Assignment - 1

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(1) Show that the function $f(z) = \frac{x^3y^5(x+iy)}{x^6+y^{10}}, z \neq 0$, $f(0)=0$ is not analytic at the origin even though it satisfy C.R. equation at origin.

$$f(z) = \frac{x^3y^5(x+iy)}{x^6+y^{10}}$$

$$f(z) = \frac{x^4y^5 + ix^3y^6}{x^6+y^{10}}$$

$$f(z) = \frac{x^4y^5}{x^6+y^{10}} + i\frac{x^3y^6}{x^6+y^{10}}$$

$$U(x,y) = \frac{x^4y^5}{x^6+y^{10}}$$

$$V(x,y) = \frac{x^3y^6}{x^6+y^{10}}$$

By definition of partial derivative

$$\frac{\partial U}{\partial x} = U_x = \lim_{h\to 0} \frac{U(x+h,y) - U(x,y)}{h}$$

[Given $f(z)=0$ at $z=0$ $\therefore U(0,0)=0$]

$$\left.\frac{\partial U}{\partial x}\right]_{at(0,0)} = U_{x(0,0)} = \lim_{h\to 0} \frac{U(h,0) - U(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3 \cdot 0}{(h^6+0)\cdot h}$$

$$= 0$$

$$\left.\frac{\partial U}{\partial y}\right]_{at(0,0)} = U_{y(0,0)} = \lim_{h\to 0} \frac{U(0,h) - U(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0 \cdot h^5}{h\cdot(0+h^{10})}$$

$$= 0$$

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$$\left.\frac{\partial v}{\partial x}\right]_{at(0,0)} = V_x(0,0) = \lim_{h\to 0} \frac{V(h,0) - V(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3 \cdot 0}{\cancel{h}(h^6+0)}$$

$$= 0$$

$$\left.\frac{\partial v}{\partial y}\right]_{at(0,0)} = V_y(0,0) = \lim_{h\to 0} \frac{V(0,h) - V(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0 \cdot h^6}{\cancel{h}(0+h^{10})}$$

$$= 0.$$

Here C.R. equation, $U_x(0,0) = V_y(0,0)$
and $V_x(0,0) = U_y(0,0)$ are satisfy

Now we check differentiability of $f(z)$ at $z=0$

$f(z)$ is differential at $z=0$ if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

Should exist and equal to $f'(z_0)$

i.e, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ should exist

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^3y^5(x+iy)}{x^6+y^{10}} - 0}{z - 0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3y^5\cancel{(x+iy)}}{x^6+y^{10}\cancel{(x+iy)}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3y^5}{x^6+y^{10}} \text{ should exist}$$

→ Along Real axis $y=0$.

$$\lim_{x\to 0} \frac{x^3 \cdot 0}{x^6+0}$$

$$= 0.$$

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→ Along Imaginary Axis, $x=0$

$$= \lim_{y\to 0} \frac{0\cdot y^5}{0+y^{10}}$$

$$= 0.$$

→ Along $x^3 = y^5$

$$\lim_{y\to 0} \frac{y^5\cdot y^5}{(y^5)^2+y^{10}}$$

$$= \lim_{y\to 0} \frac{\cancel{y^{10}}}{2\cancel{y^{10}}}$$

$$= \frac{1}{2}.$$

Since from two different path limit is different.
Therefore $f(z)$ is not differentiable at $z=0$.
$\Rightarrow$ $f(z)$ is not an analytic function at origin
Even C.R. equation are satisfy at origin.

(2) Show that if $f(z)$ is analytic and $Re f(z) = $ constant then $f(z)$ is constant.

Since the function $f(z) = u(x,y) + iv(x,y)$ is analytic, it satisfy the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also, $Re f(z) = $ constant
Therefore, $u(x,y) = c_1$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$$

By C.R. equations, $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$

Hence, $v(x,y) = c_2 = $ a real constant

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$\therefore f(z) = u(x,y) + iv(x,y) = c_1 + ic_2$
= a complex constant.

