1. Suppose that the money supply process has the form $m_t = m + \rho m_{t-1} + \varepsilon_t$, where m is a constant and $0 < \rho < 1$.

a. Show that it is possible to express m_{t+n} in terms of the known value m_t and the sequence $\{\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots, \varepsilon_{t+n}\}$.

$m_t = m + \rho m_{t-1} + \varepsilon_t$, m is constant, $0<\rho<1$

$m_1 = m + \rho m_0 + \varepsilon_1$

$m_2 = m + \rho m_1 + \varepsilon_2 = m + \rho \cdot (m + \rho m_0 + \varepsilon_1) + \varepsilon_2 = m + \rho m + \rho^2 m_0 + \rho \varepsilon_1 + \varepsilon_2$

$m_3 = m + \rho m_2 + \varepsilon_3 = m + \rho \cdot (m + \rho m + \rho^2 m_0 + \rho \varepsilon_1 + \varepsilon_2) + \varepsilon_3$

$= m + \rho m + \rho^2 m + \rho^3 m_0 + \rho^2 \varepsilon_1 + \rho \varepsilon_2 + \varepsilon_3 = m(\rho^0 + \rho^1 + \rho^2) + \rho^3 m_0 + \rho^2 \varepsilon_1 + \rho \varepsilon_2 + \varepsilon_3$

$\therefore m_t = m \sum_{i=0}^{t-1} \rho^i + \rho^t m_0 + \sum_{i=0}^{t-1} \rho^i \varepsilon_{t-i}$

If $0<\rho<1$ the term ρ^t approaches zero as t approaches infinity

Also, the infinite sum $[1, \rho, \rho^2 + \cdots]$ converges to $1/(1-\rho)$

$\therefore m_t = m/(1-\rho) + \sum_{i=0}^{\infty} \rho^i \varepsilon_{t-i}$

b. Suppose that all values of ε_{t+i} for $i > 0$ have a mean value of zero. Explain how you could use your result in part a to forecast the money supply n periods into the future.

$m_t = m + \rho m_{t-1} + \varepsilon_t$

$m_{t+1} = m + \rho m_t + \varepsilon_{t+1}$

$E[m_{t+1}] = m + \rho m_t$

$m_{t+2} = m + \rho m_{t+1} + \varepsilon_{t+2}$

$E[m_{t+2}] = m + \rho \cdot E[m_{t+1}] = m + \rho \cdot (m + \rho m_t) = m + \rho m + \rho^2 m_t$

$m_{t+3} = m + \rho m_{t+2} + \varepsilon_{t+3}$

$E[m_{t+3}] = m + \rho \cdot E[m_{t+2}] = m + \rho \cdot (m + \rho m + \rho^2 m_t) = m + \rho m + \rho^2 m + \rho^3 m_t$

$\therefore m_{t+n} = m \sum_{i=0}^{n-1} \rho^i + \rho^n m_t$

if $0<\rho<1$ the term ρ^n approaches zero as n approaches infinite

Also the infinite sum $[1, \rho, \rho^2 + \cdots]$ converges to $1/(1-\rho)$

$\therefore m_{t+n} = m/(1-\rho)$

2. Consider the stochastic difference equation

$$y_t = 0.8y_{t-1} + \varepsilon_t - 0.5\varepsilon_{t-1}$$

a. Suppose that the initial conditions are such that $y_0 = 0$ and $\varepsilon_0 = \varepsilon_{-1} = 0$. Now suppose that $\varepsilon_1 = 1$. Determine the values y_1 through y_5 by forward iteration.

