Lecture-1 Vector

1. Base (base) and Numerical Value (value)

How do we generally describe an objective object? For example, if there is something bright red in front of us, we usually have the sensation that there is something in front of us. We might call it '1 apple.' In this expression, '1' represents the quantity, while 'apple' represents the specific object. Quantities 1, 2, 3, 4 are a way of counting, separate from the object itself. Objects, on the other hand, can vary greatly. For instance, we can also define: '1 orange,' '1 pear,' '1 banana.'

For these different definitions, we use the same counting method to represent the quantity. However, there are different bases. When these bases are independent of each other, even when we were young, we knew that '1 apple + 1 apple' equals '2 apples.' The numbers cannot be added here because they correspond to different bases. Here, we can see that there exists a system like this, which is represented by both **a base** and **the corresponding counting.** For such a system, we call it **space**.

2 Vectors and Index

A vector is an entity with value and a base (or direction). In our real world, things like position, velocity, acceleration, etc., can all be represented using vectors.

$$\mathbf{u} = u\mathbf{e}$$

where, "value" u stands for the value, while "base" $\mathbf{e}$ represents the direction. To fully represent a vector, both the value and the base are indispensable.

It's worth noting that, up to this point, we haven't introduced any content related to coordinate systems. We've simply started with the material representation of the abstract world and obtained these two key features: value and base. These vectors here can essentially be mapped to all the explanations we provided in the first section. Of course, we will add several rules of operation later. Firstly, for such a space, it can have a finite or infinite number of bases. Specifically, for a space with three bases $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, we call it a three-dimensional space. In three-dimensional space, any entity can be represented as"

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = (u_1, u_2, u_3)$$

In a three-dimensional space, there exist three mutually independent bases. Just like the example of apples and bananas we mentioned earlier, here the different bases cannot be subjected to any operations for the time being. According to the rules of cardinality, we can easily derive the addition and multiplication rules satisfied by space.

The Addition Rule

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

The Multiplication rule

$$a(u_1, u_2, u_3) = (au_1, au_2, au_3)$$

3. Vector Dot Product

Why do we define vector multiplication? After all, in life, multiplying one apple by one orange has no meaning. In fact, the biggest problem is that we lack a means to express the magnitude of a vector. For example, for a vector equal to "one apple + one orange," we always want to know what quantity this vector has. We'll talk about function spaces later. For "Euclidean space," our definition of vector multiplication is:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

where δ_{ij} is Kronecker delta.

$$\begin{aligned} \delta_{ij} &= 1, \text{when } i = j \\ \delta_{ij} &= 0, \text{when } i \neq j \end{aligned}$$

Note that from now on, we will be using subscripts. It will greatly simplify our writing workload. The dot multiplication operation here satisfies the commutative property and associative law.

$$\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{e}_i$$

4. Vector Cross Product

Now, let's define the vector cross product.

$$\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_k \epsilon_{ijk}$$

where $i, j, k \in 1,2,3$. ϵ_{ijk} is the permutation symbol.

$$\epsilon_{ijk} = 1 \text{ when } i = 1, j = 2, k = 3 \text{ or } i = 2, j = 3, k = 1 \text{ or } i = 3, j = 1, k = 2$$

$$\epsilon_{ijk} = -1 \text{ when } i = 2, j = 1, k = 3 \text{ or } i = 3, j = 2, k = 1 \text{ or } i = 1, j = 3, k = 2$$

$$\epsilon_{ijk} = 0 \text{ when others}$$

5. Extraction of the Value

A practical issue is how we obtain the value of a vector. In fact,

$$u_1 = \mathbf{u} \cdot \mathbf{e}_1$$

$$u_2 = \mathbf{u} \cdot \mathbf{e}_2$$

$$u_3 = \mathbf{u} \cdot \mathbf{e}_3$$

The way to obtain this value will be used extensively in tensor operation.

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