Multi-variable Quantification of BDDs in External Memory using Nested Sweeping

Steffan Christ Sølvsten, Jaco van de Pol

????? 202?

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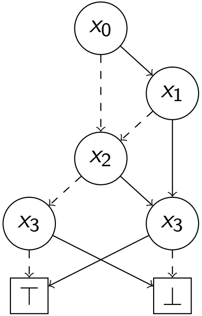
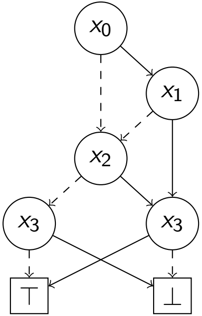
1986

Binary Decision Diagrams



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

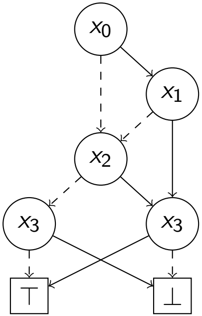
Binary Decision Diagrams



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Theorem (Bryant '86)
Given a fixed variable order, a (Reduced Ordered) Binary Decision Diagram is a unique canonical representation of a Boolean function.

Binary Decision Diagrams



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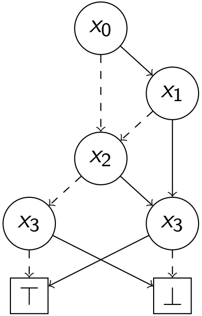

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Theorem (Bryant '86)
Given BDDs ϕ and ψ, $\phi \odot \psi$ is computible in $\mathcal{O}(|\phi| \cdot |\psi|)$ time.

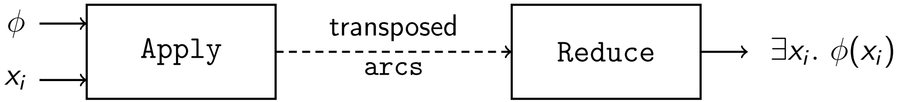
Theorem (Bryant '86)
Given BDD ϕ and Boolean b, $\phi[x_i \mapsto b]$ is computible in $\mathcal{O}(|\phi|)$ time.

Corollary
Given BDD ϕ, $\exists x_i.\ \phi(x)$ requires $\mathcal{O}(|\phi|^2)$ time.

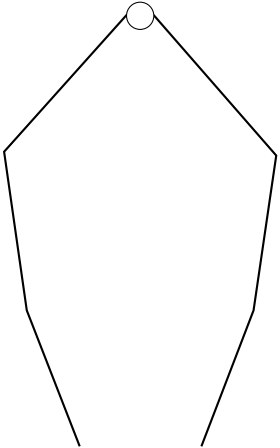
Proof.
$\exists x_i.\phi(x_i) \equiv \phi[x_i \mapsto \bot] \vee \phi[x_i \mapsto \top]$ $\square$

1996

$\exists x_i.\ \phi(x_i)$



$\exists x_i.\ \phi(x_i)$: `Apply`



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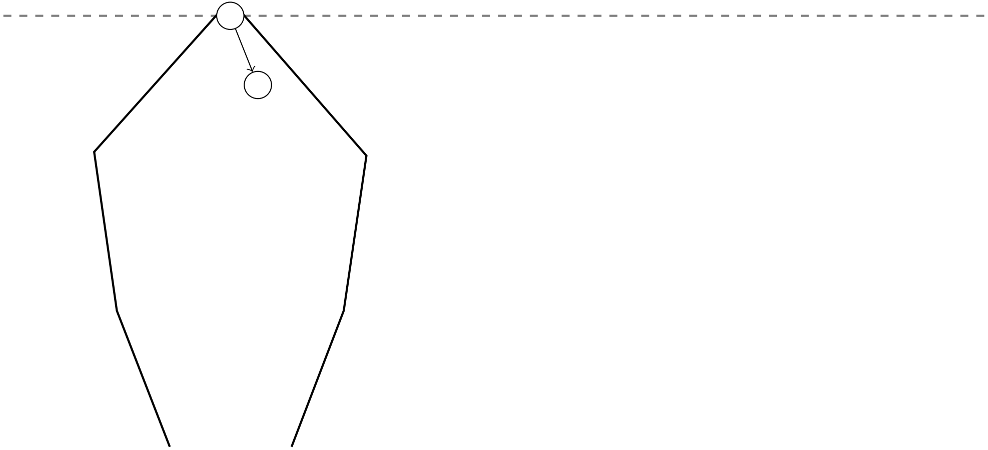

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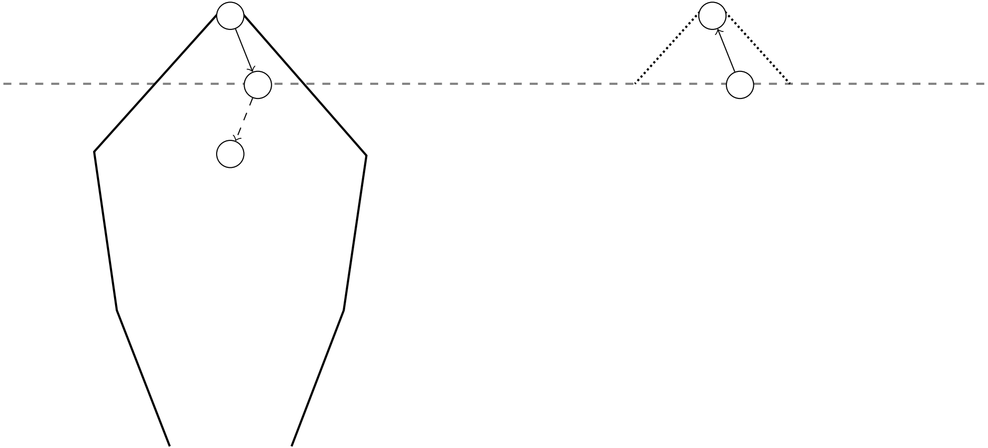

