③ a) Show that $\gamma^a_b v^b$ is purely spatial.

We have, $\gamma_{ab} = g_{ab} + n_a n_b$

$\gamma^a_b v^b$ is purely spatial

$\Rightarrow n_a \gamma^a_b v^b = 0$

$$g^{ac}\gamma_{bc} = g^{ac}g_{bc} + g^{ac}n_b n_c$$

$$\gamma^a_b = \delta^a_b + n_b n^a$$

$$n_a(\delta^a_b + n_b n^a)v^b$$

$$n_a v^a - n_a n_b n^a v^b$$

$$n_a v^a - n_b v^b = 0$$

b) $g = \det(g_{ab})$, $\gamma = \det(\gamma_{ik})$. Show that, $\sqrt{-g} = \alpha\sqrt{\gamma}$

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta^i\beta_i & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix} ; \quad g^{ab} = \frac{1}{\alpha^2}\begin{pmatrix} 1 & \beta^i \\ \beta^j & \frac{\gamma^{ij}}{\alpha^2} - \beta^i\beta^j \end{pmatrix}$$

$$g^{ab} = \frac{\text{co-factor of } g_{ab}}{\det(g_{ab})} = \frac{\text{co-factor of } g_{ab}}{g}$$

Say for, $g^{00} = \dfrac{\text{co-factor } g_{00}}{g} = \dfrac{\det(\gamma_{ik})}{g}$

$$-\frac{1}{\alpha^2} = \frac{\gamma}{g} \quad \Rightarrow \sqrt{-g} = \alpha\sqrt{\gamma}$$

---

④ Consistency, stability & convergence in the context of finite differencing methods.

Convergence:

In some appropriate norm, the norm tends to zero as we increase the grid resolution to solve the equations at those different resolutions. The scheme is convergent if it's stable & consistent.

$$\|u - u_h\| \xrightarrow[h \to \varepsilon]{} 0$$

Suppose we want to compute, $F_h(u) - F_h(U_h)$,

exact solution at grid points minus discrete numerical solution,

$$F_h(u) - F_h(U_h) = \frac{\partial F_h}{\partial u}(u - U_h) \quad \text{(Taylor expansion)}$$

$$u - U_h = \left(\frac{\partial F_h}{\partial u}\right)^{-1}\left[F_h(u) - F_h(U_h)\right]$$

Take norm:

$$\|u - U_h\| \leq \left\|\left(\frac{\partial F_h}{\partial u}\right)^{-1}\right\| \, \|F_h(u) - F_h(U_h)\|$$

A scheme is consistent if,

$\|F_h(u) - F_h(U_h)\|$ goes to zero.

$$F_h(u) = d_h + \tau_h \leftarrow \text{Truncation error}$$

$$= F_h(U_h) + \tau_h$$

$$\|u - U_h\| \leq \left\|\left(\frac{\partial F_h}{\partial u}\right)^{-1}\right\| \cdot \underbrace{\|\tau_h\|}$$

↳ depends on order of scheme & smoothness

$\Rightarrow$ a scheme is consistent if $\|\tau_h\| \to 0$ in the right norm.

But $\left\|\left(\frac{\partial F_h}{\partial u}\right)^{-1}\right\|$ could blow up! A scheme is

$\therefore$ stable, if $\left\|\left(\frac{\partial F_h}{\partial u}\right)^{-1}\right\|$ is bounded.

$$\left.\begin{array}{l} \text{Consistent: } \|\tau_h\| \to 0 \\ \text{Stable: } \left\|\left(\frac{\partial F_h}{\partial u}\right)^{-1}\right\| \leq M \end{array}\right\} \text{scheme is convergent.}$$

For FD schemes, $\tau_h \sim h^p$

spectral " , $\tau_h \sim 1 - e^h$

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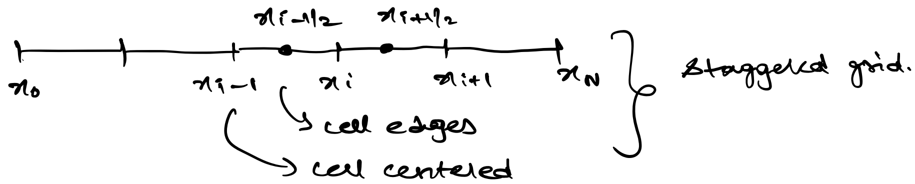
(5) Linear problem with an elliptic equation:

