

CS 5854 : Networks, Crowds, and Markets

Homework 2

Instructor: Rafael Pass TAs: Cody Freitag, Jonathan Shi

Assigned: February 28, 2019 Due: March 21, 2019, 11:59 pm

Homework Policies and Guidelines:

Submission: Your homework solutions must be **typed** and submitted as a single .pdf file in CMS. A template .tex file and .py files will be provided containing an outline for your submission. You must additionally submit a single .zip file containing all relevant files specified in the assignment for all coding problems.

Late Days: Each student may use four “late days”, each of which grants a 24-hour extension to an assignment’s due date, as desired throughout the semester; late work beyond this limit will still be accepted and graded until solutions or grades have been released for that assignment, but (unless discussed in advance with the TA and/or instructor) will have a negative impact on final grades.

Collaborations: You may work in groups of up to **4** students, but you must turn in your own individually written solutions. Every member of the group must list all other collaborators at the top of their assignment. (*Note: the maximum size of a connected component of groups must be 4. If A,B,C, and D work together, it must not be the case that A and E work together.*) Additionally, you may make use of published material, provided that you acknowledge all sources used. It is considered a violation of academic integrity to submit a problem solution that you are unable to explain orally to a member of the course staff.

How to receive credit: You must **justify all answers** to receive credit unless specified otherwise. For coding questions, please turn in complete, executable code for each part of the question that asks for an implementation, and include a .txt file containing any required outputs if not already included in the main .pdf file. *Using Python is strongly preferred.* If you want to use another language, you must first consult with the TAs. We will not grade code based on style, but we may mark down code if we are unable to understand what it is doing. You may use standard libraries to implement data structures such as graphs, but, unless otherwise specified, you may not use pre-existing implementations of any algorithms without express permission from the TAs.

Part 1: Best-Response Dynamics

1. (a) Let G_1 and G_2 be games with the same set of players and action sets for each player, and *assume that BRD converges to a PNE* in both of these games. Consider a game $G = (G_1 + G_2)$ where each player plays a single action for both G_1 and G_2 and receives utility equal to the *sum* of the utilities they would earn from G_1 and G_2 . *Will BRD converge in this game?* If so, prove it; if not, find a counterexample and argue why it doesn't converge. (*Hint:* Think of whether you can split rock-paper-scissors into two games that each converge to a strictly dominant PNE.)
- (b) Assume in a particular game $G = G_1 + G_2$ that BRD does converge. Will it necessarily converge to a state that is also a PNE in either G_1 or G_2 ? If so, prove it; if not, find a counterexample. (*Hint:* Perhaps work backwards... consider a two-action game with a strictly dominant PNE, and see if you can split it into two games each with a different strictly dominant PNE.)
2. Consider the process of “better-response dynamics” rather than BRD, where instead of choosing a player's best response (i.e. the response that maximizes their utility given others' actions), a player chooses any response that *strictly* increases their utility given other players' actions.
 - (a) Does the process of “better-response dynamics” still converge in games for which there is an ordinal potential function Φ ? Prove it or show a counterexample. (*Hint:* This is in the notes now. It suffices to reference the correct theorem.)
 - (b) Can you think of a game for which BRD converges but “better-response dynamics” might not? Show an example or justify that one doesn't exist. (*Hint:* Remember that the better-response dynamics graph can have edges that the BRD graph might not. What if those edges formed a loop? Perhaps try adding a new strictly dominant strategy to a game where BRD does not converge.)
- * **Bonus Question 1.** Recall the Traveler's Dilemma that we studied in chapter 1. We have already illustrated why BRD converges in this game; find a **weakly** ordinal potential function Φ over states of this game and use it to definitively prove the convergence of BRD. Make sure to justify that the potential function you find is weakly ordinal (you can do this via computer if you wish).
- * **Bonus Question 1.5.** (*This may be very difficult!*) Determine whether *better-response dynamics* converge for the Traveler's Dilemma. No formal proof is necessary, and you need not come up with a potential function. You are free to either use simulations, show (experimentally) that the graph contains no cycle, or find an ordinal potential function and either prove it or experimentally verify.

