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~~1.~~ Exercise 11.6

1. (a) Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 8 > 1$ (Here $L = 8 > 1$)

Therefore, by the ratio test, the series $\sum_{n=1}^{\infty} a_n$ is divergent

(b) Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.1 < 1$ (Here, $L = 0.8 < 1$)

Therefore, by the Ratio Test, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(c) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

Here $L = 1$

Therefore, the Ratio test is inconclusive, the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge

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2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \ldots$

The sequence $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}$ is a decreasing sequence

Let $a_n = 1/\sqrt{n}$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Hence the series is convergent by alternate series test

Let $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}$

Then

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} 1/n^{1/2}.$$

Now use p-series test

Here, $p = 1/2 < 1$.

By the p series test $\sum_{n=1}^{\infty} |a_n|$ is divergent

So, the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right|$ is divergent.

Therefore the series is conditionally convergent.

3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1} = \frac{(-1)^0}{5(0)+1} + \frac{(-1)^1}{5(1)+1} + \frac{(-1)^2}{5(2)+1} + \frac{(-1)^3}{5(3)+1} + \frac{(-1)^4}{5(4)+1} + \ldots$

$= 1 - \frac{1}{6} + \frac{1}{11} - \frac{1}{16} + \frac{1}{21} - \ldots$

The series is alternative

(i) $b_{n+1} \leq b_n$, $\frac{1}{5n+6} < \frac{1}{5n+1}$ for all n

(ii) $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{5n+1} = \lim_{n \to \infty} \frac{1}{n(5+1/n)}$

$= 0$

Therefore, the series is convergent.

Use limit comparison test,

Consider the series

$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{5n+1} \right| = \sum_{n=0}^{\infty} \frac{1}{5n+1}$

Let $a_n = \frac{1}{5n+1}$

Choose $b_n = 1/n$

Find the value of the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(5n+1)}{1/n}$$

$$= \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{n}{n(5+1/n)} = 1/5$$

> 0

There, the limit exists

$$\Sigma b_n = \sum_{n=1}^{\infty} 1/n$$

Here $p=1$ in this series. Hence the series is divergent because $p \leq 1$ $(p=1)$

The series of absolute values of $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$

is divergent. Hence the series is conditionally convergent.

4. $\Sigma |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1}$

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Note that

$n^3+1 > n^3$ for $n \in I$

$$\frac{1}{n^3+1} < \frac{1}{n^3} \text{ for all } n \in I$$

Note that $\sum_{n=1}^{\infty} 1/n^3$ is convergent and

$$\frac{1}{n^3+1} < 1/n^3 \text{ for all } n \in I$$

Therefore, by comparison test, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \text{ is convergent.}$$

Since the absolute valued series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1} \text{ is convergent}$$

(by comparison test) the series is called

absolutely convergent.

5. $|\sin n| \leq 1$

$$\left|\frac{\sin n}{2^n}\right| = \left|\sin n \cdot \left(1/2^n\right)\right| = |\sin n| \cdot \left|\frac{1}{2^n}\right|$$

$$\leq \left|\frac{1}{2^n}\right| = \left(\frac{1}{2}\right)^n$$

Now use comparison test

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series, having ratio $\frac{1}{2} < 1$.

Since $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

We know that $\sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right|$ and $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

are series of positive terms and $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent.

Then since the terms of the two series satisfy the inequality $\left|\frac{\sin n}{2^n}\right| \leq \left(\frac{1}{2}\right)^n$, the series $\sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right|$ is also convergent

Thus, both the series $\sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right|$ and $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ are convergent, so the series is absolutely convergent.

7. Here $a_n = n/5^n$

Replace n by $n+1$, then

$$a_{n+1} = \frac{n+1}{5^{n+1}} = \frac{n+1}{3^n \cdot 5}$$

Find the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{5^n \cdot 5}}{n/5^n} = \frac{n+1}{5^n \cdot 5} \cdot \frac{5^n}{n} = \frac{n+1}{5n}$$

$$= \frac{n(1+1/n)}{5n} = \frac{1+1/n}{5n}$$

Now take the limit:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{1+1/n}{5} \right|$$

$$= \frac{1+0}{5} = 1/5 = 0.2 < 1$$

Since the limit $L = 0.2 < 1$, the ratio test tells that the given series is convergent.

9. For the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$, get

$$a_n = (-1)^{n-1} \frac{3^n}{2^n n^3}$$

Consider the $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{(n+1)-1}\frac{3^{n+1}}{2^{n+1}(n+1)^3}}{(-1)^{n-1}\frac{3^n}{2^n n^3}}\right|$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{2^{n+1}(n+1)^3} \cdot \frac{2^n n^3}{3^n}$$

$$= \lim_{n\to\infty} \frac{3\cdot 3^n}{2\cdot 2^n(n+1)^3} \cdot \frac{2^n n^3}{3^n} =$$

$$\frac{3}{2} \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \frac{3}{2} \lim_{n\to\infty} \frac{n^3}{n^3(1+1/n)^3}$$

$$= \frac{3}{2} \lim_{n\to\infty} \frac{1}{(1+1/n)^3} = \frac{3}{2} \frac{1}{(1+0)^3}$$

$$= \frac{3}{2}\cdot 1 = \frac{3}{2} > 1$$

Since $L > 1$, by Ratio test the series

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$ is divergent

11. $a_k = 1/k!$. Then $a_{k+1} = \frac{1}{(k+1)!}$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left(\frac{1/(k+1)!}{1/k!}\right)$$

$$= \lim_{k\to\infty}\left(\frac{k!}{(k+1)!}\right) = \lim_{k\to\infty}\left(\frac{k!}{(k+1)(k!)}\right)$$

$$= \lim_{k\to\infty}\left(\frac{1}{k+1}\right) = \frac{1}{\infty} = 0 < 1$$

As $L < 1$, By the Ratio test the given series converges.

19. Let $a_n = \dfrac{n^{100}\,100^n}{n!}$

Then

$$a_{n+1} = \frac{(n+1)^{100}\,100^{n+1}}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(n+1)^{100}\,100^{n+1}}{(n+1)!}}{\dfrac{n^{100}\,100^n}{n!}}\right|$$

$$= \left|\frac{(n+1)^{100}\,100^{n+1}}{(n+1)!} \times \frac{n!}{n^{100}\,100^n}\right|$$

$$= \left|\frac{(n+1)^{100}\,100^n \cdot 100}{(n+1)n!} \times \frac{n!}{n^{100}\,100^n}\right|$$

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$$= \left| \frac{(n+1)^{100}\,100}{(n+1)} \times 1/n^{100} \right| = \left| \frac{100}{n+1} \left(\frac{n+1}{n} \right)^{100} \right|$$

$$= \left| \frac{100}{n+1} \cdot \left(\frac{n+1}{n} \right)^{100} \right|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{100}{n+1} \left(1 + 1/n \right)^{100} \right|$$

$$= \left| \lim_{n\to\infty} \frac{100}{n+1} \lim_{n\to\infty} \left(1+1/n \right)^{100} \right|$$

$$= 0 \cdot 1^{00} = 0$$

Because the limit $L = 0 < 1$, by the ratio test, the series $\sum_{n=1}^{\infty} \frac{n^{100}\,100^n}{n!}$ converges.

21. Rewrite the series as $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Assume $a_n = (-1)^{n-1} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Replace n by $n+1$

$$a_{n+1} = (-1)^{n+1-1} \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2(n+1)-1)}$$

$$= (-1)^n \frac{(n+1)!}{1\cdot 3\cdot 5 \cdots (2n+1)}$$

$$= (-1)^n \frac{(n+1)(n)!}{1\cdot 3\cdot 5 \cdots (2n+1)}$$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n \cdot \dfrac{(n+1)(n)!}{1\cdot 3\cdot 5 \cdots (2n+1)}}{(-1)^{n-1} \dfrac{n!}{1\cdot 3\cdot 5 \cdots (2n-1)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-\dfrac{(n+1)(n)!}{1\cdot 3\cdot 5 \cdots (2n+1)}}{\dfrac{n!}{1\cdot 3\cdot 5 \cdots (2n-1)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(n)!}{1\cdot 3\cdot 5 \cdots (2n+1)} \times \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)}{(2n+1)} \right| = \lim_{n\to\infty} \left| \frac{n\left(1+1/n\right)}{n\left(2+1/n\right)} \right|$$

$$= \frac{\left(1 + \lim_{1/n\to 0} \left(1/n\right)\right)}{\left(2 + \lim_{1/n\to 0} \left(1/n\right)\right)} = \frac{1+0}{2+0}.$$

$= 1/2 < 1$

Since $L < 1$, the series $\sum a_n$ is convergent absolutely by the Ratio Test.

25. Let $a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n}$$

$$= \lim_{n\to\infty} \left[\left(\frac{n^2+1}{2n^2+1}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{n^2+1}{2n^2+1}\right) = \lim_{n\to\infty} \left(\frac{n^2(1+1/n^2)}{n^2(2+1/n^2)}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1+1/n^2}{2+1/n^2}\right) = \frac{1+0}{2+0} = 1/2 < 1$$

Thus the series $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

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29. $a_n = (1+1/n)^{n^2}$

Now,

$\sqrt[n]{|a_n|} = (1+1/n)^n$

Evaluate the value of $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (1+1/n)^n = e$

So,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (e^n) = \infty$

Therefore, the series is divergent.

30. $a_n = (\arctan n)^n$

$\sqrt[n]{|a_n|} = (\arctan n)$

As, $n\to\infty$ the limit is

$\lim_{n\to\infty} (\arctan n)^n = \pi/2 > 1$

Therefore the series is divergent.

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21. Using alternative series test.

Let $a_n = \frac{1}{\ln n}$,

And

$$n < n+1$$

$$1/n > \frac{1}{n+1}$$

$$\frac{1}{\ln(n)} > \frac{1}{\ln(n+1)}$$

That is, $b_{n+1} < b_n$. Therefore, the series is monotonously decreasing.

And the limit is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n)} = \frac{1}{\lim_{n\to\infty} \ln(n)} = 1/\infty$$

$= 0$

Thus, by the alternating test series the series converges

From the series,

$$|a_n| = \left|\frac{1}{\ln n}\right|$$

From the result,

$\ln(n) < n$

$$\frac{1}{\ln(n)} > 1/n$$

That is,

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} 1/n$$

Now, the series $\sum_{n=2}^{\infty} 1/n$ is a divergent by the p-series test with $p=1$.

Therefore, the series $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ is also divergent by the comparison test.

From the definition, given series is conditionally convergent.

33. Using ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-9)^{n+1}}{(n+1)\cdot 10^{n+2}}}{\frac{(-9)^n}{n\cdot 10^{n+1}}}\right|$$

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$$= \left| \frac{(-9)^{n+1}}{(n+1)\cdot 10^{n+2}} \times \frac{n\cdot 10^{n+1}}{(-9)^n} \right|$$

$$= \left| \frac{(-9)}{(n+1)\cdot 10} \times \frac{n}{1} \right| = \frac{9}{10} \left| \frac{n}{n+1} \right|$$

$$= \frac{9}{10} \left| \frac{1}{1+1/n} \right| \quad 9/10, \ n \to \infty$$

As $9/10 < 1$

Hence the series is absolutely convergent and convergent.

35. Using the Root Test

$a_n = (n/\ln n)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{n}{\ln n}\right)^n \right|}$$

$$= \lim_{n\to\infty} \left[\left(\frac{n}{\ln n}\right)^n \right]^{1/n} = \lim_{n\to\infty} \frac{n}{\ln n}$$

The limit is an indeterminate form. So use L'Hospital Rule.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{d/dn(n)}{d/dn(\ln n)} = \lim_{n\to\infty} 1/(1/n)$$

$= \lim_{n\to\infty} n = \infty$

So, $\infty > 1$. Hence by the Root Test, the series $\sum_{n=2}^{\infty} \left(\frac{n}{\ln n}\right)^n$ is divergent.

43. (a) $a_n = 1/n^3$, $a_{n+1} = \frac{1}{(n+1)^3}$

Evaluate $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n^3}{(n+1)^3}\right|$

$= \lim_{n\to\infty} \left|\frac{n^3}{n^3(1+1/n)^3}\right| = \lim_{n\to\infty} \left|\frac{n^3}{n^3(1+1/n)^3}\right|$

$= \lim_{n\to\infty} \left|\frac{1}{(1+1/n)^3}\right| = \frac{1}{(1+0)^3} = 1$.

Therefore, the Ratio test is inconclusive

(b) $a_n = n/2^n$, $a_{n+1} = \frac{n+1}{2^{n+1}}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n+1}{2^{n+1}}}{n/2^n}\right|$

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$$\lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \frac{1}{2}\left|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\right|$$

$$= \frac{1}{2}\left|\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right| = 1/2\,(1+0)$$

$$= 1/2 < 0$$

Hence the series $\sum_{n=1}^{\infty} n/2^n$ is convergent.

i) $a_n = \frac{(-3)^{n-1}}{\sqrt{n}}$, $a_{n+1} = \frac{(-3)^n}{\sqrt{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-3)^n}{\sqrt{n+1}}}{\frac{\sqrt{n}}{(-3)^{n-1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-3}{\sqrt{n+1}}\cdot\sqrt{n}\right| = 3\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= 3\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}\cdot\sqrt{1+1/n}} = 3\lim_{n\to\infty}\frac{1}{\sqrt{1+1/n}}$$

$$= 3\cdot\frac{1}{\sqrt{1+0}} = 3 > 1$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ is divergent

(d) $a_n = \frac{\sqrt{n}}{1+n^2}$, $a_{n+1} = \frac{\sqrt{n+1}}{1+(n+1)^2}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\sqrt{n+1}}{1+(n+1)^2}}{\frac{\sqrt{n}}{1+n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{1+(1+n)^2} \cdot \frac{1+n^2}{\sqrt{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{n}\sqrt{1+1/n}}{n^2\left(1/n^2 + (1/n+1)^2\right)} \cdot \frac{n^2(1/n^2+1)}{\sqrt{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{1+1/n} \cdot (1/n^2+1)}{1/n^2 + (1/n+1)^2} \right|$$

$$= \left| \frac{\sqrt{1+0} \cdot (0+1)}{0+(0+1)^2} \right| = 1$$

Therefore, the Ratio Test is inconclusive

Exercise 11.7

1. $a_n = \dfrac{n^2-1}{n^3+1}$

The comparison series for the Limit Comparison Test,

$b_n = n^2/n^3 = 1/n$

Since $\sum a_n$ and $\sum b_n$ are series of positive terms.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2-1}{n^3+1}}{1/n} = \lim_{n\to\infty} \frac{n^2-1}{n^3+1} \times n$$

$$= \lim_{n\to\infty} \frac{n^3-n}{n^3+1} = \lim_{n\to\infty} \frac{n^3(1-1/n^3)}{n^3(1+1/n^3)}$$

$= 1 > 0$

By p-series test, the series $\sum 1/n$ is divergent since $p > 1$

Therefore, by Limit Comparison Test, the series $\sum a_n = \sum \frac{n^2-1}{n^3+1}$ is also divergent.

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2. Using comparison test,

Since,

$$\frac{n-1}{n^3+1} < n/n^3 = 1/n^2$$

And, $\sum 1/n^2$ is converging because $p = 2$ which is greater than 1.

Therefore, the series is convergent.

3. Use alternating series test

$$a_n = \frac{n^2-1}{n^3+1}$$

For all $n \geq 2$

$$n^3 > n^2$$

$$n^3+1 > n^2-1$$

Also,

$$n^2 > 1$$

Thus, the sequance a_n is a positive and monotonically decreasing sequence for $n \geq 2$

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$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^3+1}$$

$$= \lim_{x\to\infty} \left(\frac{\frac{n^2}{n^3} - 1/n^3}{n^3/n^3 + 1/n^3} \right)$$

$$= \lim_{x\to\infty} \left(\frac{1/n - 1/n^3}{1 + 1/n^3} \right) = 0$$

Hence, the second condition holds.
It can be concluded that the provided series converges.

$$\left| \sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$$

Now assume $a_n = \frac{n^2-1}{n^3+1}$

Notice that,

$$\frac{n^2-1}{n^3+1} < \frac{n^2}{n^3} = 1/n \; \cdots \text{(i)}$$

Use the limit comparison test:
$\sum 1/n$ is not convergent because $p > 1$

(by p-series test).