steady-state heat problem

$$-\frac{d}{dx}\Big[\underbrace{K(x)\frac{du}{dx}}_{\text{flux}}\Big] = \underbrace{\eta(x)}_{\text{source}} - \underbrace{\alpha(x)u(x)}_{\text{sink}}$$

$$u(0) = u(1) = 0$$

Finite volume scheme,



$$-\left(\frac{F_{i+1/2} - F_{i-1/2}}{h}\right) + \alpha_i u_i = \eta_i$$

$$F_{i+1/2} = \left(\frac{K_i + K_{i+1}}{2}\right)\left(\frac{u_{i+1} - u_i}{h}\right)$$

$$A\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_N \end{pmatrix}$$

A has contributions from $F_{i+1/2}$ through $\frac{K_i + K_{i+1}}{2}$ terms and α_i which is diagonal

Based on the boundary conditions, A transforms. If, $u(0) = u(1) = 0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ - & - & - & - & \\ \cdots & \cdots & \cdots & & 1 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Ch4 of Leveque treats inverting matrices

1) BCs
2) Solve the linear eq problem.

two families of methods
- iterative (CG) (typically larger probs)
- direct (UMFPACK) (smaller probs)

We will use matrix free method by implementing iterative method.

* When there are no exact solutions, we can come up with manufactured solutions!

Manufactured soln for the given problem:

$$-\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] = \eta(x) - \alpha(x)u(x)$$

Assuming, $k(x) = 1 + x(1-x)$

$\alpha(x) = 1$

setting $\eta(x)$ for $u(x) = \sin(\pi x)$

$$\frac{du}{dx} = \pi\cos(\pi x)$$

$$k(x)\frac{du}{dx} = \pi\cos\pi x + \pi\cos(\pi x)(x - x^2)$$

$$= \pi\cos\pi x + \pi\cos(\pi x)x - \pi\cos(\pi x)x^2$$

$$\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = -\pi^2\sin\pi x - \pi^2\sin(\pi x)x + \pi\cos(\pi x) + \pi^2\sin(\pi x)x^2 - 2\pi\cos(\pi x)x$$

$$= \pi^2\sin(\pi x)(x^2 - x - 1) + \pi\cos(\pi x)(1 - 2x)$$

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = \pi\cos(\pi x)(2x-1) + \pi^2\sin(\pi x)(1 + x - x^2)$$

$$\eta(x) = -\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) + u(x)$$

$$\eta(x) = \pi(2x-1)\cos(\pi x) - \pi^2(x(x-1) - 1)\sin(\pi x) + \sin(\pi x)$$

---

① The $\gamma = \det(\gamma_{ij})$ & trace of K_{ij}, $K = K^i_i$ have the following evolution equations:

We have,

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

Applying γ^{ij} on both sides.

$$\gamma^{ij}\partial_t\gamma_{ij} = -2\alpha\gamma^{ij}K_{ij} + \gamma^{ij}D_i\beta_j + \gamma^{ij}D_j\beta_i$$

$$\frac{1}{\gamma}\partial_t\gamma = -2\alpha K + D_i\beta^i + D_j\beta^j \quad (\because D_i\gamma^{ij} = 0)$$

$$\frac{1}{\gamma}\partial_t\gamma = -2\alpha K + 2D_i\beta^i \quad (\because j\text{'s are dummy indices})$$

$$\left[\begin{array}{l} \partial_t\gamma_{ij} = \gamma\,\mathrm{Tr}(\gamma^{ij}\partial_t\gamma_{ij}) \\ \because \gamma^{ij} = \frac{\text{cofactor}(\gamma_{ij})}{\gamma} \\ \gamma^{ij}\partial_t\gamma_{ij} = -\gamma_{ij}\partial_t\gamma^{ij} \end{array}\right]$$

$$\frac{1}{2\gamma}\partial_t \gamma = -\alpha K + D_i\beta^i$$

$$\tfrac{1}{2}\partial_t \ln\gamma = -\alpha K + D_i\beta^i$$

$$\boxed{\partial_t \ln\gamma^{1/2} = -\alpha K + D_i\beta^i}$$

We also have,

$$\partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) - D_iD_j\alpha$$
$$-8\pi\alpha(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)) + \beta^k\partial_k K_{ij}$$
$$+ K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k$$

Applying γ^{ij} on both sides.