(3) Find the value of a, b and c such that the function $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$ is analytic. Express $f(z)$ in terms of z.

$$f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$$
$$U(x,y) = -x^2 + xy + y^2$$
$$V(x,y) = ax^2 + bxy + cy^2$$

$$U_x = \frac{\partial U}{\partial x} = -2x + y$$
$$U_y = \frac{\partial U}{\partial y} = x + 2y$$
$$V_x = \frac{\partial V}{\partial x} = 2ax + by$$
$$\frac{\partial V}{\partial y} = V_y = bx + 2cy$$

By C.R. equation, $U_x = V_y$
$\Rightarrow -2x + y = bx + 2cy$
By comparing $b = -2$, and $c = \frac{1}{2}$
also, $U_y = -V_x$
$x + 2y = -2ax - by$
By comparing, $a = \frac{-1}{2}$.

Therefore, $a = \frac{-1}{2}$, $b = -2$ and $c = \frac{1}{2}$

$$f(z) = -x^2 + xy + y^2 + i\left(\frac{-x^2}{2} - 2xy + \frac{y^2}{2}\right).$$
$$= \frac{-1}{2}\left(2x^2 - 2xy - 2y^2 - 2i\left(\frac{-x^2}{2} - 2xy + \frac{y^2}{2}\right)\right)$$

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$$= \frac{-1}{2}\left(2x^2 - 2xy - 2y^2 + ix^2 + 4xyi - iy^2\right)$$

$$= \frac{-1}{2}\left(2x^2 + 4xyi - 2y^2 + ix^2 - 2xy - iy^2\right)$$

$$= \frac{-1}{2}\left(2(x^2 + 2xyi - y^2) + i(x^2 + 2xyi - y^2)\right)$$

$$= \frac{-1}{2}(2+i)(x^2 + 2xyi - y^2)$$

$$= \frac{-1}{2}(2+i)(z^2)$$

$$\boxed{f(z) = \frac{-1}{2}(2+i)z^2}$$

④ Show that $V(x,y) = e^{-x}(x\cos y + y\sin y)$ is harmonic. Find the harmonic conjugate.

$$V(x,y) = e^{-x}(x\cos y + y\sin y)$$

$$V_x = -e^{-x}(x\cos y + y\sin y) + e^{-x}(\cos y)$$

$$V_{xx} = -\left(-e^{-x}(x\cos y + y\sin y) + e^{-x}\cos y\right) - e^{-x}\cos y$$

$$= e^{-x}(x\cos y + y\sin y) - e^{-x}\cos y - e^{-x}\cos y$$

$$= e^{-x}(x\cos y + y\sin y) - 2e^{-x}\cos y \quad \text{— (A)}$$

and,

$$V_y = xe^{-x}(-\sin y) + e^{-x}(\sin y + y\cos y)$$

$$= xe^{-x}(-\sin y) + e^{-x}\sin y + e^{-x}y\cos y$$

$$V_{yy} = -xe^{-x}\cos y + e^{-x}\cos y + e^{-x}\cos y + e^{-x}y\sin y$$

$$= -e^{-x}(x\cos y + y\sin y) + 2e^{-x}\cos y \quad \text{— (B)}$$

Adding equation (A) & (B) we get,

$$V_{xx} + V_{yy} = e^{-x}(x\cos y + y\sin y) - 2e^{-x}\cos y - e^{-x}(x\cos y + y\sin y) + 2e^{-x}\cos y$$

$$\Rightarrow \boxed{V_{xx} + V_{yy} = 0}$$

$\Rightarrow V(x,y)$ is a harmonic function.