Hence, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1}$ is

conditionally convergent.

4. Let $a_n = (-1)^n \frac{n^2-1}{n^2+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1}$

$\lim_{n\to\infty} (-1)^n \frac{1-1/n^2}{1+1/n^2} = (-1)^n \frac{1-0}{1+0}$

$= (-1)^n \neq 0$.

Because $\lim_{n\to\infty} a_n \neq 0$, so by the Divergent test the series diverges.

5. Using ratio test,

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(e)^{n+1}}{(n+1)^2}}{\frac{e^n}{n^2}}\right|$

$= \frac{(e)^n \cdot e}{(n+1)^2} \times \frac{n^2}{e^n} = (e)\left(\frac{n}{n+1}\right)^2$

$$= e\left(\frac{1}{1+1/n}\right)$$

Since, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = e > 1$

Therefore, the series is divergent.

6. $a_n = \frac{n^{2n}}{(1+n)^{3n}} = \left[\frac{n^2}{(1+n)^3}\right]^n$

Now by root test

$$p = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$$

$$= \lim_{n\to\infty} \left[\frac{n^2}{(1+n)^3}\right]^{n/n} = \lim_{n\to\infty} \frac{n^2}{(1+n)^3}$$

$$p = \lim_{n\to\infty} \frac{(n^2/n^3)}{\left(\frac{(1+n)^3}{n^3}\right)}$$ [Divide by num. and den. by n^3]

$$= \lim_{n\to\infty} \frac{(1/n)}{\left[\frac{n+1}{n}\right]^3} = \lim_{n\to\infty} \frac{0}{[1+0]^3}$$

$$= 0 < 1$$

Hence by root test provided series converges.

8. Using ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n \frac{(n+1)^4}{4^{n+1}}}{(-1)^{n-1} \frac{n^4}{4^n}}\right|$$

$$= \frac{(-1)^n . (n+1)^4}{4^n . 4} \times \frac{4^n}{(-1)^n . n^4}$$

$$= 1/4 \times \frac{(n+1)^4}{n^4} = 1/4 \times \left(\frac{n+1}{n}\right)^4$$

$$= 1/4 \times (1 + 1/n)^4$$

Since,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 1/4 \times (1 + 1/n)^4$$

$$= 1/4 < 1$$

Therefore, the series is convergent

9. Using ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} \frac{\pi^{2(n+1)}}{(2(n+1))!}}{(-1)^n \frac{\pi^{2n}}{(2n)!}}\right|$$

$$= \frac{(-1)^n \cdot (-1) \cdot \pi^{2n} \cdot \pi^2}{(2n+2)(2n+1) \cdot 2n!} \times \frac{(2n)!}{\pi^{2n}(-1)^n}$$

$$= \frac{1 \cdot \pi^2}{(2n+2)(2n+1)} \times \frac{1}{1} = \pi^2 \left(\frac{1}{4n^2+6n+2} \right)$$

$0, n \to \infty$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

Therefore, the series is convergent.

10. Use the integral test

Consider $f(x) = x^2 e^{-x^3}$ where f is continuous and positive for all x

$$f'(x) = 2xe^{-x^3} + (-3x^2) x^2 e^{-x^3}$$

$$= 2xe^{-x^3} - 3x^4 e^{-x^2} = (2 - 3x^3) x e^{-x^3} < 0$$

for $x \geq 1$

Thus, the function f is a decreasing function

Consider $x^3 = t$

$3x^2 dx = dt$

And, $t \to \infty$ as $x \to \infty$

Then, $\int_1^\infty x^2 e^{-x^3} dx = \int_1^\infty e^{-t} \frac{dt}{3}$

$= \frac{1}{3} \lim_{y\to\infty} \int_1^y e^{-t}\,dt = \frac{1}{3} \lim_{y\to\infty} [-e^{-t}]_1^y$

$= \frac{1}{3} \lim_{y\to\infty} [-e^{-y} + e^{-1}] = \frac{1}{3}[0 + e^{-1}] = \frac{1}{3e}$

Thus, $\int_1^\infty x^2 e^{-x^3}\,dx$ is convergent and the series is also convergent.

11. $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{1}{3^n}$

We know that by p-series test

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent with $p = 3 > 1$

And by geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent with $r = \frac{1}{3} < 1$

Hence the given series $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$ is convergent.

18. Use the alternating series test and verify conditions (1) and (2)

The series $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}-1}$ is an alternati series, with $b_n = \dfrac{1}{\sqrt{n}-1}$

Consider

$$\sqrt{n} < \sqrt{n+1}$$
$$\sqrt{n}-1 \leq \sqrt{n+1}-1$$

$$\frac{1}{\sqrt{n+1}-1} \leq \frac{1}{\sqrt{n}-1}$$

$b_{n+1} \leq b_n$ for all n

Hence, the first condition of the Alternating Series Test is satisfied.

Now find the value $\lim_{n\to\infty} b_n$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = \frac{1}{\lim_{n\to\infty}(\sqrt{n}-1)} = 1/\infty$$

$= 0$

Thus, the second condition is satisfied.

Therefore, by the Alternating Series test the given series is convergent.

21. Using alternate series test

Let $\sum (-1)^n u_n = \sum (-1)^n \cos(1/n^2)$

$\Rightarrow u_n = \cos(1/n^2)$

For all $n = 1, 2, 3, \dots, \infty$, that implies
$0 \le 1/n^2 < 1 \Rightarrow 1/n^2 \in$ First quadrant

Therefore, $\cos(1/n^2) > 0 \Rightarrow u_n > 0$

$u_n = \cos(1/n^2) \Rightarrow u_n' = -\left[\sin(1/n^2)\right]\left(\frac{-2}{n^3}\right)$

$= 2/n^3 \sin(1/n^2) > 0 \Rightarrow u_n' > 0$

Therefore $\{u_n\}$ is an increasing sequence

Consider,

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\cos(1/n^2)\right) = \cos 0 = 1$

As $\{u_n\}$ is an increasing sequence but $\lim_{n\to\infty} u_n \neq 0$

Hence, by alternating series test the given series diverges

23. Use the Limit Comparison Test with $a_n = \tan(1/n)$ and $b_n = 1/n$ as follows:

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{\tan(1/n)}{(1/n)}$$

Let $1/n = \theta$ then $\theta \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} a_n/b_n = \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta/\cos\theta}{\theta}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta\cdot\cos\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{\cos\theta}$$

$$= (1)\cdot\left(\frac{1}{\cos(0)}\right) = (1)\cdot(1) = 1$$

From the resutt; $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$

Since, this limit exists and $\sum_{n=1}^{\infty} 1/n$ is a divergent p-series with $p=1$

The given series $\sum_{n=1}^{\infty} \tan(1/n)$ divergence by the limits comparison Test.

26. Use the ratio test, with $a_n = \frac{n^2+1}{5^n}$

We have $\frac{a_{n+1}}{a_n} = \frac{[(n+1)^2+1]}{5^{n+1}} \cdot \frac{5^n}{(n^2+1)}$

$$= \frac{[(n+1)^2+1]}{5(n^2+1)} = \frac{(1+1/n)^2 + 1/n^2}{5(1+1/n^2)} \to 1/5$$

as $n \to \infty$

So $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1/5 < 1$.

Thus, by the ratio test, the given series converges.

37. This is a form of $\sum a_n^n$, So use the root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (\sqrt[n]{2} - 1)$$

That is,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (2^{1/n} - 1) = 2^0 - 1 = 0$$

Clearly, $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

Hence, the series

$\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$ is convergent

Exercise 11.8

1. A power series is an infinite series is of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$

where, x is a variable and $\{c_n\}$ are constants called the coefficients of the series.

The sum of the series is a function
$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots c_n x^n +$
$\ldots$ whose domain is the set of all x for which the series converges.

If $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$

Generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

is called a power series centered at a.

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2. (A) The radius of convergence R is a positive number such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. We can find R generally by the ratio test.

(B) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

$$|x-a|<R$$

$$-R<x-a<R$$

$$a-R<x<a+R$$

At the endpoints of the interval ($x = a \pm R$) the series may converge or diverge.

3. Apply the ratio test.

$a_n = (-1)^n n x^n$

Then $a_{n-1} = (-1)^{n+1}(n+1)x^{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(-1)^{-n}(n+1)x^{n+1}x^{-n}}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(n+1)x}{n}\right|$$

$$= \lim_{n\to\infty} \left| (-1)\left(\frac{n+1}{n}\right) x \right|$$

$$= \lim_{n\to\infty} |(-1)| \left| \left(\frac{n+1}{n}\right) \right| |x|$$

$$= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right) \right| |x|$$

$$= \left| \left(1 + 1/\infty\right) \right| |x| = |(1+0)| |x|$$

$$= |x|$$

Using ratio test, the series is convergent, when $|x| < 1$

That means, $-1 < x < 1$

And also, the radius of convergence is $R = 1$

Now, check the convergence at end points of the interval $(-1, 1)$

If, $x = 1$, then the series becomes as follows.

$$\sum_{n=1}^{\infty} (-1)^n n x^n = \sum_{n=1}^{\infty} (-1)^n n (-1)^n = \sum_{n=1}^{\infty} (-1)^{2n} n$$

Here, the nth term is $b_n = (-1)^{2n} n$

Then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left((-1)^{2n} n\right)$$

$$= \lim_{n\to\infty} (n) \cdot \lim_{n\to\infty} (-1)^{2n} = \infty$$

So, by the limit test, the series $\sum_{n=1}^{\infty} (-1)^n nx^n$ is divergent

If, $x = 1$

$$\sum_{n=1}^{\infty} (-1)^n nx^n = \sum_{n=1}^{\infty} (-1)^n n(1)^n = \sum_{n=1}^{\infty} (-1)^n n$$

Here, the nth term is $b_n = (-1)^n n$

Let $n = 2m$

$$\sum_{n=1}^{\infty} (-1)^n n = \sum_{n=1}^{\infty} (-1)^{2m} (2m)$$

$$= \sum_{n=1}^{\infty} (-1)^{2m} (2m) = 2 \sum_{n=1}^{\infty} (-1)^{2m} m = \infty$$

Now, for even integer, the nth term becomes as $b_n = (-1)^n n$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left((-1)^{2n} n\right)$$

$$= \lim_{n\to\infty} (n) \cdot \lim_{n\to\infty} (-1)^{2n} = \infty$$

Let $n = 2m+1$

$$\sum_{n=1}^{\infty} (-1)^n n = \sum_{n=1}^{\infty} (-1)^{2m+1} (2m+1)$$

So, by the ~~lo~~ limit divergent criteria test, the series is divergent.

Hence, the Series $\sum_{n=1}^{\infty} (-1)^n \cancel{n} x^n$ is convergent in the interval $(-1, 1)$

4. Use ratio test

Take $a_n = \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}} \times \frac{\sqrt[3]{n}}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n \cdot (-1) x^n \cdot x}{\sqrt[3]{n+1}} \times \frac{\sqrt[3]{n}}{(-1)^n \cdot x^n} \right|$$

$$= \lim_{n\to\infty} \left| -x \times \left(\frac{n}{n+1}\right)^{1/3} \right|$$

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$$= \lim_{n\to\infty} \left| -x \times \left(\frac{n}{n(1+1/n)} \right)^{1/3} \right|$$

$$= \lim_{n\to\infty} \left| -x \times \left(\frac{1}{1+1/n} \right)^{1/3} \right| = |x|$$

By ratio test, series converges for $|x|<1$
Hence the radius of convergence of the given series is 1.
Clearly from the above result series converges for $|x|<1$

Take $x=-1$

Then,

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^{1/3}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^{1/3}} = \sum_{n=1}^{\infty} 1/n^{1/3}$$

This is in the form of $1/n^p$ [p-test]
if $p>1$ series converges and if $p\le 1$ series diverges

Hence the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ diverges

Take $x = 1$

Then,

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n^{1/3}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$$ which is convergent by alternate series test

Hence the interval of convergence of the given series is $(-1, 1]$

7. If $a_n = \frac{x^n}{n!}$, then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1}$$

And $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim \frac{1}{n+1} = 0 < 1$$

for all x

Thus by the ratio test, the given series converges for all values of x. Therefore, the radius of convergence $R = \infty$, and the interval of convergence is $(-\infty, \infty)$

9. Using ratio test

$$a_n = \frac{x^n}{n^4 4^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^4 4^{n+1}}}{\frac{x^n}{n^4 4^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^4 4^{n+1}} \times \frac{n^4 4^n}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{4} \cdot \frac{n^4}{(n+1)^4}\right| = \lim_{n\to\infty}\left|\frac{x}{4} \cdot \frac{n^4}{n^4(1+1/n)^4}\right|$$

$$= \lim_{n\to\infty}\left|x/4 \cdot \frac{1}{(1+1/n)^4}\right|$$

$$= \left|\frac{x}{4} \cdot \frac{1}{(1+0)^4}\right| = \left|\frac{x}{4}\right|$$

By the Ratio Test, the above series converges if $\left|\frac{x}{4}\right| < 1$

That is $|x| < 4$

That is $-4 < x < 4$

Thus, the radius of convergence is $R = 4$

The series converges in the interval $(-4, 4)$

Now test for convergence at the endpoints

If $x = 4$, then the series becomes,

$$\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n}{n^4 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

It converges by the alternating series test, as $1/n^4$ is a decreasing sequence converges to 0.

If $x = 4$, then the series becomes.

$$\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n} = \sum_{n=1}^{\infty} \frac{(4)^n}{n^4 4^n} = \sum_{n=1}^{\infty} 1/n^4$$

This is convergent by p - series Test with $p = 4 > 1$.

Therefore, the given power series converges with $-4 \leq x \leq 4$

Thus, the interval of convergence is $[-4, 4]$

11. Let $a_n = \frac{(-1)^n 4^n x^n}{\sqrt{n}}$.

Then $a_{n+1} = \frac{(-1)^{n+1} 4^{n+1} x^{n+1}}{\sqrt{n+1}}$

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So $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}4^{n+1}x^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n 4^n x^n}{\sqrt{n}}}\right|$

$= \left|\frac{(-1)^{n+1}4^{n+1}x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n 4^n x^n}\right|$

$= \left|4x\sqrt{\frac{n}{n+1}}\right| = 4\frac{1}{\sqrt{1+1/n}}|x|$

Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 4\frac{1}{\sqrt{1+1/n}}|x|$

$= 4|x|$ As $\lim_{n\to\infty} 1/n = 0$

By the ratio test, the given series converges if $4|x| < 1$ and diverges if $4|x| > 1$

So, the given series converges if $|x| < 1/4$ and diverges if $|x| > 1/4$

Therefore, the radius of convergence is $R = 1/4$

Now ; $|x| < 1/4 \iff -1/4 < x < 1/4$

Thus, the series converges in the interval $(-1/4, 1/4)$

If $x = -1/4$, then the series becomes

$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} (-1/4)^n = \sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by

p-series test with $p = 1/2 < 1$

If $x = 1/4$, then the series becomes

$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} (1/4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges

by the Alternating Series Test with $\lim_{n\to\infty} 1/\sqrt{n}$

$= 0$

Hence, the given series converges when

$-1/4 < x \leq 1/4$

Therefore, the interval of convergence is

$(-1/4, 1/4]$

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12. Use ratio test

$$a_n = \frac{(-1)^{n-1} x^n}{n \cdot 5^n}$$

Then,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{(n+1)5^{n+1}} x^{n+1}}{\frac{(-1)^{n-1} x^n}{n \cdot 5^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n}{(n+1)5^{n+1}} x^{n+1} \cdot \frac{n 5^n}{(-1)^{n-1} x^n} \right|$$

$$= \lim_{n\to\infty} \left| -\frac{n}{(n+1)5} x \right| = \left| \frac{x}{5} \right| \lim_{n\to\infty} \left| \frac{1}{(1+1/n)} \right| = \left| \frac{x}{5} \right| \left| \frac{1}{(1+0)} \right| = \left| \frac{x}{5} \right|$$

Therefore, the series converges for $|x| < 5$ and diverges if $|x| > 5$

Thus, the radius of the convergence is $R = 5$

Now, test for convergence at $(-5, 5)$

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If $x = -5$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 5^n} (-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$$

$$= -\sum_{n=1}^{\infty} 1/n$$

Observe that it is P- series with $(p = 1 \leq 1)$ so the series $\sum_{n=1}^{\infty} \frac{(-1)}{n}$ diverges.

