Contact Mechanics

1. Hertz Model

For two spheres of radius R_1 and R_2 pressed together under a load **the radius of the circle of contact** is given by

$$a_0^3 = \frac{3}{4}\pi\left(k_1 + k_2\right)\frac{R_1 R_2}{R_1 + R_2}P_0 \tag{1}$$

where,

$$k_1 = \frac{1-v_1^2}{\pi E_1}, k_2 = \frac{1-v_2^2}{\pi E_2} \tag{2}$$

where v is the Poisson ratio and E the Young modulus of each material. Resulting from local compression near to the contact region, distant points in the two spheres approach each other by a distance δ given by (i.e. the relative displacement of two origins of the two spheres. Notes: the δ here assembles the penetration depth in Boussinesq's theory)

$$\delta^3 = \frac{9}{16}\pi^2\left(k_1 + k_2\right)^2 \frac{R_1 + R_2}{R_1 R_2}P_0^2 \tag{3}$$

The distribution of pressure over the contact area is

$$p = \frac{3P_0}{2\pi a^2}\sqrt{\left(1 - \frac{r^2}{a^2}\right)} \tag{4}$$

2. JKR Model

2.1 Approximate Theory

Consider two elastic spheres in contact under zero external load. Attractive forces between the surfaces produce a finite contact radius, a, a balance eventually being established between stored elastic energy and lost surface energy. The loss in surface energy U_S is given by

$$U_S = -\pi a^2 \gamma \tag{5}$$

where γ is the energy per unit contact area (i.e. the two surfaces). The force F_S associated with this energy change is (the direction of force is the perpendicular to the rigid flat surface)

$$F_S = -dU_S/dx \tag{6}$$

where x is the movement of the bodies and is approximately the same as which is given by the Hertz equations (1) and (2) but cannot be worked out exactly from these because the attractive forces disturb the stress distributions in the bodies. Thus we may only write that

$$x \approx a^2\left(R_1 + R_2\right)/R_1 R_2 \tag{7}$$

(That is

$$x \approx a^2\left(R_1+R_2\right)/R_1R_2=\delta=a_0^2\left(\frac{R_1+R_2}{R_1R_2}\right) \tag{8}$$

However, there is no rigorous analysis in this part. I don't believe $x \approx \delta$ is true. The article also said

"The above theory can only be approximate since the Hertz equations are not valid when surface forces act at a contact

"

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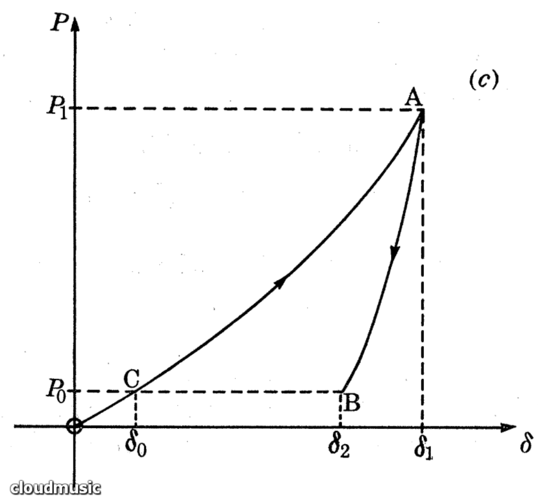
Combining all of above, we have

$$F_S=-dU_S/dx \approx \frac{\gamma\pi R_1R_2}{\left(R_1+R_2\right)} \tag{9}$$

(It is noted that F_S is only a function of the geometry and the surface property. F_S is not influenced by the elastic properties)

2.1 Rigorous Theory

(The most important energy figure is given below)



A rigorous determination of the contact equilibrium between elastic spheres involves computation of the total energy U_T in the system as a function of contact radius *a*. Equilibrium will then obtain when

$$dU_T/da=0 \tag{10}$$

Consider the situation illustrated in figure 1. When no surface forces act, the contact radius a_0 is given by the generalized Hertz equation

$$a_0^3 = RP_0/K \tag{11}$$

where $R = R_1R_2/(R_1+R_2)$ and $K = 4/3\pi(k_1+k_2)$. (It is noted that, if we exchange the radius of the spheres or the elastic modulus of the materials, R and K remain the same. This is why the Boussinesq's indenter model can be applied to study Maugis's problem)

The movement of the applied load is given by

$$\delta = a^2/R \tag{12}$$

Now if attractive forces act between the surfaces the contact radius in equilibrium will be a_1, which is greater than a_0. Although the applied load remains at P_0, an apparent Hertz load P_1 corresponding to the contact radius a_1 may be defined such that

$$a_1^3 = RP_1/K \tag{13}$$

(Under the adhesive stress, the radius of the contact area changes, however the applied load remained at P_0. B-B-B-B-B-But, we can reconsider the problem in another way: at first, Hertz problem, with the load P_0, we have the radius a_0; second, increasing the load to P_1, according to Hertz problem, we have the radius a_1; third, we add the adhesion and decrease the load from P_1 to P_0, the system remain stable. I-I-I-I-If we don't know the existence of adhesion force, we may think the apparent pressure is P_1, however, the real load is P_0. This what we defined as **apparent Hertz load**. The whole procedure can be explained by the above figure)

The total energy of this system U_T is made up of three terms, the stored elastic energy U_E, the mechanical energy in the applied load U_M and the surface energy U_S.

