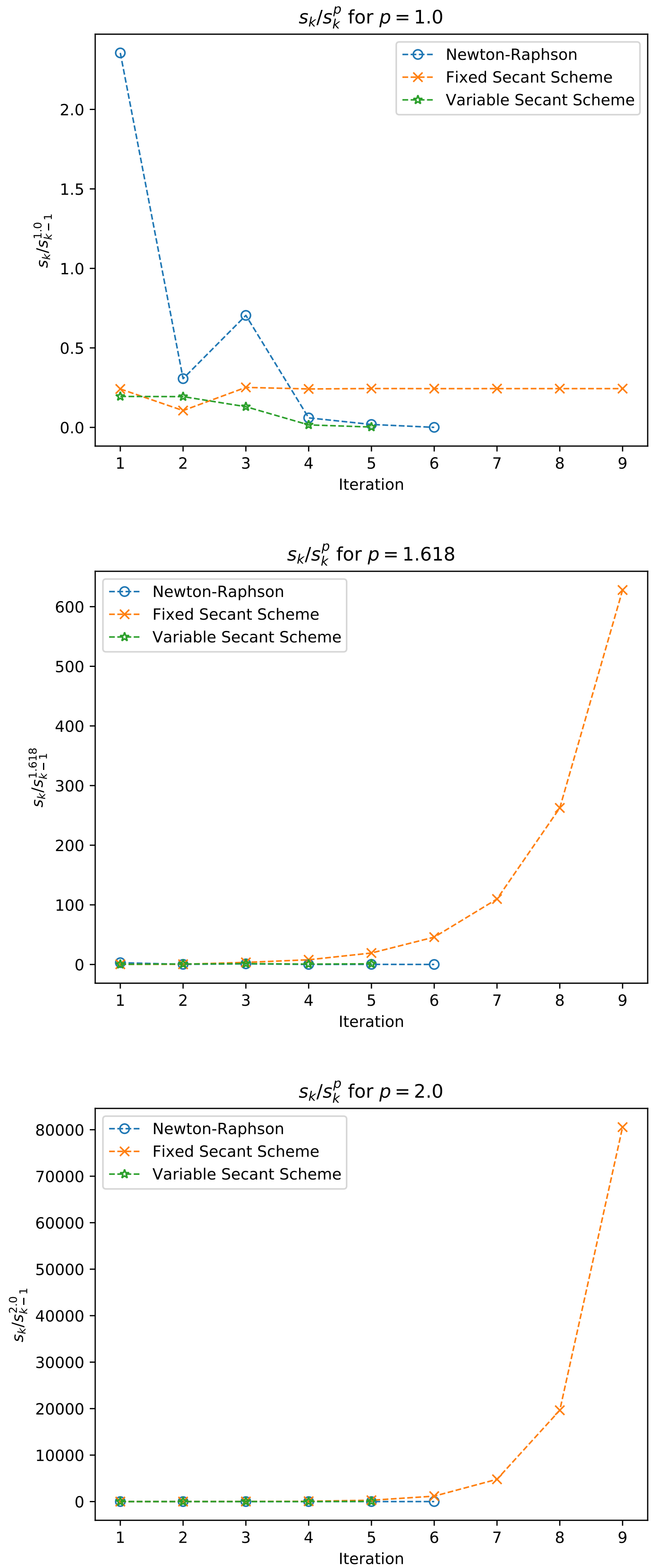

Figure 2: Comparison of relative error for different models and p.