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Q 1)

Q. 1.

since,

$$P(\mu|x) \propto P(\mu) * P(x|\mu)$$

$$P(\mu|x) \propto P(\mu) * \prod_{i=1}^{n} P(x_i|\mu)$$

$$= \frac{1}{\sqrt{2\pi\sigma_0^2}} * e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}} * \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}^{\,n+1} * \sqrt{\sigma_0^2 * \sigma^{2n}}} * e^{\left(\frac{-\mu^2+2\mu\mu_0-\mu_0^2}{2\sigma_0^2} - \sum_{i=1}^{n} \frac{x_i^2-2\mu x_i+\mu^2}{2\sigma^2}\right)}$$

$$\propto e^{\left[\frac{-\mu^2\sigma^2+2\mu\mu_0\sigma^2-\mu_0^2\sigma^2-\sigma_0^2\sum x_i^2+2\mu\sigma_0^2\sum x_i-n\mu^2\sigma_0^2}{2\sigma_0^2\sigma^2}\right]}$$

$$= e^{\left[\frac{-\mu^2(\sigma^2+n\sigma_0^2)+2\mu(\mu_0\sigma^2+\sigma_0^2\sum x)-(\mu_0^2\sigma^2+\sigma_0^2\sum x_i^2)}{2\sigma_0^2\sigma^2}\right]}$$

$$P(\mu|x) = e^{\left[\frac{-\mu^2+\frac{2\mu(\mu_0\sigma^2+\sigma_0^2\sum x)}{\sigma^2+n\sigma_0^2}-\left(\frac{\mu_0\sigma^2+\sigma_0^2\sum x}{\sigma^2+n\sigma_0^2}\right)^2+\left(\frac{\mu_0\sigma^2+\sigma_0^2\sum x}{\sigma^2+n\sigma_0^2}\right)^2}{\frac{2\sigma_0^2\sigma^2}{\sigma^2+n\sigma_0^2}}\right]} * e^{-\left(\frac{\mu_0^2\sigma^2+\sigma_0^2\sum x^2}{2\sigma_0^2\sigma^2}\right)}$$

$$P(\mu|x) \propto e^{\left[\frac{-\left(\mu-\left(\frac{\mu_0\sigma^2+\sigma_0^2\sum x_i}{\sigma^2+n\sigma_0^2}\right)\right)^2}{\frac{2\sigma_0^2\sigma^2}{\sigma^2+n\sigma_0^2}}\right]} \quad \ldots (A)$$

Q 2)

From the equation, A) the derived posterior distribution is the quadratic and a Gaussian distribution as it is proportional to the Normal form.

Q. 2.

$$e^{-\frac{(\mu-\mu_n)^2}{2\sigma_n^2}} \sim N(\mu_n, \sigma_n^2)$$

where, $\sigma_n^2 = \dfrac{\sigma_0^2\sigma^2}{\sigma^2+n\sigma_0^2}$ and

$$\mu_n = \frac{\mu_0\sigma^2+\sigma_0^2\cdot\Sigma x}{\sigma^2+n\sigma_0^2}$$

Q 3)

Q. 3) (1)

$$\sigma_n^2 = \frac{\sigma_0^2\sigma^2}{\sigma^2+n\sigma_0^2} = \frac{1}{\left(\frac{\sigma^2+n\sigma_0^2}{\sigma_0^2\sigma^2}\right)}$$

$$= \frac{1}{\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)}$$

$$= \left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)^{-1}$$

$$\boxed{\frac{1}{\sigma_n^2} = \frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}} \quad \ldots (B)$$

$$\mu_n = \frac{\mu_0\sigma^2+\sigma_0^2\,\Sigma x_i}{\sigma^2+n\sigma_0^2}$$

$$= \frac{\mu_0\sigma^2}{\sigma^2+n\sigma_0^2} + \frac{\sigma_0^2\,\Sigma x}{\sigma^2+n\sigma_0^2}$$

$$= \frac{\mu_0\sigma^2}{\sigma^2+n\sigma_0^2} + \frac{\sigma_0^2 n * \frac{\Sigma x_i}{n}}{\sigma^2+n\sigma_0^2}$$

$$\boxed{\mu_n = \frac{\mu_0\sigma^2}{\sigma^2+n\sigma_0^2} + \frac{n\bar{x}\sigma_0^2}{\sigma^2+n\sigma_0^2}} \quad \ldots (C)$$