The elastic energy U_E may be calculated by considering the idealized load displacement curve shown in figure. Neglecting surface force, the system is loaded to give a contact radius a_1 with a load P_1 (condition A) which requires energy U_1. Keeping the contact radius at a_1 the load is then reduced to P_0 to give the final state of the system (condition B), releasing energy U_2

$$U_E = U_1 - U_2 \tag{14}$$

$$U_1 = \int_0^{P_1} \frac{2}{3}\frac{P^{\frac{2}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}}dP = \frac{2}{5}\frac{P_1^{\frac{5}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}} \tag{15}$$

(Integral along the path)

For the A-B line, the load-displacement relation is the given by (reference is Johnson's model in Chapter 6)

$$\delta = \frac{2}{3}\frac{P}{Ka_1} \tag{16}$$

Therefore

$$U_2 = \int_{P_0}^{P_1} \frac{2}{3}\frac{P}{Ka_1} dP = \frac{1}{3K^{\frac{2}{3}}R^{\frac{1}{3}}}\frac{P_1^2 - P_0^2}{P_1^{\frac{1}{3}}} \tag{17}$$

Thus

$$U_E = \frac{2}{5}\frac{P_1^{\frac{5}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}} - \frac{1}{3}\frac{\left(P_1^2 - P_0^2\right)}{K^{\frac{2}{3}}R^{\frac{1}{3}}P_1^{\frac{1}{3}}} = \frac{1}{K^{\frac{2}{3}}R^{\frac{1}{3}}}\left[\frac{1}{15}P_1^{\frac{5}{3}} + \frac{1}{3}P_0^2 P_1^{-\frac{1}{3}}\right] \tag{18}$$

The mechanical potential energy U_M of the applied load is

$$\begin{aligned} U_M &= -P_0\delta_2 = -P_0\left(\delta_1 - \frac{2}{3}\frac{P_1 - P_0}{Ka_1}\right) = -P_0\left(\frac{P_1^{\frac{2}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}} - \frac{2}{3}\left(\frac{K}{RP_1}\right)^{\frac{1}{3}}\frac{P_1 - P_0}{K}\right) \\ &= \frac{-P_0}{K^{\frac{2}{3}}R^{\frac{1}{3}}}\left(\frac{1}{3}P_1^{\frac{2}{3}} + \frac{2}{3}P_0 P_1^{-\frac{1}{3}}\right) \end{aligned} \tag{19}$$

The surface energy U_S is given by

$$U_S = -\gamma\pi a_1^2 = -\gamma\pi\left(RP_1/K\right)^{\frac{2}{3}} \tag{20}$$

where γ is the energy of adhesion of both surfaces.

The total energy U_T is

$$\begin{aligned} U_T = U_E + U_M + U_S &= \frac{1}{K^{\frac{2}{3}}R^{\frac{1}{3}}}\left[\frac{1}{15}P_1^{\frac{5}{3}} + \frac{1}{3}P_0^2 P_1^{-\frac{1}{3}}\right] - \frac{1}{K^{\frac{2}{3}}R^{\frac{1}{3}}}\left(\frac{1}{3}P_0 P_1^{\frac{2}{3}} + \frac{2}{3}P_0^2 P_1^{-\frac{1}{3}}\right) \\ &-\gamma\pi\left(RP_1/K\right)^{\frac{2}{3}} \end{aligned} \tag{21}$$

Equilibrium ensues when

$$dU_T/da_1 = 0 \tag{22}$$

This is equivalent to

$$dU_T/dP_1 = 0 \tag{23}$$

That is

$$dU_T / dP_1 = \frac{P_1^{-\frac{4}{3}}}{9K^{\frac{2}{3}}R^{\frac{1}{3}}}\left[P_1^2 - P_0^2 - 2P_1P_0 + 2P_0^2 - 6\gamma\pi RP_1\right] \tag{24}$$

Therefore at equilibrium

$$P_1 = P_0 + 3\gamma\pi R \pm \sqrt{\left(P_0 + 3\gamma\pi R\right)^2 - P_0^2} \tag{25}$$

By examining the second differential of the total energy it may be shown that for equilibrium to be stable it is necessary to take the positive sign so that

$$P_1 = P_0 + 3\gamma\pi R + \sqrt{6\gamma\pi RP_0 + \left(3\gamma\pi R\right)^2} \tag{26}$$

Therefore the relationship between the radius of contact and the pressure is

$$a_1^3 = \frac{R}{K}\left(P_0 + 3\gamma\pi R + \sqrt{6\gamma\pi RP_0 + \left(3\gamma\pi R\right)^2}\right) \tag{27}$$

At zero applied load the contact area is finite and given by

$$a_1^3 = R\left(6\gamma\pi R\right)/K \tag{28}$$

When the applied load is made negative the contact radius decreases. For a real solution be obtained, we shall have

$$6\gamma\pi RP_0 + \left(3\gamma\pi R\right)^2 \geq 0 \Rightarrow -\frac{3}{2}\gamma\pi R \leq P_0 \tag{29}$$

Separation of the spheres will just occur when

$$P = -\frac{3}{2}\gamma\pi R \tag{30}$$

which we note is independent of the elastic modulus.

3. Maugis Model

As mentioned before (i.e. Eq. (11)), the problem is symmetric. Both the geometry parameters and elastic properties can be exchanged in two structures. Therefore, instead of considering an elastic sphere touches the solid surface, we can solve the problem of a solid indenter touches the elastic surface. The latter problem is thoroughly studied by Boussinesq.