Its harmonic conjugate,

$$2.\partial U = \frac{\partial U}{\partial x} . \partial x + \frac{\partial U}{\partial y} . \partial y$$

By C.R. equation, $U_x = V_y$ and $U_y = -V_x$

$$2\partial U = \frac{\partial V}{\partial y} . \partial x + \left(-\frac{\partial V}{\partial x}\right) . \partial y$$

$$2U(x,y) = \int \left(xe^{-x}(-\sin y) + e^{-x}(\sin y + y\cos y)\right) dx$$
$$- \int \left(-e^{-x}(x\cos y + y\sin y) + e^{-x}\cos y\right) \partial y$$

$$= (-xe^{-x} - e^{-x})(-\sin y) - e^{-x}(\sin y + y\cos y)$$
$$e^{-x}(x\sin y + \int -y\cos y - \int -\cos y\, dy) - e^{-x}\sin y$$

$$= \sin y . x . e^{-x} + e^{-x}\cancel{\sin y} - e^{-x}\cancel{\sin y} - e^{-x} y\cos y$$
$$+ e^{-x} . x . \sin y - y\cos y\, e^{-x} + e^{-x}\cancel{\sin y} - e^{-x}\cancel{\sin y}$$

$$= \sin y . x . e^{-x} - e^{-x} y\cos y + e^{-x} x\sin y + c$$
$$- y\cos y\, e^{-x}$$

$$= 2x\sin y\, e^{-x} - 2ye^{-x}\cos y + c$$

$$\cancel{2}U(x,y) = \cancel{2}e^{-x}(x\sin y - y\cos y) + c$$

$$\boxed{U(x,y) = e^{-x}(x\sin y - y\cos y) + c}$$

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(5) Find the analytic function $f(z)$ in terms of z whose real part is $\dfrac{\sin 2x}{\cosh 2y + \cos 2x}$

Given $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$

Then, $\dfrac{\partial u}{\partial x} = \dfrac{2\cos 2x(\cosh 2y + \cos 2x) - \sin 2x(0 - 2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$

$= \dfrac{2\cosh 2y \cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2}$

$= \dfrac{2\cosh 2y \cdot \cos 2x + 2}{(\cosh 2y + \cos 2x)^2} = \phi_1$ (say)

put $x = z$ and $y = 0$ in ϕ_1 we get,

$\phi_1(z,0) = \dfrac{2\cosh 0 \cos 2z + 2}{(\cosh 0 + \cos 2z)^2}$

$= \dfrac{2(1 + \cos 2z)}{(1 + \cos 2z)^2}$

$= \dfrac{2}{(1 + \cos 2z)}$

$= \dfrac{2}{1 + 2\cos^2 z - 1} = \dfrac{1}{\cos^2 z}$

$\phi_1(z,0) = \sec^2 z$

& $\dfrac{\partial u}{\partial y} = \sin 2x \dfrac{\partial}{\partial y}\left(\dfrac{1}{\cosh 2y + \cos 2x}\right)$

$= \left[\sin 2x \,(-1) \dfrac{1}{(\cosh 2y + \cos 2x)^2} \times \sinh 2y \cdot 2\right]$

$= \dfrac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$

$= \phi_2(x,y)$ (say)

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put $x=z$ and $y=0$ in $\phi_2(x,y)$ we get

$$\phi_2(z,0) = 0$$

Then by Milnes thomson method we get analytic function

$$f(z) = \int (\phi_1(z,0) - i\phi_2(z,0))\,dz + c$$

$$= \int \sec^2 z\,dz + c$$

$$\boxed{f(z) = \tan z + c}$$

Q6. If $f(z) = u+iv$ is an analytic function of z and $u-v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$

prove that $f(z) = \dfrac{1}{2}(1-\cot z)$ when $f\left(\dfrac{\pi}{2}\right) = 0$

Given $u-v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$

$$u-v = \frac{\cos x + \sin x - \cosh y + \sinh y}{2\cos x - 2\cosh y}$$

$$\left[\because \begin{array}{l} e^{y} = \cosh y + \sinh y \\ e^{-y} = \cosh y - \sinh y \end{array}\right]$$

$$u-v = \frac{1}{2} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} \quad \text{--- (i)}$$