Therefore, the series $\sum_{n=1}^{\infty} 1/n$ diverges at $x = -5$

If $x = 5$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} (5)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Assume $b_n = 1/n$

1. Since $\frac{1}{n+1} < 1/n$ so $b_{n+1} \leq b_n$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n = 0$

By Alternating series test converges at intervals $(-5, 5]$.

14. Let $a_n = \dfrac{x^{2n}}{n!}$ then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}}\right| =$$

$$\left|\frac{x^{2n}\cdot x^2}{(n+1)\cdot n!} \times \frac{n!}{x^{2n}}\right| = \left|\frac{x^2}{(n+1)}\right|$$

$$= |x^2| \left|\frac{1}{n+1}\right| \quad 0, n \to \infty$$

Therefore, the radius of convergence is ∞ and the interval is $(-\infty, \infty)$

15. Use ratio test

$$a_n = \frac{(x-2)^n}{n^2+1}$$

Replace n by n+1

$$a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2+1} = \frac{(x-2)^{n+1}}{n^2+1+2n+1}$$

$$= \frac{(x-2)^{n+1}}{n^2+2n+2}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(x-2)^{n+1}}{n^2+2n+2}}{\frac{(x-2)^n}{n^2+1}}$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n^2+2n+2} \times \frac{n^2+1}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty} |x-2| \frac{n^2+1}{n^2+2n+2}$$

$$= |x-2| \cdot \lim_{n\to\infty}\left(\frac{1+1/n^2}{1+\frac{2}{n}+\frac{2}{n^2}}\right)$$

$$= |x-2|$$

$$-1 < x-2 < 1$$
$$-1+2 < x-2+2 < 1+2$$
$$1 < x < 3$$

Thus, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ converges if

$1 < x < 3$

Hence, the Radius of Convergence of the Series is $R = 1$

Take $x = 1$

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Assume $b_n = \frac{1}{n^2+1}$

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$

Therefore, by an Alternating series test the series is convergent at $x = 1$

Take $x = 3$

Thus the series $= \sum_{n=0}^{\infty} \frac{1}{n^2+1}$

Since $\frac{1}{n^2+1} \leq 1/n^2$

The series $\sum_{n=0}^{\infty} 1/n^2$ is convergent

by P-series test with $P = 2 > 1$

Thus, the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ is also

convergent by the Comparison Test

So, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ converges at

$x=3$

Hence, the interval of convergence is $[1,3]$

17. Let $a_n = \frac{(x+2)^n}{2^n \ln n}$

$a_{n+1} = \frac{(x+2)^{n+1}}{2^{n+1} \ln(n+1)}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(x+2)^{n+1}}{2^{n+1} \ln(n+1)}}{\frac{(x+2)^n}{2^n \ln n}} \right|$$

$$= \left| \frac{(x+2)^n \cdot (x+2)}{2^n \cdot 2 \cdot \ln(n+1)} \times \frac{2^n \ln n}{(x+2)^n} \right|$$

$$= \left| \frac{(x+2)}{2 \cdot \ln(n+1)} \times \frac{\ln n}{1} \right|$$

$$= \frac{|x+2|}{2} \left| \frac{\ln n}{\ln(n+1)} \right|$$

$$= \frac{|x+2|}{2} \left| \frac{\ln n}{\ln\left(n\left(1+1/n\right)\right)} \right|$$

$$= \frac{|x+2|}{2} \left| \frac{\ln n}{\ln n + \ln\left(1+1/n\right)} \right|$$

$$= \frac{|x+2|}{2} \left| \frac{\ln n}{\ln n\left(1 + \frac{\ln(1+1/n)}{\ln n}\right)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x+2|}{2} \lim_{n\to\infty}$$

$$\left| \frac{1}{1 + \frac{\ln(1+1/n)}{\ln n}} \right| = \frac{|x+2|}{2} \left| \frac{1}{1+\frac{\ln(1+0)}{\infty}} \right|$$

$$= \frac{|x+2|}{2}$$

$$\frac{|x+2|}{2} < 1 \Rightarrow |x+2| < 2$$

$-2 < x+2 < 2$
$-4 < x < 0$

Check the convergence at end points of the interval $-4 < x < 0$

At $x = -4$,

$$\sum_{n=2}^{\infty} \frac{(-4+2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$$

Let $a_n = (-1)^n \frac{1}{\ln n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{1}{\ln n} = 0$$

The series converges at $x = -4$ as the limit is 0 as $n \to \infty$

At $x = 0$,

$$\sum_{n=2}^{\infty} \frac{(2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

By, the comparison test, the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges

Therefore, the radius of convergence is 2 and the interval is $[-4, 0)$

19. $a_n = \dfrac{(x-2)^n}{n^n}$, $|a_n| = \left|\dfrac{(x-2)^n}{n^n}\right|$

Use root test

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^n}\right|}$$

$$= \lim_{n\to\infty} \left|\frac{(x-2)}{n}\right| = |x-2| \lim_{n\to\infty} \left|\frac{1}{n}\right|$$

$$L = |x-2| \cdot 0 = 0$$

$$L = 0 \; ; \; L < 1$$

The above series is convergent for all real numbers by root test.
The radius of convergence is $R = \infty$.
The interval of convergence is $(-\infty, \infty)$.

23. First, use ratio test

Let $a_n = n!(2x-1)^n = (n+1)!\,(2x-1)^{n+1}$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n}\right|$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)n!}{n!} \right) |2x-1| =$$

$$\lim_{n\to\infty} (n+1)|2x-1| = \infty$$

By the Ratio Test the series diverges when $2x-1 \neq 0$

Thus, the series converges only when $2x-1=0$

$\Rightarrow$ when $x = 1/2$

Therefore, the series converges at $x = 1/2$

and radius of convergence is $R = 0$

30. (a) If compared with the series $\sum_{n=0}^{\infty} c_n =$ $\sum_{n=0}^{\infty} c_n(1)^n$ with $\sum_{n=0}^{\infty} c_n x^n$, then $x = 1$

As $-4 < x < 4$, the series $\sum_{n=0}^{\infty} c_n (1)^n$ converges.

Therefore, the series $\sum_{n=0}^{\infty} c_n$ is convergent.

(b) If compared with the series $\sum_{n=0}^{\infty} c_n 8^n$ with $\sum_{n=0}^{\infty} c_n x^n$, then $x = 8$

As $x \geq 6$, the series $\sum_{n=0}^{\infty} c_n 8^n$ diverges.

Therefore, the series $\sum_{n=0}^{\infty} c_n 8^n$ is divergent.

(c) If compared with the series $\sum_{n>0}^{\infty} c_n(-3)^n$

with $\sum_{n=0}^{\infty} c_n x^n$, then $x = -3$.

As $-4 < x < 4$, the series $\sum_{n>0}^{\infty} c_n(-3)^n$ converges.

Therefore, the series $\sum_{n>0}^{\infty} c_n(-3)^n$ is convergent.

(d) If compared with the series $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

$= \sum_{n=0}^{\infty} c_n(-9)^n$ with $\sum_{n=0}^{\infty} c_n x^n$, then $x = -9$

As $x < -6$, the series $\sum_{n>0}^{\infty} c_n(-9)^n$ diverges.

Therefore, the series $\sum_{n>0}^{\infty} (-1)^n c_n 9^n$ is divergent.

Exercise 11.9

1. If the power series $\sum c_n(x-a)^n$ has the radius of convergence $R>0$ then the function f defined by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ has the radius of convergence R.

Here $f'(x) = \sum_{n>1}^{\infty} nc_n x^{n-1}$

The radius of convergence of the series

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$\sum_{n=0}^{\infty} C_n x^n$ is 10, so the derivative must have the same radius of convergence 10. Therefore, the radius of convergence of the series $\sum_{n=1}^{\infty} n C_n x^{n-1}$ is 10.

2. Given that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$

Again, $\int \sum_{n=0}^{\infty} b_n x^n dx = \sum_{n=0}^{\infty} b_n \int x^n dx$

$= \sum_{n=0}^{\infty} b_n \frac{x^{n+1}}{n+1}$

Therefore, the integration of this power series $\sum_{n=0}^{\infty} b_n \frac{x^{n+1}}{n+1}$ must converge for $|x| < 2$, but it might not have the same interval of convergence.

3. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$

Replace x by $-x$ in the above series and simplify.

$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$.

Thus, the required power series representatio
for the function $f(x) = \frac{1}{1+x}$ is,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$$

Apply Ratio Test to series $\sum_{n=0}^{\infty} (-1)^n x^n$

$a_n = (-1)^n x^n$
$a_{n+1} = (-1)^{n+1} x^{n+1}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} |x| = |x|$$

By the Ratio Test, the series $\sum_{n=0}^{\infty} (-1)^n x^n$
is convergent if $|x| < 1$

Substitute $x = -1$ in $\sum_{n=0}^{\infty} (-1)^n x^n$

$$\sum_{n=0}^{\infty} (-1)^n (-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} ((-1)^2)^n$$

$$= \sum_{n=0}^{\infty} (1)^n$$

Since $|r| = |1| = 1 \geq 1$, by Geometric Series

Test, the series is divergent for $x = -1$

Substitute $x = 1$ in $\sum_{n=0}^{\infty} (-1)^n x^n$

$$\sum_{n=0}^{\infty} (-1)^n (1)^n = \sum_{n=0}^{\infty} (-1)^n$$

Since $|r| = |-1| = 1 \geq 1$, by Geometric Series Test, the series is divergent for $x = 1$

Hence, the required interval for convergence for the power series is $(-1, 1)$

4. $$f(x) = \frac{5}{1-4x^2} = 5\sum_{k=0}^{\infty} (4x^2)^k = 5\sum_{k=0}^{\infty} 4^k x^{2k}$$

Hence, the power series representation of $f(x)$ is $5\sum_{k=0}^{\infty} 4^k x^{2k}$

To find the interval of converges, use the ratio test as given below:

Take $u_k = 4^k x^{2k}$

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \left| \frac{4^{k+1} x^{2k+2}}{4^k x^{2k}} \right| = |4x^2|$$

By ratio test this series converges for

$|4x^2| < 1 \Rightarrow |x| < 1/2$

Check the convergence at the endpoints

Take $x = -1/2$

$$5\sum_{k=0}^{\infty} 4^k x^{2k} \Big|_{x=-1/2} = 5\sum_{k=0}^{\infty} 4^k (-1/2)^{2k}$$

$$= 5\sum_{k=0}^{\infty} 4^k (-1)^{2k} \left(1/2^{2k}\right) =$$

$$5\sum_{k=0}^{\infty} 4^k (-1)^{2k} \left(1/4^k\right) =$$

$5\sum_{k=0}^{\infty} (-1)^{2k}$ which is divergent series

Take $x = 1/2$

$$5\sum_{k=0}^{\infty} 4^k x^{2k} \Big|_{x=1/2} = 5\sum_{k=0}^{\infty} 4^k (1/2)^{2k}$$

$$= 5\sum_{k=0}^{\infty} 4^k (1)^{2k} \left(1/2^{2k}\right) =$$

$$5\sum_{k=0}^{\infty} 4^k (1)^{2k} \left(1/4^k\right) = 5\sum_{k=0}^{\infty} (1)^{2k}$$

which is also a divergent series.

Hence, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

5. $\frac{2}{3-x} = \frac{2}{3\left(1-\frac{x}{3}\right)} = \frac{2}{3}\left(1-\frac{x}{3}\right)^{-1}$

$= \frac{2}{3}\left(1+\frac{x}{3}+\left(\frac{x}{3}\right)^2+\left(\frac{x}{3}\right)^3+L+\left(\frac{x}{3}\right)^n+L\right) = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$

$= 2\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$

Thus the power series representation of $f(x)$ is $2\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$

The above series is an infinite geometric series with first term, $a = \frac{1}{3}$ and common ratio $r = \frac{x}{3}$

So, the series converges when $\left|\frac{x}{3}\right| < 1$ that is, $|x| < 3$

Therefore, the interval of convergence of the function is $(-3, 3)$

6. $\frac{4}{2x+3} = 4 \cdot \frac{1}{3\left(1+\left(\frac{2}{3}\right)x\right)}$

$$= \frac{4}{3} \cdot \frac{1}{\left(1-\left(-\frac{2x}{3}\right)\right)} = \frac{4}{3}\left(1-\left(-\frac{2x}{3}\right)\right)^{-1}$$

$$= \frac{4}{3} \sum_{n=0}^{\infty} \left(-\frac{2x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2^2}{3} \times \left(-\frac{2x}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^{n+2}}{3^{n+1}}\right) x^n =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{3^{n+1}} x^n$$

Therefore the power series of the function is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{3^{n+1}} x^n$

Let $a_n = \dfrac{(-1)^n 2^{n+2}}{3^{n+1}} x^n$

$a_{n+1} = \dfrac{(-1)^{n+1} 2^{n+3}}{3^{n+2}} x^{n+1}$

Using the ratio test

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 2^{n+3}}{3^{n+2}} x^{n+1}}{\frac{(-1)^n 2^{n+2}}{3^{n+1}} x^n} \right|$$

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$, \frac{2}{3}|x|$

So the series converges when $\left|\frac{2x}{3}\right|<1$

That is, $|x|<3/2$ this implies the interval of convergence is $(-3/2, 3/2)$

7. $f(x) = \dfrac{x^2}{x^4+16} = \dfrac{x^2}{16}\left(\dfrac{1}{1-\left(-\frac{x^4}{16}\right)}\right)$

$$= \frac{x^2}{16}\sum_{n=0}^{\infty}\left(-\frac{x^4}{16}\right)^n = \frac{x^2}{16}\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{16^n}$$

$= \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+2}}{2^{4n+4}}$ by multiplying every

term by $\dfrac{x^2}{16}$

The formula $\dfrac{1}{1-\left(-\frac{x^4}{16}\right)} = \sum_{n=0}^{\infty}\left(-\frac{x^4}{16}\right)^n$

holds only when $\left|-\frac{x^4}{16}\right|<1$

So, the formula is valid only when $\left|\frac{x^4}{16}\right|$

<1, so when $|x^4|<16$. Taking the fourth-root of both sides, we get $|x|<2$

Thus the interval of convergence is $(-2,2)$

9. $f(x) = \dfrac{x-1}{x+2} = \dfrac{(x+2)-2-1}{x+2}$

$$= \frac{x+2-3}{x+2} = \frac{x+2}{x+3} - \frac{3}{x+2}$$

$$= 1 - \frac{3}{x+2} = 1 - \frac{3}{2\left(1+\frac{x}{2}\right)}$$

$$= 1 - \frac{3}{2}\frac{1}{(1+x/2)}$$

$$= 1 - \frac{3}{2}\left(1 - (x/2) + (x/2)^2 - (x/2)^3 + \cdots\cdots\right), \ |x/2| < 1$$

$$= -\frac{1}{2} + 3\left(\frac{x}{2^2}\right) - 3\left(\frac{x^2}{2^3}\right) + 3\left(\frac{x^3}{2^4}\right) + \cdots\cdots, \ |x/2| < 1$$

$$= -\frac{1}{2} - \left(-3\left(x/2^2\right) + 3\left(\frac{x^2}{2^3}\right) - 3\left(\frac{x^3}{2^4}\right) + \cdots\cdots\right), \ |x/2| < 1$$

$= -1/2 - \left((-1)^1 3 \left(\frac{x}{2^{1+1}} \right) + (-1)^2 3 \left(\frac{x^2}{2^{2+1}} \right) + (-1)^3 3 \left(\frac{x^3}{2^{3+1}} \right) + \cdots \right), \left| \frac{x}{2} \right| < 1$

$= -1/2 - \sum_{n=1}^{\infty} (-1)^n \frac{3}{2^{n+1}} x^n , \quad |x/2| < 1$

Radius of convergence:

$-1 < x/2 < 1$

$-2 < x < 2$

Substitute $x = -2$

$-1/2 - \sum_{n=1}^{\infty} (-1)^n 3/2^{n+1} x^n = -1/2 - \sum_{n=1}^{\infty} (-1)^n 3/2^{n+1} (-2)^n = -1/2 - \sum_{n=1}^{\infty} (-1)^n \frac{3}{2^{n+1}} (-1)^n 2^n$

$= -1/2 - \sum_{n=1} 3/2$

The series is divergent because it is a constant series

So, the endpoint $x = -2$ cannot be included to the interval of convergence

Substitute $x = 2$

$$-\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^n \frac{3}{2^{n+1}} x^n =$$

$$-\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^n \frac{3}{2^{n+1}} (2)^n$$

$$= -\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^n \frac{3}{2^{n+1}} 2^n$$

$$= -\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^n \cdot \frac{3}{2} =$$

$$-\frac{1}{2}-\frac{3}{2}\sum_{n=1}^{\infty}(-1)^n$$

The series is divergent because its terms are ~~alternately~~ alternatively positive and negative.