Take the center of the tip as origin, the profile of the tip can be expressed as

$$w(r) = f\left(r/a\right) \text{ with } f\left(0\right) = 0$$

The penetration depth is

$$\delta = \int_0^1 \frac{f'(x)\,dx}{\sqrt{1-x^2}} + \frac{\pi}{2}\chi(1) \tag{31}$$

The total load is

$$P = \frac{3\pi aK}{4}\int_0^1 \chi(t)\,dt = \frac{3aK}{2}\left[\delta - \int_0^1 \frac{xf(x)\,dx}{\sqrt{1-x^2}}\right] \tag{32}$$

The stress distribution in the contact area is

$$\sigma_y(r,0) = -\frac{3K}{8a}\left[\frac{\chi(1)}{\sqrt{1-\frac{r^2}{a^2}}} - \int_{r/a}^1 \frac{\chi'(t)}{\sqrt{t^2-\frac{r^2}{a^2}}}\,dt\right], r < a \tag{33}$$

The displacement outside the contact area is

$$u_y(r,0) = \int_0^1 \frac{\chi(t)}{\sqrt{\frac{r^2}{a^2}-t^2}}\,dt, \quad r > a \tag{34}$$

where

$$\chi(t) = \frac{2}{\pi}\left[\delta - t\int_0^t \frac{f'(x)\,dx}{\sqrt{t^2-x^2}}\right] \tag{35}$$

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In the original Sneddon's paper (1965), the expression of $\chi(t)$ is

$$\chi(t) = \frac{2}{\pi}\left[\delta - \frac{d}{dt}\int_0^t \frac{xf(x)\,dx}{\sqrt{t^2-x^2}}\right]$$

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and

$$\frac{1}{K} = \frac{3}{4}\frac{1-v^2}{E} \tag{36}$$

From Eq. (32), we know

$$\int_0^1 \chi(t)\,dt = \frac{2}{\pi}\left[\delta - \int_0^1 \frac{xf(x)\,dx}{\sqrt{1-x^2}}\right] \tag{37}$$

It should be noted that, in the current system, we need to have one input (the penetration depth or the total load. Therefore, the function $\chi(t)$ can be determined implicitly with the given input)

For punches with continuous profile, Sneddon let the arbitrary rigid body displacement $\chi(1)$ equal zero to have finite stresses at the edge of the contact and used this criterion to determine δ.

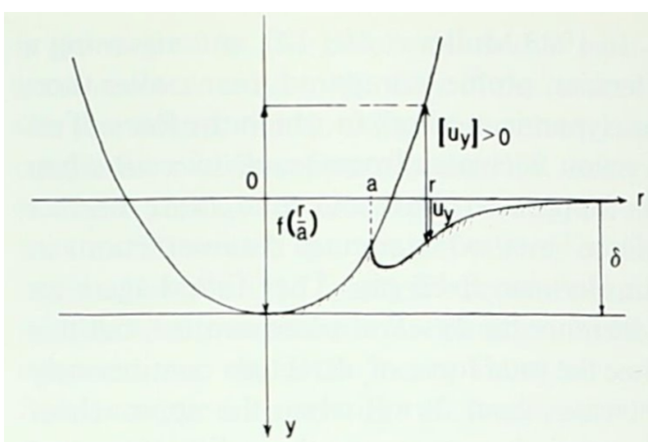
(We can assume that we apply the punch to the surface without additional constraints. The stress on the boundary at the contact must be finite. One contact radius is corresponding to one penetration depth and one loading pressure, as shown in Eq. (35). In this case, $\chi(1)$ will equal zero. However, if we keep the contact area of the punch constant, and lift the punch, the stress at the edge of the contact may not be finite any more. And $\chi(1)$ is not zero. A good example is the flat punch: to avoid any stress singularity, we can't apply any loading to the flat punch, otherwise the stress on the boundary of the flat punch is infinite. Boussinesq didn't care it and applied the loading to the flat punch anyway. He obtained his solutions with the stress singularity on the boundary of the flat punch. However, Sneddon didn't like Boussinesq's theory. He tried to prohibit any kind of stress singularity, and obtain the solutions in this case. Maugis, however, claimed that the stress singularity is the result of the adhesion force. By adjusting the value of stress intensity factor, he discussed the influence of adhesion force on the contact problem and thence get the JKR model)

Defining the stress intensity factor as

$$K_1 = -\frac{3K}{8}\chi(1)\sqrt{\frac{\pi}{a}} \tag{38}$$

The stress σ_y and the discontinuity of displacement are

$$\left[u_y\right] = f\left(\frac{r}{a}\right) - \delta + u_y(r,0) \tag{39}$$



At a distance ρ from the edge of the contact can be written

$$\sigma_y(a-\rho,0) \simeq \frac{K_I}{\sqrt{2\pi\rho}} \tag{40}$$

$$\left[u_y\right](a+\rho,0) \simeq \frac{4\left(1-v^2\right)}{E} K_I \sqrt{\frac{\rho}{2\pi}} \tag{41}$$

Which are the fracture mechanics formulae for plane strain (with a factor 2 for $\left[u_y\right]$, where K_I is the stress intensity factor.)

In the linear elastic fracture mechanics approximation, the energy release rate is thus

$$G = \frac{1}{2}\frac{1-v^2}{E}K_I^2 \tag{42}$$

Let P_1 be the apparent load that, if $\chi(1)=0$, would give the same radius of contact as the radius observed under the load P with $\chi(1)\neq 0$ (the same definition as given by JKR):

(for a flat punch $P_1 = 0$)

From Eq. (32) and Eq. (31), we get

$$P_1 - P = \frac{3aK}{2}\left(\frac{\pi}{2}\chi(1)\right) = \frac{3\pi aK}{4}\chi(1) = \sqrt{4\pi a^3}K_1 \tag{43}$$

(I think there may be some typo error in the original Maugis' paper.)

i.e.

$$G = \frac{1}{2}\frac{1-v^2}{E}K_I^2 = \frac{\left(P_1 - P\right)^2}{6\pi a^3 K} \tag{44}$$

The equilibrium is given by the Griffith relation

$$G = w \tag{45}$$

and the stability of this equilibrium by $\partial G/\partial a > 0$, the result being generally different at fixed load P and at fixed grips δ.

