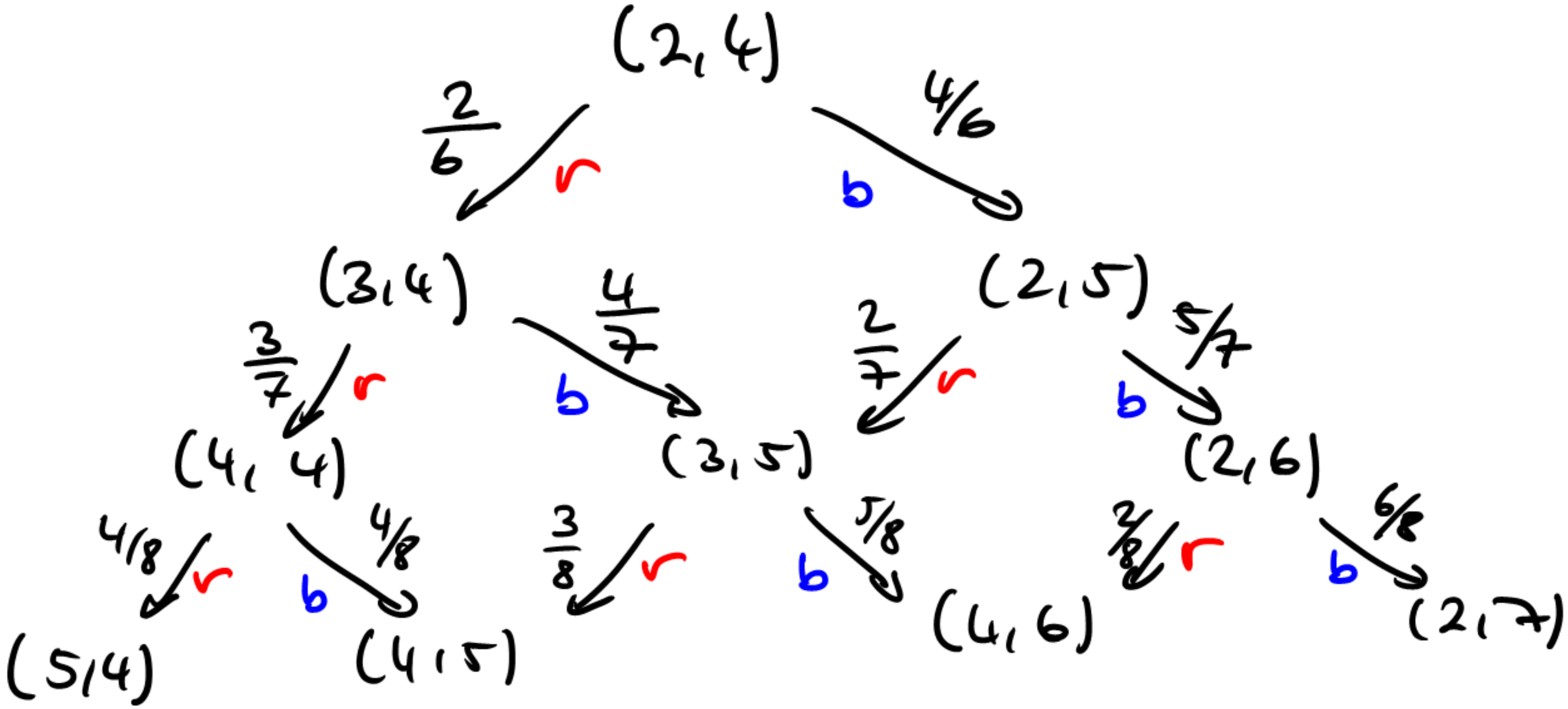
TA 2.1 Experiment



$$\Omega = \{r, b\}^3$$

(a) $\Pr[X_2 = 1] = \Pr[\{(k_1, k_2, k_3) \in \Omega \mid k_2 = r\}]$

$$= \frac{2}{6} \cdot \frac{3}{7} \cdot \left(\frac{4}{8} + \frac{4}{8}\right) + \frac{4}{6} \cdot \frac{2}{7} \cdot \left(\frac{3}{8} + \frac{5}{8}\right)$$