Part 2: Networked Coordination Games

3. Consider the following simple coordination game between two players:

	$(*, X)$	$(*, Y)$
$(X, *)$	(x, x)	$(0, 0)$
$(Y, *)$	$(0, 0)$	(y, y)

Show how we can pick x and y and then modify this payoff matrix by adding an intrinsic utility for a single player and a single choice (e.g. give player 1 some intrinsic utility for choice Y) such that the socially optimal state of the game is no longer an equilibrium. (*Hint:* Try $x = 4$ and $y = 1$. Clearly, (X, X) is the social optimum. Now, add intrinsic value to one player so that (X, X) is still the social optimum, but no longer the PNE.)

4. Consider a networked coordination game on a complete graph of five nodes. Assume for simplicity that all edges represent the same coordination game (that is, $Q_e(X, X) = Q_{e'}(X, X)$ for any pair of edges e, e' , and respectively for Y , but it is not necessarily the case that $Q_e(X, X) = Q_e(Y, Y)$), and that all nodes have the same intrinsic values for X and Y (that is, $R_i(X) = R_j(X)$ for any nodes i, j , and respectively for Y).
 - (a) Show a possible assignment of intrinsic and coordination utilities such that every pure-strategy Nash equilibrium of the resulting game must be socially optimal. Justify that this holds for your construction. (*Hint:* Make the intrinsic values huge!)
 - (b) Is there a possible assignment of intrinsic and coordination utilities such that there exists a pure-strategy Nash equilibrium with social welfare less than half of the maximum attainable social welfare? If not, prove it. If so, show such an assignment and explain why your construction does not violate Theorem 3.5 from the notes (the price of stability). (*Hint:* Theorem 3.5 only talks about the *best* equilibrium. Try to make a simple two-node game with one very good equilibrium and one bad equilibrium, and then extend what you find there to the five-node graph. It also won't be necessary to use intrinsic values for this part.)

Part 3: Cascading Behavior in Networks

5. Consider the network in Figure 1. Suppose that each node starts with the behavior Y , and has a $q = 2/5$ threshold for adopting behavior X . (That is, if at least $2/5$ of a node's neighbors have adopted X , that node will as well.)
 - (a) Let e and f form a set S of initial adopters of X . Which nodes will eventually switch to X ? (Assume that S will not participate in BRD.)
 - (b) Find a cluster of density $1 - q = 3/5$ in the part of the graph outside S which blocks X from spreading to all nodes starting from S .
 - (c) Add one node to S such that a complete cascade will occur at the threshold $q = 2/5$. Demonstrate how the complete cascade could occur (i.e. in what order nodes will switch).
6.
 - (a) Formulate a graph G , threshold q , and set S of initial adopters such that, assuming we start with S choosing X (and, importantly, able to participate in BRD) and other nodes choosing Y , we can either end up with every node in G playing X or with every node in G playing Y , depending on the order of switches in the BRD process.
 - (b) What must be true of the set density of S for the above property to hold?
 - (c) What must be true of the set density in any subset of $V \setminus S$ for the above property to hold?

Part 4: Traffic Networks

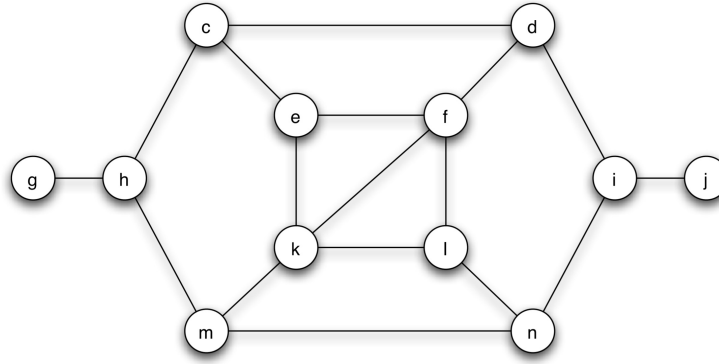


Figure 1: Question 5.