The derivation is straightforward for a flat punch $f(x)=0$. Assuming a radius of contact *a* (which implies a negative load to have a crack), we get $P_1 = 0$, (I can't understand)

$$\chi(t) = \chi(1) = \frac{2\delta}{\pi} \tag{46}$$

and

$$\delta = \frac{2P}{3aK} \tag{47}$$

$$\sigma_y(r,0) = -\frac{P}{2\pi a^2}\frac{1}{\sqrt{1-\frac{r^2}{a^2}}}, r<a \tag{48}$$

$$u_y(r,0) = \frac{1-v^2}{\pi E}\frac{P}{a}\sin^{-1}\frac{a}{r}, \quad r>a \tag{49}$$

$$G = \frac{(P_1-P)^2}{6\pi a^3 K} = \frac{3K\delta^2}{8\pi a} \tag{50}$$

The above equations were first given by Boussinesq (2). The equilibrium given by $G = w$ is always unstable either at fixed load or at fixed grips, and the adherence force is

$$P_{\min} = -\sqrt{6\pi a^3 K w} \tag{51}$$

Is a result given by Kendall.

For a spherical punch, with

$$f(x) = \frac{a^2}{2R}x^2 \tag{52}$$

We get

$$\chi(t) = \frac{2}{\pi}\left(\delta - \frac{a^2}{R}t^2\right) \tag{53}$$

$$\chi(1) = \frac{2}{\pi}\left(\delta - \frac{a^2}{R}\right) \tag{54}$$

If we take $\chi(1) = 0$ to have finite stresses we have

$$\delta = \frac{a^2}{R} \tag{55}$$

$$P = \frac{a^3 K}{R} \tag{56}$$

$$\sigma_y(r,0) = -\frac{3}{2}\frac{P}{\pi a^2}\sqrt{1-\frac{r^2}{a^2}}, \quad r<a \tag{57}$$

$$u_y(r,0) = \frac{a^2}{\pi R}\left[\sqrt{\frac{r^2}{a^2}-1} + \left(2-\frac{r^2}{a^2}\right)\sin^{-1}\frac{a}{r}\right], \quad r>a \tag{58}$$

$$\left[u_y\right] = \frac{a^2}{\pi R}\left[\sqrt{\frac{r^2}{a^2}-1} + \left(\frac{r^2}{a^2}-2\right)\cos^{-1}\frac{a}{r}\right], \quad r > a \tag{59}$$

which are the Hertz and Boussinesq results. On the other hand, if we take $\chi(1) \neq 0$, we get

$$\delta = \frac{a^2}{3R} + \frac{2P}{3aK} \tag{60}$$

$$\sigma_y(r,0) = \frac{K_I}{\sqrt{\pi a}} \frac{1}{\sqrt{1-\frac{r^2}{a^2}}} - \frac{3aK}{2\pi R}\sqrt{1-\frac{r^2}{a^2}} \tag{61}$$

$$u_y(r,0) = -\frac{2\left(1-v^2\right)}{\pi E} K_I \sqrt{\pi a}\sin^{-1}\frac{a}{r} + \frac{a^2}{\pi R}\left[\sqrt{\frac{r^2}{a^2}-1} + \left(2-\frac{r^2}{a^2}\right)\sin^{-1}\frac{a}{r}\right], \quad r > a \tag{62}$$

$$\left[u_y\right] = \frac{2\left(1-v^2\right)}{\pi E} K_I \sqrt{\pi a}\cos^{-1}\frac{a}{r} + \frac{a^2}{\pi R}\left[\sqrt{\frac{r^2}{a^2}-1} + \left(\frac{r^2}{a^2}-2\right)\cos^{-1}\frac{a}{r}\right], \quad r > a \tag{63}$$

and

$$P_1 = \frac{a^3 K}{R} \tag{64}$$

which is the Hertzian load giving the radius of contact *a*. The energy release rate is

$$G = \frac{\left(P_1 - P\right)^2}{6\pi a^3 K} = \frac{3K}{8\pi a}\left(\delta - \frac{a^2}{R}\right)^2 \tag{65}$$

and the Griffith criterion gives the equilibrium relations

$$\frac{a^3 K}{R} = P + 3\pi wR + \sqrt{6\pi wRP + \left(3\pi wR\right)^2} \tag{66}$$

$$\delta = \frac{a^2}{R} - \left(\frac{8\pi wa}{3K}\right)^{1/2} \tag{67}$$

which are the JKR results (Eq. (27)).

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Integration of Eq.(61) is

$$\int_0^a 2\pi r\sigma_y dr = \int_0^a 2\pi r\left(\frac{K_I}{\sqrt{\pi a}}\frac{1}{\sqrt{1-\frac{r^2}{a^2}}} - \frac{3aK}{2\pi R}\sqrt{1-\frac{r^2}{a^2}}\right)$$

$$= 2\pi K_I \frac{1}{\sqrt{\pi a}}a^2 - \frac{a^3 K}{R} \Rightarrow P = \frac{a^3 K}{R} - 2\pi K_I \frac{1}{\sqrt{\pi a}}a^2$$

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Under zero load, the radius of contact is given by

$$a_0^3 = \frac{6\pi w R^2}{K} \quad (68)$$

(corresponding to Eq. (28))

By slowly decreasing the load (P) (soft machine) or the displacement δ (hard machine), these equilibrium curves, $a(P)$ and $\delta(a)$, can be followed up to the instability point, given by $\partial G/\partial a = 0$. At fixed load, instability appears for

$$\partial G/\partial a = \frac{P_1^2 - P^2}{2\pi a^4 K} = 0 \quad (69)$$

(obtained by Eq. (65))

i.e.