if $t=1$

$y_1 = 0.8y_0 + \varepsilon_1 - 0.5\varepsilon_0 = \varepsilon_1 = 1$

$\therefore y_1 = 1$

if $t=2$

$y_2 = 0.8y_1 + \varepsilon_2 - 0.5\varepsilon_1 = 0.8 + \varepsilon_2 - 0.5$

$\therefore y_2 = 0.3 + \varepsilon_2$

if $t=3$

$y_3 = 0.8y_2 + \varepsilon_3 - 0.5\varepsilon_2$

$= 0.8(0.3 + \varepsilon_2) + \varepsilon_3 - 0.5\varepsilon_2$

$= 0.24 + 0.8\varepsilon_2 + \varepsilon_3 - 0.5\varepsilon_2$

$\therefore y_3 = 0.24 + 0.3\varepsilon_2 + \varepsilon_3$

if $t=4$

$y_4 = 0.8y_3 + \varepsilon_4 - 0.5\varepsilon_3$

$= 0.8(0.24 + 0.3\varepsilon_2 + \varepsilon_3) + \varepsilon_4 - 0.5\varepsilon_3$

$= 0.192 + 0.24\varepsilon_2 + 0.8\varepsilon_3 + \varepsilon_4 - 0.5\varepsilon_3$

$\therefore y_4 = 0.192 + 0.24\varepsilon_2 + 0.3\varepsilon_3 + \varepsilon_4$

if $t=5$

$y_5 = 0.8y_4 + \varepsilon_5 - 0.5\varepsilon_4$

$= 0.8(0.192 + 0.24\varepsilon_2 + 0.3\varepsilon_3 + \varepsilon_4) + \varepsilon_5 - 0.5\varepsilon_4$

$\cdot 536 + 0.192\varepsilon_2 + 0.24\varepsilon_3 + 0.8\varepsilon_4 + \varepsilon_5 - 0.5\varepsilon_4$

$\therefore y_5 = 0.1536 + 0.192\varepsilon_2 + 0.24\varepsilon_3 + 0.3\varepsilon_4 + \varepsilon_5$

$= 0.1536\varepsilon_1 + 0.192\varepsilon_2 + 0.24\varepsilon_3 + 0.3\varepsilon_4 + \varepsilon_5$

b. Trace out the time path of an ε_t shock on the entire time path of the $\{y_t\}$ sequence.

$y_4 = 0.3\times 0.8^2\varepsilon_1 + 0.3\times 0.8^1\varepsilon_2 + 0.3\times 0.8^0\varepsilon_3 + \varepsilon_4$

$y_5 = 0.3\times 0.8^3\varepsilon_1 + 0.3\times 0.8^2\varepsilon_2 + 0.3\times 0.8^1\varepsilon_3 + 0.3\times 0.8^0\varepsilon_4 + \varepsilon_5$

$\therefore y_t = 0.3\sum_{i=1}^{t-1}\varepsilon_i\, 0.8^{t-1-i} + \varepsilon_t$

3. Consider the second-order autoregressive process $y_t = a_0 + a_2 y_{t-2} + \varepsilon_t$ where $|a_2| < 1$. Find:

- $E_{t-2} y_t$

$E_{t-2}\, y_t = E_{t-2}[a_0 + a_2 y_{t-2} + \varepsilon_t] = E_{t-2}[a_0] + E_{t-2}[a_2 y_{t-2}] + E_{t-2}[\varepsilon_t] = a_0 + a_2 y_{t-2}$

$\therefore\ a_0 + a_2 y_{t-2}$

- $E_{t-1} y_t$

$E_{t-1} y_t = E_{t-1}[a_0 + a_2 y_{t-2} + \varepsilon_t] = a_0 + a_2 y_{t-2}$

$\therefore\ a_0 + a_2 y_{t-2}$

- $E_t y_{t+2}$

$E_t\, y_{t+2} = E_t[a_0 + a_2 y_t + \varepsilon_{t+2}] = a_0 + a_2 y_t$

$\therefore\ a_0 + a_2 y_t$

- $cov(y_t, y_{t-1})$

$Cov(y_t, y_{t-1}) = Cov(a_0 + a_2 y_{t-2} + \varepsilon_t,\ y_{t-1}) = Cov(a_0, y_{t-1}) + Cov(a_2 y_{t-2}, y_{t-1}) + Cov(\varepsilon_t, y_{t-1})$

$= Cov(a_2 y_{t-2}, y_{t-1}) = a_2 \cdot Cov(y_{t-2}, y_{t-1})$

$= a_2 \cdot Cov(a_0 + a_2 y_{t-4} + \varepsilon_{t-4},\ a_0 + a_2 y_{t-3} + \varepsilon_{t-1}) = a_2\, Cov(a_2 y_{t-4}, a_2 y_{t-3})$