So, the endpoint $x=2$ cannot be included to the interval of convergence.

Therefore, the series representation of the given function with interval of convergence is,

$$f(x) = -\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^n \frac{3}{2^{n+1}} x^n, \ (-2, 2)$$

15. Diff $f(x) = \ln(5-x)$ on both sides w.r.t x

$$f'(x) = \frac{d}{dx}\ln(5-x)$$

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$$= \frac{1}{5-x}\frac{d}{dx}(5-x) = \frac{-1}{5-x} = \frac{-1}{5(1-x/5)}$$

Here, $\frac{1}{(1-x/5)} = 1+(x/5)+(x/5)^2+(x/5)^3 + \ldots\ldots$

Then,

$$-1/5\,\frac{1}{(1-x/5)} = \frac{-1}{5}\left(1+\left(\frac{x}{5}\right)+\left(\frac{x}{5}\right)^2+\left(\frac{x}{5}\right)^3+\ldots\ldots\right)$$

Integrate on both sides, then

$$-1/5\int\frac{1}{\left(1-\frac{x}{5}\right)}\,dx = \int\frac{-1}{5}\left(1+\left(\frac{x}{5}\right)+\left(\frac{x}{5}\right)^2+\left(\frac{x}{5}\right)^3+\ldots\ldots\right)dx$$

$$\ln(5-x) = -1/5\int\left(1+(x/5)+(x/5)^2+(x/5)^3+\ldots\ldots\right)dx$$

$$= -1/5\left(\int 1\,dx + 1/5\int x\,dx + \int\left(\frac{x}{5}\right)^2 dx + \ldots\ldots\right) = -1/5\left(x+1/5\,(x^2/2)+1/25\,(x^3/3)+\ldots\ldots\right)+C$$

$$\ln(5-x) = C - \frac{x}{5} - \frac{x^2}{50} - \frac{x^3}{375} - \frac{x^4}{2500} - \ldots$$

When $x=0$, then

$$\ln(5-0) = C - 0/5 - 0^2/50 - 0^3/375 \ldots$$

$$\ln(5) = C$$

So, the series expansion of $\ln(5-x)$ is

$$= \ln(5) - \left[x/5 + 1/2 \left(\frac{x}{5}\right)^2 + 1/3 \left(\frac{x}{5}\right)^3 + 1/4 \left(\frac{x}{5}\right)^4 + \ldots \right)$$

$$= \ln(5) - \sum_{n=1}^{\infty} 1/n \left(\frac{x}{5}\right)^n .$$

$$= \ln(5) - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$$

Thus the power rep of $\ln(5-x)$ is,

$$\ln(5-x) = \ln(5) - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$$

Let $a_n = \dfrac{x^n}{n5^n}$

Replace n by $n+1$

$a_{n+1} = \dfrac{x^{n+1}}{(n+1)5^{n+1}}$

Then,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{(n+1)5^{n+1}}}{\dfrac{x^n}{n5^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)5^{n+1}} \times \frac{n5^n}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{nx}{(n+1)5}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\left(\frac{x}{5}\right)\right)$$

$$= \left|\left(x/5\right)\right| \lim_{n\to\infty}\frac{1}{(1+1/n)} = \left|\left(x/5\right)\right|$$

The series $\sum_{n=1}^{\infty}\dfrac{x^n}{n5^n}$ converges when $|x/5|<1$

That is,

$|x/5| < 1 \qquad |x| < 5$

Therefore $R = 5$

17. $f(x) = (x)\left(\frac{1}{(1+4x)^2}\right)$

$$\frac{1}{(1-(-4x))} = \sum_{n=0}^{\infty} (-4x)^n.$$

$$\frac{1}{1+4x} = \sum_{n=0}^{\infty} (-4)^n (x)^n$$

Diff. both sides w.r.t to x :-

$$d/dx\left[\frac{1}{1+4x}\right] = d/dx\left[\sum_{n=0}^{\infty}(-4)^n (x)^n\right]$$

$$d/dx\,(1+4x)^{-1} = \sum_{n=1}^{\infty} (-4)^n n (x)^{n-1}$$

$$-\frac{4}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^n n (x)^{n-1}$$

Divide both sides by -4.

$$\frac{1}{(1+4x)^2} = \frac{1}{-4}\sum_{n=1}^{\infty} (-4)^n n (x)^{n-1}$$

$$= (-4)^{-1}\sum_{n=1}^{\infty}(-4)^{n-1} n (x)^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} (4)^{n-1} n (x)^{n-1}$$

Multiply both sides by x.

$$\frac{x}{(1+4x)^2} = x\sum_{n=1}^{\infty}(-1)^{n-1}(4)^{n-1}n(x)^{n-1}$$

$$= \sum_{n=1}^{\infty} n(-1)^{n-1}(4)^{n-1}(x)^{n-1+1}$$

$$= \sum_{n=1}^{\infty} n(-1)^{n-1}(4)^{n-1}(x)^{n}$$

Replace n with $n+1$

$$\frac{x}{(1+4x)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}(4)^{n-1}(x)^{n}$$

$$= \sum_{n+1=1}^{\infty}(n+1)(-1)^{n+1-1}(4)^{n-1+1}x^{n+1}$$

$$= \sum_{n=0}^{\infty}(n+1)(-1)^{n}(4)^{n}x^{n+1}$$

$$= \sum_{n=0}^{\infty}(n+1)(-1)^{n}(4)^{n}x^{n+1}$$

$a_n = (n+1)(-1)^n(4)^n x^{n+1}$

Apply the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)(-1)\cdot(4)^n\cdot 4\cdot x^{n+1}\cdot x}{(n+1)\cdot(4)^n\cdot x^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{4(n+2)x}{(n+1)}\right| = |4x|\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

$$= |4x|\lim_{n\to\infty}\left|\frac{1+2/n}{1+1/n}\right| = |4x|\left|\frac{1+0}{1+0}\right|$$

$$= |4x|$$

By the ratio test the ~~se~~ series converges if $L<1$.

That is,

$|4x|<1$

$|x|<1/4$

The power series rep. for $f(x)$ is

$f(x) = \sum_{n=0}^{\infty}(n+1)(-1)^n(4)^n x^{n+1}$ and the radius of convergence for this series is

$R = 1/4$

25. $\frac{1}{1-t^8} = \sum_{n=0}^{\infty}(t^8)^n = \sum_{n=0}^{\infty}t^{8n} = 1+t^8+t^{16}+t^{24}+\dots$

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Now integrate term by term

$$\int \frac{1}{1-t^8}\,dt = t\int \frac{1}{1-t^8}\,dt$$

$$= t\int \sum_{n=0}^{\infty} t^{8n}\,dt = \int \sum_{n=0}^{\infty} t^{8n+1}\,dt$$

$$= \sum_{n=0}^{\infty} \int t^{8n+1}\,dt = C + \sum \frac{t^{8n+2}}{8n+2}$$

$|t^8| < 1$

So, this series is converging for $|t^8|<1$, that is $|t|<1$ and the radius of convergence is $R=1$

Therefore, the Indefinite integral can be expressed as a power series is $C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$ and the radius of convergence is $R=1$

27. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \; |x|<1$

Multiply $\ln(1+x)$ with x^2

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$$x^2\ln(1+x) = x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{n+2}$$

$$\int x^2 \ln(1+x)\,dx = \int \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+2}}{n}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int x^{n+2}\,dx =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\frac{x^{n+3}}{(n+3)}\right]$$

Hence, $\int x^2 \ln(1+x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+3)} x^{n+3}$,

$|x|<1$

$$a_n = \frac{(-1)^{n-1}}{n(n+3)} x^{n+3}$$

Then;

$$a_{n+1} = \frac{(-1)^{n+1-1}}{(n+1)(n+4)} x^{n+1+3}$$

Use ratio test

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n \cdot \frac{x^{n+4}}{(n+1)(n+4)}}{(-1)^{n-1} \frac{x^{n+3}}{n(n+3)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{n+4}\, n(n+3)}{x^{n+3}(n+1)(n+4)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x(1+3/n)}{(1+1/n)(1+4/n)} \right|$$

$$= |x|$$

Therefore, by Ratio Test the series converges when $|x|<1$.

Thus the radius of convergence is $R=1$

Exercise 11.10

5. The Taylor series of the function $f(x) = xe^x$ centered at the value of $a=0$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} + \frac{f''''(0)}{4!}x^4 + \dots \quad (1)$$

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$f(x) = xe^x$ $\quad f(0) = 0$

$f'(x) = xe^x + e^x$ $\quad f'(0) = 1$

$f''(x) = xe^x + 2e^x$ $\quad f''(0) = 2$

$f'''(x) = xe^x + 3e^x$ $\quad f'''(0) = 3$

$f''''(x) = xe^x + 4e^x$ $\quad f''''(0) = 4$

Substitute above values in the Taylor Series (1)

$$= 0 + 1/1!\, x + 2/2!\, x^2 + 3/3!\, x^3 + 4/4!\, x^4 + \dots$$

$$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \dots$$

Therefore, the first four nonzero terms of the Taylor series are $x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}$

7. $f(x) = \sqrt[3]{x}$ $\quad f(8) = \sqrt[3]{8} = 2$

$f'(x) = 1/3\, x^{-2/3}$ $\quad f'(8) = 1/12$

$f''(x) = -2/9\, x^{-5/3}$ $\quad f''(8) = -1/144$

$f'''(x) = \frac{10}{27} x^{-8/3}$ $\quad f'''(8) = \frac{5}{6912} \cdot 2$

$$= \frac{5}{3456}$$

$$f(x) = 2 + (1/12)(x-8) + \frac{-1/144}{2!}(x-8)^2$$

$$+ \frac{5/3456}{3!}(x-8)^3$$

$$= 2 + 1/12(x-8) - \frac{1}{288}(x-8)^2 +$$

$$\frac{5}{3456 \cdot 6}(x-8)^3$$

$$= 2 + 1/12(x-8) - \frac{1}{288}(x-8)^2 +$$

$$\frac{5}{20736}(x-8)^3$$

Therefore the first 4 terms of the Taylor Series for the function $f(x)$ is

$$2 + 1/12(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8$$

8. $f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = 1/x$ $\quad f'(1) = 1$

$f''(x) = -1/x^2$ $\quad f''(1) = -1$

$f'''(x) = -2/x^3$ $\quad f'''(1) = 2$

$f''''(x) = -6/x^4$ $\quad f''''(1) = -6$

The Taylor series is,

$$f(x) = 0 + 1/1!(x-1) - 1/2!(x-1)^2 +$$

$$2/3!(x-1)^3 - 6/4!(x-1)^4 + \dots$$

$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \dots$

Hence, the first four nonzero terms of the Taylor series are.

$(x-1)$, $\frac{1}{2}(x-1)^2$, $\frac{1}{3}(x-1)^3$, $\frac{1}{4}(x-1)^4$

11. $f(0) = (1-0)^{-2} = (1)^{-2} = 1$

$f'(x) = d/dx\,[(1-x)^{-2}] = 2(1-x)^{-3}$

$f'(0) = 2$

$f''(x) = d/dx\,[2(1-x)^{-3}] = 6(1-x)^{-4}$

$f''(0) = 6$

$f'''(x) = d/dx\,[6(1-x)^{-4}]$

$= 24(1-x)^{-5}$

$f'''(0) = 24$

$f^{IV}(x) = d/dx\,[24(1-x)^{-5}]$

$= 120(1-x)^{-6}$

$f^{IV}(0) = 120$

The Maclaurin series of function is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!} + \cdots + \frac{f^n(0)}{n!}x^n$$

$$f(x) = 1 + 2x + 6/_{2!}\, x^2 + \frac{24}{3!}x^3 + \frac{120}{4!}x^4 + \cdots + \frac{(n+1)!}{n!}x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 \cdots$$

$$f(x) = \sum_{n=0}^{\infty} (n+1)x^n$$

Use the ratio test with $a_n = (n+1)x^n$ as shown below,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$

$$= \frac{(n+2)}{(n+1)}|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(1+2/n)}{(1+1/n)}|x|$$

$$= \frac{(1+0)}{(1+0)} |x| = |x|$$

Thus converges, when

$|x| < 1$

$-1 < x < 1$

$$R = \frac{1-(-1)}{2} = 1$$

There the associated radius of convergence is $R = 1$

13. $f(x) = \cos x$

$f(0) = \cos 0 = 1$

$f'(x) = -\sin x = -\sin 0$

$f'(0) = 0$

$f''(x) = -\cos x$

$f''(0) = -\cos 0 = -1$

$f'''(x) = \sin 0 = 0$

$f^{IV}(x) = \cos x$

$f^{IV}(0) = 1$

$$f(x) = 1 + 0 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$a_n = (-1)^n \frac{x^{2n}}{(2n)!}$$

$$a_{n+1} = (-1)^{n+1} \frac{x^{2(n+1)}}{(2(n+1))!}$$

$$= (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{(2n+2)!} \times \frac{(2n)!}{x^{2n}}\right|$$

$$= \left|\frac{x^2}{(2n+2)(2n+1)}\right|$$

So, $\lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$

Thus, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ and

the radius of convergence is $R = \infty$

14. $f(x) = e^{-2x} \rightarrow f(0) = e^{-2(x)} = 1$
$f'(x) = -2e^{-2x} \rightarrow f'(0) = -2e^{-2(0)} = -2$
$f''(x) = 2^2 e^{-2x} \rightarrow f''(0) = 2^2 e^{-2(0)} = 2^2$
$f'''(x) = -2^3 e^{-2x} \rightarrow f'''(0) = -2^3 e^{-2(0)} = -2^3$

$f^n(x) = (-1)^n 2^n e^{-2x} \rightarrow f^{(n)}(0) =$

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$$(-1)^n 2^n e^{-2(0)} = (-1)^n 2^n$$

$$f(x) = 1 - 2x + \frac{2^2 x^2}{2!} - \frac{2^3 x^3}{3!} + \cdots$$

$$+ \frac{(-1)^n 2^n x^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!}$$

$$a_n = \frac{(-1)^n 2^n x^n}{n!}$$

Apply the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2x}{(n+1)} \right| = 0 < 1$$

By the ratio test $(L = 0 < 1)$ the series converges for all values of x, that is $-\infty < x < \infty$

Thus, the radius of convergence is $R = \infty$

15. $f(x) = 2^x \quad f(0) = 1$
$f'(x) = 2^x \ln 2 \quad f'(0) = \ln 2$
$f''(x) = 2^x (\ln 2)^2 \quad f''(0) = (\ln 2)^2$

$$f'''(x) = 2^x (\ln 2)^3$$
$$f'''(0) = (\ln 2)^3$$
$$f^{(n)}(x) = 2^x ((\ln 2))^n$$
$$f^{(n)}(0) = (\ln 2)^n$$

$$f(x) = 1 + (\ln 2)x + \frac{(\ln 2)^2 x^2}{2!} + \frac{(\ln 2)^3 x^3}{3!} + \dots \frac{(\ln 2^{\circ})^n x^n}{n!} + \dots$$

Hence the Maclaurin Series for $f(x) = 2^x$ is

$$2^x = 1 + \cancel{\ln}\, (\ln 2)x + \frac{(\ln 2)^2 x^2}{2!} + \frac{((\ln 2)^3 x^3}{3!} + \frac{((\ln 2))^n x^n}{n!} + \dots$$

$$a_n = \frac{((\ln 2)^n x^n}{n!}$$

Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{((\ln 2)^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{((\ln 2)^n} \right.$$

$$= \lim_{n\to\infty} \frac{(\ln 2)x}{n+1} = 0 < 1$$

It is convergent for all x

Hence the radius of convergence is ∞

URBAN EDGE

21. $f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = 1/x \qquad f'(2) = 1/2$