$$P_{\min} = -P_1 = -\frac{3}{2}\pi w R \quad (70)$$

$$a = \left(\frac{3}{2}\frac{\pi w R^2}{K}\right)^{1/3} = 0.63 a_0 \quad (71)$$

$$\delta = -\frac{a^2}{3R} = -\left(\frac{\pi^2 w^2 R}{12K^2}\right)^{1/3} \quad (72)$$

At fixed grips, instability is given by

$$\left(\frac{\partial G}{\partial a}\right)_\delta = \frac{3K}{8\pi R^2 a^2}\left(a^2 - \delta R\right)\left(3a^2 + \delta R\right) = 0 \quad (73)$$

i.e.

$$\delta_{\min} = -\frac{3a^2}{R} = -\left(\frac{3}{4}\frac{\pi^2 w^2 R}{K^2}\right)^{1/3} \quad (74)$$

$$a=\left(\frac{\pi wR^2}{6K}\right)^{1/3}=0.30a_0 \tag{75}$$

$$P=-5P_1=-\frac{5}{6}\pi wR \tag{76}$$

It is now clear that the JKR theory was, indeed, included in the general theory of elasticity, but a priori hypothesis of zero stress at the edge of the contact failed to reveal it.

- DMT Analysis

Contrary to the JKR theory, the DMT theory takes into account the interaction forces outside the contact area, but except in the self-consistent numerical computation of Muller et al, which cannot give analytical results, these interaction forces are assumed not to deform the profile, which remains Hertzian. The only useful results are that the pull-out force $2\pi wR$ is reached at zero contact radius and under zero load the radius of contact is given by

$$a_0^3=\frac{2\pi wR^2}{K} \tag{77}$$

In the thermodynamic method the attraction forces outside the contact add a fictitious load which decreases from $2\pi wR$ to πwR when the approach increases, whereas in the force method this fictitious load increases continuously from $2\pi wR$ (see the Fig. 3 in Ref. (43)). Note that if this fictitious load remained equal to $2\pi wR$, one would have

$$\frac{a^3K}{R}=P+2\pi wR \tag{78}$$

i.e.,

$$G=\frac{P_1-P}{2\pi R} \tag{79}$$

which gives correctly the pull-out force at $a=0$ and the radius of contact under zero load. That is the result we will derive below.

Note also that as in the DMT theory, the stress distribution is discontinuous. The stress is zero at $r=a_-$ and is the theoretical stress at $r=a_+$. As discussed in Appendix A it is not physically consistent to have only compressive stresses in the contact area and adhesion forces outside it.

- Crack Analysis

Let us consider an external axisymmetric crack of radius a in an infinite solid, and let us assume that a constant pressure p is applied on a length $d=c-a$. (This means that the adhesive force is applied only outside the contact area)

Lowengrub and Sneddon have derived expressions for the distribution of stresses in the neck and the elastic displacement of the crack lips in case of an axisymmetric pressure distribution $p(r)$, (currently, the Dugdale model is not applied here, and the structure is not sphere.)

$$\sigma_y(r,0) = \frac{2}{\pi}\left[\frac{g(a)}{\sqrt{a^2-r^2}} + \int_a^\infty \frac{g'(t)\,dt}{\sqrt{t^2-r^2}}\right], \quad r < a \tag{80}$$

$$u_y(r,0) = \frac{4(1-v^2)}{\pi E}\int_a^r \frac{g(t)\,dt}{\sqrt{r^2-t^2}}, \quad r > a \tag{81}$$

with

$$g(t) = \int_t^\infty \frac{sp(s)}{\sqrt{s^2-t^2}}\,ds \tag{82}$$

For a constant pressure p between a and c, one has (the Dugdale model is applied here)

$$g(t) = p\sqrt{c^2-t^2}, \quad \text{with } g(t)=0 \text{ for } t \geq c \tag{83}$$

$$g'(t) = -\frac{pt}{\sqrt{c^2-t^2}} \tag{84}$$

$$g(a) = p\sqrt{c^2-a^2} \tag{85}$$

and the integration of Eq.(80) leads to

$$\sigma_y(r,0) = \frac{2p}{\pi}\left[\sqrt{\frac{c^2-a^2}{a^2-r^2}} - tg^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}}\right] \tag{86}$$

However, the stresses are not self-equilibrated because the force

(Integrating via Mathematica, we shall have

$$\int_0^a \sigma_y 2\pi r dr = 4p\int_0^a \left[\sqrt{\frac{c^2-a^2}{a^2-r^2}} - tg^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}}\right] r dr =$$

$$4p\sqrt{c^2-a^2}\,a - 4p\int_0^a r tg^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}} dr =$$

$$4p\sqrt{c^2-a^2}\,a - 4p\left(\frac{\pi}{4}\left(a^2-c^2\right) + \frac{\left(a\sqrt{c^2-a^2}+c^2\left(\arctan\frac{\sqrt{c^2-a^2}}{a}\right)\right)}{2}\right)$$

$$= p\pi\left(c^2-a^2\right) - 2pa^2\left(\frac{c^2}{a^2}\cos^{-1}\frac{a}{c} - \sqrt{\frac{c^2}{a^2}-1}\right)$$

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$$\int_0^a \sigma_y 2\pi r dr = p\pi\left(c^2-a^2\right) - 2pa^2\left[\frac{c^2}{a^2}\cos^{-1}\frac{a}{c} - \sqrt{\frac{c^2}{a^2}-1}\right] \tag{87}$$

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this means that, the adhesive force acting outside the contact area and the holding force acting inside the contact area are not balanced. There is an additional force P' in the system

$$2pa^2\left[\frac{c^2}{a^2}\cos^{-1}\frac{a}{c} - \sqrt{\frac{c^2}{a^2}-1}\right] \tag{88}$$

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does not equilibrate the force $p\pi\left(c^2-a^2\right)$ applied inside the crack. This is due to the implicit assumption of no elastic displacement at y_∞. (We don't want this kind of force P'. Because the loading P' on a flat punch will deteriorate our analysis of the loading on a sphere punch. We only want the force $p\pi\left(c^2-a^2\right)$. That is why we need to add additional force $-P'$ to remove the influence of P')

A compressive force exists in the neck (neck is the contact area),

$$P' = +2pa^2\left[\frac{c^2}{a^2}\cos^{-1}\frac{a}{c} - \sqrt{\frac{c^2}{a^2}-1}\right] > 0 \tag{89}$$

which equilibrates the stress applied at y infinite to give $u(r,\infty)=0$.