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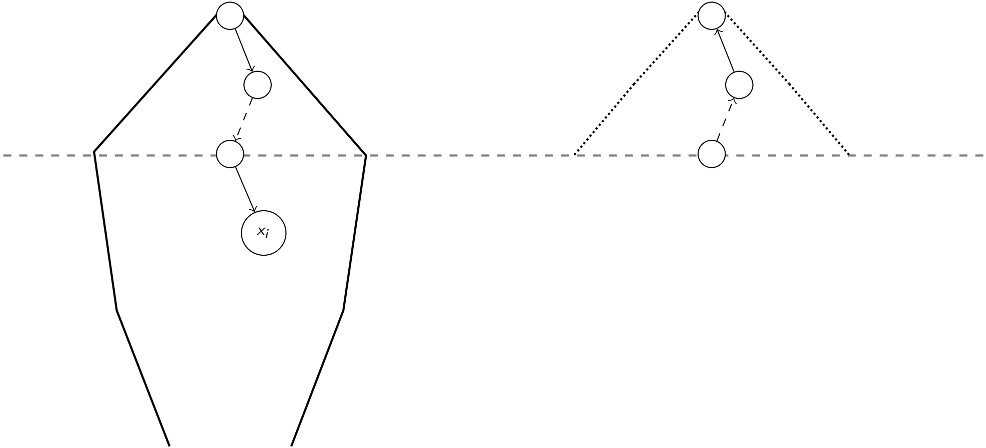

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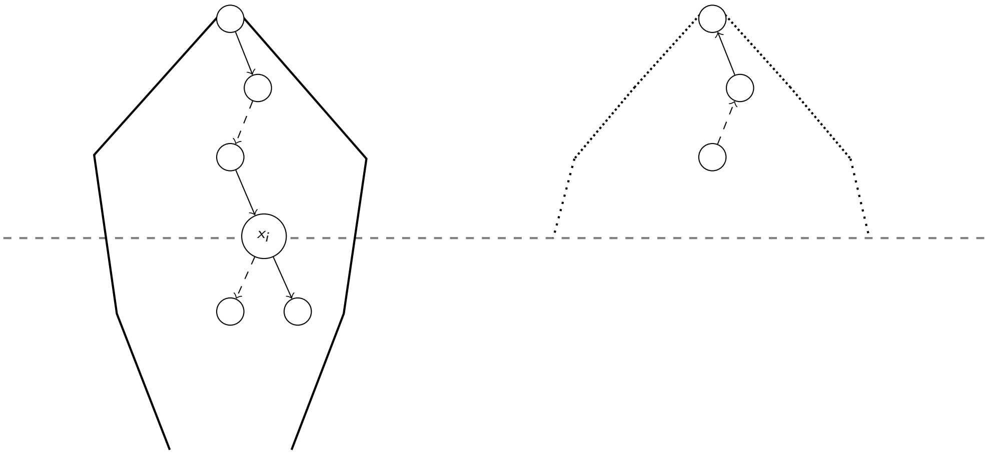

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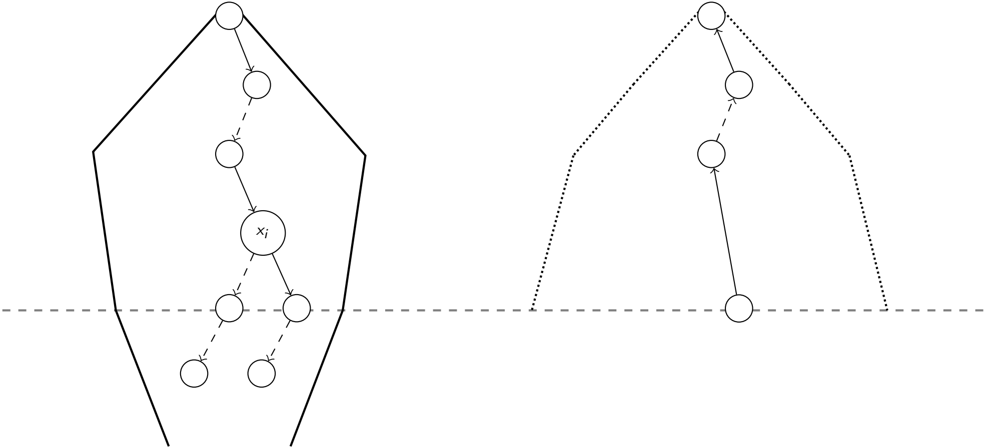

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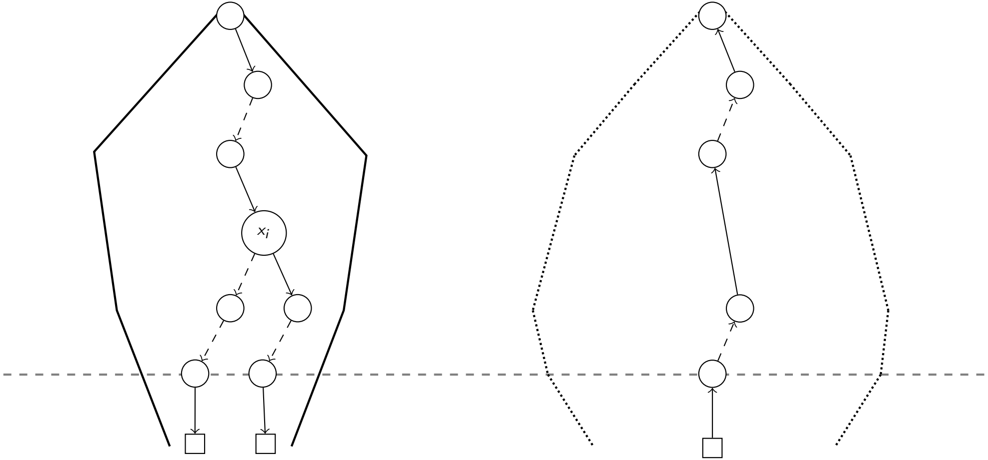

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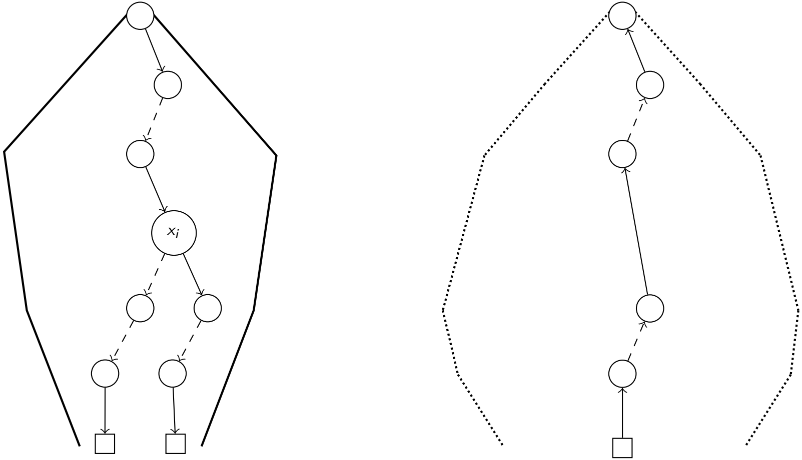
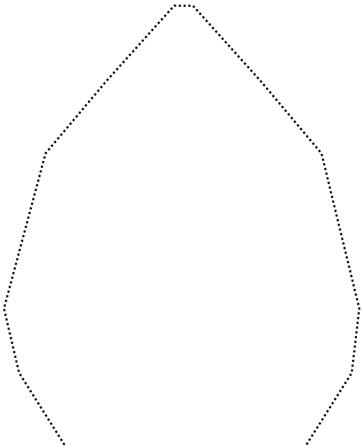