$$=$$

(b) $$\Pr[R_2 = r_0] = \Pr[\{(b,b,k) \mid k \in \{r,b\}\}]$$
$$= \frac{b_0}{r_0+b_0} \cdot \frac{b_0+1}{r_0+b_0+1} = \frac{4}{6} \cdot \frac{5}{7}$$

$$\Pr[R_2 = r_0+1] = \Pr[\{(r,b,k), (b,r,k) \mid k \in \{r,b\}\}]$$
$$= \frac{r_0}{r_0+b_0} \cdot \frac{b_0}{r_0+b_0+1} + \frac{b_0}{r_0+b_0} \cdot \frac{r_0}{r_0+b_0+1}$$
$$= 2 \cdot \frac{2 \cdot 4}{6 \cdot 7}$$

$$\Pr[R_2 = r_0+2] = \Pr[\{(r,r,k) \mid k \in \{r,b\}\}]$$
$$= \frac{r_0}{r_0+b_0} \cdot \frac{r_0+1}{r_0+b_0+1} = \frac{2 \cdot 3}{6 \cdot 7}$$

(c)

$$\Pr[X_3 = 1] = \frac{r_0 \cdot (r_0+1) \cdot (r_0+2)}{(r_0+b_0)(r_0+b_0+1)(r_0+b_0+2)}$$

$$+ 2 \cdot \frac{r_0 \cdot b_0 \cdot (r_0+1)}{(r_0+b_0)(r_0+b_0+1)(r_0+b_0+2)}$$

