Problem 3 :

Schwarzschild spacetime in isotropic coordinates,

$$ds^2 = -\left(\frac{1-M/2r}{1+M/2r}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2)$$

In 3+1, we have,

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

This gives us by comparing,

$$\alpha = \frac{1-M/2r}{1+M/2r}$$

$$\beta^i = 0$$

$$\gamma_{ij} = \left(1+\frac{M}{2r}\right)^4 \mathrm{diag}(1, r^2, r^2\sin^2\theta)$$

Next we have,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t \gamma_{ij} = 0 \quad (\because \gamma_{ij} \text{ is time-independent})$$

$$\beta^i = 0$$

$$\Rightarrow -2\alpha K_{ij} = 0$$

$$K_{ij} = 0$$

If the lapse & shift satisfy the following coupled set of hyperbolic equations, then the coordinates are Harmonic.

$$(\partial_t - \beta^j\partial_j)\alpha = -\alpha^2 K \quad -①$$

$$(\partial_t - \beta^j\partial_j)\beta^i = -\alpha^2(\gamma^{ij}\partial_j \ln\alpha + \gamma^{jk}\Gamma^i_{jk}) \quad -②$$

We have,

$$\alpha = \frac{1-2M/r}{1+2M/r}, \quad \beta^i = 0, \quad K_{ij} = 0 = K = \gamma^{ij}K_{ij}$$

$$① \Rightarrow (\partial_t - 0\partial_j)\left(\frac{1-2M/r}{1+2M/r}\right) = -\alpha^2 0$$

$$\partial_t\left(\frac{1-2M/r}{1+2M/r}\right) = 0$$

$$0 = 0$$

② $\Rightarrow$ $(\partial_t - \beta^j\partial_j)0 = -\alpha^2(\gamma^{ij}\partial_j \ln\alpha + \gamma^{jk}\Gamma^i_{jk})$

$\Rightarrow$ $\gamma^{ij}\partial_j \ln\alpha = -\gamma^{jk}\Gamma^i_{jk}$

$$\left.\begin{aligned} \gamma^{rj}\partial_j \ln\alpha &= -\gamma^{jk}\Gamma^r_{jk} \\ \gamma^{\theta j}\partial_j \ln\alpha &= -\gamma^{jk}\Gamma^\theta_{jk} \\ \gamma^{\phi j}\partial_j \ln\alpha &= -\gamma^{jk}\Gamma^\phi_{jk} \end{aligned}\right\} - ③$$

But, $\alpha = f(r) \Rightarrow \partial_\theta\alpha = \partial_\phi\alpha = 0$

also, γ^{ij} is diagonal

③ $\Rightarrow$ $\gamma^{rr}\partial_r \ln\alpha = -(\gamma^{rr}\Gamma^r_{rr} + \gamma^{\theta\theta}\Gamma^r_{\theta\theta} + \gamma^{\phi\phi}\Gamma^r_{\phi\phi})$

$0 = -(\gamma^{rr}\Gamma^\theta_{rr} + \gamma^{\theta\theta}\Gamma^\theta_{\theta\theta} + \gamma^{\phi\phi}\Gamma^\theta_{\phi\phi})$

$0 = -(\gamma^{rr}\Gamma^\phi_{rr} + \gamma^{\theta\theta}\Gamma^\phi_{\theta\theta} + \gamma^{\phi\phi}\Gamma^\phi_{\phi\phi})$

($\because \gamma^{ij}$ is diagonal)

Here, $\Gamma^\theta_{rr} = \Gamma^\theta_{\theta\theta} = \Gamma^\phi_{rr} = \Gamma^\phi_{\theta\theta} = \Gamma^\phi_{\phi\phi} = 0$

$\Gamma^r_{rr} = \dfrac{-2M}{Mr+2r^2}$, $\Gamma^r_{\theta\theta} = \dfrac{(M-2r)r}{M+2r}$, $\Gamma^r_{\phi\phi} = -\dfrac{r(-M+2r)\sin^2\theta}{M+2r}$

$$\gamma^{rr}\Gamma^r_{rr} = \left(1+\frac{M}{2r}\right)^{-4}\left(\frac{-2M}{Mr+2r^2}\right) = \left(1+\frac{M}{2r}\right)^{-4}\frac{1}{2r^2}\left(\frac{-2M}{\frac{M}{2r}+1}\right)$$

$$\gamma^{\theta\theta}\Gamma^r_{\theta\theta} = \left(1+\frac{M}{2r}\right)^{-4}\frac{1}{r^2}\left(\frac{(M-2r)r}{M+2r}\right) = \left(1+\frac{M}{2r}\right)^{-4}\frac{1}{2r^2}\left(\frac{M-2r}{\frac{M}{2r}+1}\right)$$

$$\gamma^{\phi\phi}\Gamma^r_{\phi\phi} = -\left(1+\frac{M}{2r}\right)^{-4}\frac{1}{r^2\cancel{\sin^2\theta}}\,\frac{r(-M+2r)\cancel{\sin^2\theta}}{M+2r} = \left(1+\frac{M}{2r}\right)^{-4}\frac{1}{2r^2}\,\frac{(M-2r)}{\frac{M}{2r}+1}$$

