Lecture-2 Tensor

1. Einstein Notation

Before introducing tensors, let's first talk about Einstein notation. For the equation shown below

$$s = a_1 b_1 + a_2 b_2 + a_3 b_3$$

In Einstein notation, it is written as

$$s = a_i b_i$$

In this case, the subscript is repeated more than once among the variables being multiplied, indicating addition operation. Additionally, for the following system of equations:

$$s_1 = a_1 b_1 c_1 + a_2 b_2 c_1 + a_3 b_3 c_1$$

$$s_2 = a_1 b_1 c_2 + a_2 b_2 c_2 + a_3 b_3 c_2$$

$$s_3 = a_1 b_1 c_3 + a_2 b_2 c_3 + a_3 b_3 c_3$$

We have

$$s_j = a_i b_i c_j$$

2. Space and Tensor

In the previous chapter, we introduced that space is composed of 'value' and 'base.' In three-dimensional space, vectors have three bases. Similarly, tensors must also be represented through the values and the bases

$$\mathbf{T} = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

where, $i, j = 1,2,3$, T_{ij} is the value, $\mathbf{e_i} \otimes \mathbf{e_j}$ is the base.

Scalars are well understood by everyone, such as temperature, mass, for example. Scalars are fundamentally direction independent. Vectors, on the other hand, involve spatial directions, such as position and electric field lines, and so on. When we say that position is a vector, what we mean is that to represent the position of an object, you need to provide both a base and the corresponding value simultaneously. So, what is a tensor? Let's assume there's a kind of physical quantity for which, to evaluate it, you need to first determine one direction, then another direction, and only then can you provide the value of this physical quantity based on direction 1 + direction 2.

The most straightforward example is washing a car. Imagine you have a cubic car. It's being sprayed with water in various directions. You want to describe the state of the car being sprayed with water. But this state is clearly not just a single number. To introduce it to others, you must be more specific in your expression. First, you need to specify a surface, such as the one corresponding to the left door of the car. Let's assume that the vertical line on this surface is $\mathbf{e_1}$. You know that water is being sprayed on this surface. However, this surface can be sprayed with water from various directions. You want to know the water flow in the direction or, in other words, the velocity component of the water sprayed towards the left door in the direction.

So, you have this precise expression: "In the $\mathbf{e_1}$ direction of the car, the amount of water sprayed from the direction$\mathbf{e_2}$ is T_{12}. To fully describe the car's water-receiving state, we need the following physical quantities:

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

The quantity in this water-sprayed state is simultaneously composed of two directions at its base. We call this quantity a second-order tensor. Of course, similarly, we can define third-order, fourth-order, and so on tensors. Readers familiar with continuum mechanics will also know that this definition is the definition of Cauchy stress."

3. Tensor Base

The above analysis only provides a physical explanation of tensors. However, is there any deeper physical significance to this bidirectional base? We know that for vectors, a very important property is that if a physical quantity is a vector, it is independent of the coordinate system. For example, a physical quantity is

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$$

where the base is $\mathbf{e_1}$ and $\mathbf{e_2}$. If we consider another coordinate, we have

$$\mathbf{u} = 0.5u_1\mathbf{e}_1 + u_n\mathbf{e}_n$$

where

$$u_n \mathbf{e}_n = (u_2 \mathbf{e}_2 + 0.5 u_1 \mathbf{e}_1)$$

Here, the new base is $\mathbf{e_1}$ and $\mathbf{e_n}$, corresponding to the new values. But no matter how we linearly combine different bases, the vector we represent is always the same; it does not change because of a coordinate transformation. Therefore, the same applies to tensors. A set of tensors can be written as

$$\mathbf{T} = t_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

In another coordinate, we have

$$\mathbf{T} = T_{ij} \mathbf{E}_i \otimes \mathbf{E}_j$$

where every $\mathbf{E_i}$ in the new coordinate is a linear superposition of $\mathbf{e_j}$ in the old coordinate.

4. Coordinate Transformation Matrix

Since we know $\mathbf{E_i} \leftrightarrow \mathbf{e_j}$ is a linear combination, this implies that we can transform one set of coordinate systems into another set of coordinate systems. This is not difficult at all. According to the transformation relationship,

$$\mathbf{e}_i = F_{ij} \mathbf{E}_j$$

Leading to

$$\mathbf{T} = t_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = t_{ij} (F_{im} \mathbf{E}_m) \otimes (F_{jn} \mathbf{E}_n)$$

Therefore

$$\mathbf{T} = F_{mi}^T t_{ij} F_{jn} \mathbf{E}_m \otimes \mathbf{E}_n$$

The transformation matrix is

$$T_{mn} = F_{mi}^T t_{ij} F_{jn}$$

In matrix form, we have

$$T = F^T t F$$

where F is the coordinate transformation matrix. To be frank, the definition of the coordinate transformation matrix here is not very useful for most people. However, this method of performing coordinate transformation by linear combinations of the base can greatly enhance your accuracy in analyzing tensor problems.

5. Tensor Operation

Some textbooks may define tensors in a precise (and crude) way as "mathematical quantities needed to map one set of vectors to another set of vectors." Personally, I believe this is the worst definition! To understand this definition, we need to grasp the concepts of "initial vectors" and "resultant vectors," as well as their corresponding transformation operations. Every time you want to use this definition, you must mentally construct initial vectors and resultant vectors. Because of this complex definition, it will make your future tensor analysis endeavors painful, no matter what tensor operations you do, such as tensor addition or contraction, you'll always have to keep building sets of vectors and their corresponding transformation relationships in your mind. It's truly a torture for your brain cells. Therefore, in my personal opinion, the best way to understand it is to define the essence of tensors through the "value" and "base" methods discussed earlier. Based on this foundation, we can then define mathematical operations. Returning to our question, tensors, like vectors, can undergo addition and multiplication operations.

$$(T_{ij}\mathbf{E}_{ij} + S_{ij}\mathbf{E}_{ij}) = (T_{ij} + S_{ij})\mathbf{E}_{ij}$$

$$a(T_{ij}\mathbf{E}_{ij}) = aT_{ij}\mathbf{E}_{ij}$$

It satisfies the commutative property and the associative property. (Not listing them again).

6. Tensor Operations: Dot Product and Contraction.

The most important operation for tensors is the dot product, contraction, and similar operations. I must say that there are many names for this operation in the textbooks. Some call it "dot product," some call it "vector contraction," some call it "single dot product," some call it "double dot product," and some call it "contraction." We don't care about the specific names; understanding the underlying principles is what matters.

- Vector Contraction

This is easy to understand; it means combining two vectors to form a tensor.

$$(a_1\mathbf{e}_1) \otimes (a_2\mathbf{e}_2) = a_1 a_2 \mathbf{e}_1 \otimes \mathbf{e}_2$$

- Contraction

In English, this generally refers to performing an operation between a tensor and a vector, resulting in a final vector.

$$T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{e}_k = T_{ij}\mathbf{e}_i \delta_{jk}$$

- Single Dot Product

This is taking the dot product of the two middle vectors. The remaining two vectors continue to be added together as new tensors.

$$T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \cdot S_{mn}\mathbf{e}_m \otimes \mathbf{e}_n = T_{ij}S_{mn}\mathbf{e}_i \otimes \mathbf{e}_n \delta_{jm}$$

- Double Dot Product

$$T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \cdot S_{mn}\mathbf{e}_m \otimes \mathbf{e}_n = T_{ij}S_{mn}\delta_{im}\delta_{jn}$$

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