Let, $f(z) = u+iv$ & $if(z) = u - iv$

$\therefore (1+i)f(z) = (u-v) + i(u+v)$

$G(z) = U + iV$

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Given, $V = u - v = \frac{1}{2} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)}$

Then $\frac{\partial V}{\partial x} = \frac{1}{2} \frac{(\cos x - \cosh y)(\cos x) - (\sin x + \sinh y)(-\sinh y)}{(\cos x - \cosh y)^2}$

$= \phi_1(x, y)$ (say)

Then, $\frac{\partial V}{\partial y} = \frac{1}{2} \cdot \frac{(\cos x - \cosh y)(\cosh y) - (\sin x + \sinh y)(-\sinh y)}{(\cos x - \cosh y)^2}$

$= \phi_2(x, y)$ (say)

Then, put $x = z$, and $y = 0$ in ϕ_1 we get

$\phi_1(z, 0) = \frac{1}{2} \frac{(\cos z - \cos 0)(\cos z) - (\sin z + \sin 0)(-\sin z)}{(\cos z - \cos 0)^2}$

$= \frac{1}{2} \cdot \frac{-\cos z + \cos^2 z + \sin^2 z}{(\cos z - 1)^2}$

$= \frac{1}{2} \cdot \frac{1 - \cos z}{(\cos z - 1)^2}$

$= \frac{1}{2} \frac{1}{(1 - \cos z)}$

Now, put $x = z$ and $y = 0$ in ϕ_2 we get

$\phi_2(z, 0) = \frac{1}{2} \cdot \frac{(\cos z - \cos 0)\cos 0 - (\sin z + \sin 0)(-\sin 0)}{(\cos z - \cos 0)^2}$

$= \frac{1}{2} \frac{(\cos z - 1)}{(\cos z - 1)^2}$

$= \frac{1}{2} \frac{1}{(\cos z - 1)}$

$= \frac{1}{2} \frac{(-1)}{(1 - \cos z)}$

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By Milne's thomson theorem,

$$G(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + c$$

$$= \int \left[\frac{1}{2}\cdot\frac{1}{(1-\cos z)} - i\,\frac{1}{2}\left(\frac{-1}{(1-\cos z)}\right)\right] dz + c$$

$$= \frac{1}{2}(1+i)\int \frac{dz}{1-\cos z} + c$$

$$= \frac{1+i}{2}\int \frac{dz}{1-1+2\sin^2\frac{z}{2}} + c$$

$$= \frac{1+i}{2}\times\frac{1}{2}\int \operatorname{cosec}^2\frac{z}{2}\,dz + c$$

$$= \frac{1+i}{4}\left[\frac{-\cot\frac{z}{2}}{\frac{1}{2}}\right] + c$$

$$G(z) = -\frac{(1+i)}{2}\cot\frac{z}{2} + c$$

$$\boxed{f(z) = \frac{-1}{2}\cot\frac{z}{2} + \frac{c}{1+i}} \quad \text{--- } (A)$$

put $z = \frac{\pi}{2}$ in (A) we get

$$f\left(\frac{\pi}{2}\right) = \frac{-1}{2}\cot\frac{\pi}{4} + \frac{c}{1+i}$$

$$0 = \frac{-1}{2} + \frac{c}{1+i}$$

$$c = \frac{1+i}{2}$$

Hence, $\boxed{f(z) = \frac{-1}{2}\cot\frac{z}{2} + \frac{1}{2}}$

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7) If $f(z)$ is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Given $f(z)$ is analytic. Then $f(z)$ satisfy C.R. equation,

$$\left. U_x = V_y \;\&\; U_y = -V_x \right\} \text{--- (i)}$$

Also, If $f(z)$ is analytic then u & v are harmonic function,

$$\left. U_{xx} + U_{yy} = 0 \;\&\; V_{xx} + V_{yy} = 0 \right\} \text{--- (ii)}$$

