Problem 2 :

Amplification factor, $R(z) = \frac{|y^{(n+1)}|}{|y^{(n)}|}$

$$= \left|\frac{y(1)}{y(0)}\right| = y(1)$$

$$(I - zA)K = z\, y^{(n)}$$
$$y^{(n+1)} = y^{(n)} + b^T K$$

$$y^{(n+1)} = 1 + b^T z\, (I - zA)^{-1} (y^{(n)})$$

$$y^{(n+1)} = 1 + (z)\,(b^T)\,(I - zA)^{-1}\,(y^{(n)})$$

$$y^{(n+1)} = 1 + \left(z\, b^T (I - zA)^{-1}\right) y^{(n)}$$

$$R(z) = 1 + \left(z b^T (I - zA)^{-1}\right) y^{(n)}$$

$$y^{(n)} = (1, 1, 1, 1, \cdots)^T$$

Problem 3:

a) $\vec{\xi}_\varphi$ - Killing vector is present in Spacetime

Consider, $T^{\alpha\beta} = (\rho + P) u^\alpha u^\beta + P g^{\alpha\beta}$

$$j = (\rho + P) W^2 V_\varphi, \quad V_\varphi = \frac{1}{W} \gamma^\mu_\varphi u_\mu$$
$$= (\rho + P) W^2 \left(\frac{1}{W} \gamma^\mu_\varphi V_\mu\right)$$
$$j = (\rho + P) W \gamma^\mu_\varphi V_\mu V_\varphi$$

$$J^\alpha = T^{\alpha\beta} \xi_\beta \; ; \quad \vec{u} = W(\vec{n} + \vec{v})$$
$$W = -\vec{n} \cdot \vec{u}$$

$$J^\alpha = [(\rho + P) u^\alpha u^\beta + P g^{\alpha\beta}]\, \xi_\beta$$

$$= (\rho + P) u^\alpha u^\beta \xi_\beta + P g^{\alpha\beta} \xi_\beta$$

$$J^\alpha = (\rho + P) u^\alpha u^\beta \xi_\beta + P \xi_\varphi^\alpha$$

$$j = J^\alpha n_\alpha$$

$$= (\rho + P) u^\alpha n_\alpha u^\beta \xi_\beta + P \xi_\varphi^\alpha n_\alpha$$

$$= (\rho + P)(W) W (n^\beta + v^\beta) \xi_{\varphi\beta} + \cancelto{0}{P \xi_\varphi^\alpha n_\alpha} \qquad \left(\because u^\alpha n_\alpha = \underbrace{W n^\alpha n_\alpha}_{=-W}\right) \; (\because \xi_\varphi^\alpha n_\alpha = 0)$$

$$= (\rho + P)(W^2)\, v^\beta \xi_{\varphi\beta}$$

But, $\xi_\varphi^\alpha = (\partial\phi)^\alpha$

$j = (\rho + P) W^2 V_\varphi$

from slides, of NR class: $\frac{d}{dt}\int_\Omega J^\alpha n_\alpha \sqrt{\gamma}\, d^3x = \frac{d}{dt}\int_\Omega j \sqrt{\gamma}\, d^3x = \int_{\partial\Omega} J^i v_i \sqrt{\gamma}\, d^3x$

b) $\nabla_\alpha J^\alpha = 0$; $\frac{1}{\alpha\sqrt{\gamma}}\partial_\mu(\alpha\sqrt{\gamma}\, J^\mu) = 0$

$$\int \nabla_\alpha J^\alpha \sqrt{\gamma_2}\, d^2x = \int_{t_0}^{t_1}\int_z \partial_\alpha(\sqrt{\gamma_2}\, J^\alpha)\, dt\, dz$$

$$= \left[\int_z \alpha_z J^0 \sqrt{h}\, dz\right]_{t_0}^{t_1}$$

$$+ \int_{t_0}^{t_1} dt \int_z \partial_i(\sqrt{\gamma_2}\, J^i)\, dz$$

$$\Rightarrow \frac{d}{dt}\int_z \alpha_z J^\alpha n_\alpha \sqrt{h}\, dz = \int_z \partial_i(\sqrt{\gamma_2}\, J^i)\, dz$$

Here, γ_2 is the metric of t-z spacetime

and $\gamma_2^{ab} = h^{ab} + n^a n^b$, 1+1 split into t & z

Here, $\int_z \alpha_z J^\alpha n_\alpha \sqrt{h}\, dz$ is the integrated angular momentum along z.