$\exists x_i.\ \phi(x_i)$: `Reduce`



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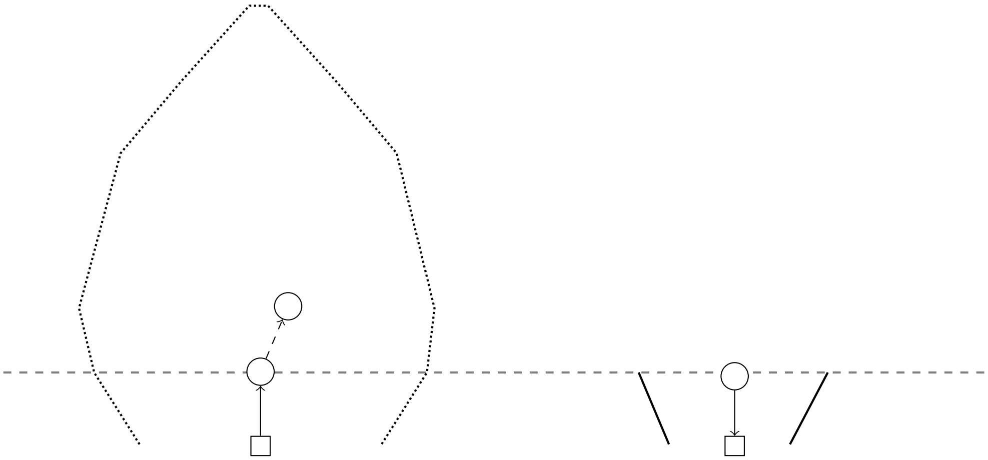

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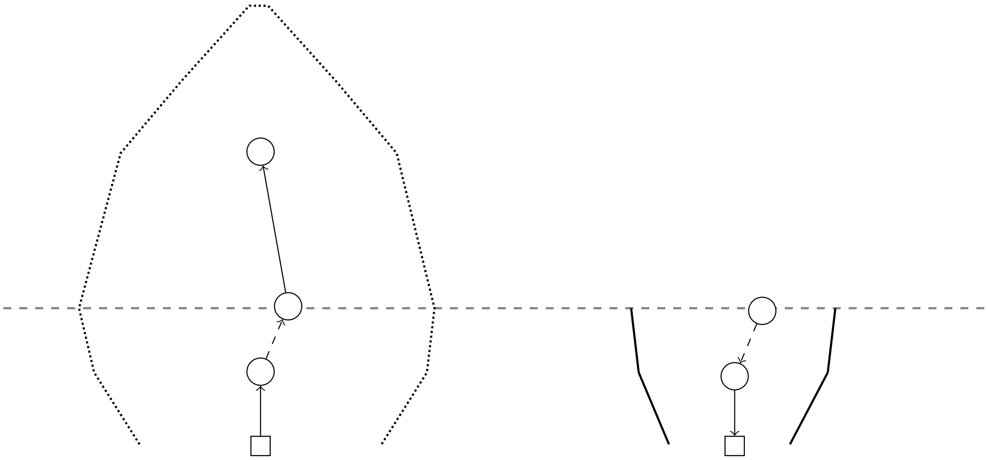

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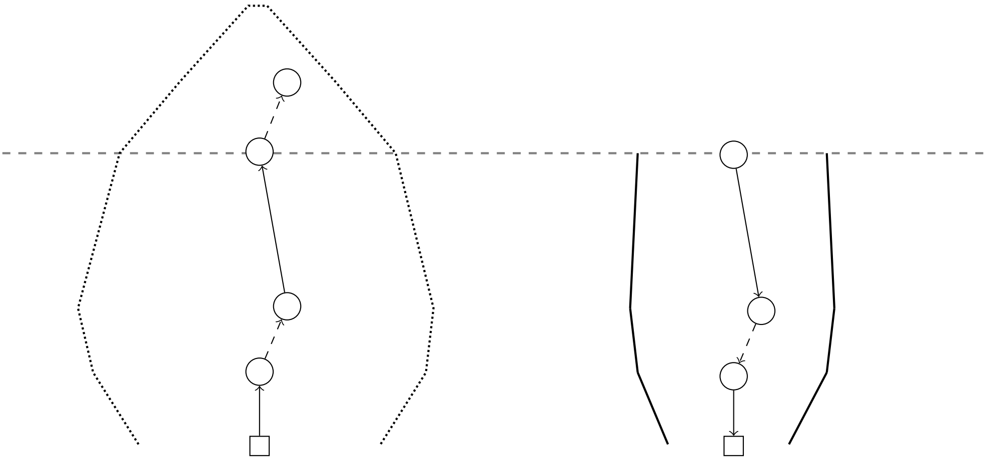

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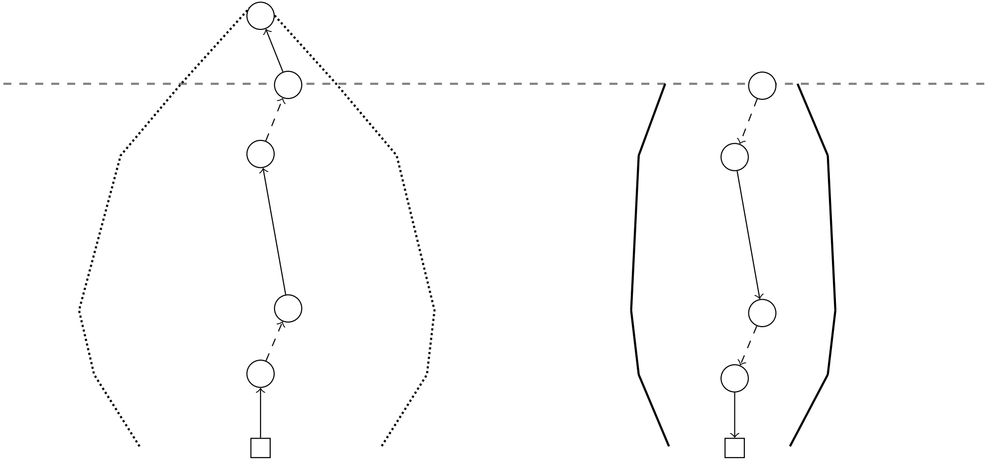