$$\Rightarrow \left(1+\frac{M}{2r}\right)^{-4}\partial_r \ln\left(\frac{1-2M/r}{1+2M/r}\right) = -\left(\left(1+\frac{M}{2r}\right)^{-5}\frac{1}{2r^2}\left(-\cancel{2M}+\cancel{M}-2r+\cancel{M}-2r\right)\right)$$

$$\partial_r \ln\left(\frac{1-2M/r}{1+2M/r}\right) = \left(1+\frac{M}{2r}\right)^{-1}\frac{(4r)}{2r^2}$$

$$= \frac{2}{r}\left(\frac{1}{1+\frac{M}{2r}}\right) = \frac{2}{r+\frac{M}{2}} = \frac{1}{\frac{r}{2}+\frac{M}{4}}$$

$$\partial_r \ln\left(\frac{1-2M/r}{1+2M/r}\right) = \left(\frac{1+2M/r}{1-2M/r}\right)\left(\frac{\left(1+\frac{2M}{r}\right)\left(\frac{2M}{r^2}\right)+\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^2}\right)}{(1+2M/r)^2}\right)$$

$$= \frac{\frac{M}{r^2}(4)}{(1-2M/r)(1+2M/r)}$$

$$= \frac{\frac{M}{r^2}(\cancel{4})}{\cancel{4}\left(\frac{1}{2}-\frac{M}{r}\right)\left(\frac{1}{2}+\frac{M}{r}\right)} = \frac{M/r^2}{\left(\frac{1}{4}-\frac{M^2}{r^2}\right)}$$

$$= \frac{M}{(r^2/4 - M^2)}$$

$$= \frac{1}{\left(\frac{r^2}{4M} - M\right)}$$

Problem 2:

The volume of spatial slices is given by,

$$\mathrm{vol}(S) = \int_S d^3x \sqrt{\gamma}$$

$$\delta\,\mathrm{vol}(S) = \int_S d^3x \left(\frac{\partial\sqrt{\gamma}}{\partial n^a} \delta n^a \frac{\delta d^a}{\partial n^a} + \frac{\partial\sqrt{\gamma}}{\partial b^a} \delta b^a \frac{\delta d^a}{\partial b^a} \right)$$

$$= \int_S d^3x \left(\mathcal{L}_n \sqrt{\gamma} c + \mathcal{L}_b \sqrt{\gamma} \right) \quad \begin{pmatrix} \because d^a = cn^a + b^a \\ n^a b_a = 0 \end{pmatrix}$$

But we have,

$$\mathcal{L}_n \sqrt{\gamma} = -\gamma^{1/2} K$$

$$\mathcal{L}_b \sqrt{\gamma} = b^k \partial_k \gamma^{1/2} + \gamma^{1/2} \partial_k b^k$$

$$= 0 \, ?$$

$$= \int_S d^3x \left(-\gamma^{1/2} K c \right)$$

$$\Rightarrow \delta\,\mathrm{vol}(S) = \int_S d^3x \sqrt{\gamma} \, (-cK)$$

$K = 0 \Rightarrow \delta\,\mathrm{vol}(S) = 0$, extremizes the volume.

Problem 1:

a) $$\nabla^2 \psi(r) = \psi^4(r) f(r)$$

If $\psi(r) = \frac{r^2}{1+r^3}$, $f(r) = ?$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\nabla^2 \psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{r^2}{1+r^3}\right)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{(1+r^3)^2(2r) - (r^2)(3r^2)}{(1+r^3)^2}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2[(1+r^6+2r^3)(2r) - 3r^4]}{(1+r^3)^2}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2(2r+2r^7+4r^4-3r^4)}{(1+r^3)^2}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{2r^3+2r^9+r^6}{(1+r^3)^2}\right)$$

$$= \frac{\frac{1}{r^2}\left[(1+r^3)^2(6r^2+18r^8+6r^5) - (2r^3+2r^9+r^6)(2(1+r^3)(3r^2))\right]}{(1+r^3)^4}$$

$$= \frac{1}{r^2}\frac{[(1+r^3)(6r^2+18r^8+6r^5) - (2r^3+2r^9+r^6)(6r^2)]}{(1+r^3)^3}$$

$$= \frac{1}{r^2(1+r^3)^3}[6r^2+18r^8+6r^5+6r^5+18r^{11}+6r^8 + 12r^{11} - 12r^{11} - 6r^8]$$

$$= \frac{1}{r^2(1+r^3)^4}[-12r^8 - 6r^5 + 6r^2]$$

$$= \frac{1}{(1+r^3)^4}[-12r^6 - 6r^3 + 6]$$

$$= \frac{6}{(1+r^3)^4}[-2r^6 - r^3 + 1]$$

$$f(r) = \psi^{-4}(r)\,\nabla^2\psi(r)$$

$$= \left(\frac{1+r^3}{r^2}\right)^4 \frac{6}{(1+r^3)^4}\left[-2r^6 - r^3 + 1\right]$$

$$f(r) = \frac{6}{r^8}\left[1 - r^3 - 2r^6\right]$$