AA 7.1

$N \sim \text{Bin}(120, \frac{1}{2})$, $\mathbb{E}[N] = 60$, $\text{Var}[N] = 30$

ⓐ (i) $\Pr[N > k] = \Pr[N \geq k+1] \leq \dfrac{\mathbb{E}[N]}{k+1} \overset{!}{\leq} 0.1$

$$\rightsquigarrow \quad 600 \overset{!}{\leq} k+1$$

$$\rightsquigarrow \quad \underline{k = 599} \quad (\ldots)$$

(ii) $\Pr[N > k] = \Pr[N \geq k+1] \leq \Pr[\,|N-60| \geq \underset{k>59}{(k-59)}\,]$

$$\leq \frac{30}{(k-59)^2} \overset{!}{\leq} 0.1$$

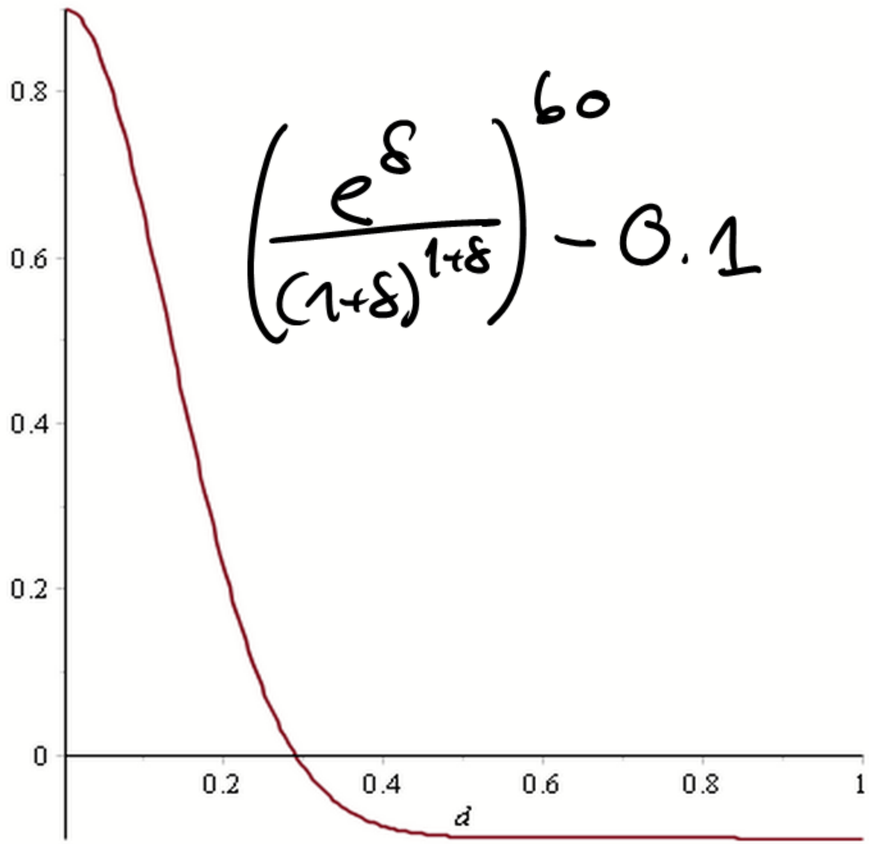
$$\rightsquigarrow \quad (k-59)^2 \geq 300 \overset{(k>59)}{\rightsquigarrow} k \geq 59 + \sqrt{300} \approx 76{,}32\ldots$$

$$\rightsquigarrow \quad \underline{k = 77}$$

iii) $\Pr[N \geq (1+\delta)60] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{60} \overset{!}{\leq} 0.1$

für $(1+\delta)60 = k+1$ mit $\delta > 0$

$\rightsquigarrow \quad k = \lceil 60 \cdot \delta + 59 \rceil$



$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{60} - 0.1$

$\rightarrow \delta \in [0.289, 0.29]$

$\delta \approx 0.28955\ldots$

$\rightsquigarrow \quad k = 77$

(b) $\Pr[|X - \frac{n}{2}| \geq \delta] = \Pr[X \geq \frac{n}{2} + \delta] + \Pr[X \leq \frac{n}{2} - \delta]$

$$\Pr[X \leq \frac{n}{2} - \delta] = \Pr[n - X \geq \frac{n}{2} + \delta] = \Pr[X \geq \frac{n}{2} + \delta]$$

Da: $Y = n - X$ zählt die Misserfolge, während X die Erfolge zählt. Bei $p = 1/2$ besitzen beide dieselbe Dichte.