Since, $f(z) = u + iv$

where, $z = x + iy$

$$\left. \& \; |f(z)|^2 = u^2 + v^2 \right\} \text{--- (3)}$$

Now, $Re\, f(z) = u$

$$\frac{\partial}{\partial x} u^2 = 2u \frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2} u^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u \frac{\partial^2 u}{\partial x^2} \text{--- (4)}$$

Similarly, $\frac{\partial^2}{\partial y^2} u^2 = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u \frac{\partial^2 u}{\partial y^2}$ --- (5)

Adding equation (4) & (5)

$$\frac{\partial^2}{\partial x^2} u^2 + \frac{\partial^2}{\partial y^2} u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

From equation (ii)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial q}\right)^2\right]$$

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By C.R. equation

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2\right]$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

since,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2|f'(z)|^2$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\mathrm{Re}\, f(z)|^2 = 2|f'(z)|^2 \quad — (6)$$

Similarly,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v^2 = 2|f'(z)|^2 \quad — (7)$$

Adding (6) & (7) we get.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 4|f'(z)|^2$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Q8. Find an analytic function $f(z)$ such that $Re|f'(z)| = 3x^2 - 4y - 3y^2$ and $f(1+i) = 0$ & $f'(0) = 0$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$Re|f'(z)| = \frac{\partial u}{\partial x} = 3x^2 - 4y - 3y^2 = \phi_1 \text{ (say)}$$

$$\phi_1(z,0) = 3z^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 4y - 3y^2$$

$$u = \int (3x^2 - 4y - 3y^2)\,dx + c$$

$$u = x^3 - 4xy - 3xy^2 + c$$

$$\frac{\partial u}{\partial y} = -4x - 6xy = \phi_2 \text{ (say)}$$

$$\phi_2(z,0) = -4z$$

By Milne's thomson method,

$$f(z) = \int \phi_1(z,0)\,dz \cancel{+} - i\int \phi_2(z,0)\,dz + c_1$$

$$f(z) = \int 3z^2\,dz - i\int -4z\,dz + c_1$$

$$f(z) = z^3 + i2z^2 + c_1$$

given, $f(1+i) = 0$

$$f(z) = z^2(z + 2i) + c_1$$

$$f(1+i) = (1+i)^2(1+i+2i) + c_1$$

$$0 = 2i(3i+1) + c_1$$

$$\cancel{0 = -6+2i+c_1} \Rightarrow 0 = -6 + 2i + c_1$$

$$\cancel{c_1 = 6+2i} \qquad c_1 = 6 - 2i$$

Then, $f(z) = z^3 + 2iz^2 + 6 - 2i$

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Q(9). Find the image of $|z-3i|=3$ under the transformation $w=\frac{1}{z}$.

Given, $w=\frac{1}{z}$

$$z=\frac{1}{\omega}$$

$$x+iy=\frac{1}{u+iv}\times\frac{u-iv}{u-iv}$$

$$x=\frac{u}{u^2+v^2} \text{ and } y=\frac{-v}{u^2+v^2}$$

Now, Given circle is $|z-3i|=3$

or, $|(x+iy)-3i|^2=9$

or, $|x+i(y-3)|^2=9$

or, $x^2+(y-3)^2=9$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+\left(\frac{-v}{u^2+v^2}-3\right)^2=9$$

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{u^2+v^2}+\cancel{9}+\frac{6v}{u^2+v^2}=\cancel{9}$$

$$\frac{u^2+v^2}{(u^2+v^2)^2}+\frac{6v}{u^2+v^2}=0$$

$$\frac{1}{u^2+v^2}+\frac{6v}{u^2+v^2}=0$$

$$\boxed{1+6v=0}$$

which is the required image of the circle under $w=\frac{1}{z}$

Q(10) Find the bilinear transformation which maps the points $z=0, 1, \infty$ into the points $w = i, 1, -i$ respectively.

Let $z_1=0,\ z_2=1,\ z_3=\infty$

and $w_1=i,\ w_2=1,\ w_3=-i$.