$f''(x) = -1/x^2 \qquad f''(2) = -1/2^2$

$f'''(x) = 2/x^3 \qquad f'''(2) = 2!/2^3$

$f^{(4)}(x) = -2\cdot3/x^4 \qquad f^{(4)}(2) = -3!/2^4$

$f^{(5)}(x) = \dfrac{2\cdot3\cdot4}{x^5} \qquad f^{(5)}(2) = \dfrac{4!}{2^5}$

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \dots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \dots$$

$$= \ln 2 + 1/2\,(x-2) - \frac{1}{2^2\cdot2}(x-2)^2 + \frac{1}{2^3\cdot3}(x-2)^3 - \frac{1}{2^4 4}(x-2)^4 + \dots$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n\cdot n}(x-2)^n$$

Thus, the Taylor series for $f(x)$ centered at $a=2$ is $f(x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n\cdot n}(x-2)^n$

$$a_n = \frac{(-1)^{n+1}(x-2)^n}{2^n\cdot n}$$

$$a_{n+1} = \frac{(-1)^{n+2}(x-2)^{n+1}}{2^{n+1}(n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1} \times 2^n \cdot n}{2^{n+1}(n+1)(-1)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x-2}{2}\cdot\frac{n}{n+1}\right| =$$

$$\lim_{n\to\infty}\left|\frac{x-2}{2}\cdot\frac{n}{n(1+1/n)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x-2}{2}\cdot\frac{1}{1+1/n}\right|.$$

$$= \left|\frac{x-2}{2}\cdot\frac{1}{1+0}\right| = \left|\frac{x-2}{2}\right|$$

By, the ratio test, the above series converges if $\left|\frac{x-2}{2}\right| < 1$

That is $|x-2| < 2$

Thus the radius of convergence is $R = 2$

25. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a^2)$

$+ \dots \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$ (1)

$f(x) = \sin x$ $\quad f(\pi) = 0$

$f'(x) = \cos x$ $\quad f'(\pi) = -1$

$f''(x) = -\sin x$ $\quad f''(\pi) = 0$

$-2)^n$

$f'''(x) = -\cos x \qquad f'''(\pi) = -(-1) = 1$

$f^{(4)}(x) = -(-\sin x)$
$= \sin x \qquad f^{(4)}(\pi) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(\pi) = -1$

Substitute in (1)

$$f(x) = 0 + (-1)(\cancel{\pi}x - \pi) + 0/2!\,(x-\pi)^2 + 1/3!\,(x-\pi)^3 + 0/4!\,(x-\pi)^4 + \ldots$$

$$= -(x-\pi) + 1/3!\,(x-\pi)^3 + \ldots$$

$$= \frac{-(x-\pi)}{1!} + 1/3!\,(x-\pi)^3 + \ldots$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n+1}}{(2n+1)!}$$

Use ratio test

Let $a_n = (-1)^{n+1} \dfrac{(x-\pi)^{2n+1}}{(2n+1)!}$

$a_{n+1} = (-1)^{n+2} \dfrac{(x-\pi)^{2n+3}}{(2n+3)!}$

By ratio test, find $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| (-1)^{n+1} \frac{(x-\pi)^{2n+3}}{(2n+3)!\,(-1)^{n+1}} \frac{(2n+1)!}{(x-\pi)^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \frac{(x-\pi)^2 (2n+1)!}{(2n+3)(2n+2)(2n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \frac{(x-\pi)^2}{(2n+3)(2n+2)} \right| =$$

$$\lim_{n\to\infty} \frac{(x-\pi)^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x$$

The given series converges for all values of x

Hence, the radius of convergence $R = \infty$

35. $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

, $R = 1$ from the table.

Replacing x by x^2

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} =$$

$$= (x^2) - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \ldots$$

$\arctan x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \ldots$

$\pi)^{2n+1}$

Thus the Maclaurin series for $f(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}, R=1$

36. The Maclaurin series of $f(x) = \sin(x)$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad (1)$

Replacing x by $\frac{\pi x}{4}$ in (1), to obtain

$\sin\left(\frac{\pi x}{4}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$= \frac{\left(\frac{\pi x}{4}\right)}{1!} - \frac{\left(\frac{\pi x}{4}\right)^3}{3!} + \frac{\left(\frac{\pi x}{4}\right)^5}{5!} - \frac{\left(\frac{\pi x}{4}\right)^7}{7!} + \ldots$

$t_n = a + (n+1-1)d$
$= 1 + n(2)$
$= 2n+1$

The series is reduced into

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!} x^{2n+1}$$

39. $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (2)$$

Substitute $1/2x^2$ for x in (2)

$$\cos(1/2x^2) = 1 - 1/4 \times x^4/2! + 1/16 \times \frac{x^8}{4!} - 1/64 \times x^{12}/6! + \dots \quad (3)$$

Multiply the above series by x.

$$x\cos(1/2x^2) = \left(x - 1/4 \times \frac{x^5}{2!} + \frac{1}{16} \times \frac{x^9}{4!} - 1/64 \times x^{13}/6! + \dots\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}} \cdot \frac{x^{4n+1}}{(2n)!}$$

Therefore the Maclaurin series of the function $f(x) = x\cos(\frac{1}{2}x^2)$ is,

$$x\cos(\tfrac{1}{2}x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2^{2n}(2n)!} \quad \text{and}$$

Radius of convergence $R = \infty$

67. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^{-x^2}\cos x = \left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{6} + \dots\right) \times$$

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^2}{1} + \frac{x^4}{2}$$

$$- \frac{x^6}{24} + \frac{x^6}{720} + \frac{x^4}{2} - \frac{x^6}{4} + \frac{x^8}{48}$$

$$- \frac{x^{10}}{1440} \dots$$

$$= 1 - \frac{x^2}{2} - x^2 + \frac{x^4}{24} + \frac{x^4}{2} + \frac{x^4}{2}$$

$$- \frac{x^6}{720} - \frac{x^6}{4} - \frac{x^6}{24} + \ldots$$

$$= 1 - 3/_2 x^2 + \frac{25}{24} x^4 \ldots$$

$$= 1 - 3/_2 x^2 + \frac{25}{24} x^4$$

Hence, the first three terms in the series are $1 - 3/_2 x^2 + \frac{25}{24} x^4$

73. $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x, evaluating this series at $-x^4$ gives $e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!}$, justifying the last equivalence

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = e^{-x^4}$$

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77. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Then $$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sin(\pi/4)$$

$$= 1/\sqrt{2} = \frac{\sqrt{2}}{2}$$

80. $$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Take $x = 1/2$ then

$$\tan^{-1}(1/2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}}$$

$$\Rightarrow \tan^{-1}(1/2) = 1/2 - \frac{1}{3(2^3)} + \frac{1}{5(2^5)} - \frac{1}{7(2^7)} + \dots$$

Hence $$1/2 - \frac{1}{3(2^3)} + \frac{1}{5(2^5)} - \frac{1}{7(2^7)} + \dots = \tan^{-1}(1/2)$$

Excercise 11.11

3. $f(x) = e^x$ $\qquad$ $f(1) = e$

$f'(x) = \frac{d}{dx} e^x = e^x$ $\qquad$ $f'(1) = e$

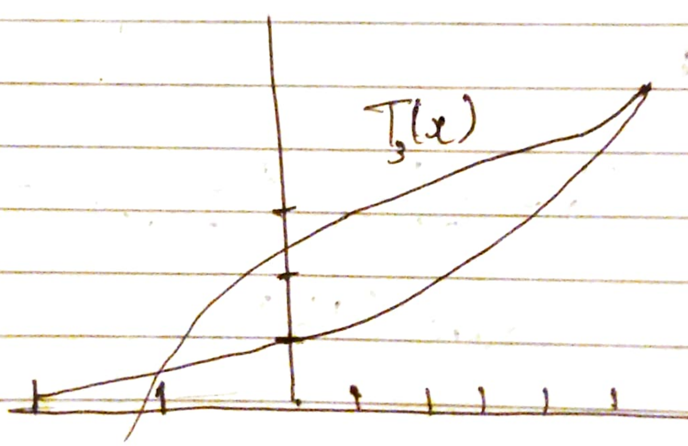
$f''(x) = \frac{d}{dx} e^x = e^x$ $\qquad$ $f''(1) = e$

$f'''(x) = \frac{d}{dx} e^x = e^x$ $\qquad$ $f'''(1) = e$

$$T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Substitute $a = 1$ and derivatives

$$T_3(x) = e + e(x-1) + \frac{1}{2}e(x-1)^2 + \frac{1}{6}e(x+1)^3$$



4. $f(x) = \sin x$ $\quad f(\pi/6) = 1/2$

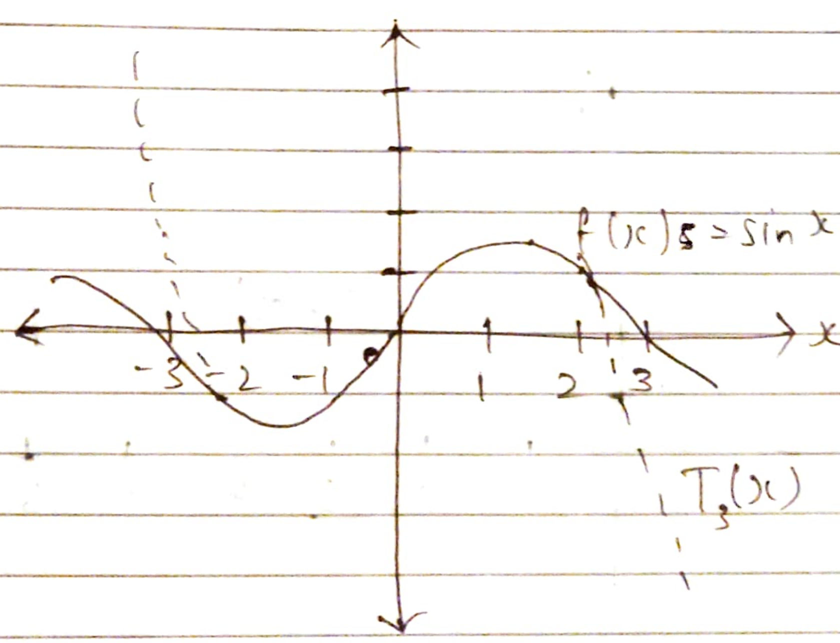
$f'(x) = \cos x$ $\quad f'(\pi/6) = \sqrt{3}/2$

$f''(x) = -\sin x$ $\quad f''(\pi/6) = -1/2$

$f'''(x) = -\cos x$ $\quad f'''(\pi/6) = -\sqrt{3}/2$

Substitute the values in Taylor Series

$$f(x) = 1/2 + \left(\frac{\sqrt{3}}{2}\right)\left(x - \frac{\pi}{6}\right) - 1/4\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$$



5. $f(x) = \cos x \qquad f(\pi/2) = 0$

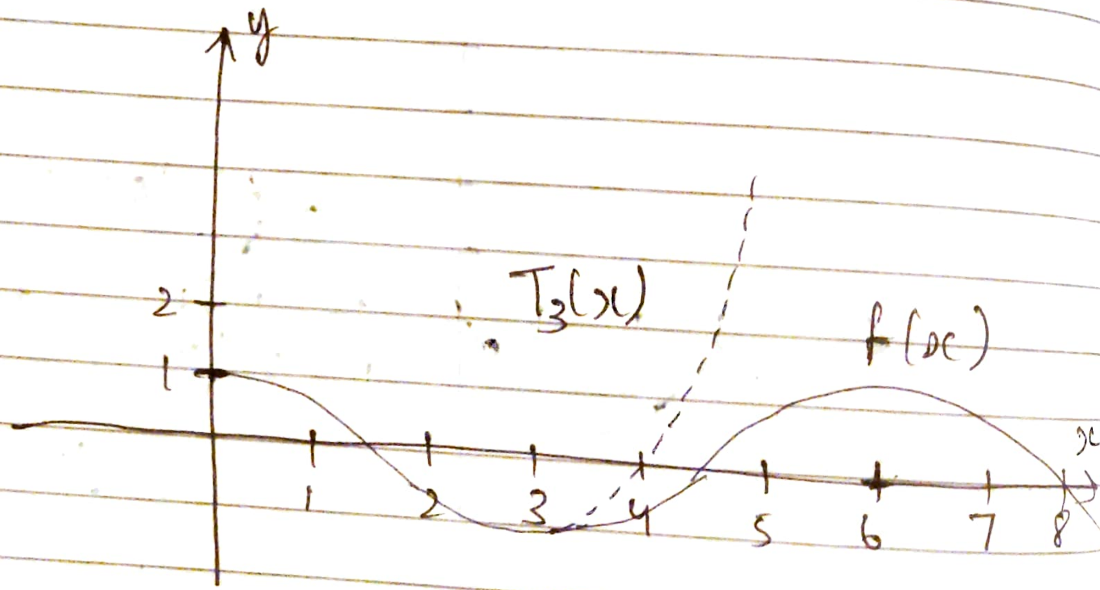
$f'(x) = -\sin x \qquad f'(\pi/2) = -1$

$f''(x) = -\cos x \qquad f''(\pi/2) = 0$

$f'''(x) = \sin x \qquad f'''(\pi/2) = 1$

$$T_3(x) = f(\pi/2) + \frac{f'(\pi/2)}{1!}\left(x - \frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2!}\left(x - \frac{\pi}{2}\right) + \frac{f'''(\pi/2)}{3!}\left(x - \pi/2\right)^3 = -\left(x - \pi/2\right) + 1/6\left(x - \pi/2\right)^3$$

$$\cos x \approx T_3(x) = -\left(x - \frac{\pi}{2}\right) + 1/6\left(x - \frac{\pi}{2}\right)^3$$



6. $f(x) = e^{-x} \sin x$

$f(0) = e^{0} \sin(0) = 0$. For $a = 0$,

$T_3(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3$ ......(1)

$f'(x) = d/dx \, [e^{-x} \sin x]$

$= -e^{-x}(\sin x) + e^{-x}(\cos x)$

$= e^{-x}(\cos x - \sin x)$

$f'(0) = 1$

$f''(x) = d/dx \, [e^{-x}(\cos x - \sin x)]$

$= -e^{-x} \cos x + e^{-x} \sin x - e^{-x} \sin x - e^{-x} \cos x$

$f''(x) = -2e^{-x} \cos(x)$

$f''(0) = -2$

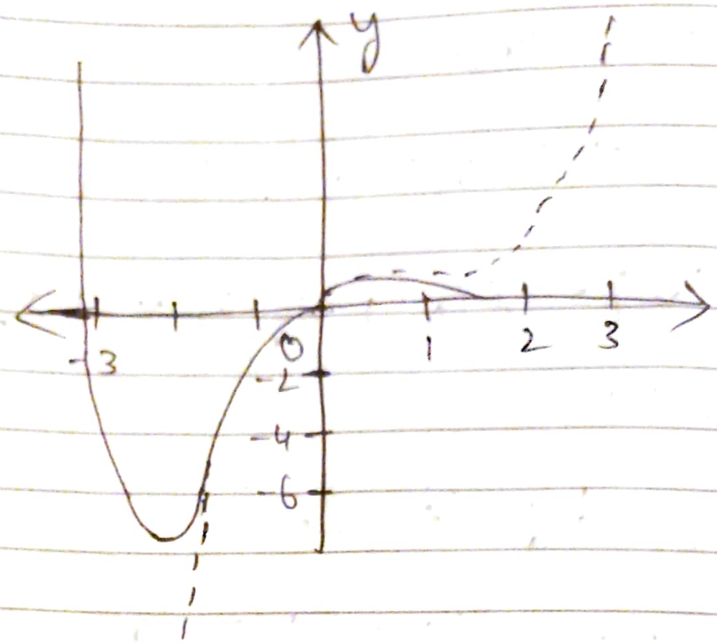
$f'''(x) = 2 \frac{d}{dx} [e^{-x} \cos x]$

$f'''(x) = 2e^{-x} \cos(x) + 2e^{-x} \sin(x)$

$f'''(0) = 2$

$T_3(x) = 0 + 1/1! \, x - 2/2! \, x^3 + 2/3! \, x^3$

$= x - x^2 + 1/3 \, x^3$



7. $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$+ \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots \dots (1)$