$\therefore\ a_2^{i}(y_0, y_1) = 0$

- $cov(y_t, y_{t-2})$

$Cov(y_t, y_{t-2}) = Cov(a_0 + a_2 y_{t-2} + \varepsilon_t,\ y_{t-2}) = Cov(a_2 y_{t-2}, y_{t-2}) = a_2\, Cov(y_{t-2}, y_{t-2})$

$= a_2\, Var(y_{t-2})$

$Var(y_{t-2}) = Var(a_0 + a_2 y_{t-4} + \varepsilon_{t-2}) = a^2 Var(y_{t-4}) + \sigma^2$

$= a_2^5 \cdot Var(y_{t-6}) + a_2 \sigma^2 + a_2^3 \sigma^2 + \cdots$

$= a_2^{i}\ Var(y_t) + a_2 \sigma^2 + \cdots$

$\therefore\ a_2^{i}\ Var(y_t) + a_2 \sigma^2 + a_2^3 \sigma^2 + \cdots$

4. [bonus points] There are often several representations for the identical time-series process. In the next, the standard equation for an AR(1) model is given by $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$.

a. Show that equivalent representations are i. $(y_t - \bar{y}) = a_1(y_{t-1} - \bar{y}) + \varepsilon_t$ where $\bar{y}$ is the unconditional mean of the $\{y_t\}$ series and ii.$y_t = a_0/(1 - a_1) + \mu_t$ where $\mu_t = a_1\mu_{t-1} + \varepsilon_t$.

i. $(y_t - \bar{y}) = a_1(y_{t-1} - \bar{y}) + \varepsilon_t$

$(y_t - \bar{y})^2 = a_1(y_{t-1} - \bar{y})(y_t - \bar{y}) + \varepsilon_t(y_t - \bar{y})$

$E[(y_t - \bar{y})^2] = a_1 E[(y_t - \bar{y})(y_{t-1} - \bar{y})]$

$\sigma^2 = a_1 \cdot \dfrac{\sigma^2 \cdot a_1}{1 - a_1^2}$

$1 = \dfrac{a_1^2}{1 - a_1^2}$

$1 - a_1^2 = a_1^2$

$2a_1^2 = 1$

$a_1^2 = \frac{1}{2}$

$a_1 = \sqrt{\frac{1}{2}}$

ii. $y_t = a_0/(1 - a_1) + \mu_t$

$\mu_t = a_1\mu_{t-1} + \varepsilon_t$

$\mu_{t-1} = a_1\mu_{t-2} + \varepsilon_{t-1}$

$\mu_{t-2} = a_1\mu_{t-3} + \varepsilon_{t-2}$

$y_t = a_0/(1-a_1) + a_1\mu_{t-1} + \varepsilon_t$

$= a_0/(1-a_1) + a_1(a_1\mu_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$

$= a_0/(1-a_1) + a_1^2\mu_{t-2} + a_1\varepsilon_{t-1} + \varepsilon_t$

$= a_0/(1-a_1) + a_1^2(a_1\mu_{t-3} + \varepsilon_{t-2}) + a_1\varepsilon_{t-1} + \varepsilon_t$

$= a_0/(1-a_1) + a_1^3\mu_{t-3} + a_1^2\varepsilon_{t-2} + a_1\varepsilon_{t-1} + \varepsilon_t$

$\therefore y_t = a_0/(1-a_1) + a_1^n\mu_{t-n} + \sum_{i=0}^{n-1} a_1^i \varepsilon_{t-i}$

if $a_1 = \sqrt{\frac{1}{2}}$ the term a_1^n approaches zero as n approaches infinite

$y_t = a_0/(1-a_1) + \sum_{i=0}^{t} a_1^i \varepsilon_{t-i}$

$\therefore y_t = a_0/(1-a_1) + \sum_{i=0}^{t} a_1^i \varepsilon_{t-i}$

b. In Chapter 1, we considered several models with a deterministic time trend. For example,

$y_t = a_0 + a_1 y_{t-1} + a_2 t + \varepsilon_t$ where $|a_1| < 1$. Explain why the y_t sequence is not stationary. Also, explain why the y_t sequence is stationary about the trend line $a_0 + a_1 t$. What does it mean to say that the y_t sequence is trend stationary?