$$+ \frac{b_0 \cdot (b_0+1) \cdot r_0}{(r_0+b_0)(r_0+b_0+1)(r_0+b_0+2)}$$

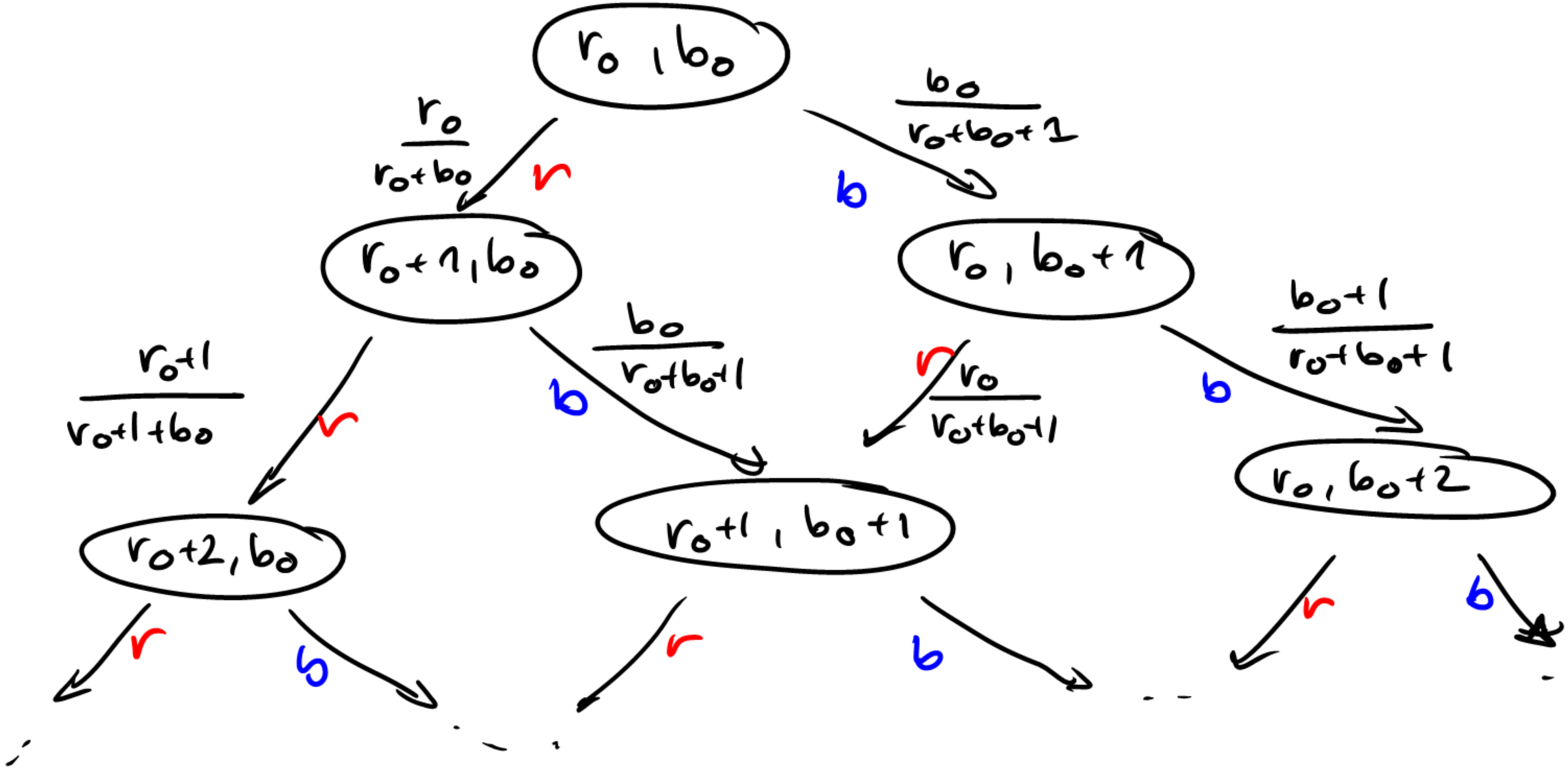
$$= \frac{2 \cdot 3 \cdot 4 + 2 \cdot 2 \cdot 4 \cdot 3 + 4 \cdot 5 \cdot 2}{6 \cdot 7 \cdot 8}$$

$$\Pr[X_3 = 0] = 1 - \Pr[X_3 = 1] =$$

$$\mathbb{E}[R_3] = \mathbb{E}[R_2 + X_3] = \mathbb{E}[R_2] + \mathbb{E}[X_3]$$

$$= \mathbb{E}[R_2] + \Pr[X_3 = 1]$$

(d) Experimentverlauf allgemein



- R_k nimmt Werte in $\{r_0, \dots, r_0+k\}$ an.
- Für $x \in \{r_0, \dots, r_0+k\}$ ist $[R_k = x]$ durch alle Wörter (= Pfade) der Länge k mit genau x "r" und k-x "b" gegeben.

- Anzahl solcher Pfade: $\binom{k}{x}$

- Jeder solche Pfad hat die W'keit:

$$\frac{r_0\,(r_0+1)\cdot\ldots\cdot(r_0+x)\cdot b_0\cdot(b_0+1)\cdot\ldots\cdot(b_0+k-1-x)}{(r_0+b_0)\,(r_0+b_0+1)\cdot\ldots\cdot(r_0+b_0+k-1)}$$

- Wegen Kommutativität von $\langle\mathbb{R},\cdot,1\rangle$ spielt die Reihenfolge, in der die roten Kugeln gezogen werden, keine Rolle.

$$\rightsquigarrow \Pr[R_k = x+r_0] = \binom{k}{x}\frac{\prod_{i=0}^{x}(r_0+i)\prod_{j=0}^{k-1-x}(b_0+j)}{\prod_{i=0}^{k-1}(r_0+b_0+i)}$$

(für $k\le n$)

HA 2.2

(a) $\Pr[X_1 = i, X_2 = j]$ gesucht.