$f(x) = \ln x \qquad f(1) = 0$

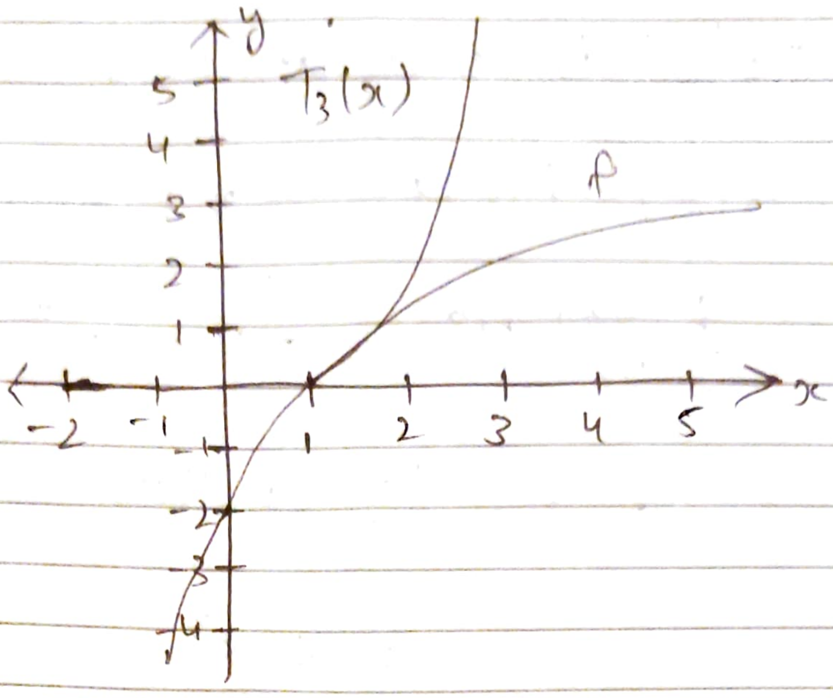
$f'(x) = 1/x \qquad f'(1) = 1$

$f''(x) = -1/x^2 \qquad f''(1) = -1$

$f'''(x) = 2/x^3 \qquad f'''(1) = 2$

Substitute in (1)

$T_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ //



9. $f(x) = xe^{-2x}$ $\qquad f(0) = 0$

$f'(x) = -2xe^{-2x} + e^{-2x}$

$f'(0) = 1$

$f''(x) = 2\frac{d}{dx}(xe^{-2x}) + \frac{d}{dx}(e^{-2x})$

$= 2(-2xe^{-2x} + e^{-2x}) - 2e^{-2x}$

$f''(x) = 4xe^{-2x} - 4e^{-2x}$

$f''(0) = -4$

$f'''(x) = 4\, d/dx\,(xe^{-2x}) - 4\, d/dx\,(e^{-2x})$

$= 4(-2xe^{-2x} + e^{-2x}) - 4(-2e^{-2x})$

$= 12e^{-2x} - 8xe^{-2x}$

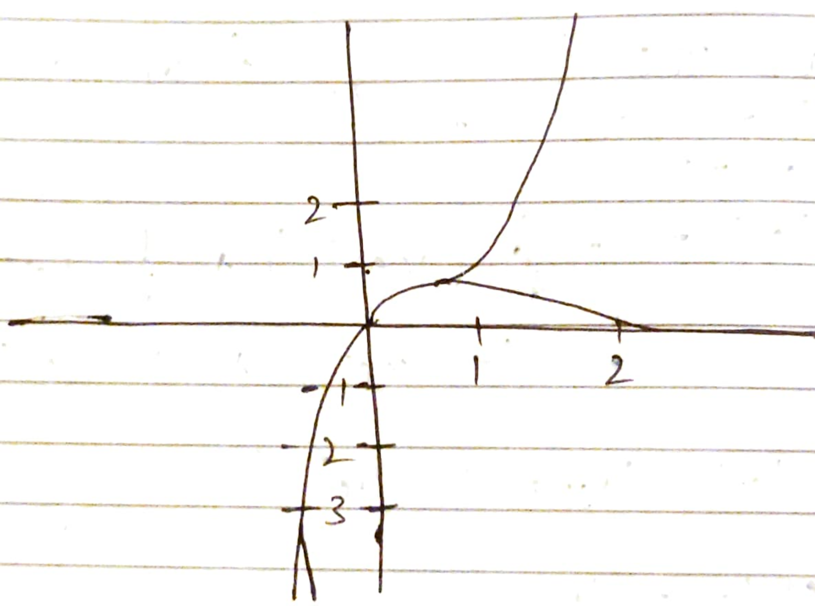
$f'''(0) = 12$

$T_3(x)$ for $f(x)$ at $a=0$

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + x - \frac{4x^2}{2!} + \frac{12x^3}{3!}$$

$$= x - \frac{4x^2}{2} + \frac{12x^3}{6}$$

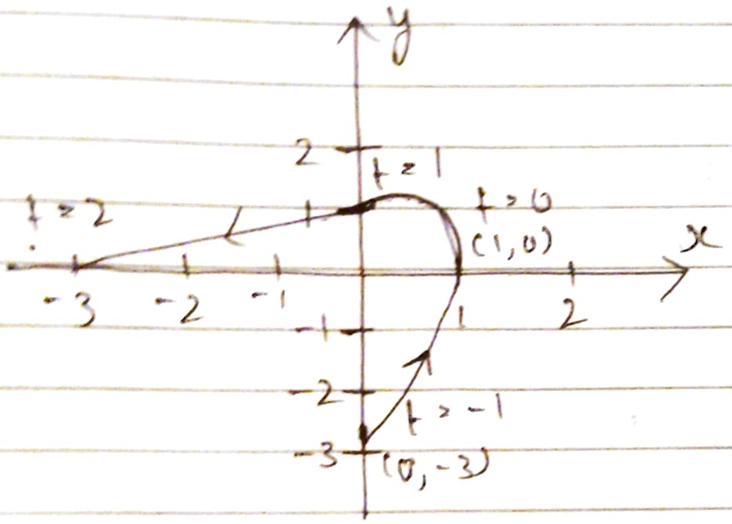
$$= x - 2x^2 + 2x^3$$



Exercise 10.1

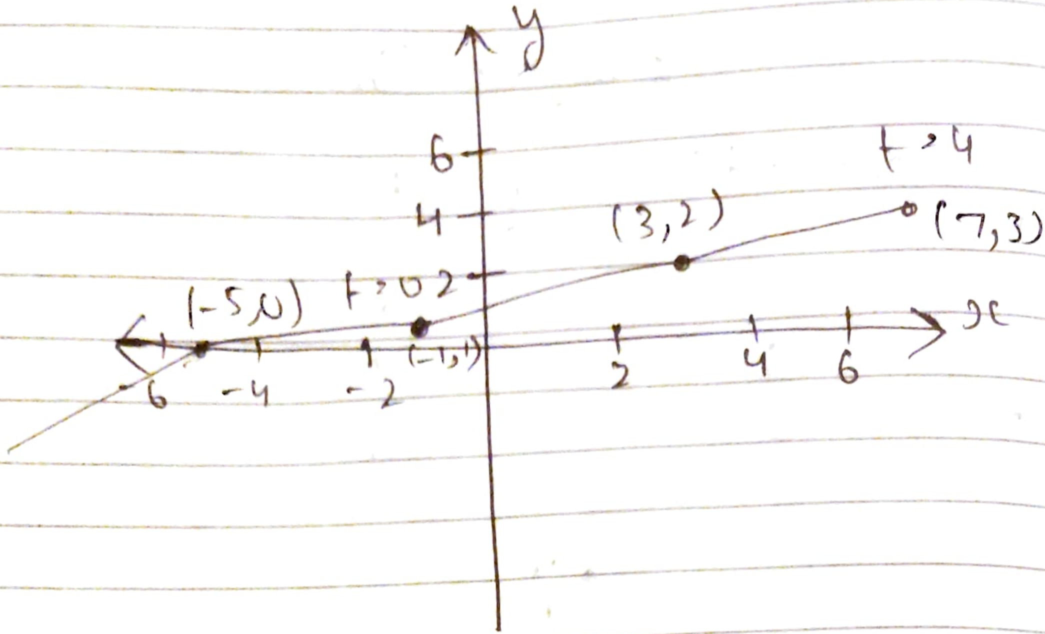
1.

t	$x = 1 - t^2$	$y = 2t - t^2$
-1	0	-3
0	1	0
1	0	1
2	-3	0



3. (a)

t	$x = 2t - 1$	$y = \frac{1}{2}t + 1$
-4	-9	-1
-2	-5	0
0	-1	1
2	3	2
4	7	3



(b) $x = 2t - 1$

$x + 1 = 2t$

$\frac{x+1}{2} = t$

$x/2 + 1/2 = t$

$y = 1/2\, t + 1$

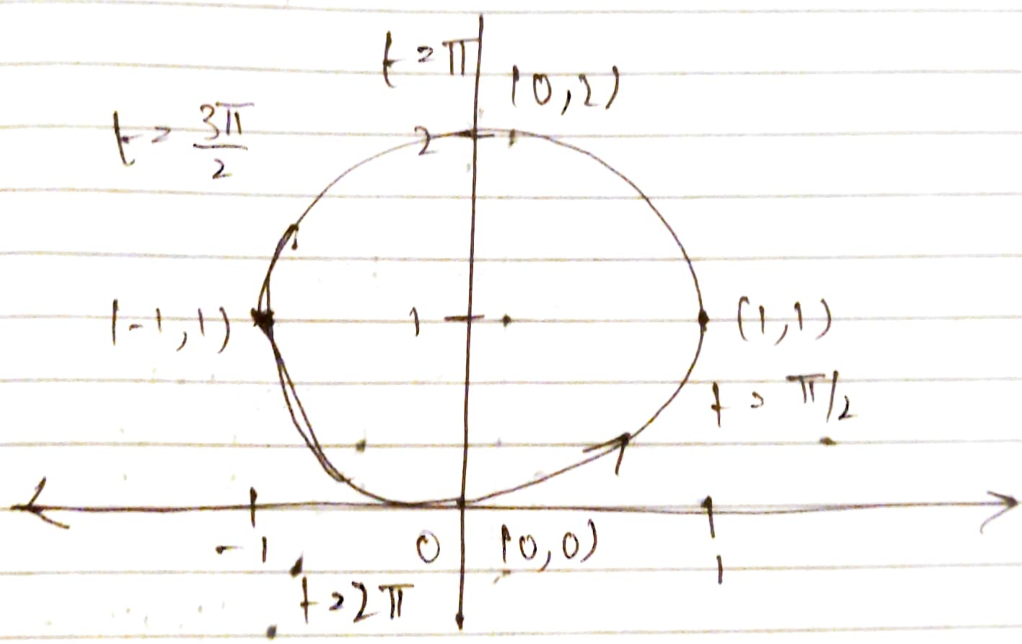
$y = 1/2(x/2 + 1/2) + 1$

$y = x/4 + 1/4 + 1$

$y = x/4 + 5/4$

8. (a)

t	$x = \sin t$	$y = 1 - \cos t$
0	0	0
$\pi/2$	1	1
π	0	2
$3\pi/2$	-1	1
2π	0	0



(b) $x^2 = \sin^2 t$

$y^2 = (1 - \cos t)^2$

$y^2 = 1 + \cos^2 t - 2\cos t$

$x^2 + y^2 = (\sin t)^2 + (1 - \cos t)^2$

$= \sin^2 t + 1 - 2\cos t + \cos^2 t$

$= (\sin^2 t + \cos^2 t) + 1 - 2\cos t$

$x^2 + y^2 = 2 - 2\cos t$

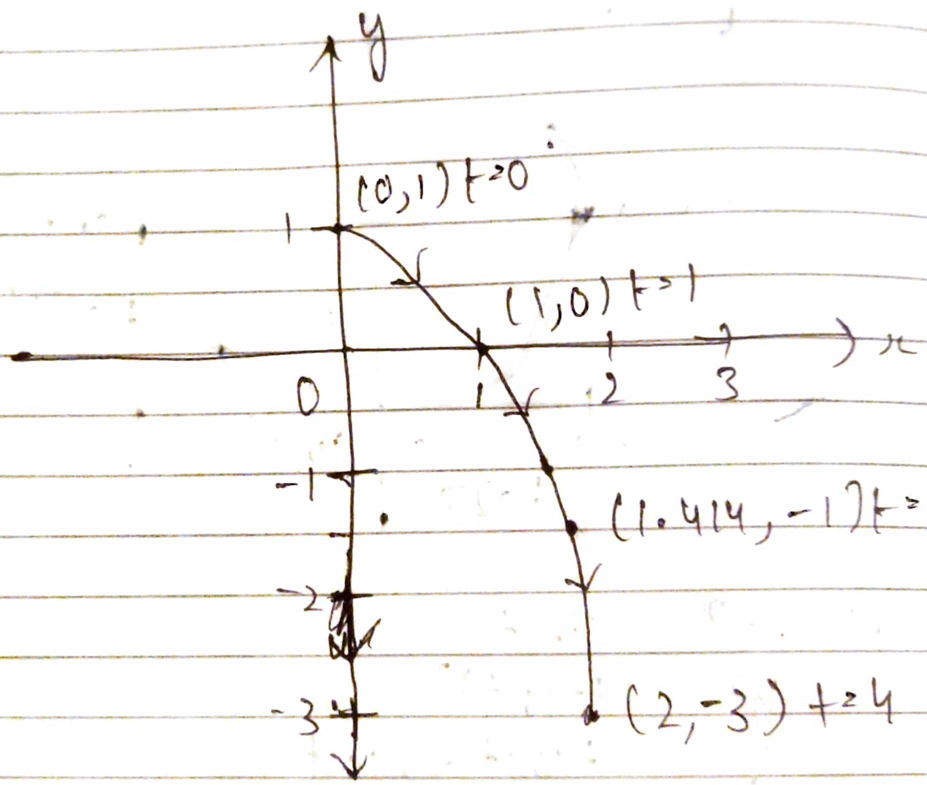
$= 2(1 - \cos t)$

$= 2y$

$x^2 + y^2 = 2y$

9. (a)

t	x	y
0	0	0
1	1	0
2	1.414	-1
4	2	-3
9	3	-8



(b) $x = \sqrt{t}$

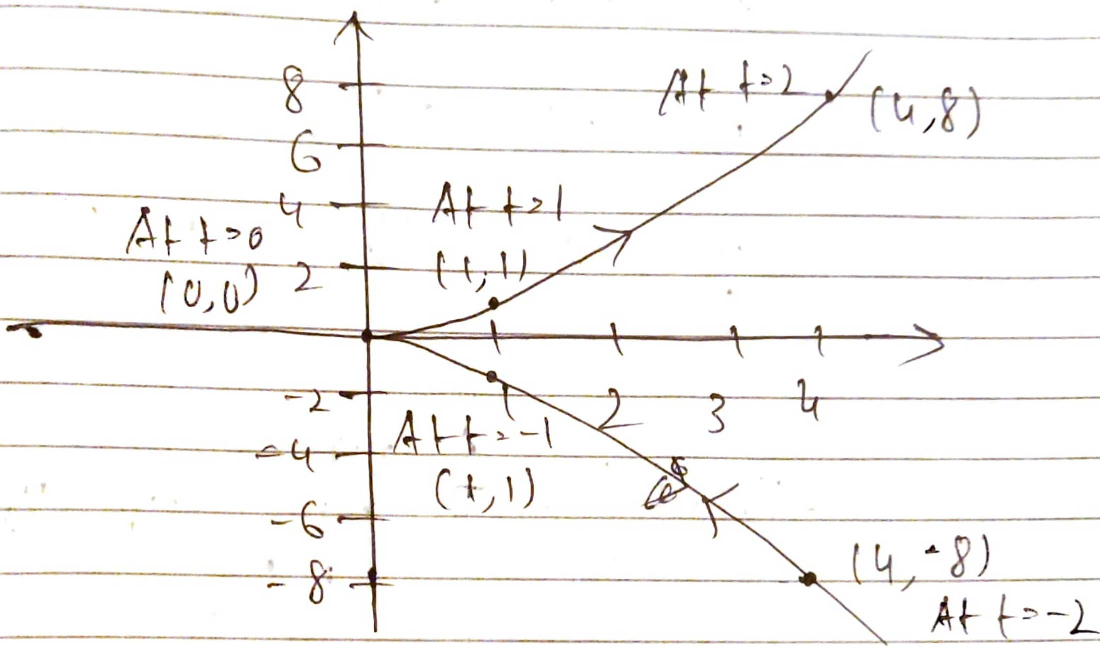
$x^2 = t, \; x \geq 0$

~~Sub~~ in :

