General Relativity: Brill waves

1.CHRISTOFFEL CONNECTION

Number of dimensions:

In[115]:=
```
n = 3
```
Out[115]= 3

Coordinate system:

In[116]:=
```
coord = {ρ, ϕ, z}
```
Out[116]= $\{\rho, \phi, z\}$

Metric Input:

In[120]:=
```
ClearAll[metric]
metric = {{e^q[ρ,z], 0, 0}, {0, ρ^2, 0}, {0, 0, e^q[ρ,z]}};
MatrixForm[metric]
```
Out[122]//MatrixForm=

$$\begin{pmatrix} e^{q[\rho,z]} & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & e^{q[\rho,z]} \end{pmatrix}$$

Inverse Metric:

In[125]:=
```
ClearAll[inversemetric]
inversemetric = Inverse[metric];
```

In[127]:=
```
MatrixForm[inversemetric]
```
Out[127]//MatrixForm=

$$\begin{pmatrix} e^{-q[\rho,z]} & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & e^{-q[\rho,z]} \end{pmatrix}$$

In[128]:=
```
metric.inversemetric // Simplify
```
Out[128]= {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

Christoffel Connection code:

In[129]:=
```
ClearAll[christ]
christ[a_, b_, c_] := Simplify[
    christ[a, b, c] = Sum[1/2 inversemetric[[a, d]] (D[metric[[d, c]], coord[[b]]] +
        D[metric[[d, b]], coord[[c]]] - D[metric[[b, c]], coord[[d]]]), {d, 1, 3}]];
```

$$\text{christ}[a, b, c] = \sum_{d=1}^{3} \frac{1}{2} \text{inversemetric}[[a, d]] \left(D[\text{metric}[[d, c]], \text{coord}[[b]]] + D[\text{metric}[[d, b]], \text{coord}[[c]]] - D[\text{metric}[[b, c]], \text{coord}[[d]]]\right)$$

In[131]:=
```
nzcc := Table[If[UnsameQ[christ[i, j, k], 0],
    {ToString[Γ[coord[[i]], coord[[j]], coord[[k]]]] → christ[i, j, k]}],
   {i, 1, n}, {j, 1, n}, {k, 1, n}];
```

In[132]:=
```
nzcc = Flatten[nzcc, 2];

nzcc = DeleteCases[nzcc, Null];
```

In[134]:=
```
TableForm[nzcc]
```

Out[134]//TableForm=

$\Gamma[\rho, \rho, \rho] \to \frac{1}{2} q^{(1,0)}[\rho, z]$

$\Gamma[\rho, \rho, z] \to \frac{1}{2} q^{(0,1)}[\rho, z]$

$\Gamma[\rho, \phi, \phi] \to -e^{-q[\rho,z]} \rho$

$\Gamma[\rho, z, \rho] \to \frac{1}{2} q^{(0,1)}[\rho, z]$

$\Gamma[\rho, z, z] \to -\frac{1}{2} q^{(1,0)}[\rho, z]$

$\Gamma[\phi, \rho, \phi] \to \frac{1}{\rho}$

$\Gamma[\phi, \phi, \rho] \to \frac{1}{\rho}$

$\Gamma[z, \rho, \rho] \to -\frac{1}{2} q^{(0,1)}[\rho, z]$

$\Gamma[z, \rho, z] \to \frac{1}{2} q^{(1,0)}[\rho, z]$

$\Gamma[z, z, \rho] \to \frac{1}{2} q^{(1,0)}[\rho, z]$

$\Gamma[z, z, z] \to \frac{1}{2} q^{(0,1)}[\rho, z]$

2. Riemann Curvature

In[135]:=
```
riemann := riemann =
   Simplify[Table[D[christ[i, j, l], coord[[k]]] - D[christ[i, j, k], coord[[l]]] +
      Sum[christ[s, j, l] × christ[i, k, s] - christ[s, j, k] × christ[i, l, s],
       {s, 1, n}], {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}]];
```

In[136]:=
```
listriemann := Table[If[UnsameQ[riemann[[i, j, k, l]], 0],
    {ToString[R[i, j, k, l]] → riemann[[i, j, k, l]]}],
   {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, k - 1}];
```

In[137]:=
```
TableForm[Partition[DeleteCases[Flatten[listriemann], Null], 2],
 TableSpacing → {2, 2}]
```

Out[137]//TableForm=

$R[1,2,2,1] \to -\frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(1,0)}[\rho,z]$	$R[1,2,3,2] \to \frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(0,1)}[\rho,z]$
$R[1,3,3,1] \to \frac{1}{2}\left(q^{(0,2)}[\rho,z]+q^{(2,0)}[\rho,z]\right)$	$R[2,1,2,1] \to \frac{q^{(1,0)}[\rho,z]}{2\rho}$
$R[2,1,3,2] \to -\frac{q^{(0,1)}[\rho,z]}{2\rho}$	$R[2,3,2,1] \to \frac{q^{(0,1)}[\rho,z]}{2\rho}$
$R[2,3,3,2] \to \frac{q^{(1,0)}[\rho,z]}{2\rho}$	$R[3,1,3,1] \to \frac{1}{2}\left(-q^{(0,2)}[\rho,z]-q^{(2,0)}[\rho,$
$R[3,2,2,1] \to -\frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(0,1)}[\rho,z]$	$R[3,2,3,2] \to -\frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(1,0)}[\rho,z]$

3. Ricci Scalar

In[138]:=
```
ricci :=
  ricci = Simplify[Table[Sum[riemann[[i, j, i, l]], {i, 1, n}], {j, 1, n}, {l, 1, n}]];
```

