Homework Set 5

You may consult books and notes. You may discuss these problems with your class mates as described in the syllabus, but you should write your solution independently. You may not search/ask for the solutions on online services, such as Math Stack Exchange. Solutions should include all passages and a detailed explanation of each step. You should submit your solutions electronically to Canvas. For problems that require programming, you should submit your solution in the form of a python script. This script should work with Anaconda Python and may not use Python libraries not included in the standard Anaconda distribution.

Problem 1 (6 pts). Write your own Dormand-Prince RK45 implementation in Python.

1. Test your implementation in the case of the simple scalar equation

$$\dot{y} = \lambda\, y,$$

with $\lambda \in \mathbb{C}$. Compare the estimate of the numerical error provided by the RK45 scheme with the actual error. How do these results change if you use local extrapolation? Experiment with different initial time step sizes.

2. Modify the TOV solver to use the RK45 integrator. Compare the density profile obtained with the naive scheme we have implemented at the beginning of the course with that obtained with the new solver.

Problem 2 (4 pts). Regions of absolute stability.

Write a Python script to plot the stability region for the Runge-Kutta methods defined by the following Butcher tableaus:

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \qquad \begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ 1/2 & 1/4 & 1/4 & \\ \hline & 1/6 & 1/6 & 2/3 \end{array}$$

(From left to right: classical RK4, 4th order Gauss-Legendre, Strong Stability Preserving RK3.)

[Hint. Write a code performing one step of the integration of $\dot{z}(t) = \lambda\, z(t)$, $z(t) = 1$ with $\Delta t = 1$.]

Problem 3 (4 pts). Angular momentum conservation.

Consider a spacetime with a rotational symmetry and let $\vec{\partial}_\varphi$ be the Killing vector associated with the symmetry. Consider a perfect fluid with stress energy tensor

$$T^{\alpha\beta} = (\rho + p)\, u^\alpha\, u^\beta + p\, g^{\alpha\beta},$$

where ρ and p are, respectively, the energy density and the pressure, and u^α is the four-velocity.

1. Show that the specific angular momentum density of the fluid

$$j = (\rho + p)\, W^2\, v_\varphi,$$

is conserved, W being the Lorentz factor of the fluid.

2. Consider an accretion disk in cyclindrical coordinates. Derive the equation of evolution for the vertically integrated angular momentum.

Problem 4 (4 pts). Valencia formulation.

Consider a perfect fluid with stress energy tensor

$$T^{\mu\nu} = (\rho + p)\, u^\mu\, u^\nu + p\, g^{\mu\nu}.$$

Show that the Valencia formulation of the equation of motion reads

$$\begin{aligned} \partial_t(\sqrt{\gamma}\, S_i) + \partial_j\, [\alpha\, \sqrt{\gamma}\, (S_i{}^j + S_i\, n^j)] &= \frac{1}{2}\, \alpha\, \sqrt{\gamma}\, T^{\mu\nu}\, \partial_i\, g_{\mu\nu}, \\ \partial_t\, (\sqrt{\gamma}\, E) + \partial_j\, [\alpha\, \sqrt{\gamma}\, (S^j + E\, n^j)] &= \alpha\, \sqrt{\gamma}\, (K_{ij}\, S^{ij} - S^j\, \partial_j \log \alpha), \end{aligned}$$

where

$$E = (\rho + p)\, W^2 - p, \qquad S_\mu = (\rho + p)\, W^2\, v_\mu, \qquad S_{\mu\nu} = S_\mu\, v_\nu + p\, \gamma_{\mu\nu}.$$

Bonus Problem (2 pts). Solve for extra points.

Show that

$$\frac{1}{2}\, \alpha\, \sqrt{\gamma}\, T^{\mu\nu}\, \partial_i\, g_{\mu\nu} = \alpha\, \sqrt{\gamma} \left(\frac{1}{2}\, S^{jk}\, \partial_i\, \gamma_{jk} + \frac{1}{\alpha}\, S_k\, \partial_i\, \beta^k - E\, \partial_i \log \alpha \right).$$

This is the form of the hydrodynamics RHS implemented in the Whisky code.