$y = 1 - x^2, \; x \geq 0$

10. (a)

t	$x = t^2$	$y = t^3$
-2	4	-8
-1	1	-1
0	0	0
1	1	1
2	4	8



(b) $x = t^2$

$t = \sqrt{x}$

Sub $t = \sqrt{x}$ in $y = t^3$

$y = t^2(t) = x(\sqrt{x}) = x^{3/2}$

$y = x^{3/2}$

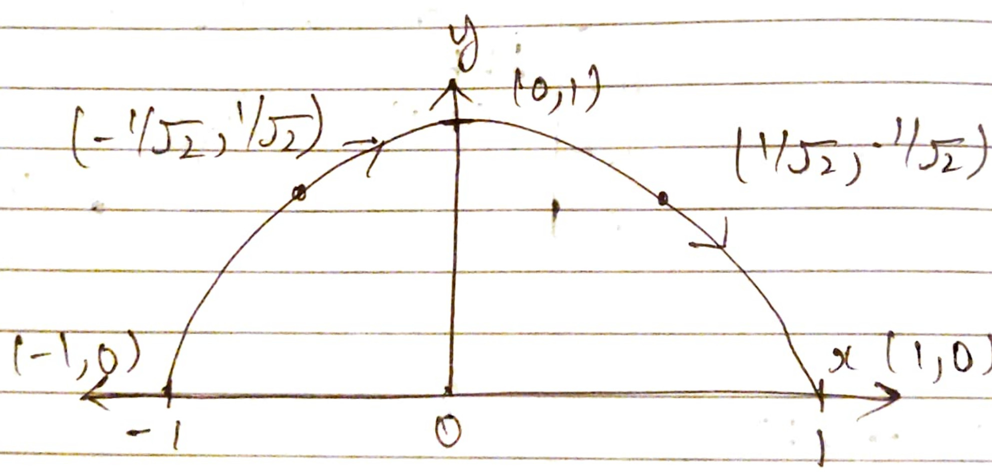
11. (a) $x^2 + y^2 = (\sin \theta/_2)^2 + (\cos \theta/_2)^2$

$= \sin^2 \theta/_2 + \cos^2 \theta/_2 = 1$

Therefore the cartesian equation of the curve is $x^2 + y^2 = 1, \; y \geq 0$

(b)

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	-1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1
y	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0



13. $y = \csc t = \dfrac{1}{\sin t} = 1/x \quad [\sin t = x]$

Therefore the Cartesian equation of the curve is $y = 1/x$

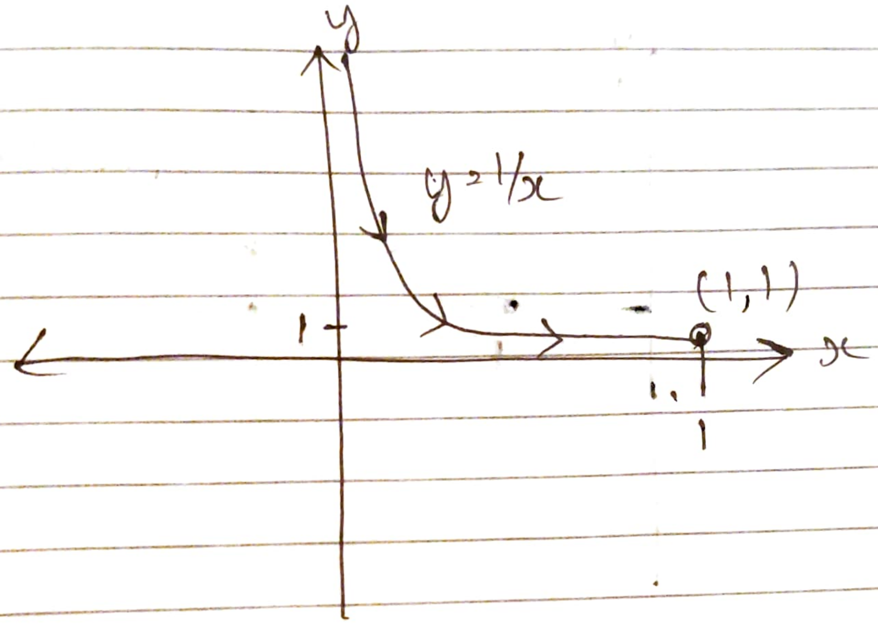
Since $0 < t < \pi/2$

Also, $\sin(0) = 0$ and $\sin(\pi/2) = 1$

This implies $x \in (0,1)$ is the domain of the equation $y = 1/x$

$y = 1/x$, $y \in (1, \infty)$ is the range

(b)



15. (a) $y = \ln t$

$e^y = e^{\ln t}$

$e^y = t$

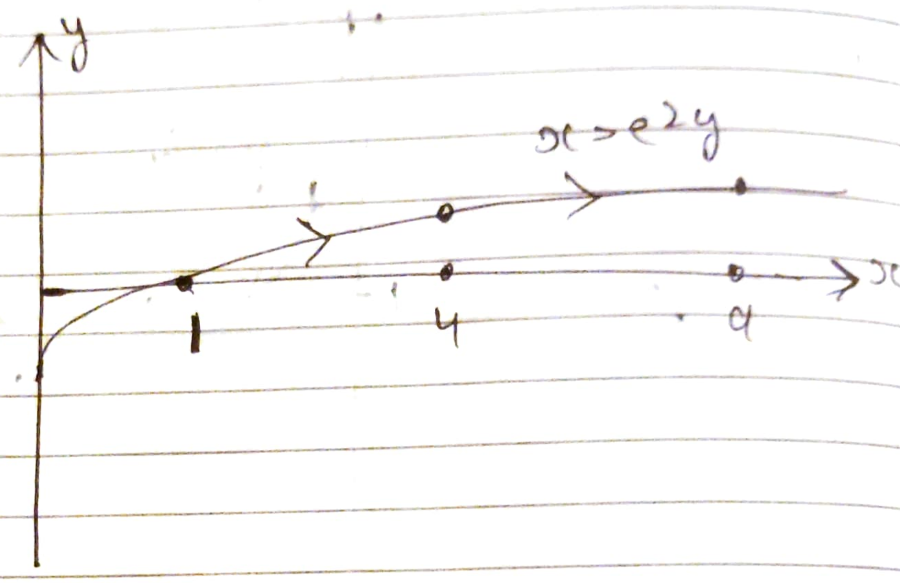
Substitute $e^y = t$ in $x = t^2$

$x = t^2 = (e^y)^2 = e^{2y}$

Thus, the required Cartesian equation of the curve is $x = e^{2y}$

(b)

t	$x = t^2$	$y = \ln t$
1	1	0
2	4	0.693
3	9	1.099
4	16	1.386



17. $\cosh t = \dfrac{e^t + e^{-t}}{2}$ $\sinh t = \dfrac{e^t - e^{-t}}{2}$

Then,

$x + y = \sinh t + \cosh t$

$= \dfrac{e^t - e^{-t} + e^t + e^{-t}}{2} = \dfrac{2e^t}{2} = e^t \quad \ldots(1)$

And

$x - y = \sinh t - \cosh t$

$$= \frac{e^t - e^{-t} - (e^t + e^{-t})}{2} = \frac{-2e^{-t}}{2}$$

$$= -e^{-t} \quad \ldots (2)$$

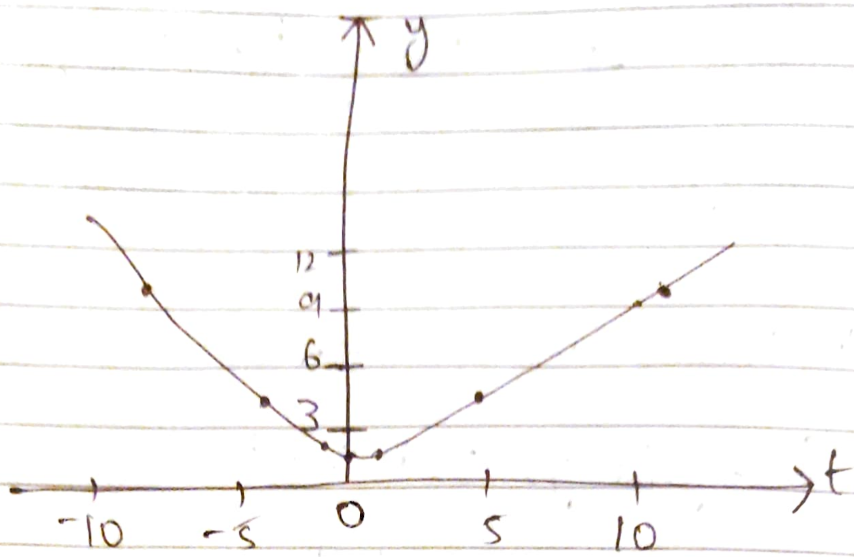
Multiplying eq. (1) and (2)

$$(x+y)(x-y) = e^t(-e^{-t})$$

$$x^2 - y^2 = -e^{t-t} = -e^0 = -1$$

$$\Rightarrow y^2 - x^2 = 1$$

t	$x = \sinh t$	$y = \cosh t$
-4	-27.290	27.380
-3	-10.018	10.068
-2	-3.627	3.762
-1	-1.175	1.543
0	0	1
1	1.175	1.543
2	3.627	3.762
3	10.018	10.068
4	27.290	27.608



19. When $t = 1$,

$x = 5 + 2\cos\pi$

$= 5 + 2(-1) = 3$

$y = 3 + 2\sin\pi$

$= 3 + 2 \cdot 0 = 3$

Thus: $(x, y) = (3, 3)$

When $t = 2$

$x = 5 + 2\cos 2\pi$

$= 5 + 2(1) = 7$

$y = 3 + 2\sin 2\pi$

$= 3 + 2 \cdot 0 = 3$

Thus $(x, y) = (7, 3)$

$x = 5 + 2\cos\pi t$

$x - 5 = 2\cos\pi t$

$\frac{x-5}{2} = \cos\pi t$

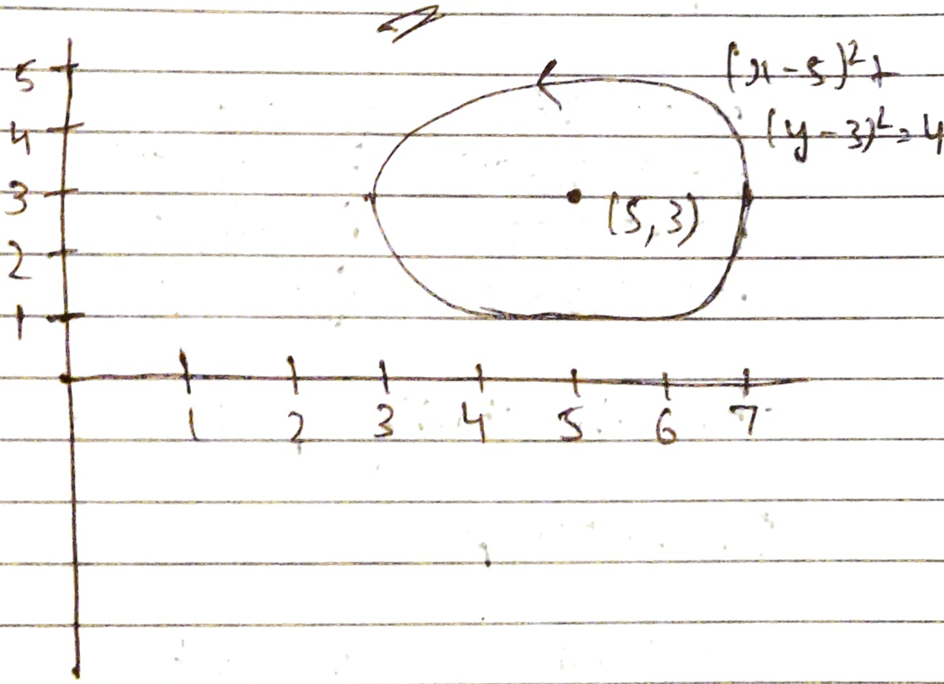
$y = 3 + 2\sin\pi t$

$y - 3 = 2\sin\pi t$

$\frac{y-3}{2} = \sin\pi t$

$\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cancel{\cos^2\pi t + \sin^2\pi t}$

$\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$



21. $\frac{x}{5} = \sin t$

$\frac{y}{2} = \cos t$

By trigonometric identity,

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{25} + \frac{y^2}{4} = 1 \quad \ldots (1)$$

This represents an ellipse with x-intercepts $x = \pm 5$ and the y-intercepts $y = \pm 2$

For $t = -\pi$

$x = 5\sin t$ $\qquad$ $y = 2\cos t$
$x = 5\sin(-\pi)$ $\qquad$ $y = 2\cos(-\pi)$
$x = -5\sin\pi$ $\qquad$ $y = 2\cos\pi$
$x = -5(0) = 0$ $\qquad$ $y = 2(-1) = -2$

Thus, the starting point is $(x, y) = (0, -2)$

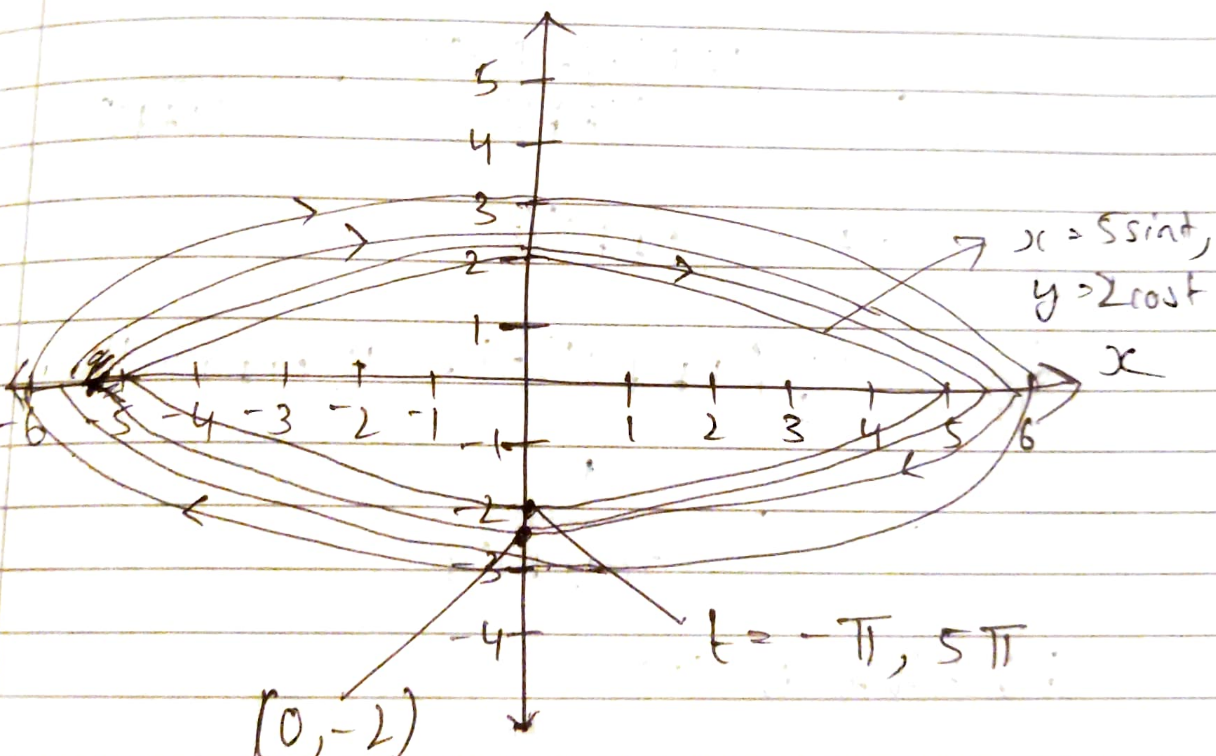
For $t = 5\pi$,

$x = 5\sin t$ $\qquad$ $y = 2\cos t$
$x = 5\sin(5\pi)$ $\qquad$ $y = 2\cos(5\pi)$
$x = 5\sin(2(2\pi) + \pi)$ $\qquad$ $y = 2\cos(2(2\pi) + \pi)$
$x = 5\sin\pi$ $\qquad$ $y = 2\cos\pi$
$x = 5(0)$ $\qquad$ $y = 2(-1)$

Thus, the ending point is $(x, y) = (0, -2)$

Since, t increases from $-\pi$ to 5π and $5\pi - (-\pi) = 6\pi = 3(2\pi)$

Therefore, the particle moves 3 times clockwise around the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$, starting and ending at $(0, -2)$.



