Problem 1:

initial data ansatz:

$$dl^2 = \psi^4 (e^{q}(d\rho^2 + dz^2) + \rho^2 d\phi^2) \quad , \quad q = q(\rho, z)$$

the co-ordinates are (ρ, ϕ, z)

the metric would be, $\gamma_{ij} = \psi^4 \begin{pmatrix} e^{q} & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & e^{q} \end{pmatrix}$

The Hamiltonian constraint with $\gamma^{ij} = \psi^{-4} \bar{\gamma}^{ij}$,

$$8\bar{D}^2\psi - \psi\bar{R} - \psi^5 K^2 + \psi^5 K_{ij}K^{ij} = -16\pi \psi^5 \rho$$

moment of time symmetry, $\partial_t \gamma_{ij} = 0$ & $\beta^i = 0$

we have, evolution equation,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\Rightarrow 0 = -2\alpha K_{ij} + 0 + 0$$

$$\Rightarrow K_{ij} = 0$$

$$\Rightarrow K = 0$$

Vacuum $\Rightarrow \rho = s^i = 0$, the H.C. becomes,

$$8\bar{D}^2\psi - \psi\bar{R} = 0$$

$$\bar{D}^2\psi = \frac{1}{8}\psi\bar{R}$$

$$\bar{\gamma}^{ij} = \psi^{-4}\gamma^{ij}$$

$$= \begin{pmatrix} e^{q} & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & e^{q} \end{pmatrix} \quad , \quad q = q(\rho, z)$$

For this conformal metric, the Ricci scalar is,

$$\bar{R} = -e^{-q}\left(\frac{\partial^2 q}{\partial z^2} + \frac{\partial^2 q}{\partial \rho^2}\right) \qquad \text{(from Mathematica)}$$

$$\bar{D}^2\psi = \bar{\gamma}^{ij}D_iD_j\psi = \bar{\gamma}^{ij}\partial_i\partial_j\psi - \bar{\gamma}^{ij}\Gamma^k_{ij}\partial_k\psi$$

$\bar{\gamma}^{ij}$ is diagonal

$$\bar{D}^2\psi = \bar{\gamma}^{\rho\rho}\partial^2_\rho\psi - \bar{\gamma}^{\rho\rho}\Gamma^k_{\rho\rho}\partial_k\psi + \bar{\gamma}^{\phi\phi}\partial^2_\phi\psi - \bar{\gamma}^{\phi\phi}\Gamma^k_{\phi\phi}\partial_k\psi$$

$$+ \bar{\gamma}^{zz}\partial^2_z\psi - \bar{\gamma}^{zz}\Gamma^k_{zz}\partial_k\psi$$

$$= \bar{\gamma}^{\rho\rho}\partial^2_\rho\psi + \bar{\gamma}^{\phi\phi}\partial^2_\phi\psi + \bar{\gamma}^{zz}\partial^2_z\psi$$

$$= e^{-q}\left[\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi + \partial_z^2\psi + e^{q}\partial_\phi^2\psi\right]$$

$$\bar{D}^2\psi = \frac{1}{8}\psi\bar{R}$$

$$\Rightarrow \cancel{e^{-q}}\underbrace{\left[\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi + \partial_z^2\psi + e^{q}\partial_\phi^2\psi\right]}_{\nabla^2\psi} = \frac{-1}{8}\psi\,\cancel{e^{-q}}\left(\frac{\partial^2 q}{\partial z^2} + \frac{\partial^2 q}{\partial \rho^2}\right)$$

$$\nabla^2\psi = -\frac{\psi}{8}\left(\frac{\partial^2 q}{\partial z^2} + \frac{\partial^2 q}{\partial \rho^2}\right)$$

Problem 2

conformal tranverse-tracelens (CTT) decomposition gives the following form for Hamiltonian constraint.

$$(\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_j K = 8\pi\psi^{10}S^i$$

