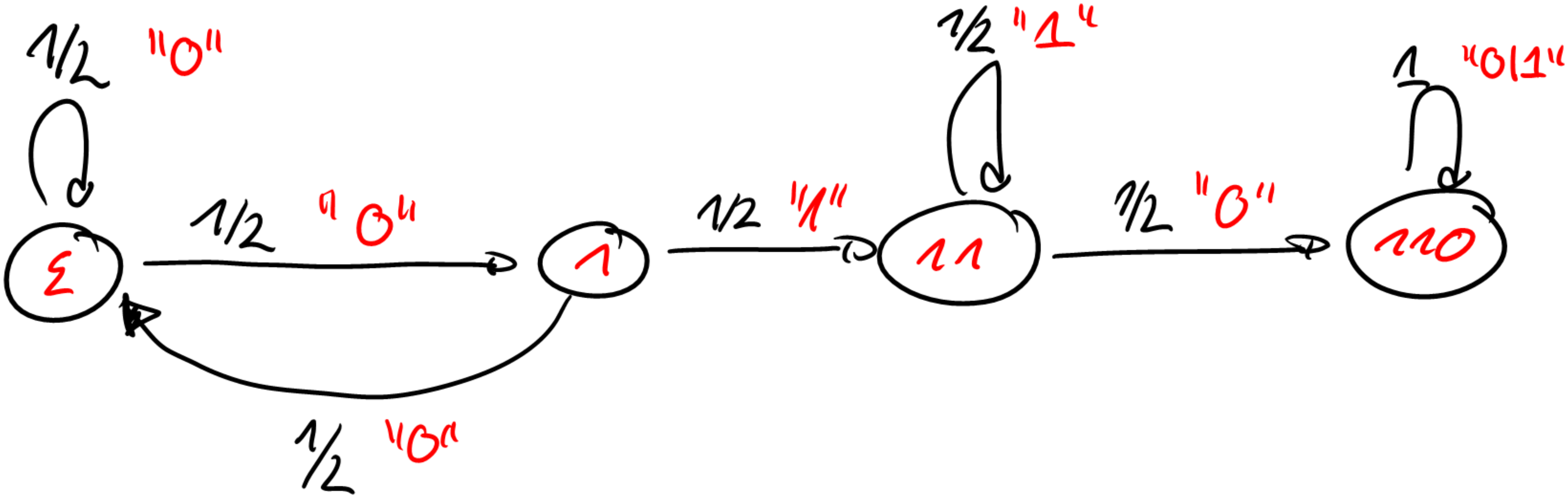
HA 4.1

(a)



⑤ <u>Wie in TA 3.1:</u>

$$z_k := \left(\Pr[Z_k = \varepsilon], \Pr[Z_k = 1], \Pr[Z_k = 11]\right)$$

<u>Beachte:</u> $\Pr[Z_k = d] = 1 - z_k \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$z_{k+1} = z_k \underbrace{\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}}_{M}$$

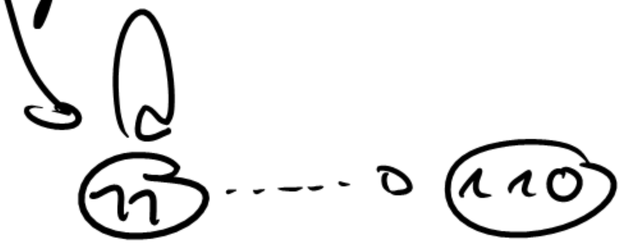
$M \leadsto M = Q \cdot J \cdot Q^{-1}$

$$Q = \begin{pmatrix} -1 & \bar{\Phi} & \Phi \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Phi = \frac{1+\sqrt{5}}{2}$$

$$\bar{\Phi} = \frac{1-\sqrt{5}}{2}$$

$$D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \boxed{\begin{matrix} \frac{\bar{\Phi}}{2} & 0 \\ 0 & \frac{\Phi}{2} \end{matrix}} \end{pmatrix}$$

gehört zu



gehört zu

$$Q^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 0 & 1 \\ -1 & \Phi & -1 \\ 1 & -\bar{\Phi} & 1 \end{pmatrix}$$

