Homework Problems for Chapter 4

1. Trapezoid and Simpson's Methods

Given function $f(x) = e^{-x}$, we study different numerical approximations to the integral

$$\int_{0.0}^{0.8} f(x)\,dx.$$

We will use the values of $f(x)$ at the points $0.0, 0.2, 0.4, 0.6, 0.8$. Generate the data set before you start the numerical integration. Use 6-digits accuracy.

(a). Write out the trapezoid rule and compute the numerical integration with 6 digits.

(b). Write out the Simpson's rule and compute the numerical integration with 6 digits.

(c). What is the exact value of the integral? What is the absolute error by using trapezoid and Simpson's rule? Which method is better?

(d). The error formula for the trapezoid rule with $n+1$ points yields

$$E_T(f;h) = -\frac{b-a}{12}h^2 f''(\xi), \quad h = \frac{b-a}{n},$$

for some $\xi \in (a,b)$. The error for Simpson's rule with (2n+1) points yields

$$E_S(f;h) = -\frac{b-a}{180}h^4 f^{(4)}(\xi'), \quad h = \frac{b-a}{2n},$$

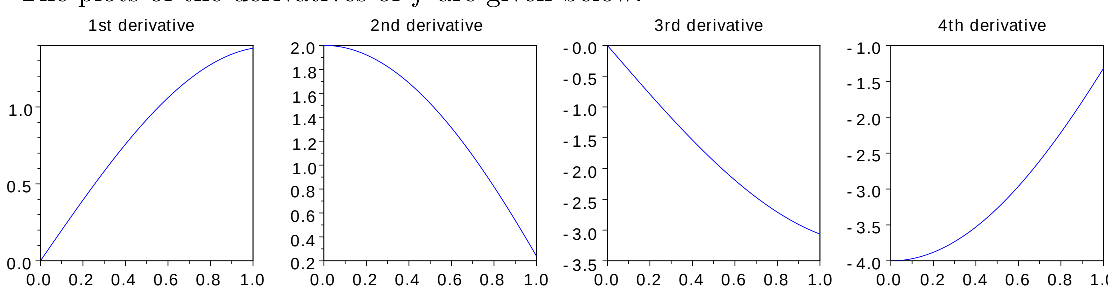
for some $\xi' \in (a,b)$. If we wish the absolute value of the error to be smaller than 10^{-4}, how many points would be needed for each method?

2. Simpson's rule.

Consider the Simpson's rule for the integral $\int_0^1 f(x)\,dx$, with the given data set:

x	0.00	0.25	0.50	0.75	1.00
$f(x)$	0.00	0.06	0.24	0.51	0.84

The plots of the derivatives of f are given below:



(a). Compute the numerical integration using Simpson's rule with $h = 0.25$.

(b) If the error tolerance is 10^{-6}, how many points must one use for Simpson's rule with uniform grid?

3. Trapezoid rule and Romberg algorithm.

(a) Compute the trapezoid rule approximations for the integral $\int_{-1}^{1} 3x^2\, dx$ with $n = 1$ and $n = 2$.

(b) Apply Romberg algorithm to the results in part (a) to get a better value. Did you get the exact value? Would you like to comment on it?

(c) Find positive values w and a such that the following rule is exact for polynomials of degree ≤ 3

$$\int_{-1}^{1} f(x)\, dx \approx w f(-a) + w f(a).$$

4. Trapezoid Rule in Matlab

Preparation: Use the `help` in Matlab to learn how to use the function `feval`.

Write a Matlab function which computes the integral by trapezoid rule. Your function should be used by the following command in Matlab command window:

```
>> v=trapezoid('funItg',a,b,n)
```

where `funItg.m` is the name of the file of the function $f(x)$, and `a,b` is the interval, and `n` is the number of sub-intervals (i.e., n+1 will be the number of points.)

Test your function on problem 1. Write a script that computes for $n = 4, 8, 16, 32, 64, 128$. Compute also the absolute error for each n, and make a plot of the absolute error against n. (Use `loglog` to plot.) How does the error change when n is doubled? Do you expect this from the error estimate? Write your comments.

What to hand in? Submit your function in the file `trapezoid.m`, your script file, the plot of error, and your comments.

