CMPSC/Math 451, Numerical Computation

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Chebychev nodes: equally distributing the error

Type I: including the end points.

For interval $[-1,1]$: $\bar{x}_i = \cos(\frac{i}{n}\pi), \quad i = 0, 1, \cdots, n$

For interval $[a,b]$: $\bar{x}_i = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos(\frac{i}{n}\pi), \quad i = 0, 1, \cdots,$

One can show that

$$\max_{a \le x \le b} \left\{ \prod_{k=0}^{n} |x - \bar{x}_k| \right\} = 2^{-n} \le \max_{a \le x \le b} \left\{ \prod_{k=0}^{n} |x - x_k| \right\}$$

where x_k is any other choice of nodes.

Error bound: $$|e(x)| \le \frac{1}{(n+1)!} \left| f^{(n+1)}(x) \right| 2^{-n}.$$

Example Consider the same example with uniform nodes, $f(x) = \sin \pi x$. With Chebyshev nodes, we have

$$|e(x)| \;\le\; \frac{1}{(n+1)!}\pi^{n+1}2^{-n}.$$

The corresponding table for errors:

n	error bound	measured error
4	1.6×10^{-1}	1.15×10^{-1}
8	3.2×10^{-4}	2.6×10^{-4}
16	1.2×10^{-11}	1.1×10^{-11}

The errors are much smaller!

Type II: Chebyshev nodes can be chosen strictly inside the interval $[a, b]$:

$$\bar{x}_i = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{2i+1}{2n+2}\pi\right), \quad i = 0, 1, \cdots, n$$

Discussion:
For large n, polynomials are heavy to deal with.

In general, interpolation polynomials do not converge to the function as $n \to \infty$.

For small intervals, the error with polynomial interpolation is small.

Conclusion: Better to use piecewise polynomial interpolation. – next chapter.