$$\rightsquigarrow \Pr[Z_k \neq d]$$

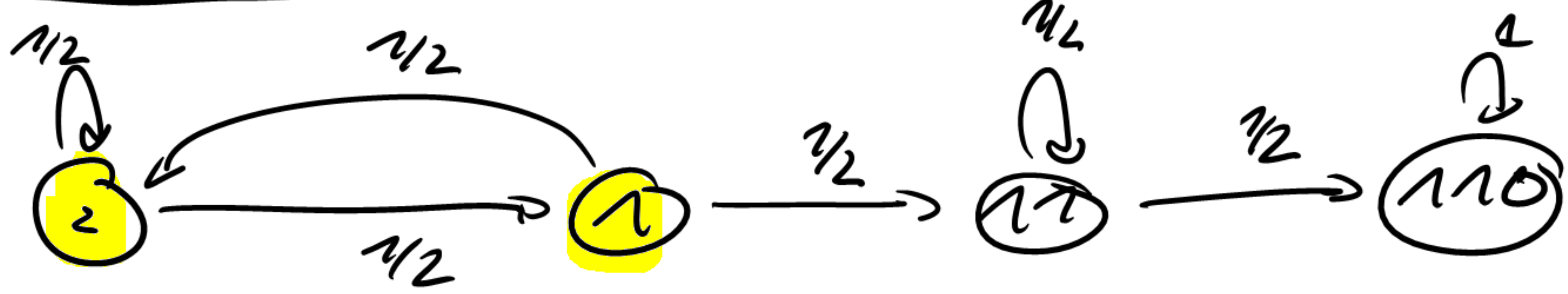
$$= \langle z_k , (1,1,1) \rangle$$

$$= \underbrace{(1,0,0)}_{= z_0} \cdot Q \cdot J^k \cdot Q^{-1} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= (-1, \bar{\Phi}, \Phi) \cdot 2^{-k} \cdot \begin{pmatrix} 1 & & \\ & \bar{\Phi}^k & \\ & & \Phi^k \end{pmatrix} \begin{pmatrix} \sqrt{5} \\ \Phi - 2 \\ 2 - \bar{\Phi} \end{pmatrix} \cdot \frac{1}{\sqrt{5}}$$

$$\rightarrow \Pr[W \geq n] = \Pr[Z_{n-1} \neq d]$$

Alternativ



- $\Pr[Z_0 = \varepsilon] = 1$

- $\Pr[Z_{k+1} = \varepsilon] = \frac{1}{2} \Pr[Z_k = \varepsilon] + \frac{1}{2} \Pr[Z_k = 1]$

  $\Pr[Z_{k+1} = 1] = \frac{1}{2} \Pr[Z_k = \varepsilon]$

$\rightsquigarrow \Pr[Z_{k+2} = \varepsilon] = \frac{1}{2} \Pr[Z_{k+1} = \varepsilon] + \frac{1}{4} \Pr[Z_k = \varepsilon]$

$\rightsquigarrow$ Gewichtete Fibonacci-Zahlen

- Induktion zeigt:

$$\Pr[z_k = \varepsilon] = \frac{F_{k+1}}{2^k} \text{ mit } \begin{array}{l} F_k = 0 \ \forall k < 0 \\ F_0 = 0 \\ F_1 = 1 \\ F_{k+1} = F_k + F_{k-1} \end{array}$$

- $\leadsto \Pr[z_k = 1] = \frac{1}{2} \Pr[z_{k-1} = \varepsilon] = \frac{F_k}{2^k}$

- $\Pr[z_k = 11] = \frac{1}{2} \Pr[z_{k-1} = 11] + \frac{1}{2} \Pr[z_{k-1} = 1]$

$$= \frac{F_{k-1}}{2^k} + \frac{F_{k-2}}{2^k} + \ldots + \frac{F_0}{2^k}$$

$$= \frac{F_{k+1} - 1}{2^k}$$

$\leadsto \Pr[Z_k = 110]$

$$= 1 - \frac{F_{k+1}}{2^k} - \frac{F_k}{2^k} - \frac{F_{k+1} - 1}{2^k}$$

$$= 1 - \frac{F_{k+3}}{2^k} + 2^{-k}$$

mit $F_k = \dfrac{\Phi^k - \bar{\Phi}^k}{\Phi - \bar{\Phi}}$, $\Phi = \dfrac{1+\sqrt{5}}{2}$, $\bar{\Phi} = \dfrac{1-\sqrt{5}}{2}$