Maximal slicing, $K = 0$

Vacuum, $S^i = \rho = 0$

$$\Rightarrow (\bar{\Delta}_L W)^i = 0$$

$\bar{\gamma}_{ij} = \eta_{ij}$ - conformal flat

$$\partial^j\partial_j W^i + \frac{1}{3}\partial^i\partial_j W^j = 0$$

a) given, $W^i = -\frac{1}{r^2}\bar{\varepsilon}^{ij}{}_k\, l^k J_j$ & $\bar{D}_k J_i = \partial_k J_i = 0$

$$(\bar{\Delta}_L W)^i = \partial^j\partial_j\left(-\frac{1}{r^2}\bar{\varepsilon}^{im}{}_k\, l^k J_m\right) + \frac{1}{3}\partial^i\partial_j\left(-\frac{1}{\rho^2}\bar{\varepsilon}^{jn}{}_k\, l^k J_n\right)$$

$$\eta_{\alpha k}\bar{\varepsilon}^{ij\alpha} = \bar{\varepsilon}^{ij}{}_k$$

$$\bar{\varepsilon}^{ij}{}_k = \begin{cases} 1 & (\text{even } i,j,k) \\ -1 & (\text{odd } i,j,k) \\ 0 & (i=j \text{ or } j=k \text{ or } i=k) \end{cases}$$

cartesian, $1,2,3 \to x,y,z$

$$(\bar{\Delta}_L W)^x = \partial^j\partial_j\left(-\frac{1}{\rho^2}\bar{\varepsilon}^{xm}{}_k\, l^k J_m\right) + \frac{1}{3}\partial^x\partial_i\left(-\frac{1}{r^2}\bar{\varepsilon}^{in}{}_k\, l^k J_n\right)$$

The terms, $\partial_i J_j = \partial_i \bar{\varepsilon}^{jk}{}_l = 0$

$\Rightarrow$ l^k can be only l^y & l^z since $\bar{\varepsilon}^{xm}{}_x = 0$ in first term.

$$= \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xm}_{y}\, l^y \bar{J}_m\right) + \frac{1}{3}\partial^x\partial_i\left(-\frac{1}{r^2}\bar{\epsilon}^{in}_{k}\, l^k \bar{J}_n\right)$$
$$+ \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xm}_{z}\, l^z \bar{J}_m\right)$$

$$= \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xz}_{y}\, l^y \bar{J}_z\right) + \frac{1}{3}\partial^x\partial_i\left(-\frac{1}{r^2}\bar{\epsilon}^{in}_{k}\, l^k \bar{J}_n\right)$$
$$+ \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xy}_{z}\, l^z \bar{J}_y\right)$$

$$(\bar{\Delta}_L W)^x = \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xz}_{y}\, l^y \bar{J}_z\right) + \frac{1}{3}\partial^x\partial_x\left(-\frac{1}{r^2}\bar{\epsilon}^{xn}_{k}\, l^k \bar{J}_n\right)$$
$$+ \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xy}_{z}\, l^z \bar{J}_y\right) + \frac{1}{3}\partial^x\partial_y\left(-\frac{1}{r^2}\bar{\epsilon}^{yn}_{k}\, l^k \bar{J}_n\right)$$
$$+ \frac{1}{3}\partial^x\partial_z\left(-\frac{1}{r^2}\bar{\epsilon}^{zn}_{k}\, l^k \bar{J}_n\right)$$

$$(\bar{\Delta}_L W)^x = \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xz}_{y}\, l^y \bar{J}_z\right) + \frac{1}{3}\partial^x\partial_x\left(-\frac{1}{r^2}\bar{\epsilon}^{xy}_{z}\, l^z \bar{J}_y\right)$$
$$+ \partial^{j}\partial_j\left(-\frac{1}{r^2}\bar{\epsilon}^{xy}_{z}\, l^z \bar{J}_y\right) + \frac{1}{3}\partial^x\partial_y\left(-\frac{1}{r^2}\bar{\epsilon}^{yz}_{x}\, l^x \bar{J}_z\right)$$
$$+ \frac{1}{3}\partial^x\partial_z\left(-\frac{1}{r^2}\bar{\epsilon}^{zx}_{y}\, l^y \bar{J}_x\right) + \frac{1}{3}\partial^x\partial_x\left(-\frac{1}{r^2}\bar{\epsilon}^{xz}_{y}\, l^y \bar{J}_z\right)$$
$$+ \frac{1}{3}\partial^x\partial_y\left(-\frac{1}{r^2}\bar{\epsilon}^{yx}_{z}\, l^z \bar{J}_x\right) + \frac{1}{3}\partial^x\partial_z\left(-\frac{1}{r^2}\bar{\epsilon}^{zy}_{x}\, l^x \bar{J}_y\right)$$