Beachte: • $\Pr[X_1 > 0] = 1$; es gibt stets einen ersten Block

• Falls $i+j < n$, so gibt es mindestens einen dritten Block, d.h. man betrachtet Elementarereignisse der Form:

$$0^i 1^j 0 \ldots \quad \text{oder} \quad 1^i 0^j 1 \ldots$$

Damit:

• $\Pr[X_1 = i, X_2 = n-i] = \frac{|\{0^i 1^{n-i}, 1^i 0^{n-i}\}|}{2^n} = 2^{1-n}$
für $i \in \{1, 2, \ldots, n\}$

• $\Pr[X_1 = i, X_2 = j] = \frac{|\{0^i 1^j 0x, 1^i 0^j 1x \mid x \in \{0,1\}^{n-i-j-1}\}|}{2^n} = 2^{-i-j}$
für $j \in \{1, 2, \ldots, n-1-i\}$

(b) $\Pr[X_2 = 0] = 2^{1-n}$

$$\Pr[X_2 = j] = \Pr[X_1 = n-j, X_2 = j]$$

$$(j \in \{1, \dots, n-1\}) \quad + \sum_{i=1}^{n-j-1} \Pr[X_1 = i, X_2 = j]$$

$$= 2^{1-n} + \sum_{i=1}^{n-j-1} 2^{-i-j}$$

$$= 2^{1-n} + 2^{-j} \sum_{i=1}^{n-j-1} 2^{-i}$$

$$= 2^{1-n} + 2^{-j} \cdot \sum_{i=0}^{n-j-2} 2^{-(i+1)}$$

$$= 2^{1-n} + 2^{-j-1} \frac{1 - 2^{-n+j+1}}{1 - 1/2}$$

$$= 2^{-j}$$

$$\mathbb{E}[X_2] = \sum_{j=1}^{n-1} j \cdot 2^{-j} = \sum_{j=0}^{n-1} j \cdot 2^{-j} = 2^{-1} \sum_{j=0}^{n-1} j \left(\tfrac{1}{2}\right)^{j-1}$$

$\rightsquigarrow$ mittels: $$\sum_{j=0}^{n-1} z^j = \frac{1-z^n}{1-z}$$

und $$\sum_{j=0}^{n-1} j z^{j-1} = \frac{-n z^{n-1}}{1-z} + \frac{1-z^n}{(1-z)^2}$$

(c)

Alle $\omega \in \{0,1\}^n$ mit $N(\omega)=k$ erhält man,

indem man sich zunächst aussucht,

ob ω mit 0 oder 1 beginnt, und danach

die $k-1$ Positionen wählt, an denen die ersten $k-1$ Blöcke enden.

Dabei darf der $(k-1)$. Block natürlich nicht bei Position n enden.

$$\rightsquigarrow \underset{(k \in [n])}{\Pr[N=k]} = \frac{2 \cdot \binom{n-1}{k-1}}{2^n} = \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1}$$

$\rightsquigarrow$ $\boxed{N-1}$ $\overset{!}{\text{ist}}$ bin$(n-1; 1/2)$-verteilt

$$\circ\ \mathbb{E}[N] = \underbrace{\mathbb{E}[N-1]}_{=\frac{1}{2}(n-1)} + \underbrace{\mathbb{E}[1]}_{=1} = \frac{n+1}{2}$$

$$\circ\ \mathrm{Var}[N] = \mathbb{E}\left[\,|N-\mathbb{E}[N]|^2\right] = \mathbb{E}\left[|N-1-\mathbb{E}[N-1]|^2\right]$$

$$= \mathrm{Var}[N-1]$$

$$\mathrm{Var}[N-1] = \mathbb{E}\left[(N-1)^2\right] - \underbrace{\mathbb{E}[N-1]^2}_{=\left(\frac{1}{2}(n-1)\right)^2} = \underline{\underline{\frac{n-1}{4}}}$$

$$\mathbb{E}\left[(N-1)^2\right] = \sum_{k=0}^{n-1} k^2 \binom{n-1}{k} \left(\frac{1}{2}\right)^{n-1}$$