$$\gamma^{ij}\partial_t K_{ij} = \alpha(\gamma^{ij}R_{ij} - 2\gamma^{ij}K_{ik}K^k{}_j + K\gamma^{ij}K_{ij})$$
$$-\gamma^{ij}D_iD_j\alpha - 8\pi\alpha(\gamma^{ij}S_{ij} - \tfrac{1}{2}\gamma^{ij}\gamma_{ij}(S-\rho))$$
$$+ \gamma^{ij}\beta^k\partial_k K_{ij} + \gamma^{ij}K_{ik}\partial_j\beta^k + \gamma^{ij}K_{kj}\partial_i\beta^k$$

$$\partial_t\gamma^{ij}K_{ij} - K_{ij}\partial_t\gamma^{ij} = \alpha(R + K^2 - 2K^j{}_kK^k{}_j) - D^2\alpha$$
$$-8\pi\alpha(S - \tfrac{1}{2}(S-3\rho)) + \beta^k\partial_k K$$
$$+ K^j{}_k\partial_j\beta^k + K^i{}_j\partial_i\beta^k$$

$$\partial_t K = \alpha(R + K^2 - 2K^j{}_kK^k{}_j) - D^2\alpha + \frac{8\pi\alpha}{2}(S-3\rho) + \beta^i D_i K$$
$$+2\alpha K_{ij}K^{ij} - K_{ij}\cancel{D^i\beta^j} - K_{ij}\cancel{D^i\beta^i} + K^j{}_{\cancel{k}}\cancel{\partial_j}\beta^k$$
$$+ K^i{}_j\cancel{\partial_i\beta^k}$$

$$= -2\alpha\cancel{K^j{}_k K^k{}_j} + 2\alpha\cancel{K_{ij}K^{ij}} + \alpha(R+K^2) - D^2\alpha$$
$$+ 4\pi\alpha S - 12\pi\alpha\rho + \beta^i D_i K$$

Using Hamiltonian constraint.

$$R + K^2 = 16\pi\rho + K_{ij}K^{ij}$$

$$= 16\pi\alpha\rho + \alpha K_{ij}K^{ij} - D^2\alpha + 4\pi\alpha S - 12\pi\alpha\rho + \beta^i D_i K$$

$$\boxed{\partial_t K = -D^2\alpha + \alpha(K_{ij}K^{ij} + 4\pi(\rho + S)) + \beta^i D_i K}$$

---

② The Schwarzschild spacetime lineelement in polar Painleve-Gullstrand coordinates.

$$ds^2 = -dt^2 + \left(dr + \left(\frac{2M}{r}\right)^{1/2} dt\right)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -dt^2 + dr^2 + \left(\frac{2M}{r}\right) dt^2 + 2\left(\frac{2M}{r}\right)^{1/2} dr\,dt + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= \left(-1 + \frac{2M}{r}\right) dt^2 + dr^2 + 2\left(\frac{2M}{r}\right)^{1/2} dr\,dt + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad - (1)$$

3+1 metric:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

γ_{ij} is flat $\Rightarrow \gamma_{ij} = \eta_{ij}$

$$= -\alpha^2 dt^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dr + \beta^1 dt \\ d\theta + \beta^2 dt \\ d\phi + \beta^3 dt \end{pmatrix} \begin{pmatrix} dr + \beta^1 dt \\ d\theta + \beta^2 dt \\ d\phi + \beta^3 dt \end{pmatrix}$$

$$= -\alpha^2 dt^2 + \begin{pmatrix} dr + \beta^1 dt & r^2 d\theta + r^2\beta^2 dt & r^2\sin^2\theta\, d\phi + r^2\sin^2\theta\, \beta^3 dt \end{pmatrix} \begin{pmatrix} dr + \beta^1 dt \\ d\theta + \beta^2 dt \\ d\phi + \beta^3 dt \end{pmatrix}$$

$$= -\alpha^2 dt^2 + \big(dr^2 + 2\beta^1 dr\,dt + \beta^{1^2} dt^2 + r^2 d\theta^2 + 2r^2\beta^2 d\theta\, dt$$
$$r^2\beta^{2^2} dt^2 + r^2\sin^2\theta\, d\phi^2 + 2r^2\sin^2\theta\, \beta^3 d\phi\, dt$$
$$+ r^2\sin^2\theta\, \beta^{3^2} dt^2 \big)$$

$$= (-\alpha^2 + \beta^{1^2} + r^2\beta^{2^2} + r^2\sin^2\theta\, \beta^{3^2})\, dt^2 + dr^2 + 2\beta^1 dr\,dt + 2r^2\beta^2 d\theta\, dt +$$
$$2r^3 \sin^2\theta\, d\phi\, dt + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

a) Comparing (1) & (2),

$$\beta^2 = \beta^3 = 0, \quad \beta^1 = \left(\frac{2M}{r}\right)^{1/2}, \quad \alpha = 1$$