$y_t = a_0 + a_1 y_{t-1} + a_2 t + \varepsilon_t$, $|a_1| < 1$

$y_1 = a_0 + a_1 y_0 + a_2 + \varepsilon_1$

$y_2 = a_0 + a_1 y_1 + 2a_2 + \varepsilon_2 = a_0 + a_1(a_0 + a_1 y_0 + a_2 + \varepsilon_1) + 2a_2 + \varepsilon_2 = a_0 + a_1 a_0 + a_1^2 y_0 + a_1 a_2 + a_1\varepsilon_1 + 2a_2 + \varepsilon_2$

$y_3 = a_0 + a_1 y_2 + 3a_2 + \varepsilon_3 = a_0 + a_1(a_0 + a_1 a_0 + a_1^2 y_0 + a_1 a_2 + a_1\varepsilon_1 + 2a_2 + \varepsilon_2) + 3a_2 + \varepsilon_3$

$= a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_0 + a_1^2 a_2 + a_1^2\varepsilon_1 + 2a_2 a_1 + a_1\varepsilon_2 + 3a_2 + \varepsilon_3$

$= (a_0 + a_1 a_0 + a_1^2 a_0) + a_1^3 y_0 + (a_1^2\varepsilon_1 + a_1\varepsilon_2 + \varepsilon_3) + (1 \cdot a_1^2 a_2 + 2a_1^1 a_2 + 3 \cdot a_1^0 \cdot a_2)$

$y_t = a_0 \cdot \sum_{i=0}^{t-1} a_1^i + a_1^t y_0 + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_2(1 \cdot a_1^2 + 2 \cdot a_1^1 + 3 \cdot a_1^0)$

$y_t = a_0 \cdot \sum_{i=0}^{t-1} a_1^i + a_1^t y_0 + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_2 \sum_{i=1}^{t} t \cdot a_1^{t-i}$

$\therefore a_2 \sum_{i=1}^{t} t \cdot a_1^{t-i}$ 가 t에 따라 변하기 때문에 not stationary.

if $|a_1| < 1$ the term a^t approaches zero as t approaches infinite

$y_t = a_0/(1-a_1) + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_2 \sum_{i=1}^{t} t \cdot a_1^{t-i}$

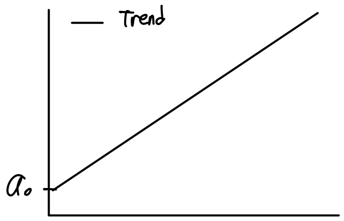
$y_t = a_0 + a_1 t$

$y_0 = a_0$

$y_1 = a_0 + a_1$

$y_2 = a_0 + 2a_1$

$y_3 = a_0 + 3a_1$



∴ Trend stationary는 Trend를 제거하면 정상성을 띄는 것을 의미한다.

Lab 1. The file QUARTERLY.CSV contains a number of series including the U.S. index of industrial production (*indprd*). unemployment rate (*urate*), and producer price index for industrial goods (*finished*). All of the series run from 1960Q1 to 2012Q4.

a. Construct the growth rate of the series as $y_t = log(indprod_t) - log(indprod_{t-1})$. Since the first few autocorrelations suggest an AR(1), estimate $y_t = \alpha + \beta y_{t-1} + \varepsilon_t$.

b. Estimate *urate* as an AR(2) process including an intercept. By defining $y_t = urate_t$, you should find values of α, β, and γ of $y_t = \alpha + \beta y_{t-1} + \gamma y_{t-2} + \varepsilon_t$.