Q 4)
From equation C) weights can be derived as below

Q.4

$$w_0 = \frac{\sigma^2}{\sigma^2 + n\sigma_0^2} \quad \text{and}$$

$$w_1 = \frac{n\sigma_0^2}{\sigma^2 + n\sigma_0^2}$$

Q 5)
Above equations, w0 and w1 indicate the weights are inversely proportional to their variances.

Q 6)
The sum of weights w0 and w1 is equal to the

Q.6.

$$w_0 + w_1 = \frac{\sigma^2}{\sigma^2 + n\sigma_0^2} + \frac{n\sigma_0^2}{\sigma^2 + n\sigma_0^2}$$

$$= \frac{\sigma^2 + n\sigma_0^2}{\sigma^2 + n\sigma_0^2}$$

$$= 1$$

$$\boxed{w_0 + w_1 = 1}$$

Q 7)

Q.7

$$W_0 = \frac{\sigma^2}{\sigma^2 + n\sigma_0^2} = \frac{1}{\left(1 + \frac{n\sigma_0^2}{\sigma^2}\right)}$$

$$W_1 = \frac{n\sigma_0^2}{\sigma^2 + n\sigma_0^2} = \frac{1}{1 + \frac{\sigma^2}{n\sigma_0^2}}$$

As the weight w0's numerator is 1 and denominator is one plus somethings so anyhow it is going to be between 0 and 1.

Similarly, for w1, its values lie between 1 and 0.

Hence, both weight's values lie between 1 and 0.

Q 8)

From previous answers, it can be inferred weight values lies between 1 and 0. The maximum values of $\mu_n = \mu_0 + \bar{x}$. But since both weights are inversely dependent on n, the edge case of 0 and 1 are not possible. Hence, μ_n will always lie between μ_0 and $\bar{x}$.

Q 9)

Q.9

$$P(x_{new} \mid X) = \int P(x_{new} \mid \mu)\, P(\mu \mid X)\, d\mu$$

since,

$$x_{new} = (x_{new} - \mu) + \mu$$

where, $x_{new} - \mu \sim N(0, \sigma^2)$ (Normal distributⁿ)

and $\mu \sim N(\mu_n, \sigma_n^2)$ (Normal distributⁿ)

$$x_{new} = (x_{new} - \mu) + \mu$$

$$P(x_{new} \mid X) \sim N(0, \sigma^2) + N(\mu_n, \sigma_n^2)$$

Hence, $N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$

$$\boxed{P(x_{new} \mid X) \sim N(\mu_n, \sigma^2 + \sigma_n^2)}$$

Q 10)

```
import numpy as np
import scipy.stats as st
from matplotlib import pyplot as plt
sample = 20
line = np.linspace(0, 10, sample)
m0 = 4
sd0 = 0.8
d_prior = st.norm(m0, sd0).pdf(line)
mx = 6 sdx = 1.5

d_sample = st.norm(mx, sdx).pdf(line)
x_t = st.norm(mx,sdx).rvs(sample)
variance = 1/((1/sd0**2)+(sample/sdx**2))
mean = variance * ((m0/sd0**2)+(np.mean(x_t)*sample/sdx**2) )
print("Posterior distibution:\nMean =",round(mean,3)) print("Variance =",round(variance,3))
d_posterior = st.norm(mean, np.sqrt(variance)).pdf(line)
plt.figure(figsize=(20,20)) plt.plot(line,d_prior,"r-",label='Prior Distribution')
plt.plot(line,d_sample,"g-",label='Sample Distribution') plt.plot(line,d_posterior,"b-
",label='Posterior Distribution')
plt.legend(loc='upper right')
plt.title('Probability Density Plot') plt.ylabel('Probability Density')
plt.xlabel('X')
plt.show()
```

```
Posterior distibution:
Mean = 5.727
Variance = 0.096
```