Problem 4:

$T_{\alpha\beta} = (\rho+P) u_\alpha u_\beta + P g_{\alpha\beta}$; $u_\alpha = W(n_\alpha + V_\alpha)$

$$E = n^\alpha n^\beta T_{\alpha\beta}$$

$$= n^\alpha n^\beta \left[(\rho+P) u_\alpha u_\beta + P g_{\alpha\beta}\right]$$

$$= (\rho+P)(W^2) + P\, n^\alpha n^\beta g_{\alpha\beta}$$

$$E = (\rho+P)(W^2) - P$$

$$S_\mu = -\gamma^\alpha_\mu n^\beta T_{\alpha\beta}$$

$$= \gamma^\alpha_\mu \left[(\rho+P) W u_\alpha + P n^\beta g_{\alpha\beta}\right]$$

$$= (\rho+P) W^2 \gamma^\alpha_\mu (n_\alpha + u_\alpha) + P \gamma^\alpha_\mu n_\alpha$$

$$S_\mu = (\rho+P) W^2 V_\mu$$

$$S_{\mu\nu} = \gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}T_{\alpha\beta}$$

$$= (\rho+P)W^2 v_\mu v_\nu + P\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}g_{\alpha\beta}$$

$$S_{\mu\nu} = S_\mu v_\nu + P\gamma_{\mu\nu} \qquad (\because S_\mu = (\rho+P)W^2 v_\mu)$$

Based on Relativistic Hydrodynamics, Rezolla et al book, pg no: 366, 7.3.3

Four divergence of $T^{\mu\nu}$ has the following relation,

$$\nabla_\mu T^{\mu\nu} = g^{\nu\lambda}\left[\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,T^{\mu}{}_{\lambda}) - \frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}\right] \quad \text{(Relativity on curved manifolds (sec: 2.10))}$$

we have conservation of energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,T^{\mu}{}_{\lambda}) = \frac{1}{2}T^{\mu\lambda}\partial_\nu g_{\mu\lambda}$$

$$\partial_t(\sqrt{-g}\,T^{0}{}_{\lambda}) + \partial_j[\sqrt{-g}\,T^{j}{}_{\lambda}] = \frac{1}{2}\sqrt{-g}\,T^{\mu\lambda}\partial_\nu g_{\mu\lambda}$$

restricting λ to only spatial indexes

$$\partial_t(\alpha\sqrt{\gamma}\,T^{0}_{i}) + \partial_j[\alpha\sqrt{\gamma}\,T^{j}_{i}] = \frac{1}{2}\alpha\sqrt{\gamma}\,T^{\mu\nu}\partial_i g_{\mu\nu}$$

We have,

$$T^{\mu\nu} = En^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu}$$

$$T^{\mu}_{\nu} = g_{\nu\alpha}T^{\mu\alpha}$$

$$= Eg_{\nu\alpha}n^\mu n^\alpha + S^\mu g_{\nu\alpha}n^\alpha + g_{\nu\alpha}S^\alpha n^\mu + g_{\nu\alpha}S^{\mu\alpha}$$

$$T^{\mu}_{\nu} = -En^\mu n_\nu + S^\mu n_\nu + S_\nu n^\mu + S^{\mu}_{\nu}$$

$$T^{0}_{j} = -E\cancel{n^0 n_j}^{0} + \cancel{S^0 n_j}^{0} + S_j n^0 + \cancel{S^0_j}^{0} \qquad (\because n_j = 0 \;\&\; S^0_j = 0)$$

$$T^{0}_{i} = S_i n^0 = \frac{S_i}{\alpha} \qquad (\because n^0 = \alpha^{-1})$$

$$T^{j}_{i} = -E\cancel{n^j n_i}^{0} + \cancel{S^j n_i}^{0} + S_i n^j + S^{j}_{i}$$

$$= S_i n^j + S^{j}_{i}$$

$$\Rightarrow \partial_t\left(\alpha\sqrt{\gamma}\frac{S_i}{\alpha}\right) + \partial_j\left(\alpha\sqrt{\gamma}\,(S^j_i + S_i n^j)\right) = \frac{1}{2}\alpha\sqrt{\gamma}\,T^{\mu\nu}\partial_i g_{\mu\nu}$$

$$\boxed{\partial_t(\sqrt{\gamma}\,S_i) + \partial_j[\alpha\sqrt{\gamma}\,(S^j_i + S_i n^j)] = \frac{1}{2}\alpha\sqrt{\gamma}\,T^{\mu\nu}\partial_i g_{\mu\nu}}$$