(c) $\mathbb{E}[W] = \sum_{n\geq 1} \Pr[W \geq n]$

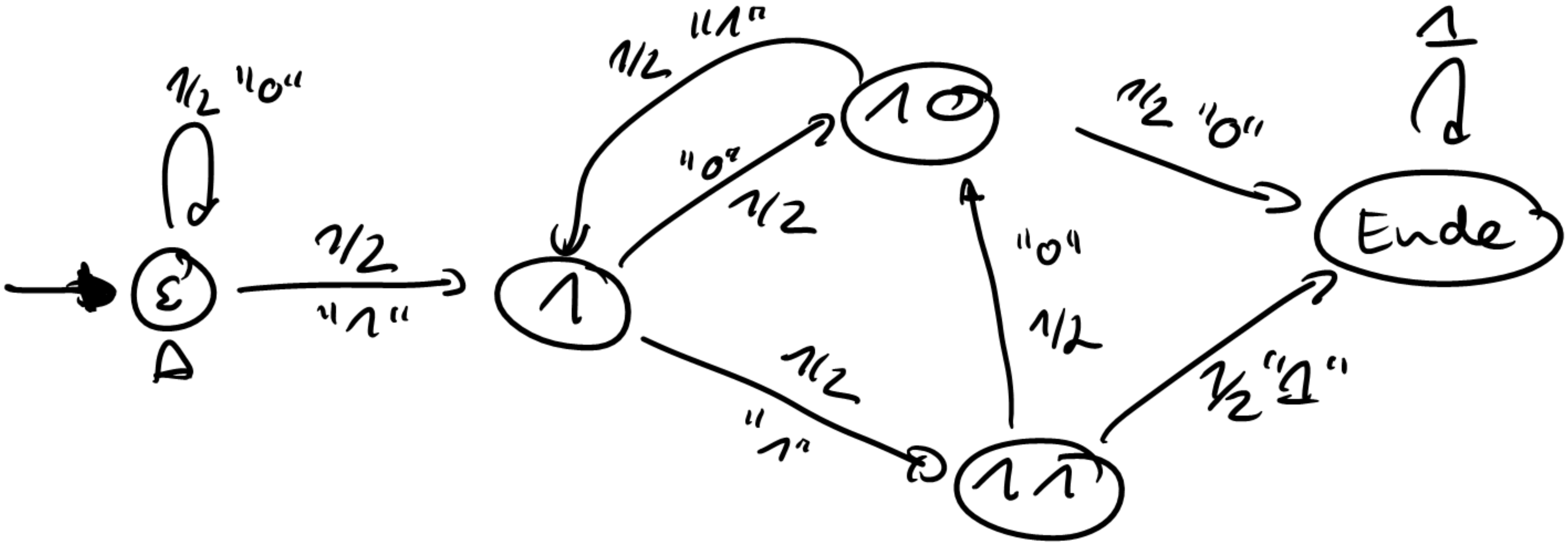
110, nicht d (Überbleipsel aus TA 3)

$$= \sum_{n\geq 0} \Pr[z_n \neq d]$$

$$= \frac{1}{\sqrt{5}} \left(-1, \overline{\Phi}, \Phi\right) \underbrace{\begin{pmatrix} \sum_{n\geq 0} 2^{-n} \\ \sum_{n\geq 0} (2^{-1}\overline{\Phi})^n \\ \sum_{n\geq 0} (2\Phi)^n \end{pmatrix}}_{\begin{pmatrix} (1-\frac{1}{2})^{-1} \\ (1-\overline{\Phi}/2)^{-1} \\ (1-\Phi/2)^{-1} \end{pmatrix}} \begin{pmatrix} \sqrt{5} \\ \Phi - 2 \\ 2 - \overline{\Phi} \end{pmatrix}$$

$$= 8$$

(d)



(c) Da das Spiel mit W'keit 1 endet, reicht es eine Gewinnw'keit zu bestimmen.