$$(\bar{\Delta}_L W)^x = \partial^{j}\partial_j\left(-\frac{1}{r^2} l^y \bar{J}_z\right) + \frac{1}{3}\partial^x\partial_x\left(-\frac{1}{r^2} l^z \bar{J}_y\right)$$
$$+ \partial^{j}\partial_j\left(-\frac{1}{r^2} l^z \bar{J}_y\right) + \frac{1}{3}\partial^x\partial_y\left(-\frac{1}{r^2} l^x \bar{J}_z\right)$$
$$+ \frac{1}{3}\partial^x\partial_z\left(-\frac{1}{r^2} l^y \bar{J}_x\right) + \frac{1}{3}\partial^x\partial_x\left(+\frac{1}{r^2} l^y \bar{J}_z\right)$$
$$+ \frac{1}{3}\partial^x\partial_y\left(+\frac{1}{r^2} l^z \bar{J}_x\right) + \frac{1}{3}\partial^x\partial_z\left(+\frac{1}{r^2} l^x \bar{J}_y\right)$$

l^i is a normal vector

$\Rightarrow$ curl of $l^i = 0$

$$(\bar{\Delta}_L W)_x = \bar{J}_z\, \partial^2_x\left(-\frac{l^y}{r^2}\right) + \bar{J}_z\, \partial^2_y\left(-\frac{l^y}{r^2}\right) + \bar{J}_z\, \partial^2_z\left(-\frac{l^y}{r^2}\right)$$
$$+ \frac{\bar{J}_y}{3}\partial^2_x\left(-\frac{l^z}{r^2}\right) + \bar{J}_y\, \partial^2_x\left(-\frac{l^z}{r^2}\right) + \bar{J}_y\, \partial^2_y\left(-\frac{l^z}{r^2}\right)$$
$$+ \bar{J}_y\, \partial^2_z\left(-\frac{l^z}{r^2}\right) + \frac{\bar{J}_z}{3}\partial^2_x\left(\frac{l^y}{r^2}\right)$$

b) $$W^i = -\frac{1}{r^2}\bar{\epsilon}^{ij}_{k}\, l^k \bar{J}_j$$

Here, $l^k = \frac{x^k}{r}$, in spherical coordinates, $l^k = (1,0,0)$

$$w^i = -\frac{1}{r^2}\bar{\varepsilon}^{irk} J_k$$

$w^r = 0$ (property of $\bar{\varepsilon}^{ijk}$)

$$w^\theta = -\frac{1}{r^2}(1) J_\phi - \frac{1}{r^2}(0) J_\theta = -\frac{J_\phi}{r^2}$$

lly, $w^\phi = \frac{-J_\theta}{r^2} = \frac{-J}{r^2}$, $J = |J_\theta|$

$$\bar{A}^L_{r\phi} = \gamma^{rr}\gamma^{\phi\phi}\frac{6}{r^3} l^{(}{}_{r}\bar{\varepsilon}_{\phi)}{}^{kl} J_k l_l = \frac{3J}{r}\sin^2\theta$$

c) given, $w^\phi = -\frac{J}{r^3}$, $w^r = w^\theta = 0$

$\bar{A}^L_{r\phi} = \frac{3J}{r^2}\sin^2\theta$ & $\bar{A}^L_{ij} = 0$ except for $ij = r\phi$

In Boyer-Lindquist coordinates, Kerr is,

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta\, dt\, d\phi}{\rho^2}$$
$$+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right)d\phi^2$$
$$\Delta = r^2 - 2Mr + a^2$$
$$\rho^2 = r^2 + a^2\cos^2\theta$$

Asymptotically, the only non-vanishing component of K_{ij} is $K_{r\phi} = \frac{3Ma}{r^2}\sin^2\theta$

But, we have, $K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ik}K$

maximal slicing ($K = 0$) & $A_{r\phi}$ is the only non-zero in spherical polar.

$$\Rightarrow K_{r\phi} = A_{r\phi} = \frac{3J}{r^2}\sin^2\theta$$

$$\Rightarrow \frac{3J}{r^2}\sin^2\theta = \frac{3Ma}{r^2}\sin^2\theta$$

$$\therefore J = Ma$$

$$\boxed{A_{r\phi} = \frac{3Ma}{r^2}\sin^2\theta} \quad \text{--- (i)}$$

from Hamiltonian constraint,

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5\rho$$

again, vacuum, maximal slicing & only $A_{r\phi}$ non-zero,

$$8\nabla^2\psi - 0 - 0 + \psi^{-7}\left(\frac{9M^2a^2}{r^4}\sin^4\theta\right) = 0$$

$$\boxed{\psi^7\nabla^2\psi = \frac{9}{8}\frac{M^2a^2}{r^2}\sin^4\theta} \quad \text{-(ii)}$$

So, the $A_{r\phi}$ & ψ given (ii) gives solution agrees with asymptotic Kerr solution in Boyer-Lindquist coordinates.