If we substitute, $n^\alpha = (\alpha^{-1}, -\alpha^{-1}\beta^i) \Rightarrow n^j = -\frac{\beta^j}{\alpha}$

$$\partial_t(\sqrt{\gamma}S_i) + \partial_j[\alpha\sqrt{\gamma}(S^j{}_i - S_i\frac{\beta^j}{\alpha})] = \frac{1}{2}\alpha\sqrt{\gamma}\,T^{\mu\nu}\partial_i g_{\mu\nu}$$

$$\boxed{\partial_t(\sqrt{\gamma}S_i) + \partial_j[\sqrt{\gamma}(\alpha S^j{}_i - S_i\beta^j)] = \frac{1}{2}\alpha\sqrt{\gamma}\,T^{\mu\nu}\partial_i g_{\mu\nu}}$$

Next we have,

$$\nabla_\mu(T^{\mu\nu}n_\nu) = (\cancel{\nabla_\mu T^{\mu\nu}}^{\,0})n_\nu + T^{\mu\nu}\nabla_\mu n_\nu \qquad (\because \text{Liebniz rule \& conservation})$$

we have,

$$\nabla_j A^i = \frac{1}{\sqrt{-g}}\partial_j(\sqrt{-g}A^i) \qquad \left(\begin{array}{l}\text{Relativity on}\\ \text{curved manifolds}\\ \text{(sec: 2.10}\end{array}\right)$$

$$\nabla_\mu(T^{\mu\nu}n_\nu) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,T^{\mu\nu}n_\nu)$$

$$\Rightarrow \quad \partial_\mu(\alpha\sqrt{\gamma}\,T^{\mu\nu}n_\nu) = \sqrt{-g}\,T^{\mu\nu}\nabla_\mu n_\nu$$

$$\partial_t(\alpha\sqrt{\gamma}\,T^{0\nu}n_\nu) + \partial_i(\alpha\sqrt{\gamma}\,T^{i\nu}n_\nu) = \alpha\sqrt{\gamma}\,T^{\mu\nu}\nabla_\mu n_\nu$$

$$T^{0\nu}n_\nu = (En^0n^\nu + S^0n^\nu + S^\nu n^0 + S^{0\nu})n_\nu$$

$$= \frac{E}{\alpha}n^\nu n_\nu + S^0\cancel{n^\nu n_\nu}^{\,0} + S^\nu\cancel{n^0 n_\nu}^{\,0} + S^{0\nu}\cancel{n_\nu}^{\,0}$$

$$= -\frac{E}{\alpha}$$

$$T^{i\nu}n_\nu = (En^in^\nu + S^in^\nu + S^\nu n^i + S^{i\nu})n_\nu$$

$$= -En^i - S^i + S^\nu n^i\cancel{n_\nu}^{\,0} + S^{i\nu}\cancel{n_\nu}^{\,0}$$

$$\partial_t(\alpha\sqrt{\gamma}(-\frac{E}{\alpha})) + \partial_i(\alpha\sqrt{\gamma}(-En^i - S^i)) = \alpha\sqrt{\gamma}\,T^{\mu\nu}\nabla_\mu n_\nu$$

$$\partial_t(\sqrt{\gamma}E) + \partial_j(\alpha\sqrt{\gamma}(S^j + En^j)) = -\alpha\sqrt{\gamma}\,T^{\mu\nu}\nabla_\mu n_\nu$$

we have,

$$K_{\mu\nu} = -\nabla_\mu n_\nu + n^a n_\mu \nabla_a n_\nu \qquad (\text{eq } 2.52,\ \text{Baumgarte})$$

$$n^\mu\nabla_\mu n_\nu = a_\nu = D_\nu \ln\alpha \qquad (\text{Ex: } 2.13 \text{ of Baumgarte})$$

$$\Rightarrow \quad K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu$$

$$\Rightarrow \quad \nabla_\mu n_\nu = -K_{\mu\nu} - n_\mu a_\nu$$

$$-T^{\mu\nu}\nabla_\mu n_\nu = (En^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu})(K_{\mu\nu} + n_\mu a_\nu)$$

$$= E n^\mu n^\nu \cancelto{0}{K_{\mu\nu}} + s^\mu n^\nu \cancelto{0}{K_{\mu\nu}} + s^\nu n^\mu \cancelto{0}{K_{\mu\nu}} + s^{\mu\nu} K_{\mu\nu}$$

$$+ E n^\mu \cancelto{0}{n^\nu n_\mu a_\nu} + s^\mu \cancelto{0}{n^\nu n_\mu a_\nu} + s^\nu n^\mu n_\mu a_\nu + s^{\mu\nu} \cancelto{0}{n_\mu a_\nu}$$