Now the mobius transformation which maps $z_1 z_2 z_3$ into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)}{(z_1-z_2)}\frac{(z_2-z_3)}{(z_3-z)}$$

$$\frac{(w-i)(1+i)}{(i-1)(-i-w)} = \frac{z\cdot \cancel{z_3}\left(\frac{z_2}{z_3}-1\right)}{(-1)\cancel{z_3}\left(1-\frac{z}{z_3}\right)}$$

$$\frac{(w-i)(1+i)}{(i-1)(-i-w)} = +z$$

$$(w-i)+i(w-i) = z(i-1)(-i-w)$$

$$w-i+iw+1 = z(i(-i-w)-1(-i-w))$$

$$w-i+iw+1 = z(1-iw+i+w)$$

$$w-i+iw+1 = z-izw+zi+zw$$

$$w+iw+1+izw-zi-zw = z+i$$

$$\frac{(w-i)(1+i)^2}{(1+i)(i-1)(-i-w)} = z$$

$$\frac{2i(w-i)}{-2(-i-w)} = z$$

$$\frac{i(w-i)}{i+w} = z$$

$$iw+1 = zi+zw$$

$$w(i-z) = zi-1$$

$$w = \frac{zi-1}{i-z} = \frac{zi^2+i^2}{i+zi^2}$$

$$\boxed{w = \frac{z+i}{1+zi}}$$

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(11) Find the image of the real axis of the z-plane on the w-plane by the transformation $w = \frac{1}{z+i}$.

we have $w = \frac{1}{z+i}$

$$z + i = \frac{1}{w}$$

where $z = x+iy$ and $w = u+iv$

$$x+iy+i = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$x+i(y+1) = \frac{u-iv}{u^2+v^2}$$

Equating of real and img part we get

$$x = \frac{u}{u^2+v^2}, \quad y+1 = \frac{-v}{u^2+v^2}$$

Equation of real axis on z-plane is $y=0$

Therefore $0+1 = \frac{-v}{u^2+v^2}$

$$\Rightarrow \quad u^2+v^2 = -v$$

Therefore $u^2+v^2+v=0$ which is circle in w-plane.

Comparing this equation with equation of circle.

$u^2+v^2+2gu+2fv+c=0$

Therefore, $g=0$, $f=\frac{1}{2}$, $c=0$

Center $(-g,-f) = (0,-\frac{1}{2})$ and radius =

$$= \sqrt{g^2+f^2-c} = \frac{1}{2}$$

$\therefore$ real axis of z-plane is mapped to circle in w-plane.

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Q(12) Show that the transformation maps a circle $w = \frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ onto the straight line $4u+3=0$.

Given, $x^2+y^2-4x=0$

$$x^2-4x+2^2-2^2+y^2=0$$

$$\boxed{(x-2)^2+y^2=2^2}$$

centre $(2,0)$, radius $=2$

$\Rightarrow$ $|z-2|=2$ —— (i)

Now,

$$w=\frac{2z+3}{z-4}$$

$$wz-4w=2z+3$$

$$wz-2z=4w+3$$

$$z=\frac{4w+3}{w-2} \quad \text{—— (ii)}$$

By equation (i)

$$\left|\frac{4w+3}{w-2}-2\right|=2$$

$$\left|\frac{4w+3-2w+4}{w-2}\right|=2$$

$$\left|\frac{2w+7}{w-2}\right|=2$$

$$|2w+7|=2|w-2|$$

$$|2(u+iv)+7|=2|u+iv-2|$$

$$|(2u+7)+i2v|=2|(u-2)+iv|$$

$$(2u+7)^2+2v^2=4(u-2)^2+4v^2$$

$$\cancel{4u^2}+49+28u+\cancel{4v^2}=\cancel{4u^2}-16u+16+\cancel{4u^2}$$

$$44u+33=0$$

$$\boxed{4u+3=0}$$