$\therefore \alpha = 1, \quad \beta^i = \left(\frac{2M}{r}\right)^{1/2} e^i, \quad \gamma_{ij} = \eta_{ij}$, where $e^i = (1,0,0)$

b) We also have,

$$\Omega_a = \nabla_a t$$
$$w_a = \alpha\, \Omega_a$$
$$n^a = -g^{ab} w_b$$
$$K_{ab} = -\nabla_a n_b - n_a D_a \ln\alpha$$
$$\alpha = 1$$
$$\Omega_a = \nabla_a t$$

$$\Rightarrow w_a = \nabla_a t = (1,0,0,0)$$

$$n^a = -g^{ab} w_b$$

$$= \begin{pmatrix} 1-\frac{2M}{r} & \left(\frac{2M}{r}\right)^{1/2} & 0 & 0 \\ \left(\frac{2M}{r}\right)^{1/2} & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$n^a = \begin{pmatrix} 1-\frac{2M}{r} \\ \left(\frac{2M}{r}\right)^{1/2} \\ 0 \\ 0 \end{pmatrix}$$

$$K_{ab} = -\nabla_a n_b$$

$$= \left(\frac{2M}{r^3}\right)^{1/2} \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$K_{ij} = \left(\frac{2M}{r^3}\right)^{1/2} \left(\eta_{ij} - \frac{3}{2} l_i l_j\right) \quad -③$$

$$K = \text{trace } K = \gamma^{ij} K_{ij} = \left(\frac{2M}{r^3}\right)^{1/2} \left(-\frac{1}{2}+1+1\right) = \frac{3}{2}\left(\frac{2M}{r^3}\right)^{1/2} \quad -④$$

$R_{ij} = 0$ (from Mathematica)
(also, $\gamma_{ij} = \eta_{ij}$)

$R = 0$

c) Schwarzschild $\Rightarrow T^{ab} = 0$

$$\rho = n_a n_b T^{ab} = 0$$

$$S^i = -\gamma^{ij} n^a T_{aj} = 0$$

$$S_{ij} = \gamma_{ia}\gamma_{jb} T^{ab} = 0$$

$$S = \gamma^{ij} S_{ij} = 0$$

$$G_{ab} = 8\pi T_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

Hamiltonian constraint:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$R = 0, \rho = 0$$

$$K^2 = K_{ij}K^{ij}$$

This is true from K & K_{ij} in ③ & ④

momentum constraint:

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

$$S^i = 0$$

$$K^{ij} - \gamma^{ij}K = \left(\frac{2M}{r^3}\right)^{1/2}(\eta^{ij} - \tfrac{3}{2}l^i l^j) - \tfrac{3}{2}\left(\frac{2M}{r^3}\right)^{1/2}(\eta^{ij})$$

$$= -\left(\frac{2M}{r^3}\right)^{1/2}\left(\tfrac{1}{2}\eta^{ij} + \tfrac{3}{2}l^i l^j\right)$$

$$-\tfrac{1}{2}D_j\left(\left(\frac{2M}{r^3}\right)^{1/2}(\eta^{ij} + l^i l^j)\right) = 0$$

evolution equation for K_{ij}:

$$\partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) - D_iD_j\alpha$$
$$- 8\pi\alpha(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)) + \beta^k\partial_k K_{ij}$$
$$+ K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k$$

$$0 = (R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) + 0 + 0 + \beta^k\partial_k K_{ij}$$
$$+ K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k$$

$$= R_{ij} - 2K_{ik}K^k{}_j + KK_{ij} + \beta^r\partial_r K_{ij} + K_{i\gamma}\partial_j\beta^\gamma + K_{\gamma j}\partial_i\beta^\gamma$$

$$= 0$$

evolution equation for γ_{ij}:

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t\gamma_{ij} = 0$$

$$\Rightarrow 0 = -2K_{ij} + D_i\beta_j + D_j\beta_i$$

$$K_{ij} = \tfrac{1}{2}(D_i\beta_j + D_j\beta_i)$$