$(\because n^\nu a_\nu = 0)$

$$= s^{ij} K_{ij} - s^i a_i \qquad (\because n^\mu n_\mu = -1)$$

$$\boxed{\partial_t(\sqrt{\gamma} E) + \partial_j [\alpha \sqrt{\gamma} (s^j + E n^j)] = \alpha \sqrt{\gamma} (K_{ij} s^{ij} - s^i \partial_i \ln \alpha)}$$

$$n^j = -\frac{\beta^j}{\alpha}$$

$$\boxed{\partial_t(\sqrt{\gamma} E) + \partial_j [\sqrt{\gamma} (\alpha s^j - \beta^j E)] = \alpha \sqrt{\gamma} (K_{ij} s^{ij} - s^i \partial_i \ln \alpha)}$$

Bonus Problem :

$$\frac{1}{2} \alpha \sqrt{\gamma} T^{\mu\nu} \partial_i g_{\mu\nu} = \frac{1}{2} \alpha \sqrt{\gamma} [E n^\mu n^\nu + s^\mu n^\nu + s^\nu n^\mu + s^{\mu\nu}] \partial_i g_{\mu\nu}$$

$$= \alpha \sqrt{\gamma} \left[\frac{1}{2} E n^\mu n^\nu \partial_i g_{\mu\nu} + s^\mu n^\nu \partial_i g_{\mu\nu} + s^\nu n^\mu \partial_i g_{\mu\nu} + s^{\mu\nu} \partial_i g_{\mu\nu} \right]$$

We have the relation,

$$\partial_i g_{\mu\nu} = \Gamma^k_{j\nu} g_{\mu k} + \Gamma^k_{j\mu} g_{k\nu}$$

$$E n^\mu n^\nu \partial_i g_{\mu\nu} = E \Gamma^k_{i\nu} n^\mu n^\nu g_{\mu k} + E \Gamma^k_{i\mu} n^\mu n^\nu g_{k\nu}$$

$$= E \Gamma^k_{i\nu} n_k n^\nu + E \Gamma^k_{i\mu} n_k n^\mu$$

$$= 2 E \Gamma^k_{i\mu} n_k n^\mu$$

$$\Gamma^k_{ij} n_k n^j = n_j \nabla^j_i = \partial_i \ln \alpha \ ?$$

$$E n^\mu n^\nu \partial_i g_{\mu\nu} = 2 E \partial_i \ln \alpha$$

$$s^\mu n^\nu \partial_i g_{\mu\nu} + s^\nu n^\mu \partial_i g_{\mu\nu} = 2 s^\mu n^\nu \partial_i g_{\mu\nu}$$

$$n^\nu \partial_i g_{\mu\nu} = \partial_i (n^\nu g_{\mu\nu}) - g_{\mu\nu} \partial_i n^\nu \qquad (\because \nu = i, \text{ only spatial indices})$$

$$= \partial_i (\cancelto{0}{n_\mu}) - g_{\mu\nu} \partial_i n^\nu \qquad (\because n_i = 0)$$

$$2 s^\mu g_{\mu\nu} \partial_i n^\nu = -2 s_\nu \partial_i n^\nu = -2 s_j \partial_i n^j$$

$$n^j = -\frac{\beta^j}{\alpha}$$

$$S^\mu n^\nu \partial_i g_{\mu\nu} = 2 S^\mu g_{\mu\nu} \partial_i n^\nu = \frac{2 S_j \partial_i \beta^j}{\alpha}$$

$S^{\mu\nu}\partial_i g_{\mu\nu}$, restricting to only spatial indices

$$S^{\mu\nu}\partial_i g_{\mu\nu} = S^{jk}\partial_i \gamma_{jk} \qquad (\because g_{ij} = \gamma_{ij})$$

$$\Rightarrow \frac{1}{2}\alpha\sqrt{\gamma}\, T^{\mu\nu}\partial_i g_{\mu\nu} = \alpha\sqrt{\gamma}\left[\frac{1}{2}E n^\mu n^\nu \partial_i g_{\mu\nu} + \frac{S^\mu n^\nu \partial_i g_{\mu\nu}}{2} + \frac{S^\nu n^\mu \partial_i g_{\mu\nu}}{2} + \frac{S^{\mu\nu}\partial_i g_{\mu\nu}}{2}\right]$$

$$= \alpha\sqrt{\gamma}\left[\frac{1}{2} S^{jk}\partial_i \gamma_{jk} + \frac{1}{2}\frac{1}{\alpha} 2 S_j \partial_i \beta^j - \frac{2E\partial_i \ln\alpha}{2}\right]$$

$$\boxed{\therefore \frac{1}{2}\alpha\sqrt{\gamma}\, T^{\mu\nu}\partial_i g_{\mu\nu} = \alpha\sqrt{\gamma}\left(\frac{1}{2} S^{jk}\partial_i \gamma_{jk} + \frac{1}{\alpha} S_j \partial_i \beta^j - E\partial_i \ln\alpha\right)}$$