Genauer: X = X_1+...+X_n mit X_1,...,X_n unahb. Ber(1/2)-verteilt.
Nach VL: Dann auch 1-X_1,...,1-X_n unahb;
Offensichtlich 1-X_i ebenfalls Ber(1/2)-verteilt
Somit: Y= (1-X_1) + ... + (1-X_n) = n - X ebenfalls Bin(n,1/2)-verteilt.

Damit: $\Pr[N \geq \frac{n}{2} + \delta] = \frac{1}{2} \Pr[|N - \frac{n}{2}| \geq \delta] \leq \frac{n/4}{2 \cdot \delta^2}$ $\square$

(c) $\Pr[N > k] = \Pr[N \geq k+1]$

$$= \Pr[N \geq 60 + (k-59)] \quad (k > 59)$$

$$\leq \frac{120}{8(k-59)^2} \overset{!}{\leq} 0.1$$

$$\rightsquigarrow (k-59)^2 \geq 150 \overset{(k>59)}{\rightsquigarrow} k \geq 59 + \sqrt{150}$$

$$\approx 71{,}24\ldots$$

$$\rightsquigarrow k = \underline{72}$$

e

$\rightsquigarrow \underline{k^* = 67}$

Untitled (1)* - [Server 1] - Maple 16

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

$$evalf\left(sum\left(\mathrm{binomial}(120, k), k = 73..120\right) \cdot 2^{-120}\right)$$

$$0.01104135790$$

$$evalf\left(sum\left(\mathrm{binomial}(120, k), k = 67..120\right) \cdot 2^{-120}\right)$$

$$0.1176016840$$

$$evalf\left(sum\left(\mathrm{binomial}(120, k), k = 68..120\right) \cdot 2^{-120}\right)$$

$$0.08532260657$$

HA 7.2 $\quad S = \sum_{i=1}^{N} X_i \quad , W_S = \mathbb{N}_0$

ⓐ $G_S(z) = \mathbb{E}[z^S]$

$$= \sum_{n=0}^{\infty} \Pr[N=n]\, \mathbb{E}[z^S \mid N=n]$$

$$= \sum_{n=0}^{\infty} \Pr[N=n]\, \mathbb{E}[z^{X_1+\dots+X_n}]$$

$\Pr[S=k \mid N=n]$

$= \Pr[X_1+\dots+X_n=k \mid N=n]$

$= \Pr[X_1+\dots+X_n=k]$,

da $N, X_1, \dots, X_n$ unabh.

$$= \sum_{n=0}^{\infty} \Pr[N=n]\, \mathbb{E}[z^{X_1}] \cdot \ldots \cdot \mathbb{E}[z^{X_n}]$$

$X_1, \dots, X_n$ unab $\Rightarrow z^{X_1}, \dots, z^{X_n}$ unabh.

$$= \sum_{n=0}^{\infty} \Pr[N=n]\, G_{X_1}(z)^n$$

$X_1, \dots, X_n$ identisch verteilt

Für $z \in [0,1]$ gilt auch $G_{X_1}(z) \in [0,1]$.

Setze $y := G_{X_1}(z)$.

$$\leadsto \sum_{n=0}^{\infty} P_r[N=n]\, G_{X_1}(z)^n$$

$$= \sum_{n=0}^{\infty} P_r[N=n]\, y^n$$

$$= G_N(y)$$

$$= G_N(G_{X_1}(z)) \quad \square$$

⑤ $N \sim Poi(\lambda) \Rightarrow G_N(z) = e^{\lambda(z-1)}$

$X_1 \sim Ber(p) \Rightarrow G_{X_1}(z) = (1-p) + pz$

$$\overset{ⓐ}{\Rightarrow} G_S(z) = e^{\lambda((1-p)+pz-1)}$$

$$= e^{\lambda - \lambda p + \lambda p z - \lambda}$$

$$= e^{\lambda p(z-1)}$$

$\leadsto$ $G_S(z)$ ist die erzeugende Funktion zu $Poi(\lambda p)$

$\leadsto$ $S \sim Poi(\lambda p)$ auf Grund der Eindeutigkeit.

