Runge-Kutta TOV

The Dormand-Prince method

Consider an ODE in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y).$$

Let y^n be the solution at time t_n. Then the Dormand-Prince method constructs two approximations of the solution y^{n+1} at $t_n + \Delta t$ as follows.

1. For $i = 1, \ldots, 7$ compute the slopes

$$k_i = f\left(t_n + c_i\,\Delta t, y_n + \Delta t \sum_{j=1}^{s} a_{ij}\,k_j\right).$$

2. The 4th and 5th order approximations of y^{n+1} are obtained as

$$y^\star = y_n + \Delta t \sum_{i=1}^{6} b_i^\star\,k_i, \qquad y^{\star\star} = y_n + \Delta t \sum_{i=1}^{7} b_i^{\star\star}\,k_i.$$

3. The local error is estimated as $e = \|y^{\star\star} - y^\star\|$

   - If $\mathrm{tol}/2^5 < e < \mathrm{tol}$, then $y^{n+1} \leftarrow y^\star$;
   - If $e < \mathrm{tol}/2^5$, then $y^{n+1} \leftarrow y^\star$ and $\Delta t \leftarrow 2\,\Delta t$;
   - If $e > \mathrm{tol}$, then $\Delta t \leftarrow \Delta t/2$ and the current values of k_i, $y^\star$, and $y^{\star\star}$ are disregarded.

The Butcher tableau $\begin{array}{c|c} c & a \\ \hline & b \end{array}$ for the Dormand Prince method is

$$\begin{array}{c|ccccccc}
0 & & & & & & & \\
1/5 & 1/5 & & & & & & \\
3/10 & 3/40 & 9/40 & & & & & \\
4/5 & 44/45 & -56/15 & 32/9 & & & & \\
8/9 & 19372/6561 & -25360/2187 & 64448/6561 & -212/729 & & & \\
1 & 9017/3168 & -355/33 & 46732/5247 & 49/176 & -5103/18656 & & \\
1 & 35/384 & 0 & 500/1113 & 125/192 & -2187/6784 & 11/84 & \\
\hline
 & 35/384 & 0 & 500/1113 & 125/192 & -2187/6784 & 11/84 & 0 \\
 & 5179/57600 & 0 & 7571/16695 & 393/640 & -92097/339200 & 187/2100 & 1/40
\end{array}$$

The first row of the b coefficients is $b^\star$, while the second row is $b^{\star\star}$.

Copy-pastable values of these coefficients are available at
`https://en.wikipedia.org/wiki/Dormand-Prince_method`

Class work

Problem 1 (6 pts). Write your own Dormand-Prince RK45 implementation in Python.

1. Test your implementation in the case of the simple scalar equation

$$\dot{y} = \lambda\, y,$$

with $\lambda \in \mathbb{C}$. Compare the estimate of the numerical error provided by the RK45 scheme with the actual error. How do these results change if you use local extrapolation? Experiment with different initial time step sizes.

2. Modify the TOV solver to use the RK45 integrator. Compare the density profile obtained with the naive scheme we have implemented at the beginning of the course with that obtained with the new solver.