7. There are two cities, A and B, joined by two routes which pass through towns C and D respectively. There are 120 travelers who begin in city A and must travel to city B, and may take the following roads:
- a local street from A to C with travel time $15 + x$, where x is the number of travelers using it,
 - a highway from C to B with travel time 90,
 - a highway from A to D with travel time 90, and
 - a local street from D to B with travel time $15 + y$, where y is the number of travelers using it.
- (a) Draw the network described above and label the edges with their respective travel times. The network should be a directed graph (assume that all roads are one-way). Note: you may just describe the graph if you are typing the assignment up.
 - (b) Find the Nash equilibrium values of x and y . (Show that this is the equilibrium.)
 - (c) The government now adds a new *two-way* road connecting the nodes where local streets and highways meet. This new road is extremely efficient and requires no travel time. Find the new Nash equilibrium.
 - (d) What happens to the total travel time as a result of the availability of the new road? (You don't need to explain, a calculation is fine.)
 - (e) Now suppose that the government, instead of closing the new road, decides to assign routes to travelers to shorten the total travel time. Find the assignment that minimizes the total travel time, and determine the total travel time using this assignment. (*Hint:* It is possible to achieve a total travel time less than the original equilibrium. Remember that, with the new road, there are now four possible routes that each traveler can take.)

Part 5: Experimental Evaluations

These problems will make use of the data set available at:
[\[http://snap.stanford.edu/data/egonets-Facebook.html\]](http://snap.stanford.edu/data/egonets-Facebook.html).

In particular, please refer to the file “facebook_combined.txt.gz”; it contains a text file listing all 88,234 edges (*undirected* edges representing Facebook friendship) in a sampled 4,039-node network (nodes are numbered 0 to 4038). It will be extremely useful, for problems 8 and 9, to be able to input a graph presented in such a format into your code.

8. The following exercise concerns finding the largest number of *edge-disjoint* paths (i.e. paths which do not share any edges) between two given nodes u and v in a graph.
 - (a) First, implement an algorithm that, given a *directed* graph, a source s , a sink t , and edge capacities over each edge in E , computes the maximum flow from s to t . Turn in your code and verify that your algorithm works on a few simple test cases. In particular, test your algorithm on the graphs in Figures 5.1 and 5.3 from the lecture notes and submit the output.
 - (b) Explain how, given some unweighted directed graph, we can use this maximum-flow algorithm to determine the maximum number of edge-disjoint paths between two given nodes u and v . (*Hint*: See the notes if you are confused about how to do this.)
 - (c) In order to run your algorithm on the undirected Facebook graph, we need to transform it into a directed graph. How can we accomplish this transformation? (Namely, how should we set our edge capacities, and how should we build our edges in the directed graph?) Briefly justify why the approach you choose will suffice to find edge-disjoint paths in the undirected graph.
 - (d) Now run this version of the algorithm several times (at least 100, preferably 1,000) on the Facebook data, selecting two random nodes each time, to produce a rough average of the number of edge-disjoint paths that exist between two random nodes on the graph.
 - (e) Intuitively, what is the meaning of your answer to part (d)?
9. (a) Consider the contagion examples that we observed in chapter 4 of the notes. Given an *undirected* graph, a set of early adopters S , and a threshold q (such that a certain choice X will spread to a node if more than q fraction of its neighbors are playing it), produce an algorithm that permanently infects the set S of early adopters with X and then runs BRD on the remaining nodes to determine whether, and to what extent, the choice will cascade through the network. (*Note*: “BRD” in this case is simply the process of iteratively deciding whether there is a node that will switch its choice and performing this switch.)
 Once again, turn in your code and verification that your algorithm works on a few simple test cases. In particular, include the output on the two examples from Figure 3.1 in the notes; let S be the set of nodes choosing X in the figure, and, for each of the two graphs, include one example with a complete cascade and one without one (and specify what value of q you used for each).
 - (b) Run your algorithm several (100) times on a fairly small random set of early adopters ($k = 10$) with a low threshold ($q = 0.1$) on the Facebook data set. What happens? Is there a complete cascade? If not, how many nodes end up being “infected” on average?
 - (c) Run your algorithm several (10) times with different values of q (try increments of 0.05 from 0 to 0.5), and with different values of k (try increments of 10 from 0 to 250). Observe and record the rates of “infection” under various conditions. What conditions

on k and q are likely to produce a complete cascade in this particular graph, given your observations?

- * **Bonus Question 2.** (Optional, extra credit awarded depending on quality of solution.) Design an algorithm that, given a graph and a threshold q , finds (an approximation of) the smallest possible set of early adopters that will cause a complete cascade. Try running it on the Facebook data with different values of q and seeing how large a set we need.