In[139]:=
```
listricci := Table[If[UnsameQ[ricci[[j, l]], 0],
    {ToString[R[j, l]] → ricci[[j, l]]}], {j, 1, 3}, {l, 1, j}];
```

In[140]:=
```
TableForm[Partition[DeleteCases[Flatten[listricci], Null], 2],
 TableSpacing → {2, 2}]
```

Out[140]//TableForm=

$R[1,1] \to -\frac{\rho\, q^{(0,2)}[\rho,z]-q^{(1,0)}[\rho,z]+\rho\, q^{(2,0)}[\rho,z]}{2\rho}$	$R[3,1] \to \frac{q^{(0,1)}[\rho,z]}{2\rho}$

In[141]:=
```
scalar = Simplify[Sum[inversemetric[[i, j]] × ricci[[i, j]], {i, 1, n}, {j, 1, n}]]
```

Out[141]= $-e^{-q[\rho,z]}\left(q^{(0,2)}[\rho,z]+q^{(2,0)}[\rho,z]\right)$

4. Einstein Tensor

In[142]:=
```
einstein := einstein = Simplify[ricci - (1/2) scalar * metric];
```

In[143]:=
```
listeinstein := Table[If[UnsameQ[einstein[[j, l]], 0],
    {ToString[G[j, l]] → einstein[[j, l]]}], {j, 1, n}, {l, 1, j}];
```

In[144]:=
```
TableForm[Partition[DeleteCases[Flatten[listeinstein], Null], 2],
 TableSpacing → {2, 2}]
```

Out[144]//TableForm=

$G[1,1] \to \frac{q^{(1,0)}[\rho,z]}{2\rho}$	$G[2,2] \to \frac{1}{2} e^{-q[\rho,z]} \rho^2 \left(q^{(0,2)}[\rho,z] + q^{(2,0)}[\rho,z]\right)$
$G[3,1] \to \frac{q^{(0,1)}[\rho,z]}{2\rho}$	$G[3,3] \to -\frac{q^{(1,0)}[\rho,z]}{2\rho}$
{}	

5. Results:

Christoffel connection:

In[145]:=
```
TableForm[nzcc]
```

Out[145]//TableForm=

$\Gamma[\rho,\rho,\rho] \to \frac{1}{2} q^{(1,0)}[\rho,z]$

$\Gamma[\rho,\rho,z] \to \frac{1}{2} q^{(0,1)}[\rho,z]$

$\Gamma[\rho,\phi,\phi] \to -e^{-q[\rho,z]} \rho$

$\Gamma[\rho,z,\rho] \to \frac{1}{2} q^{(0,1)}[\rho,z]$

$\Gamma[\rho,z,z] \to -\frac{1}{2} q^{(1,0)}[\rho,z]$

$\Gamma[\phi,\rho,\phi] \to \frac{1}{\rho}$

$\Gamma[\phi,\phi,\rho] \to \frac{1}{\rho}$

$\Gamma[z,\rho,\rho] \to -\frac{1}{2} q^{(0,1)}[\rho,z]$

$\Gamma[z,\rho,z] \to \frac{1}{2} q^{(1,0)}[\rho,z]$

$\Gamma[z,z,\rho] \to \frac{1}{2} q^{(1,0)}[\rho,z]$

$\Gamma[z,z,z] \to \frac{1}{2} q^{(0,1)}[\rho,z]$

Riemann Tensor:

In[146]:=
```
TableForm[Partition[DeleteCases[Flatten[listriemann], Null], 2],
 TableSpacing → {2, 2}]
```

Out[146]//TableForm=

$R[1,2,2,1] \to -\frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(1,0)}[\rho,z]$	$R[1,2,3,2] \to \frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(0,1)}[\rho,z]$
$R[1,3,3,1] \to \frac{1}{2}\left(q^{(0,2)}[\rho,z] + q^{(2,0)}[\rho,z]\right)$	$R[2,1,2,1] \to \frac{q^{(1,0)}[\rho,z]}{2\rho}$
$R[2,1,3,2] \to -\frac{q^{(0,1)}[\rho,z]}{2\rho}$	$R[2,3,2,1] \to \frac{q^{(0,1)}[\rho,z]}{2\rho}$
$R[2,3,3,2] \to \frac{q^{(1,0)}[\rho,z]}{2\rho}$	$R[3,1,3,1] \to \frac{1}{2}\left(-q^{(0,2)}[\rho,z] - q^{(2,0)}[\rho,$
$R[3,2,2,1] \to -\frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(0,1)}[\rho,z]$	$R[3,2,3,2] \to -\frac{1}{2} e^{-q[\rho,z]} \rho\, q^{(1,0)}[\rho,z]$

Ricci Scalar:

In[147]:=
```
scalar = Simplify[Sum[inversemetric[[i, j]] × ricci[[i, j]], {i, 1, n}, {j, 1, n}]]
```

Out[147]= $-e^{-q[\rho,z]}\left(q^{(0,2)}[\rho,z] + q^{(2,0)}[\rho,z]\right)$

Einstein Tensor:

In[148]:=
```
TableForm[Partition[DeleteCases[Flatten[listeinstein], Null], 2],
 TableSpacing → {2, 2}]
```

Out[148]//TableForm=

$G[1,1] \to \frac{q^{(1,0)}[\rho,z]}{2\rho}$	$G[2,2] \to \frac{1}{2} e^{-q[\rho,z]} \rho^2 \left(q^{(0,2)}[\rho,z] + q^{(2,0)}[\rho,z]\right)$
$G[3,1] \to \frac{q^{(0,1)}[\rho,z]}{2\rho}$	$G[3,3] \to -\frac{q^{(1,0)}[\rho,z]}{2\rho}$