Keeping the radius a constant, let us apply at infinity a load $-P'$ (i.e. a traction force). This leads to a flat punch displacement, and introduces in the neck a Boussinesq stress distribution (Eq. (48)), (it is flat punch because, for the current crack model, the indenter is flat (not sphere))

$$\sigma_y = \frac{P'}{2\pi a}\frac{1}{\sqrt{a^2-r^2}} > 0 \tag{90}$$

Adding the two stress distributions Eq.(86) and Eq.(90), we obtain

(this is the total stress distribution)

$$\sigma_T(r,0) = \frac{p}{\pi}\left[\frac{1}{\sqrt{a^2-r^2}}\left(\sqrt{c^2-a^2}+\frac{c^2}{a}\cos^{-1}\frac{a}{c}\right)-2tg^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}}\right] \tag{91}$$

and one can verify that this stress distribution equilibrates the force with the force $p\pi\left(c^2-a^2\right)$ at infinity being zero.

Introducing the stress intensity factor (i.e. the coefficient of $\frac{1}{\sqrt{a^2-r^2}}$)

$$K_m = \frac{p}{\sqrt{\pi a}}\left[\sqrt{c^2-a^2}+\frac{c^2}{a}\cos^{-1}\frac{a}{c}\right] \tag{92}$$

Eq.(91) becomes

$$\sigma_T(r,0) = \frac{K_m}{\sqrt{\pi a}}\frac{a}{\sqrt{a^2-r^2}} - \frac{2p}{\pi}tg^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}} \tag{93}$$

Integration of Eq.(81) leads to

$$u_y(r,0) = \frac{4\left(1-\nu^2\right)}{\pi E}\left[\frac{p}{a}\sqrt{c^2-a^2}\sqrt{r^2-a^2} - pc^2\int_a^{\min(r,c)}\frac{\sqrt{r^2-t^2}}{t^2\sqrt{c^2-t^2}}dt\right], \quad r > a \tag{94}$$

The remaining integral being an elliptic integral.

However, we have to take into account the elastic displacement due to the flat punch displacement under the load $-P'$. Under that load, the profile of the surface in the Boussinesq theory, Eq.(49), is

$$u_B(r,0) = -\frac{1-\nu^2}{\pi E}\frac{P'}{a}\sin^{-1}\frac{a}{r} \tag{95}$$

and the penetration of the punch is (Eq. (47))

$$\delta_B = \frac{2P}{3aK} = -\frac{1-\nu^2}{\pi E}\frac{P'}{a}\frac{\pi}{2} < 0 \tag{96}$$

Taking the origin of the displacements at the crack tip, one has

(the displacement is the displacement after rigid penetration)

$$u'_y = \delta_B - u_B = -\frac{1-\nu^2}{\pi E}\frac{P'}{a}\frac{\pi}{2} + \frac{1-\nu^2}{\pi E}\frac{P'}{a}\sin^{-1}\frac{a}{r} = -\frac{1-\nu^2}{\pi E}\frac{P'}{a}\cos^{-1}\frac{a}{r} \tag{97}$$

That is

$$u'_y = -\frac{1-\nu^2}{\pi E}\frac{P'}{a}\cos^{-1}\frac{a}{r} = \frac{1-\nu^2}{\pi E}2pa\left[\sqrt{\frac{c^2}{a^2}-1} - \frac{c^2}{a^2}\cos^{-1}\frac{a}{c}\right]\cos^{-1}\frac{a}{r} \tag{98}$$

We have now to add this displacement to the displacement given by Eq., taking acre of the signs. The load $-P'$ at infinity increases the gap u_y so that

$$u_{yT} = u_y - u'_y \tag{99}$$

i.e.

$$\begin{aligned} &u_{yT} = u_y - u'_y = \\ &\frac{2(1-\nu^2)}{\pi E}K_m\sqrt{\pi a}\cos^{-1}\frac{a}{r} + \\ &\frac{4(1-\nu^2)}{E}\frac{p}{\pi a}\left[\sqrt{c^2-a^2}\left(\sqrt{r^2-a^2} - a\cos^{-1}\frac{a}{r}\right) - ac^2\int_a^{\min(r,c)}\frac{\sqrt{r^2-t^2}}{t^2\sqrt{c^2-t^2}}dt\right] \end{aligned} \tag{100}$$

which describes the opening of the crack due to the internal pressure p.

- Dugdale Analysis

It is well known since Barenblatt that the stresses at a crack tip reach the theoretical stress of the material, which is for a Lennard-Jones potential

$$\sigma_{th} = 1.03\frac{w}{Z_0} \tag{101}$$

(where Z_0 is the equilibrium separation of the atoms) and then decrease to zero over a distance d, called the cohesive zone. The stress intensity factor due to this inner loading cancels the stress intensity factor due to the outer loading so that singularities disappear and the deformed crack has a sharp tip. The LEFM approximation can only be used when the cohesive zone is small compared to a characteristic length, for example the crack length. Expressed in that manner the problem is self-consistent, the stress distribution in the crack depending on the crack shapes, which in turn depend on the Dugdale model, in which the stresses are assumed constant (σ_0) over a distance d, which is determined by the cancellation of stress intensity factors. In the original theory σ_0 was the yield stress of the material. We take there the theoretical stress.

