Surendra Padamata
988072780

1) a) A (1,1) tensor: A^a_b

$$A^{a'}_{b'} = M^{a'}_c A^c_d M^d_{b'}$$

where, $M^{a'}_c, M^d_{b'}$ are general change of basis matrices. If both the old & new basis are coordinate basis, i.e. a coordinate change,

$$M^{a'}_c = \frac{\partial x^{a'}}{\partial x^c} \quad \& \quad M^d_{b'} = \frac{\partial x^d}{\partial x^{b'}}$$

$$A^{a'}_{b'} = \frac{\partial x^{a'}}{\partial x^c}\frac{\partial x^d}{\partial x^{b'}} A^c_d$$

<u>Background:</u> By chain rule

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu$$

$$V = V^\mu \partial_\mu = V^{\mu'}\partial_{\mu'} = V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu$$

$$\Rightarrow V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

Ily, $\underline{\omega}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\underline{\omega}_\mu \quad (\because dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}dx^\mu)$

b) Lorentz transformations preserve the spacetime interval.

We have, $x' = \Lambda x$

$$\Delta s^2 = (\Delta x')^T \eta (\Delta x') \quad \text{in } x'$$
$$= (\Delta x)^T \Lambda^T \eta \Lambda (\Delta x)$$
$$\Delta s^2 = (\Delta x)^T \eta (\Delta x) \quad \text{in } x$$

But, $\Delta s^2 = (\Delta x')^T \eta (\Delta x') = (\Delta x)^T \eta (\Delta x)$ }

$$\Rightarrow \Lambda^T \eta \Lambda = \eta$$

or $\eta_{ab} = \Lambda^{c'}_a \Lambda^{d'}_b \eta_{c'd'}$

For a boost in x-direction, the Lorentz transformation is,

$$x' = \gamma(x - \beta t)$$
$$t' = \gamma(t - \beta x)$$

where we used Geometric units, c = 1

$$\Lambda = \begin{pmatrix} \frac{\partial t'}{\partial t} & \frac{\partial x'}{\partial t} \\ \frac{\partial t'}{\partial x} & \frac{\partial x'}{\partial x} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$\Lambda^T = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$\eta' = \Lambda^T \eta \Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma & -\beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma^2 + \beta^2\gamma^2 & +\beta\gamma^2 - \beta\gamma^2 \\ \beta\gamma^2 - \beta\gamma^2 & -\beta\gamma^2 + \gamma^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma^2(1-\beta^2) & 0 \\ 0 & \gamma^2(1-\beta^2) \end{pmatrix}$$

But, $1 - \beta^2 = \frac{1}{\gamma^2}$

$$= \begin{pmatrix} -\gamma^2/\gamma^2 & 0 \\ 0 & \gamma^2/\gamma^2 \end{pmatrix}$$

$$\eta' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \eta$$

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4. Show that

$$D_2 u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}$$

provides 2nd order approximation to $u'(x)$.

Here, over the interval containing the point of interest $\bar{x}$, the $D_2 u(\bar{x})$ provides a centered approximation to derivative of $u(x)$ at $\bar{x}$.

$$D_2 u(\bar{x}) = \frac{1}{2}\left(D_+ u(\bar{x}) + D_- u(\bar{x})\right)$$

where, $D_+ u(\bar{x}) = \dfrac{u(\bar{x}+h) - u(\bar{x})}{h}$

$D_- u(\bar{x}) = \dfrac{u(\bar{x}) - u(\bar{x}-h)}{h}$

are one-sided approximations.

The error in $D_2 u(\bar{x})$ would be,

$D_2 u(\bar{x}) - u(\bar{x})$

$$D_2 u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}$$

Taylor series of $u(\bar{x} \pm h)$ is:

$$u(\bar{x} \pm h) = u(\bar{x}) \pm h u'(x) + \frac{h^2}{2} u''(\bar{x}) \pm \frac{h^3}{6} u'''(x) + \cdots$$

$$\begin{aligned}
u(\bar{x}+h) - u(\bar{x}-h) &= u(\bar{x}) + h u'(\bar{x}) + \frac{h^2}{2} u''(\bar{x}) + \frac{h^3}{6} u'''(\bar{x}) + \cdots \\
&\quad - \left(u(\bar{x}) - h u'(\bar{x}) + \frac{h^2}{2} u''(\bar{x}) - \frac{h^3}{6} u'''(\bar{x}) + \cdots \right) \\
&= u(\bar{x}) + h u'(\bar{x}) + \frac{h^2}{2} u''(\bar{x}) + \frac{h^3}{6} u'''(\bar{x}) + \cdots \\
&\quad - u(\bar{x}) + h u'(\bar{x}) - \frac{h^2}{2} u''(\bar{x}) + \frac{h^3}{6} u'''(\bar{x}) - \cdots \\
&= 2h u'(\bar{x}) + \frac{2h^3}{6} u'''(\bar{x}) + O(h^5) + \cdots
\end{aligned}$$

$$\frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h} = u'(\bar{x}) + \frac{h^2}{6} u'''(\bar{x}) + O(h^5) + \cdots$$

error: $\dfrac{u(\bar{x}+h) - u(\bar{x}-h)}{2h} - u'(\bar{x}) = \dfrac{h^2}{6} u'''(\bar{x}) + O(h^5) + \cdots$

$\therefore$ The error is of order h^2.

Implying, $D_2 u(\bar{x})$ is a second order approximation method.

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2. Poor man's version of numerical relativity.

Oppenheimer-Snyder spherical dust collapse

line element:

$$ds^2 = -d\tau^2 + a^2 (d\chi^2 + \sin^2\chi \, d\Omega^2)$$

proper time, τ - time coordinate from the onset of collapse.

χ - Lagrangian or comoving radial coordinate

$$a = \frac{1}{2} a_m (1+\cos\eta)$$
$$\tau = \frac{1}{2} a_m (\eta + \sin\eta)$$

where, $\eta \in [0, \pi]$

a_m is a free parameter fixed by matching interior & exterior metric solutions.

χ, θ, ϕ & τ - are synchronous, Gaussian or geodesic coordinates

Exterior spacetime metric is, Schwarzschild

$$ds^2 = -\left(1 - \frac{2M}{r_s}\right) dt^2 + \left(1 - \frac{2M}{r_s}\right)^{-1} dr_s^2 + r_s^2 d\Omega^2$$

$$R = \frac{1}{2} R_0 (1+\cos\eta)$$
$$\tau = \left(\frac{R_0^3}{8M}\right)^{1/2} (\eta + \sin\eta)$$

Matching both solutions,

$$a_m = \left(\frac{R_0^3}{2M}\right)^{1/2}$$

$$\sin\chi_0 = \left(\frac{2M}{R_0}\right)^{1/2}$$

$$R_2 = \left(\frac{M_2}{M_1}\right)^{1/3} (R_1)$$ (mass-fractions & radius relation)

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3. TOV equations:

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad \text{- gravitational mass}$$

$$\frac{dP}{dr} = -\frac{\rho m}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi P r^3}{m}\right)\left(1 - \frac{2m}{r}\right)^{-1}$$

$$\frac{dm_0}{dr} = 4\pi r^2 \rho_0 (1 - 2m/r)^{-1/2}$$

(baryonic or rest-mass)

initial conditions:

$m = 0, \ P = P_c, \ r = 0$

supply a EOS to convert ρ in above equations into some $f(P)$

for example, polytrope,

$$P = K\rho_0^{\Gamma}$$