$$= \left(\frac{1}{2}\right)^{n-1} \left[\sum_{k=0}^{n-1} k^2 \binom{n-1}{k} z^k\right]_{z=1}$$

$$= \left(\frac{1}{2}\right)^{n-1} \left((n-1)(n-2)\,2^{n-3} + (n-1)\,2^{n-2}\right) = \overbrace{\frac{(n-1)(n-2)}{4}}^{\frac{(n-1)^2}{4} - \frac{n-1}{4}} + \frac{n-1}{2}$$

Alternative Lösung:

$$[N(\omega)=k] = \{ a^{i_1} \bar{a}^{i_2} \dots x^{i_k} \mid i_1 + \dots + i_k = n, \; i_1, i_2, \dots, i_k > 0, \; a \in \{0,1\}, \; \bar{a} = 1-a \}$$

$$\begin{pmatrix} x = a \text{ für } k \equiv_2 0 \\ x = \bar{a} \text{ für } k \equiv_2 1 \end{pmatrix}$$

$$= \{ (a, i_1, i_2, \dots, i_k) \mid a \in \{0,1\}, \; i_1 + \dots + i_k = n, \; i_1 > 0, \dots, i_k > 0 \}$$

$$|[N=k]| = 2 \cdot \binom{(n-k)+(k-1)}{k-1} = 2\binom{n-1}{k-1}$$

$n-k$ Striche, $k-1$ Kommata:

$(1,2,1,3)$

$\rightsquigarrow (\varepsilon, \mathbf{|}, \varepsilon, ||)$

TA 2.3

- $A = |X_1 - X_2|$

$$W_A = \{0, 1, 2, 3\}$$

$x_1 = x_2 \quad x_2$

$$\Pr[A=0] = \Pr[X_1 = X_2] = \frac{4 \cdot 4 \cdot}{4^3} = \frac{1}{4}$$

$$\Pr[A=1] = \Pr[X_1 = X_2 + 1] + \Pr[X_2 = X_1 + 1]$$

$$= 2 \Pr[\underset{2,3,4}{X_1} = \underset{1,2,3}{X_2} + 1] = \frac{2 \cdot 3 \cdot 4}{4^3} = 3/8$$

$$\Pr[A=2] = 2 \cdot \Pr[X_1 = X_2 + 2] = \frac{2 \cdot 2 \cdot 4}{4^3} = \frac{1}{4}$$

$$\Pr[A=3] = 2 \cdot \Pr[X_1 = X_2 + 3] = \frac{2 \cdot 1 \cdot 4}{4^3} = \frac{1}{8}$$

$$\mathbb{E}[A] = 1 \cdot \frac{3}{8} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{13}{8}$$

$$\mathbb{E}[A^2] = 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4} = \frac{29}{8}$$

$$\mathrm{Var}[A^2] = \mathbb{E}[A^2] - \mathbb{E}[A]^2 = \frac{63}{64}$$

- $B := X_3^{A} = X_3^{|X_1 - X_2|}$

! Unabhängigkeit wurde u.U. noch nicht behandelt. Insofern muss man hier noch "geschickt" abzählen.

$$W_B = \{1, 2, 2^2, 2^3, 3, 3^2, 3^3, 4, 4^2, 4^3\}$$

$$\Pr[B = 1] = \Pr[X_3 = 1 \vee A = 0]$$

$$= \Pr[X_3 = 1 \vee X_1 = X_2]$$

$$= \Pr[X_3 = 1] + \Pr[X_1 = X_2] - \Pr[1 = X_1 = X_2 = X_3] = \frac{3}{8}$$

$$\Pr[B = 2] = \Pr[X_3 = 1 \wedge A = 1]$$

$$= \Pr[X_3 = 1 \wedge X_2 = X_3 + 1]$$

$$+ \Pr[X_3 = 1 \wedge X_2 + 1 = X_3]$$

$$= 2 \cdot \Pr[X_3 = 1 \wedge \overset{2,3,4}{X_2} = \overset{1,2,3}{X_3} + 1]$$

$$= \frac{2 \cdot 3}{4^3} = 3/32$$

usw.