Changing the internal pressure p to $-\sigma_0$ and adding Eq.(61) and Eq.(93),

$$K_1 = -\frac{3K}{8}\chi(1)\sqrt{\frac{\pi}{a}} = -K_m = \frac{\sigma_0}{\sqrt{\pi a}}\left[\sqrt{c^2-a^2}+\frac{c^2}{a}\cos^{-1}\frac{a}{c}\right] \tag{102}$$

This relation fixes the value c/a of as a function of material property σ_0 and K_I (which is a function of a and P).

The stress distribution in the neck reduces to (i.e the neck here seems to be the contact area)

$$\sigma_y(r,0) = -\frac{3K}{2\pi R}\sqrt{a^2-r^2}+\frac{2\sigma_0}{\pi}tg^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}} \tag{103}$$

with $\sigma_y(a,0)=\sigma_0$, so that $K_I+K_m=0$ ensures the continuity of stresses.

When $c/a \to 1$, Eq.(102) shows that

$$\frac{K_I}{\sigma_0\sqrt{\pi a}} \to 0 \tag{104}$$

$$\sqrt{c^2-a^2} \simeq \frac{K_I\sqrt{\pi a}}{2\sigma_0} \tag{105}$$

This last equation can be written

$$d \simeq \frac{\pi K_I^2}{8\sigma_0^2} \tag{106}$$

Which is the classic small scale yielding approximation in fracture mechanics. The stresses distribution becomes

$$\sigma_y(r,0) \simeq -\frac{3K}{2\pi R}\sqrt{a^2-r^2}+\frac{K_I}{\sqrt{\pi a}}\frac{a}{\sqrt{a^2-r^2}} \tag{107}$$

which is the JKR stress distribution.

When $c/a \to \infty$, Eq. (102) show that

$$\frac{K_I}{\sigma_0\sqrt{\pi a}} \sim \frac{c^2}{2a^2} \to \infty \tag{108}$$

and the stress distribution becomes

$$\sigma_y(r,0) \simeq -\left(\frac{3K}{2\pi R}+\frac{2\sigma_0}{\pi c}\right)\sqrt{a^2-r^2}+\sigma_0 \tag{109}$$

Since it is σ_0 that tends towards zero in Eq. (108), the limit is a Hertzian distribution as assumed in the DMT theory.

Elastic Displacement

The air gap in the JKR theory is given by Eq.(63). Intermolecular forces reduce it. Changing p to $-\sigma_0$, adding Eq. (63) and Eq.(100), and using $K_m + K_I = 0$, we get

$$\begin{aligned}\left[u_y\right] &= \frac{a^2}{\pi R}\left[\sqrt{\frac{r^2}{a^2}-1}+\left(\frac{r^2}{a^2}-2\right)\cos^{-1}\frac{a}{r}\right]+ \\ &\frac{4\left(1-\nu^2\right)}{E}\frac{\sigma_0}{\pi a}\left[ac^2\int_a^{\min(r,c)}\frac{\sqrt{r^2-t^2}}{t^2\sqrt{c^2-t^2}}dt-\sqrt{c^2-a^2}\left(\sqrt{r^2-a^2}-a\cos^{-1}\frac{a}{r}\right)\right]\end{aligned} \tag{110}$$

When $c \to 0$, Eq.(110) reduces to

$$\left[u_y\right] = \frac{a^2}{\pi R}\left[\sqrt{\frac{r^2}{a^2}-1}+\left(\frac{r^2}{a^2}-2\right)\cos^{-1}\frac{a}{r}\right] \tag{111}$$

which is the Hertzian air gap, Eq.(59).

When $c \to a$, one can see that the limit of Eq.(110) is the JKR profile. In effect, using the second theorem of the mean value with $a < \xi < c$, we can write

$$ac^2\int_a^c \frac{\sqrt{r^2-t^2}}{t^2\sqrt{c^2-t^2}}dt = ac^2\sqrt{r^2-\xi^2}\int_a^c\frac{dt}{t^2\sqrt{c^2-t^2}} = \sqrt{r^2-\xi^2}\sqrt{c^2-a^2} \tag{112}$$

which tends toward $\sqrt{r^2-a^2}\sqrt{c^2-a^2}$ when $c \to a$.

It is also easy to see that the curve given by Eq.(110) has a vertical tangent at $r = c$. This behavior comes from the integral. Derivating under the integral either for or we get

$$\frac{dI(r)}{dr} = r\int_a^{\min(r,c)}\frac{dt}{t^2\sqrt{r^2-t^2}\sqrt{c^2-t^2}} \tag{113}$$

and for

$$\left(\frac{dI}{dr}\right)_{r=c} = c\left[-\frac{1}{c^2 t}+\frac{1}{2c^3}Ln\left|\frac{c+t}{c-t}\right|\right]_a^c \tag{114}$$

which is infinite. (what is the function I(r)?)