5. Simpson's Rule in Matlab

Write a Matlab function which computes the integral by Simpson's rule. Your function should be used by the following command in Matlab command window:

```
>> v=Simpson('funItg',a,b,n)
```

where `funItg.m` is the name of the file of the function $f(x)$, and `a, b` is the interval, and `n` is the number of sub-intervals (i.e., 2n+1 will be the number of points.)

Test your function on problem 1. Write a script that computes, for $n = 2, 4, 8, 16, 32, 128$, the absolute error for each n, and a plot (with `loglog`) of the absolute error against n. How does the error change when n doubled? Compare the results with trapezoid rule and comment.

What to hand in? Your function in `Simpson.m`, your script file, the plot of error, and your comments.

6. Romberg Algorithm in Matlab

Preparation: Use `helpdesk` in Matlab to learn how to use the functions `feval` and `quad`.

(a). Write a Matlab function that computes the Romberg integration. One should be able to call the function by:

```
>> R=romberg('f',a,b,n)
```

where `f` is the name of the function where $f(x)$ is implemented, and `a` and `b` defines the integrating interval, and `n` is the size of your Romberg table. The function should return the whole Romberg table. The best approximation of the interval would be the value in `R(n,n)`.

You may follow the pseudo-code in the lecture notes, or implement your own.

You may check your code against the result of the simulation in Section 4.8, to make sure that your code works.

(b). Use your `romberg` to compute the integrals

$$\text{i)} \qquad \int_0^{\pi} \sin(x)dx \qquad (= 2)$$

$$\text{ii)} \qquad \int_0^{1} \sqrt{x}dx \qquad (= 2/3)$$

Compute also the errors. The exact values of the integrals are given in the parentheses above. Print the errors along the diagonal of the table, and note how it changes along the diagonal of the table. Use `format short e` in Matlab to display the error data.

(c). Explain why Romberg algorithm works poorly for the last integral.

(d). Use Matlab functions `quad` and `quadl` to compute the integrals in b). Use `1e-9` as tolerance for both integrations. Mark your observation.

What to hand in? The Matlab file `romberg.m`, the file for your function `f.m`, a script file that does b) and d), and the Romberg tables your get in b).

7. Numerical Integration and Extrapolation

Consider the function

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Note that f is a continuous function for all t. Let

$$I(x) = \int_0^x f(t)dt, \qquad \text{and} \qquad J = I(1).$$

(a). Compute approximations to J by trapezoid rule with 1,3 and 9 equal intervals. Try to use fewest possible time of computing $f(t)$. Show which formula you use.

(b). Derive a Richardson extrapolation algorithm which uses trapezoid approximations from a), to obtain better approximations to J. Show which formula you use. Generate the corresponding Romberg triangle.

8. Gaussian Quadrature and Beyond

(a). Consider the Gaussian Quadrature rule with 4 points on the interval $[-1, 1]$,

$$\int_{-1}^{1} f(x)\,dx \approx a_1 f(x_1) + a_2 f(x_2) + a_3 f(x_3) + a_4 f(x_4)$$

where

$$\begin{aligned} x_1 &= -\sqrt{\frac{1}{7}(3 - 4\sqrt{0.3})}, \\ x_2 &= -\sqrt{\frac{1}{7}(3 + 4\sqrt{0.3})}, \\ x_3 &= \sqrt{\frac{1}{7}(3 - 4\sqrt{0.3})}, \\ x_4 &= \sqrt{\frac{1}{7}(3 + 4\sqrt{0.3})}, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{1}{2} + \frac{1}{12}\sqrt{\frac{10}{3}}, \\ a_2 &= \frac{1}{2} - \frac{1}{12}\sqrt{\frac{10}{3}}, \\ a_3 &= \frac{1}{2} + \frac{1}{12}\sqrt{\frac{10}{3}}, \\ a_4 &= \frac{1}{2} - \frac{1}{12}\sqrt{\frac{10}{3}}. \end{aligned}$$

Show that the rule is exact for all polynomials of degree ≤ 7.

(b). Construct a rule of the form

$$\int_{-1}^{1} f(x)\,dx \approx a_1 f(-0.5) + a_2 f(0) + a_3 f(0.5)$$

that is exact for all polynomials of degree ≤ 2; that is, determine the values for a_1, a_2, a_3.