Data Load

In [2]:
```
import pandas as pd
import numpy as np
import statsmodels.api as sm

data = pd.read_csv("QUARTERLY.csv")
```

Data preprocessing

In [3]:
```
data['IndProd_t-1'] = data['IndProd'].shift(1)
data['yt_inprod'] = np.log(data['IndProd']) - np.log(data['IndProd_t-1'])
```

In [4]:
```
data.head(5)
```

Out[4]:

	Date	FFR	Tbill	Tb1yr	r5	r10	PPINSA	Finished	CPI	CPICORE	...	M2SA	M2NSA	Unemp	IndProd	RGDP	Potent	Deflator	Curr	IndProd_t-1
0	1960-01-01	3.93	3.87	4.57	4.64	4.49	31.67	33.20	29.40	18.92	...	896.1	299.40	5.13	23.93	2845.3	2824.2	18.521	31.830	NaN
1	1960-04-01	3.70	2.99	3.87	4.30	4.26	31.73	33.40	29.57	19.00	...	903.3	300.03	5.23	23.41	2832.0	2851.2	18.579	31.862	23.93
2	1960-07-01	2.94	2.36	3.07	3.67	3.83	31.63	33.43	29.59	19.07	...	919.4	305.50	5.53	23.02	2836.6	2878.7	18.648	32.217	23.41
3	1960-10-01	2.30	2.31	2.99	3.75	3.89	31.70	33.67	29.78	19.14	...	932.8	312.30	6.27	22.47	2800.2	2906.7	18.700	32.624	23.02
4	1961-01-01	2.00	2.35	2.87	3.64	3.79	31.80	33.63	29.84	19.17	...	948.9	317.10	6.80	22.13	2816.9	2934.8	18.743	32.073	22.47

5 rows × 21 columns

a

coef const가 알파, coef ar.L1이 베타
yt = 알파 + 베타 yt-1 + 엡실론 t

In [5]:
```
model = sm.tsa.arima.ARIMA(data['yt_inprod'], order = (1,0,0))
result = model.fit()
result.summary()
```

Out[5]:

SARIMAX Results

Dep. Variable:	yt_inprod	**No. Observations:**	212
Model:	ARIMA(1, 0, 0)	**Log Likelihood**	621.539
Date:	Wed, 29 Mar 2023	**AIC**	-1237.079
Time:	23:08:49	**BIC**	-1227.009
Sample:	0	**HQIC**	-1233.009
	- 212		
Covariance Type:	opg		

	coef	std err	z	P>|z|	[0.025	0.975]
const	0.0065	0.002	2.607	0.009	0.002	0.011
ar.L1	0.6066	0.041	14.746	0.000	0.526	0.687
sigma2	0.0002	1e-05	16.110	0.000	0.000	0.000

Ljung-Box (L1) (Q):	0.92	**Jarque-Bera (JB):**	77.49
Prob(Q):	0.34	**Prob(JB):**	0.00
Heteroskedasticity (H):	0.37	**Skew:**	-0.09
Prob(H) (two-sided):	0.00	**Kurtosis:**	5.96

b

In [46]:
```
model1 = sm.tsa.arima.ARIMA(data['Unemp'], order = (2,0,0))
result1 = model1.fit()
result1.summary()
```

Out[46]:

SARIMAX Results

Dep. Variable:	Unemp	**No. Observations:**	212
Model:	ARIMA(2, 0, 0)	**Log Likelihood**	-5.541
Date:	Wed, 22 Mar 2023	**AIC**	19.082
Time:	13:13:49	**BIC**	32.508
Sample:	0	**HQIC**	24.508
	- 212		
Covariance Type:	opg		

	coef	std err	z	P>|z|	[0.025	0.975]
const	6.0881	0.568	10.724	0.000	4.975	7.201
ar.L1	1.6404	0.037	43.886	0.000	1.567	1.714
ar.L2	-0.6780	0.039	-17.568	0.000	-0.754	-0.602
sigma2	0.0604	0.005	12.972	0.000	0.051	0.070

Ljung-Box (L1) (Q):	0.05	**Jarque-Bera (JB):**	79.56
Prob(Q):	0.83	**Prob(JB):**	0.00
Heteroskedasticity (H):	0.65	**Skew:**	0.89
Prob(H) (two-sided):	0.07	**Kurtosis:**	5.41