$$\Pr[B = 2^2] = \Pr[X_3 = 2 \wedge A = 2]$$

$$+ \Pr[X_3 = 4 \wedge A = 1]$$

$$\frac{4}{4^3} + \frac{3}{4^3} = \frac{7}{4^3}$$

$$= \begin{cases} = \Pr[X_3 = 2 \wedge (X_1 = X_2 + 2 \vee X_2 = X_1 + 2)] \\ + \Pr[X_3 = 4 \wedge (X_1 = X_2 + 1 \vee X_2 = X_1 + 1)] \end{cases}$$

$$C := \min\{X_1, X_3\}$$

$$W_C = \{1, 2, 3, 4\}$$

$$\Pr[C \geq k] = \Pr[X_1 \geq k \wedge X_3 \geq k]$$

$$= \frac{|([4]\setminus[k-1]) \times [4] \times ([4]\setminus[k-1])|}{4^3}$$

$$= \begin{cases} 0 & \text{, für } k \geq 5 \\ \frac{(4-k+1)^2}{4^2} & \text{, für } k \in [4] \\ 1 & \text{, für } k \leq 0 \end{cases}$$

$$\Pr[C = k] = \Pr[C \geq k] - \Pr[C \geq k-1]$$

$$\mathbb{E}[C] = \sum_{k=1}^{4} k \Pr[C=k]$$

$$= \sum_{k=1}^{4} k \left(\Pr[C \geq k] - \Pr[C \geq k+1] \right)$$

$$= 1 \cdot \left(\Pr[C \geq 1] - \Pr[C \geq 2] \right)$$

$$+ 2 \left(\Pr[C \geq 2] - \Pr[C \geq 3] \right)$$

$$+ 3 \cdot \left(\Pr[C \geq 3] - \Pr[C \geq 4] \right)$$

$$+ 4 \left(\Pr[C \geq 4] - \underbrace{\Pr[C \geq 5]}_{=0} \right)$$

$$= \Pr[C \geq 1] + \Pr[C \geq 2] + \Pr[C \geq 3] + \Pr[C \geq 4]$$

$$= \frac{1}{4^3} \sum_{k=1}^{4} k^2 = \frac{1}{4^3} \frac{4 \cdot 5 \cdot 9}{6} = \frac{15}{48}$$

• $D = A/C$ mit $A = |X_1 - X_2|$, $C = \min\{X_1, X_3\}$

$$W_D \subseteq \left\{ a/c \;\middle|\; a \in \{0,1,2,3\}, c \in [4] \right\}$$

<u>Bsp. i</u> $\Pr[D = 1] = \Pr[A = C \wedge \underbrace{C \neq 0}_{\text{gilt immer}}]$

$$\overset{P}{=} \sum_{k \in \underbrace{W_A \cap W_C}_{=\{1,2,3\}}} \Pr[A = C = k]$$

$X_2 = 1$ $\quad$ $X_1 = 1, X_3$ bel

• $\Pr[A = C = 1] = \Pr[A = 1 \wedge C = 1 \wedge X_1 = C]$

$\qquad + \Pr[A = 1 \wedge C = 1 \wedge X_1 \neq C]$

$X_1 >$

$X_2 > 1 \wedge X_3 = 1$

$=$

$$\ldots = \Pr[X_2 = 1 \land X_1 = 1] = \frac{|\{(1,1,x_3) \mid x_3 \in [4]\}|}{4^3}$$

$$+ \Pr[\cancel{A} = 1 \land X_3 = 1 \land X_1 > 1]$$

$$X_2 = X_1 + 1$$
$$\lor X_2 = X_1 - 1$$

$$= \frac{4}{4^3} + \Pr[X_2 = \overbrace{X_1 + 1 \land X_3 = 1 \land X_1}^{\text{nur } X_1 \in \{2,3\}} \in \{2,3,4\}]$$