HA 7.3

ⓐ $G_a(z) = \frac{1}{2} z G_a(z) + \frac{1}{2} z G_b(z)$ (angegeben)

$$G_b(z) = \frac{1}{4} z G_a(z) + \frac{1}{4} z G_c(z) + \frac{1}{2} z \underbrace{G_d(z)}_{=1}$$

$$G_c(z) = \frac{1}{2} z G_c(z) + \frac{1}{2} z G_b(z)$$

Begründung (nicht verlangt)

$Z_i \hat{=}$ Zustand nach i Schritten

- $\mathbb{E}_a[s^N] = \Pr_a[Z_1 = a]\, \mathbb{E}_a[s^N \mid Z_1 = a]$
$$+ \Pr_a[Z_1 = b]\, \mathbb{E}_a[s^N \mid Z_1 = b]$$

nach VL.

- Schon bekannt: $\Pr_a[Z_1 = a] = \delta(a,a)$, $\Pr_a[Z_1 = b] = \delta(a,b)$
- $\mathbb{E}_a[s^N \mid Z_1 = a] = \sum_{\pi \in [a \rightsquigarrow d]^{d=1} \cap [Z_1 = a]} z^{N(\pi)} \Pr[\pi]$

Wie in TA 4.2: $[a \rightsquigarrow d]^{d=1} \cap [Z_1 = a] = a[a \rightsquigarrow d]^{d=1}$

$$\leadsto \sum_{\pi \in a[a \leadsto d]^{d=1}} z^{N(\pi)} \; P_r[\pi]$$

$$= \sum_{\substack{\pi' \in [a \leadsto d]^{d=1} \\ \pi = a\pi'}} z^{N(a\pi')} \; P_r[a\pi']$$

$$\left[N(a\pi') = 1 + N(\pi') \; ; \; \underset{da\pi' \in [a \leadsto d]^{d=1}}{P_r[a\pi'] = \delta(a,a) P_r[\pi']}, \right]$$

$$= \sum_{\pi' \in [a \leadsto d]^{d=1}} z^{1+N(\pi')} \cdot P_r[\pi']$$

$$= \mathbb{E}_a[z^{1+N}] \; . \quad \left| \; \mathbb{E}_a[z^N \mid Z_1 = b] = \mathbb{E}_b[z^{1+N}] \right.$$

analog.

6 Lösen des LGS in $\{G_a(z), G_b(z), G_c(z)\}$:

$$G_a(z) = \frac{1}{2} z\, G_a(z) + \frac{1}{2} z\, G_b(z)$$

$$G_b(z) = \frac{1}{4} z\, G_a(z) + \frac{1}{4} z\, G_c(z) + \frac{1}{2} z$$

$$G_c(z) = \frac{1}{2} z\, G_c(z) + \frac{1}{2} z\, G_b(z)$$

$$\leadsto \quad G_a(z) = G_c(z) = \frac{1/2 \cdot z}{1 - 1/2 \cdot z}\, G_b(z)$$

$$\leadsto \quad G_b(z) = \frac{1/4\, z^2}{1 - 1/2\, z}\, G_b(z) + 1/2\, z$$

$$= \frac{1/2\, z}{1 - \frac{1/4\, z^2}{1 - 1/2\, z}}$$

Test :

Untitled (1)* - [Server 1] - Maple 16

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

> restart

$G_b(z) := \frac{1}{2}\cdot\frac{z}{1-\frac{1}{4}\cdot\frac{z^2}{1-\frac{1}{2}\cdot z}}$

$$z \rightarrow \frac{1}{2}\,\frac{z}{1-\frac{1}{4}\,\frac{z^2}{1-\frac{1}{2}z}} \qquad (1)$$

$G_b(1)$

$$1 \qquad (2)$$

$eval\left(diff\left(G_b(z), z\right), [z=1]\right)$

$$4 \qquad (3)$$

$G_a(z) := \frac{1}{2}\cdot\frac{z}{1-\frac{1}{2}\cdot z}\cdot G_b(z)$

$$z \rightarrow \frac{1}{2}\,\frac{z\,G_b(z)}{1-\frac{1}{2}z} \qquad (4)$$

$G_a(1)$

$$1 \qquad (5)$$

$eval\left(diff\left(G_a(z), z\right), [z=1]\right)$

$$6 \qquad (6)$$

Ready

C:\Program Files\Maple 16 Memory: 2.0M Time: - Math Mode

EN 98% 15:59