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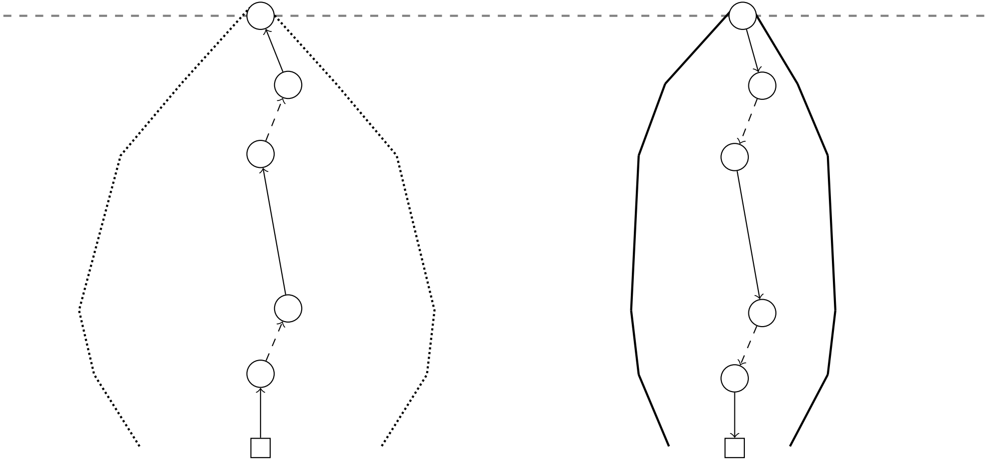

$\exists x_i.\ \phi(x_i)$

Theorem (Lars Arge '96)
Given BDDs ϕ and ψ, $\phi \odot \psi$ is computible in $\mathcal{O}(sort(|\phi| \cdot |\psi|))$ time and I/Os.

Theorem (Sølvsten et al. '22)
Given BDD ϕ and Boolean b, $\phi[x_i \mapsto b]$ is computible in $\mathcal{O}(sort(|\phi|))$ time and I/Os.

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Given BDD ϕ and Boolean b, $\phi[x_i \mapsto b]$ is computible in $\mathcal{O}(sort(|\phi|))$ time and I/Os.

Corollary (Sølvsten et al. '22)
Given BDD ϕ, the time and I/O complexity of quantification is

- $\mathcal{O}(sort(|\phi|^2))$ *for a single variable.*
- $\mathcal{O}(sort(|\phi|^{2^k}))$ *for k variables.*

2022

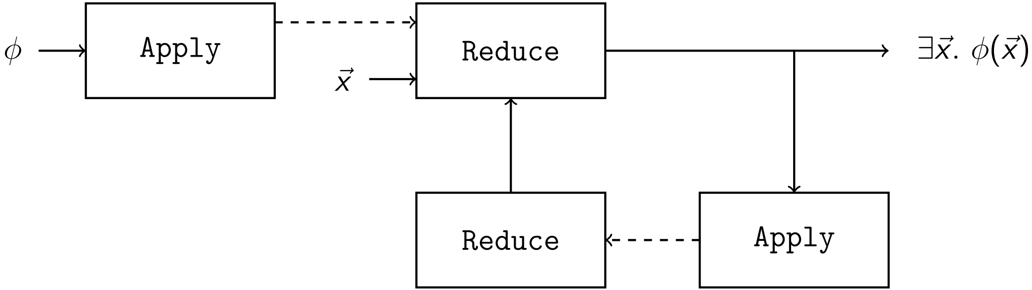
Adiar

I/O-efficient Decision Diagrams

github.com/ssoelvsten/adiar

2025

$\exists \vec{x}.\ \phi(\vec{x})$



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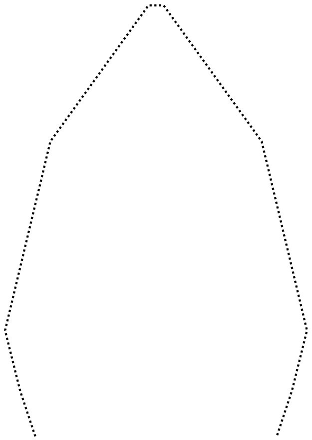

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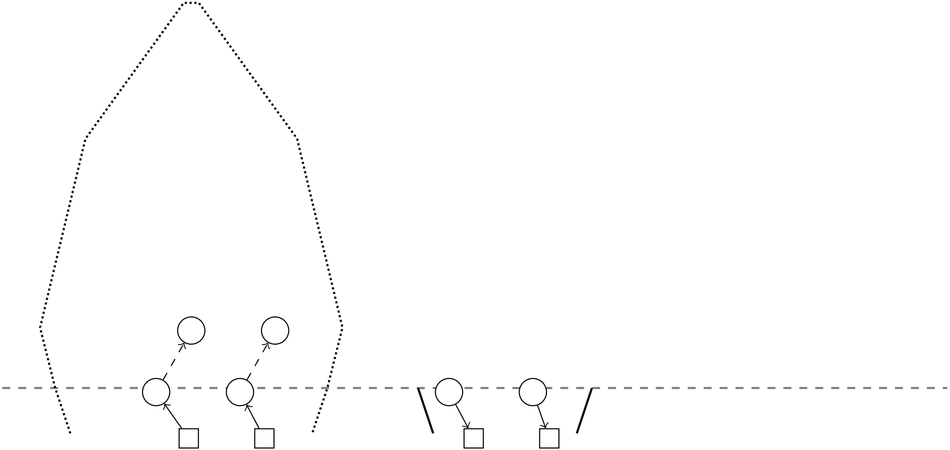

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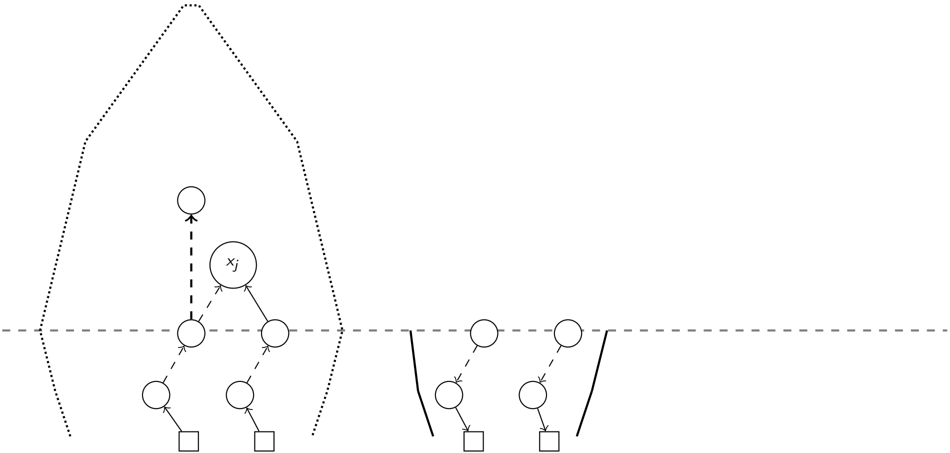