Sei M_x : "Maxi gewinnt von x aus".

$$\leadsto \Pr[M_\varepsilon] = \frac{1}{2}\Pr[M_\varepsilon] + \frac{1}{2}\Pr[M_1]$$

$$= \Pr[M_1]$$

$$\Pr[M_1] = \frac{1}{2}\Pr[M_{10}] + \frac{1}{2}\Pr[M_{11}]$$

$$\Pr[M_{10}] = \frac{1}{2}\Pr[M_1] + \frac{1}{2}\cdot 1$$

$$\Pr[M_{11}] = \frac{1}{2}\Pr[M_{10}] + \frac{1}{2}\cdot \emptyset$$

Damit

$$\Pr[M_\Sigma] = \Pr[M_1] = \frac{3}{4}\Pr[M_{10}]$$

$$= \frac{3}{8} + \frac{3}{8}\Pr[M_1]$$

$$= \frac{3}{5}$$

$\rightsquigarrow$ Also hat Maxi einen Vorteil.

HA4.2

(a) Sei A das Ereignis, dass die Komponente A funktioniert.

System funktioniert entspricht:

$$A \wedge (C \vee D) \vee B \wedge (A \wedge (C \vee D) \vee D)$$

$$\equiv A \wedge (C \vee D) \vee B \wedge D$$

$$\rightsquigarrow \Pr[A \wedge (C \vee D) \vee B \wedge D \mid \neg A] = \frac{\Pr[\neg A \wedge B \wedge D]}{\Pr[\neg A]}$$

$\neg A, B, D$ unabh. $\Pr[B] \Pr[D] = (1-p)^2$

(b) $\Pr[A \wedge ((C \vee D) \vee B \wedge D) \mid A]$

$$= \frac{\Pr[A \wedge ((C \vee D) \vee A \wedge B \wedge D]}{\Pr[A]}$$

$$= \frac{\Pr[A \wedge (C \vee D)]}{\Pr[A]}$$

A, $C \vee D$ unabh.

$$= \Pr[C \vee D] = 1 - \Pr[\neg C]\Pr[\neg D]$$

$$= 1 - p^2$$

Damit

$$\Pr[A \wedge (C \vee D) \vee B \wedge D]$$

$$= p(1-p)^2 + (1-p)(1-p^2)$$

$$= p - 2p^2 + p^3 + 1 - p^2 - p + p^3$$

$$= \underline{\underline{1 - 3p^2 + 2p^3}}$$

(c) • $\Pr[N=4 \mid A \wedge (C \vee D) \vee B \wedge D] = 0$

• $\Pr[N=3 \mid A \wedge (C \vee D) \vee B \wedge D] = 0$

<u>Da:</u> Damit das System funktioniert, müssen mindestens 2 Komponenten funktionieren.

- $\Pr[N=2 \mid A\wedge(C\vee D) \vee B\wedge D]$

$$= \frac{\Pr[A\wedge C\wedge\neg B\wedge\neg D \vee A\wedge D\wedge\neg B\wedge\neg C \vee B\wedge D\wedge\neg A\wedge\neg C]}{\Pr[A\wedge(C\vee D)\vee B\wedge D]}$$

$$= \frac{3p^2(1-p)^2}{1-3p^2+2p^3}$$

- $\Pr[N=0 \mid A\wedge(C\vee D) \vee B\wedge D]$

$$= \frac{\Pr[A\wedge B\wedge C\wedge D]}{1-3p^2+2p^3} = \frac{(1-p)^4}{1-3p^2+2p^3}$$

- $\Pr[N=1 \mid A \wedge (C \vee D) \vee B \wedge D]$

$$= 1 - \frac{3p^2(1-p)^2 + (1-p)^4}{1-3p^2+2p^3}$$

$$= \frac{4p(1-p)^3}{1-3p^2+2p^3}$$