$$+ \Pr[X_2 = X_1 - 1 \land X_3 = 1 \land X_1 \in \{2,3,4\}]$$

$$= \frac{4}{4^3} + \frac{2}{4^3} + \frac{3}{4^3} = \frac{9}{4^3}$$

- $\Pr[A=2 \wedge C=2]$

$$= \Pr[A=2 \wedge C=2 \wedge X_1=2 \wedge X_3 \geq 2]$$

$X_2 = 4$ $\qquad$ $X_3 \in \{2,3,4\}$

$$+ \Pr[A=2 \wedge C=2 \wedge X_1>2 \wedge X_3=2]$$

$X_1 \in \{3,4\}$

$X_2 = X_1 - 2$

$$= \frac{3}{4^3} + \frac{2}{4^3} = \frac{5}{4^3}$$

$$\Pr[A = 3 \wedge C = 3]$$

$$= \Pr[A = 3 \wedge C = 3 \wedge X_1 = 3 \wedge X_3 \geq 3]$$

$x_2 \in \{0,3\} \cap W_{x_2} = \emptyset$

$$+ \Pr[A = 3 \wedge C = 3 \wedge X_1 > 3 \wedge X_3 = 3]$$

$X_1 = 4$

$X_2 = 1$

$$= \frac{1}{4^3} \quad \rightsquigarrow \Pr[D = 1] = \frac{15}{4^3}$$

$$\Pr\left[D = \tfrac{1}{2}\right] = \Pr[A=1 \wedge C=2] + \Pr[A=2 \wedge C=4]$$

$$= \Pr[A=1 \wedge X_1=2 \wedge X_3=2]$$
$$+ \Pr[A=1 \wedge X_1=2 \wedge X_3>2]$$
$$+ \Pr[A=1 \wedge X_1>2 \wedge X_3=2]$$
$$+ \Pr[A=2 \wedge X_1=X_3=4]$$

$$= \frac{1}{4^3} + \frac{2}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} = \frac{5}{4^3}$$

usw.

(b)

$$\boxed{B = X_3^{A} = X_3^{|X_1 - X_2|}, \quad D = \frac{A}{C} = \frac{|X_1 - X_2|}{\min\{X_1, X_3\}}}$$

$$W_B \subseteq \{1, 2, 2^2, 2^3, 3, 3^2, 3^3, \cancel{4}, 4^2, 4^3\}$$

$$W_D \subseteq \{ \tfrac{a}{c} \mid a \in \{0,1,2,3\}, c \in \{1,2,3,4\}\}$$

Bsp.: $\Pr[B = 3, D = 1/2]$

$$= \Pr[X_3 = 3 \wedge A = 1 \wedge C = 2]$$

$X_1 = 2$

$X_2 = 3 \vee X_2 = 1$

$$= \Pr[X_1 = 2, X_2 \in \{1,3\}, X_3 = 3] = \frac{2}{4^3} = \underline{\underline{\frac{1}{32}}}$$

$$B = X_3^{A},\quad A = |X_1 - X_2|$$
$$D = \frac{A}{C},\quad C = \min\{X_1, X_3\}$$

$$\Pr[\ B = 4\ ,\ D = 1/3]$$

$$= \Pr[\ (X_3 = 4 \wedge A = 1 \vee X_3 = 2 \wedge \underline{A = 2}) \wedge \underline{A = 1} \wedge C = 3]$$

$$= \Pr[\ X_3 = 4 \wedge A = 1 \wedge C = 3]$$

$$= \Pr[X_3 = 4 \wedge A = 1 \wedge X_1 = 3]$$

$$= \Pr[X_3 = 4 \wedge X_2 \in \{2, 4\} \wedge X_1 = 3] = \frac{1}{32}$$

wow.