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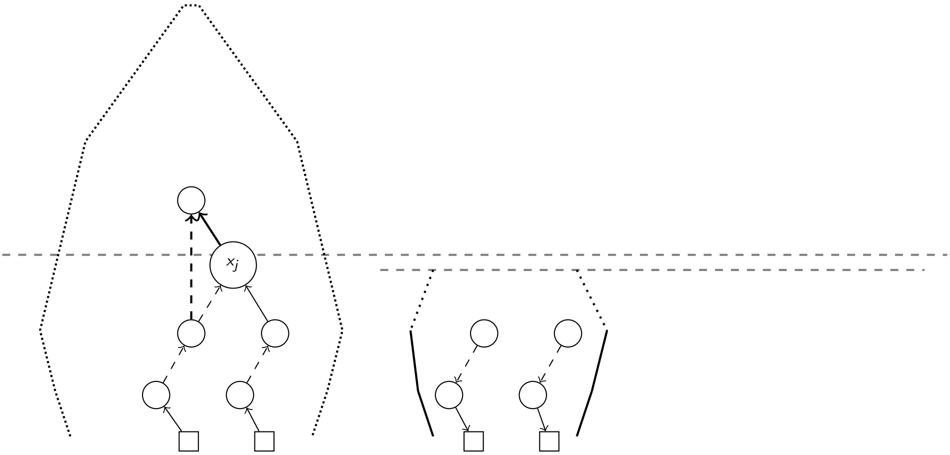

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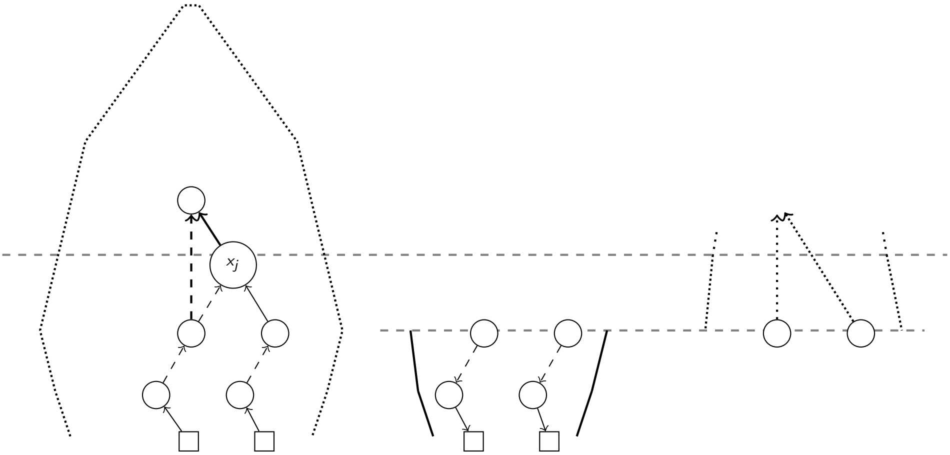

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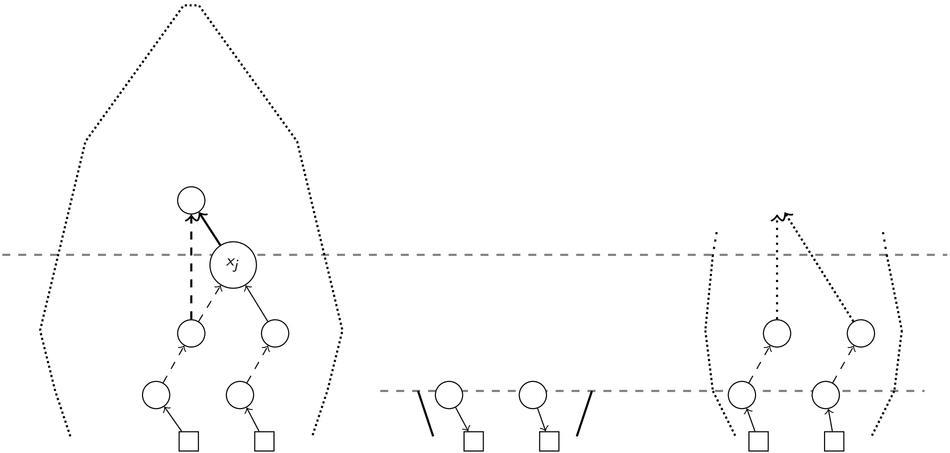

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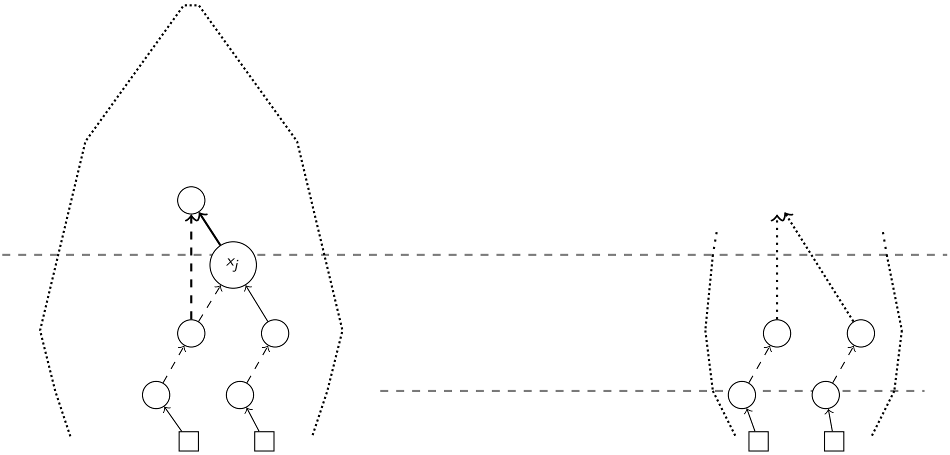