The penetration δ of the sphere is the sum of the JKR penetration, Eq.(60), and the penetration due to molecular forces outside the area of contact. Replacing p with $-\sigma_0$ in Eq.(96) we get

$$\delta = \frac{a^2}{3R} + \frac{2P}{3aK} + \frac{1-v^2}{E}\sigma_0 a\left[\frac{c^2}{a^2}\cos^{-1}\frac{a}{c} - \sqrt{\frac{c^2}{a^2}-1}\right] \tag{115}$$

Inserting this relation into Eq.(102), we can see that the condition of cancellation of stress singularities reduces to a condition on the penetration δ: (with the help of Eq.(36) and Eq.(42))

$$\begin{aligned}\delta &= \frac{a^2}{R} - \left(\frac{8\pi wa}{3K}\right)^{1/2} + \frac{1-v^2}{E}\sigma_0 a\left[\frac{c^2}{a^2}\cos^{-1}\frac{a}{c} - \sqrt{\frac{c^2}{a^2}-1}\right] \\ &= \frac{a^2}{R} - \frac{8\sigma_0}{3K}\sqrt{c^2-a^2}\end{aligned} \tag{116}$$

At this stage c/a is given as a function of (P,a) or (δ,a), but the equilibrium curves $\delta(a)$, $a(p)$, and $P(\delta)$ cannot be drawn since a relation linking c to σ_0 and a is still missing. This closure relation is given by the Griffith relation $G = w$ based on an energy balance. (I think the relationship $\left[u_y\right](c) = d$ can help to determine c. Why do we need the energy release rate).

3.1. Further Calculation

According to the Rice J-integral, the energy release rate in a Dugdale model is simply

Where is the crack opening displacement (COD), i.e., the crack opening for $r = c$ at the end of the cohesive zone. Making $r = c$ in Eq.(110), letting $m = c/a$, and recalling that

$$\cos^{-1}\frac{a}{c} = tg^{-1}\sqrt{\frac{c^2}{a^2}-1} \tag{117}$$

We have

$$\begin{aligned}G = \frac{\sigma_0 a^2}{\pi R}\left\{\left[\sqrt{m^2-1} + \left(m^2-2\right)tg^{-1}\sqrt{m^2-1}\right]\right\} + \\ \frac{4\left(1-v^2\right)}{\pi E}\sigma_0^2 a\left[\sqrt{m^2-1}tg^{-1}\sqrt{m^2-1} - m + 1\right]\end{aligned} \tag{118}$$

Recalling that

…

As a verification let us integrate the stress distribution, Eq.(103), over the area of contact:

$$I=\int_0^a 2\pi\sigma_y r dr=\int_0^a 2\pi r\left[-\frac{3K}{2\pi R}\sqrt{a^2-r^2}+\frac{2\sigma_0}{\pi}tg^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}}\right]dr$$

$$=-\frac{a^3K}{R}+4\sigma_0\int_0^a r\times \mathrm{tg}^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}}dr$$

$$=-\sigma_0\pi\left(c^2-a^2\right)-\frac{a^3K}{R}+2\sigma_0 a^2\left(\sqrt{\frac{c^2}{a^2}-1}+\frac{c^2}{a^2}\cos^{-1}\frac{a}{c}\right)$$

$$=-P_{\mathrm{jkr}}-\sigma_0\pi\left(c^2-a^2\right)$$

3.2. Elliptic integral

Our aim is to simplify the calculation of the integration

$$\int_a^{\min(r,c)}\frac{\sqrt{r^2-t^2}}{t^2\sqrt{c^2-t^2}}dt$$

For $r=c$,

$$\int_a^{\min(r,c)}\frac{\sqrt{r^2-t^2}}{t^2\sqrt{c^2-t^2}}dt=\int_a^c\frac{1}{t^2}dt=\left[-t^{-1}\right]_a^c=\frac{1}{a}-\frac{1}{c}$$

For $r<c$, let

$$I_1=\int_a^r\frac{\sqrt{r^2-t^2}}{t^2\sqrt{c^2-t^2}}dt$$

And make $t=1/x$. We get

$$I_1=\frac{r}{c}\int_{r^{-1}}^{a^{-1}}\frac{\sqrt{x^2-r^{-2}}}{\sqrt{x^2-c^{-2}}}dx$$

For convenience, we define

$$v=1/a,\alpha=1/r,\beta=1/c$$

We shall have

$$I_1=\frac{r}{c}\int_\alpha^v\sqrt{\frac{x^2-\alpha^2}{x^2-\beta^2}}dx$$

Its solution is

$$I_1=\frac{r}{c}\left[v\sqrt{\frac{v^2-\alpha^2}{v^2-\beta^2}}-\alpha E(\phi,k)\right]=\frac{1}{a}\sqrt{\frac{r^2-a^2}{c^2-a^2}}-\frac{1}{c}E(\phi,k)$$

with

$$\phi=\sin^{-1}\sqrt{\frac{v^2-\alpha^2}{v^2-\beta^2}}=\sin^{-1}\left(\frac{c}{r}\sqrt{\frac{r^2-a^2}{c^2-a^2}}\right)\quad k=\frac{\beta}{\alpha}=\frac{r}{c}$$

For $r>c$, let

$$I_2 = \int_a^c \frac{\sqrt{r^2 - t^2}}{t^2 \sqrt{c^2 - t^2}} dt$$

and make again $t = 1/x$. We get

$$I_2 = \frac{r}{c} \int_{c^{-1}}^{a^{-1}} \frac{\sqrt{x^2 - r^{-2}}}{\sqrt{x^2 - c^{-2}}} dx$$

For convenience, we define

$$u = 1/a, \alpha' = 1/c, \beta' = 1/r$$

The integral can be rewritten as

$$I_2 = \frac{r}{c} \int_{\alpha'}^{u} \frac{\sqrt{x^2 - \beta'^2}}{\sqrt{x^2 - \alpha'^2}} dx$$

Its value is

$$I_2 = \frac{1}{c} \left[\frac{r^2}{ac} \sqrt{\frac{c^2 - a^2}{r^2 - a^2}} + \frac{r^2 - c^2}{rc} F(\phi, k) - \frac{r}{c} E(\phi, k) \right]$$

with

$$k = \frac{\beta'}{\alpha'} = \frac{c}{r}$$

$$\phi = \sin^{-1}\left(\frac{r}{c} \sqrt{\frac{c^2 - a^2}{r^2 - a^2}} \right)$$

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