Lecture-1 Differential Notation

As of the author's compilation of this note on September 16, 2023, solutions to partial differential equations based on machine learning methods are emerging rapidly. However, in the current computational resources and algorithmic frameworks, machine learning methods are still far from replacing finite element methods. Machine learning, in solving problems related to PDEs, still faces serious limitations in terms of speed, accuracy, complexity, and more. Moreover, from the author's perspective, the future of solving PDEs with ML algorithms will require the integration of certain finite element frameworks. Therefore, learning the principles of finite elements is still relevant for most students in mechanics, civil engineering, structural engineering, and applied mathematics.

This chapter's theme is differential operators. Unlike differential operators for continuous functions, differential operators for discrete arrays involve operations on discrete data.

- ## Discretization of Coordinate

For a given function $u(x)$,and the uniform mesh $x_i = ih$, we have

$$u_i = u(x = x_i)$$

For any two functions $u(x)$ and $v(x)$, we have

$$(uv)_i = u_i v_i$$

In the current chapter, we don't adopt any summation rules.

- ## Operator

For a given series u_i, we define the operator E^q as

$$Eu_i = u_{i+1}$$

$$E^{-1}u_i = u_{i-1}$$

$$E^0 u_i = u_i$$

where E is the forward operator, E^{-1} is the backward operator, E^0 is the identical operator. The forward, backward and central difference operators are defined as

$$D^{+} = \frac{E - E^{0}}{h}$$

$$D^{-} = \frac{E^{0} - E^{-1}}{h}$$

$$D^{0} = \frac{E - E^{-1}}{2h} = \frac{1}{2}(D^{+} + D^{-})$$

Considering the exponential function e^{iwx}. According to the Taylor expansions, we have

$$hD^{+}e^{iwx} = \left(iwh + O(w^2h^2)\right)e^{iwx}$$

$$hD^{-}e^{iwx} = \left(iwh + O(w^2h^2)\right)e^{iwx}$$

$$hD^{0}e^{iwx} = \left(iwh + O(w^3h^3)\right)e^{iwx}$$

Therefore D^{+} and D^{-} are first order estimate of $\frac{\partial}{\partial x}$, the central difference operator D^{0} is the second order estimate。 Similarly, we know that $D^{+}D^{-}$ is the second order estimate of $\frac{\partial^2}{\partial x^2}$.

- **Norm**

For any two series u_i and v_i, the norm <> is defined as

$$\langle u, v\rangle = \sum_{i=0}^{n} \bar{u}_i v_i$$

or

$$\langle u, v\rangle_h = h\sum_{i=0}^{n} \bar{u}_i v_i$$

It's worth noting that we define the norm here in order to determine whether the numerical theme will explode in future calculations. This is often very important for assessing the stability of an algorithm. For the norm of the same array, we can express it as: "

$$|u| = \langle u, u\rangle = \sum_{i=0}^{n} \bar{u}_i v_i$$

In addition to defining a vector, we can also define the norm of a matrix. Its definition is somewhat unique.

$$|A| = \max|Au| \text{ where } |u| = 1$$

This definition is a bit not straightforward. For example, for any given matrix, it's quite difficult to search for this 'u' that maximizes |Au|. For now, we will ignore this aspect. We can verify

$$|Au| \leq |A||u|$$

$$|A+B| \leq |A| + |B|$$

$$|AB| \leq |A||B|$$

In the end, we will define the norm of and operator.

$$\|Q\| = \frac{\|Qu\|_h}{\|u\|_h} = \sup\|Qu\|_h \text{ when } \|u\|_h = 1$$

For any periodic function $\|u\|_h$, we have

$$\|E^q\| = 1$$

According to the inequility, we have

$$\|D^+\| = \left\|\frac{E - E^0}{h}\right\| \leq \left\|\frac{E}{h}\right\| + \left\|\frac{E^0}{h}\right\| = \frac{2}{h}$$

$$\|D^0\| = \left\|\frac{E^{+1} - E^{-1}}{2h}\right\| \leq \left\|\frac{E^{+1}}{h}\right\| + \left\|\frac{E^{-1}}{h}\right\| = \frac{1}{h}$$

Reference

Gustafsson, B., Kreiss, H.O. and Oliger, J., 2013. Time dependent problems and difference methods (Vol. 2). New York: Wiley.

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