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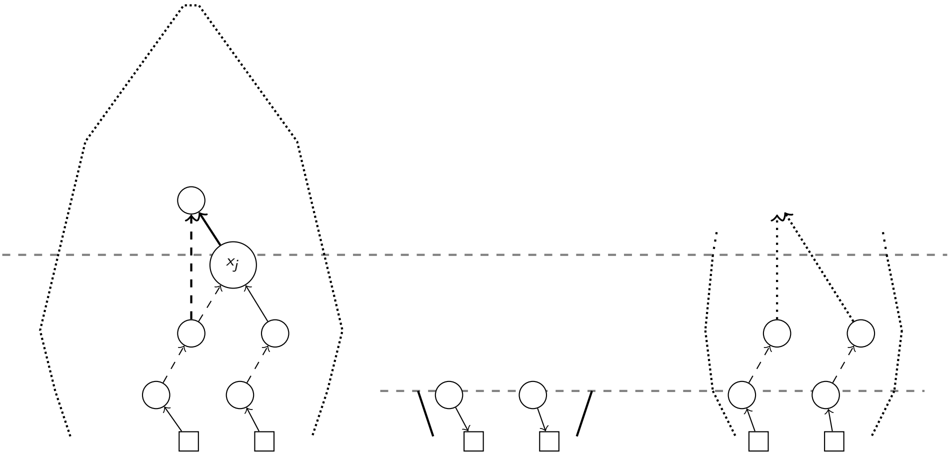

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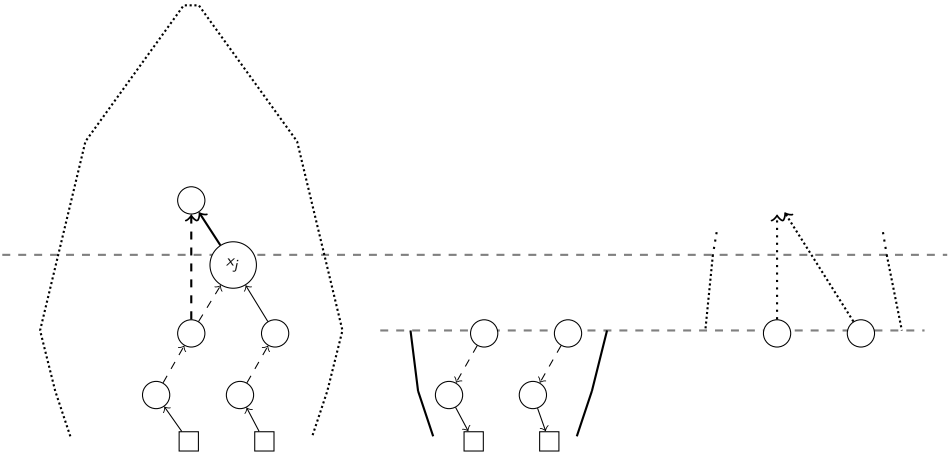

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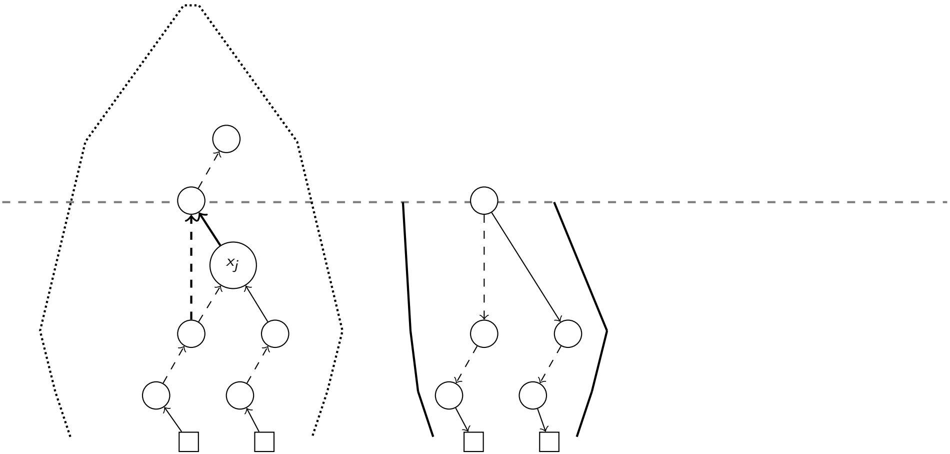

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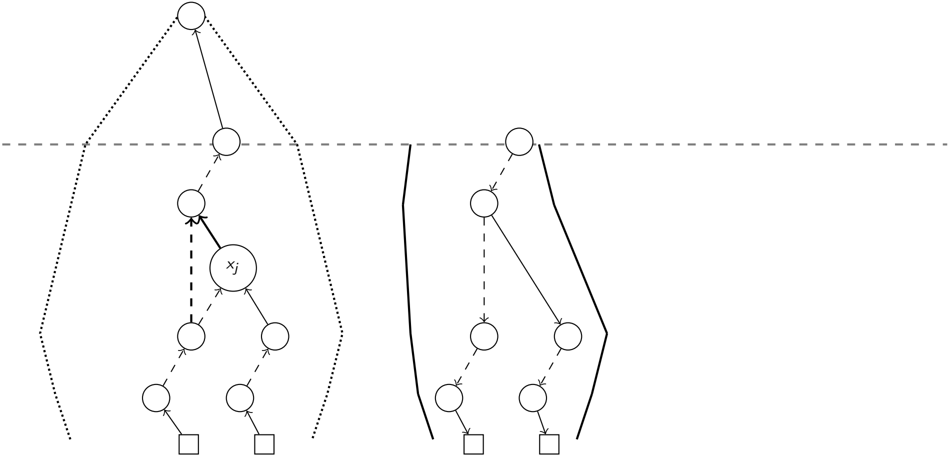

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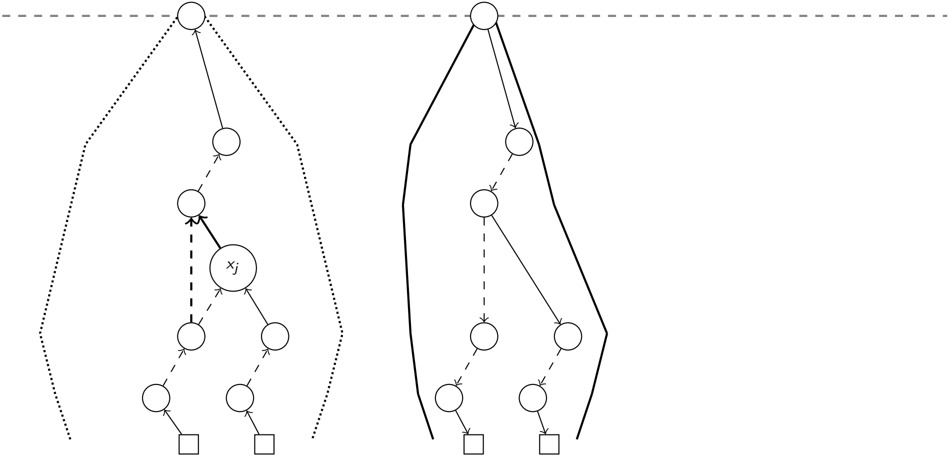

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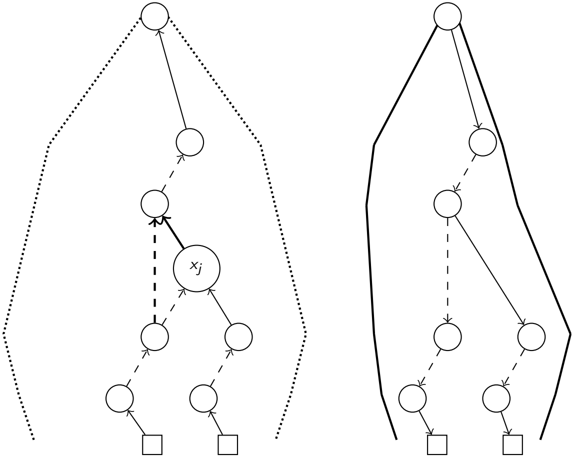


Theorem (Sølvsten et al. '25)

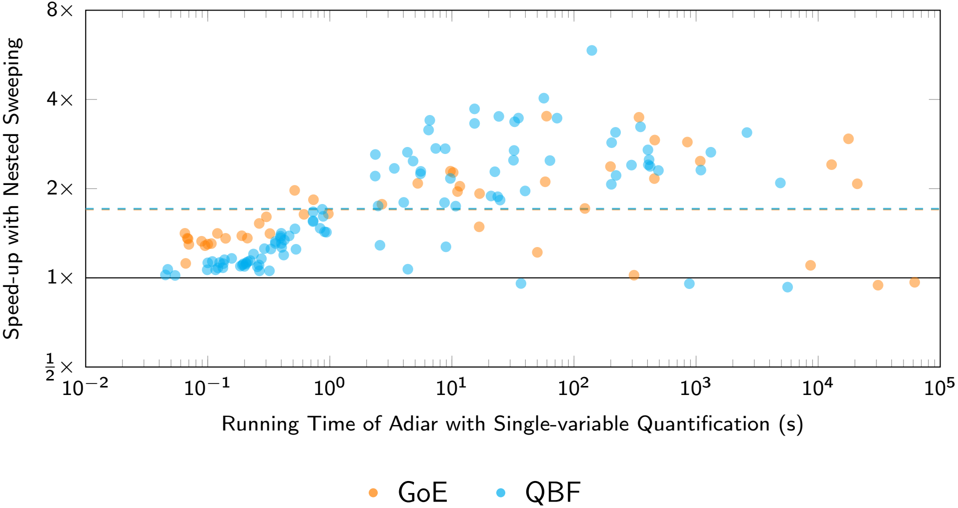
Given BDD ϕ, the quantification of k variables, $\exists \vec{x}.\ \phi(\vec{x})$, is computible in $\mathcal{O}(\text{sort}(|\phi|^{2^k}))$ time and I/Os.

Benchmarks

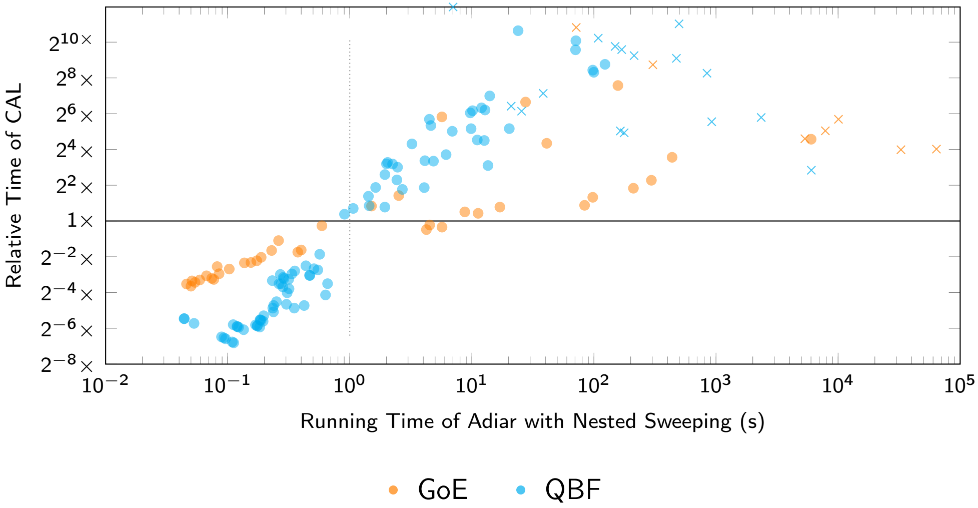
Garden-of-Eden
Given dimensions $N_1, N_2 \in \mathbb{N}$, determine whether there exists in Conway's *Game of Life* an initial state of size $N_1 \times N_2$ that is a *Garden of Eden*, i.e. is otherwise unreachable.

Quantified Boolean Formula
Determine whether a Boolean formula $\exists \vec{x_1} \forall \vec{x_2} \ldots \exists \vec{x_k}.\ \phi(\vec{x_1}, \vec{x_2}, \ldots, \vec{x_k})$ (or any order of quantifiers) evaluates to $\top$ or $\bot$. For inputs, we use the two-player games from:
Irfansha Shaik and Jaco van de Pol: "*Concise QBF encodings for games on a grid (extended version)*". arXiv (2023).

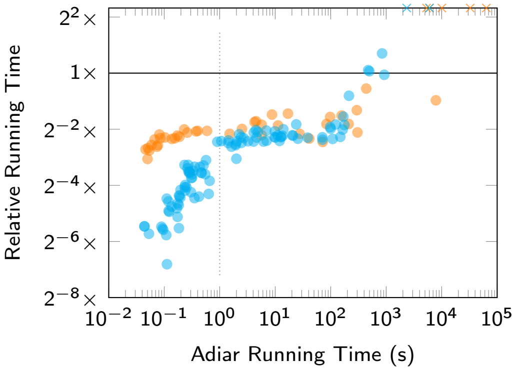
Benchmarks : Single vs. Nested Quantification



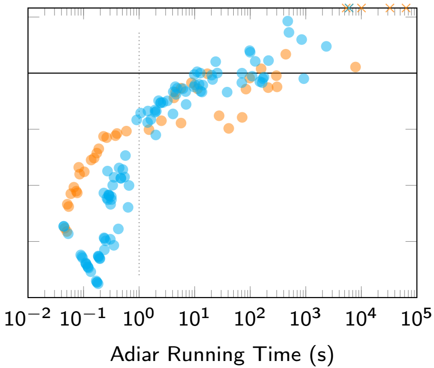
Benchmarks : Comparison to CAL



Benchmarks : Comparison to BuDDy and CUDD



(a) BuDDy



(b) CUDD

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Adiar

github.com/ssoelvsten/adiar

ssoelvsten.github.io/adiar

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Nested Sweeping Framework

New BDD algorithms:

- [x] Multi-variable Quantification
- [x] Relational Product
- [ ] Functional Composition
- [ ] Variable Reordering

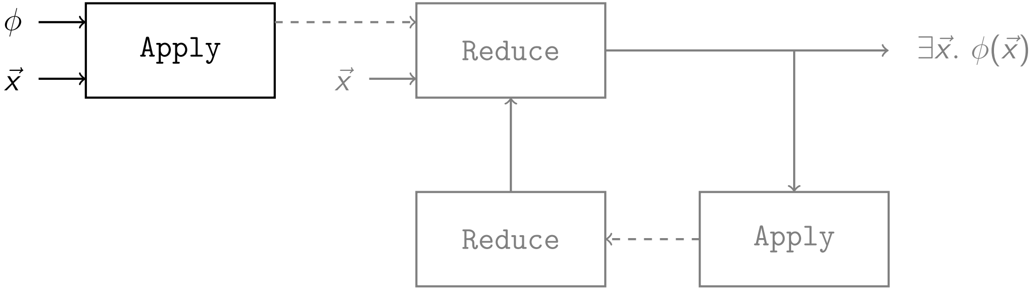
Other Decision Diagrams:

- [ ] Quantum Multi-valued Decision Diagrams
- [ ] Polynomial Boolean Rings

Optimisations for Nested Sweeping

- **Leave terminal arcs out of nested sweeps.**
  Required to satisfy invariants in nested algorithms from [TACAS 22].
- **Bail-out of Inner Sweep if all edges are subtree-preserving.**
  In practice, 75.6% of all levels are skippable (median of 81.9% for each benchmark).
- **Use a sorted list as a bridge from the outer to the inner sweep.**
  Postpones initialising data structures for the nested sweep until it is invoked.
  - More memory available for the outer `Reduce` sweep.
  - Sorting once and then merging on-the-fly with a priority queue can be faster than maintaining a larger priority queue.
  - This enables the *levelised priority queues* [TACAS* 22], *levelised cuts* [ATVA 23], and *levelised random access* [SPIN 24] optimisations.

Better Transposition



Better Transposition: Deepest Variable Quantification

$$bdd\_exists(f, \max(\vec{x}))$$

Motivation
Includes the first nested sweep inside of the transposition step.

In Practice
Slows down computation time on average by 4.7%.

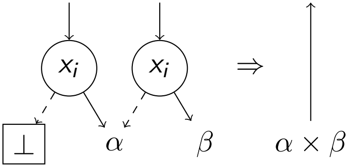
Better Transposition: Pruning $\top$ Siblings

(a) Pruning due to $\top$.

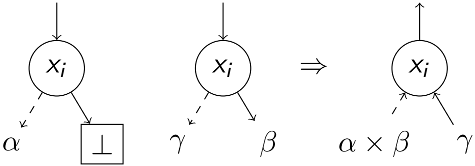
(b) Pruning due to $\bot$.

In Practice
If it prunes subtrees, running time can improve up to 21%. Otherwise, it adds an overhead of up to 2%.

Better Transposition: Partial Quantification



(a) Fully quantified pair of nodes.



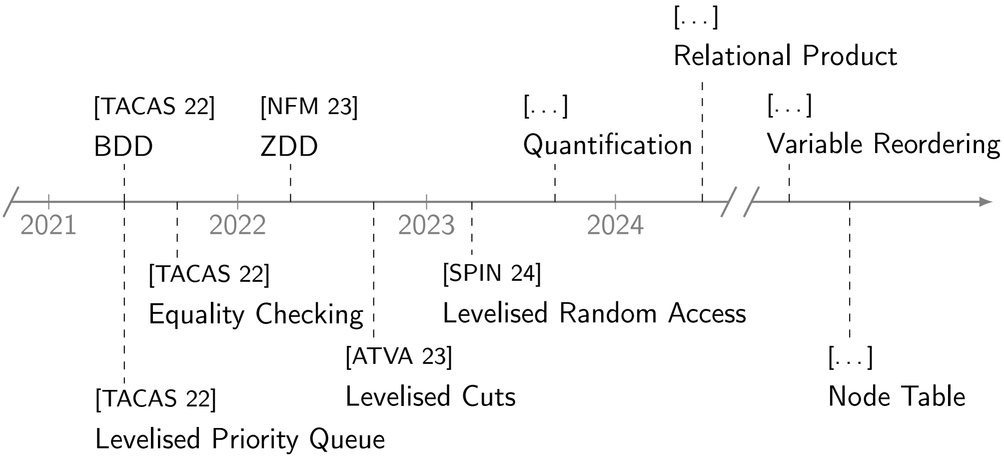
(b) Partially quantified pair of nodes.

In Practice
In some instances, it improves performance by a factor of $\sim 2\times$.

In others, it slows down by $\sim 2\times$.

Observation
Instances improved by $\top$ *Pruning* were disjoint from *Partial Quantification*.

	Single Quantification		Nested Sweeping	
	LOC	# Tests	LOC	# Tests
`nested_sweeping.h`	–	–	1287	104
`quantify`	548	84	1152	152
`core/...`	326	–	904	–
`bdd/...`	122	64	157	108
`zdd/...`	100	20	20	44


[...]
Relational Product
[TACAS 22]
BDD
[NFM 23]
ZDD
[...]
Quantification
[...]
Variable Reordering
2021
2022
2023
2024
[TACAS 22]
Equality Checking
[SPIN 24]
Levelised Random Access
[ATVA 23]
Levelised Cuts
[...]
Node Table
[TACAS 22]
Levelised Priority Queue