URBAN EDGE

Exercise 10.2

1. $dy/dx = \dfrac{(dy/dt)}{(dx/dt)}$ ..... (1)

$$\frac{dx}{dt} = \frac{(1+t)\cdot d/dt\,(t) - t\cdot d/dt\,(1+t)}{(1+t)^2}$$

$$= \frac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$y = \sqrt{1+t} = (1+t)^{1/2}$$

$$\frac{dy}{dt} = 1/2\,(1+t)^{1/2-1} = 1/2\,(1+t)^{-1/2}$$

$$= \frac{1}{2(1+t)^{1/2}}$$

Substitute in (1)

$$\frac{dy}{dx} = \frac{1}{2(1+t)^{1/2}} \times \frac{(1+t)^2}{1} = 1/2\,(1+t)^{2-1/2}$$

$$= 1/2\,(1+t)^{3/2}$$

3. $\frac{dx}{dt} = \frac{d}{dt}(t^3+1) = 3t^2$

$\frac{dy}{dt} = \frac{d}{dt}(t^4+t) = 4t^3+1$

$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) / \left(\frac{dx}{dt}\right) = \frac{4t^3+1}{3t^2}$

The slope at $t=-1$ is,

$m = \left(\frac{dy}{dx}\right)_{t=-1} = \frac{4(-1)^3+1}{3(-1)^2} = \frac{-4+1}{3}$

$= -1$

The value of x when $t=-1$

$x = t^3+1 = -1+1 = 0$

The value of y when $t-1$

$y = t^4+t' = 1-1 = 0$

The point when $t=-1$ is $(x,y) = (0,0)$

The equation to the tangent at $t=-1$ is:

$y-0 = -1(x-0)$

$y = -x$

7. (a) $\frac{dx}{dt} = \frac{d}{dt}(1+\ln t) = 1/t$

$\frac{dy}{dt} = \cancel{2t}\ d/dt\,(t^2+2) = 2t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{dy}{dx} = \frac{2t}{1/t} = 2t^2$

Substite (1, 3) in x

$x = 1 + \ln t$
$1 = 1 + \ln t$
$t = e^0 = 1$

At $t = 1$

$\frac{dy}{dx} = 2t^2 = 2(1)^2 = 2$

Therefore, the slope of the tangent line at point (1, 3) is, $m = 2$

The equation of the tangent line at (1, 3) is,

$y - 3 = 2(x - 1)$
$y - 3 = 2x - 2$
$y = 2x + 1$

(b) $x = 1 + \ln t$
$\ln t = x - 1$
$t = e^{-x-1}$

Sub in y

$y = (e^{x-1})^2 + 2 = 2 + e^{2x-2}$

The slope at (1,3) is,

$\frac{dy}{dx} = 2e^{2x-2}$

$\left.\frac{dy}{dx}\right|_{(1,3)} = 2e^{2(1)-2} = 2$

So, $m = 2$

$y - y_1 = m(x - x_1)$
$y - 3 = 2(x - 1)$
$y - 3 = 2x - 2$
$y = 2x + 1$

10. $x = \sin \pi t$
$\frac{dx}{dt} = \pi \cos \pi t$
$y = t^2 + t$
$\frac{dy}{dt} = 2t + 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{\pi\cos\pi t}$$

$\frac{dy}{dx}$ at $t = 1$ is

$$= \frac{2(1)+1}{\pi\cos\pi(1)} = -3/\pi = m$$

At $t = 1$,

$x(t) = \sin\pi t$

$x(1) = 0$

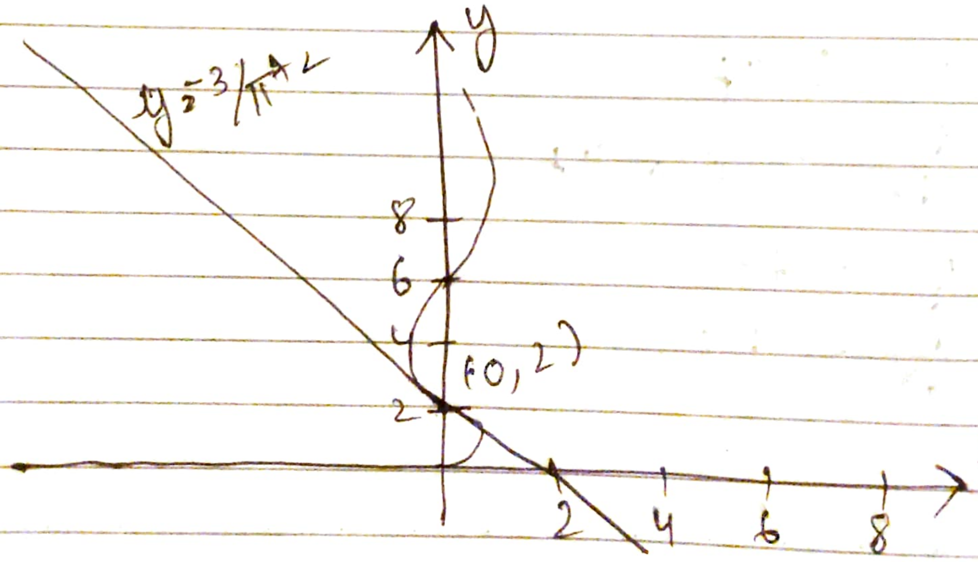
$y(t) = t^2 + 1$

$y(1) = 2$

The equation of the tangent

$$y - 2 = -3/\pi\,(x - 0)$$

$$y = -3/\pi\, x + 2$$



URBAN EDGE

11. $\frac{dx}{dt} = \frac{d}{dt}(t^2+1) = 2t$

$\frac{dy}{dt} = \frac{d}{dt}(t^2+t) = 2t+1$

$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(1+\frac{1}{2t}\right)$

$= \left(-\frac{1}{2t^2}\right) \times \left(\frac{1}{dx/dt}\right)$

$= \left(-\frac{1}{2t^2}\right) \times \left(\frac{1}{2t}\right) = -\frac{1}{4t^3}$

$\frac{d^2y}{dx^2} > 0$ Solve for t

$-\frac{1}{4t^3} > 0$

$\frac{1}{4t^3} < 0$

$t < 0$

Hence, the function f is concave upward for all the values of $t<0$ of $t \in (-\infty, 0)$

URBAN EDGE

15. $x = t - \ln t$

$dx/dt = 1 - 1/t$

$y = t + \ln t$

$\frac{dy}{dt} = 1 + 1/t$

$dy/dx = \frac{dy/dt}{dx/dt} = \frac{1 + 1/t}{1 - 1/t} = \frac{t+1}{t-1}$

$\frac{d^2y}{dx^2} = \frac{d/dt\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{d/dt\left(\frac{t+1}{t-1}\right)}{1 - 1/t}$

$= \frac{\frac{(t-1)-(t+1)}{(t-1)^2}}{\frac{(t-1)}{t}} = \frac{-2t}{(t-1)^3}$

The value of t for which the curve is concave upward is $0<t<1$

17. $\frac{dy}{dt} = \frac{d}{dt}(t^3 - 3) = 2t$

$\frac{dy}{dt} = 0 \Rightarrow 2t = 0 \Rightarrow t = 0$

Put $t = 0$ in x

$x = (0)^3 - 3(0) = 0$

$y = (0)^2 - 3 = -3$

Hence, the curve has horizontal tangent at $(0, -3)$.

$$\frac{dx}{dt} = \frac{d}{dt}(t^3 - 3t)$$

$$= 3t^2 - 3$$

$$\frac{dx}{dt} = 0$$

$3t^2 - 3 = 0$

$3t^2 = 3$

$t^2 = 1$

$t = \pm 1$

For $t = 1$,

$x = (1)^3 - 3(1) = -2$

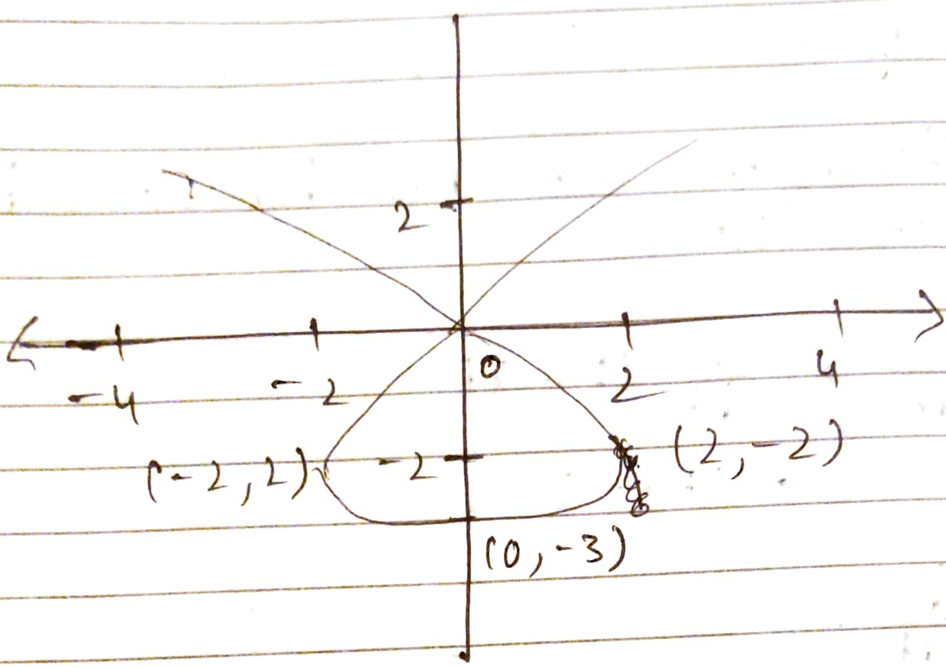
And $y = (1)^2 - 3 = -2$

For $t = -1$

$x = (-1)^3 - 3(-1) = 2$

And $y = (-1)^2 - 3 = -2$

Hence, the curve has vertical tangent at $(-2,-2)$ and $(2,-2)$



25. $x = \cos t$, $y = \sin t \cos t$

Then, $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = -\sin^2 t +$

$\cos^2 t = \cos 2t$

Let the tangent pass through $(0,0)$

$x = 0$

$\Rightarrow \cos t = 0$

$\Rightarrow t = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}$

When $t = \pi/2$

$$\frac{dx}{dt} = -\sin(\pi/2) = -1$$

and $dy/dt = \cos(2\pi/2) = -1$

Then the equation of the tangent at $(0,0)$ is $y = x$

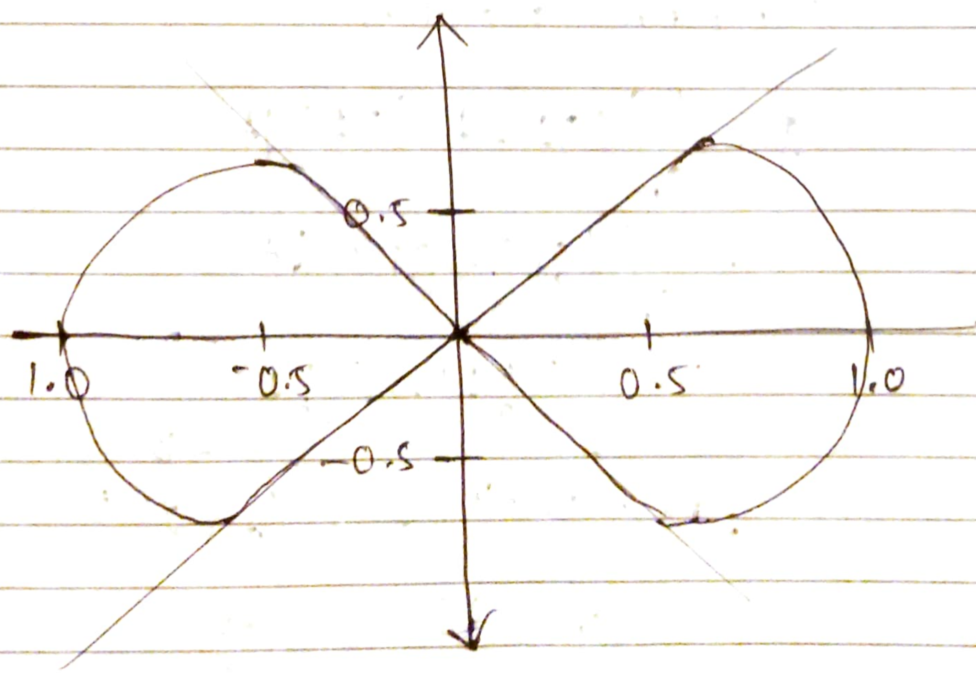
When $t = 3\pi/2$

$dx/dt = -\sin(3\pi/2) = 1$

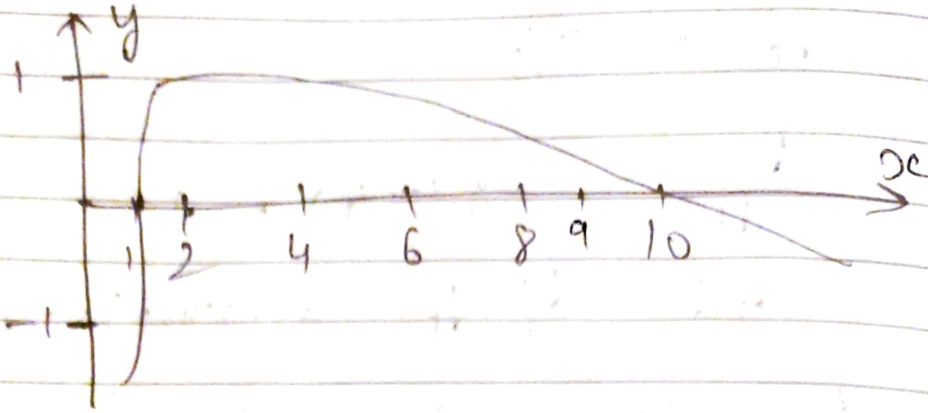
and $dy/dt = \cos(3\pi) = -1$

so $dy/dx = -1$

The ~~equation of the~~ curve has two tangents, $y = x$ and $y = -x$ at $(0,0)$



33.



$$A = \int_{\alpha}^{\beta} g(t)\, f'(t)\, dt$$

$f(t) = t^3 + 1$ and $g(t) = 2t - t^2$

$f'(t) = 3t^2$

Let $y = 0$

$2t - t^2 = 0$

$t(2 - t) = 0$

$t = 0$ or $t = 2$

Therefore $t \in (0, 2)$

$$A = \int_0^2 (2t - t^2)\, 3t^2\, dt$$

$$= \int_0^2 (6t^3 - 3t^4)\, dt$$

$$= \left[\frac{3}{2}t^4 - \frac{3}{5}t^5\right]_0^2$$

$$= \frac{3}{2}(2)^4 - \frac{3}{5}(2)^5$$

$$= 24 - \frac{96}{5} = \frac{24}{5}$$

URBAN EDGE

37. $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ ...... (1)

$\frac{dx}{dt} = d/dt\,(t + e^{-t}) = 1 - e^{-t}$

$\frac{dy}{dt} = \frac{d}{dt}(t - e^{-t}) = 1 + e^{-t}$

$= \cancel{d/dt\,(t) + (d/dt\,(e^{-t})}$

$(dx/dt)^2 = (1 - e^{-t})^2 = 1 + e^{-2t} - 2e^{-t}$

$(dy/dt)^2 = (1 + e^{-t})^2 = 1 + e^{-2t} + 2e^{-t}$

$\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)^2 = (1 + e^{-2t} - 2e^{-t}) + (1 + e^{-2t} + 2e^{-t})$

$= 1 + e^{-2x} + 1 + e^{-2t} = 2 + 2e^{-2t}.$

Substitute in (1)

$L = \int_0^2 \sqrt{2 + 2e^{-2t}}\, dt$

41. $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = 1+3t^2$ $\qquad$ $y = 4+2t^3$

$dx/dt = 6t$ $\qquad$ $dy/dt = 6t^2$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2$

$= 36t^2 + 36t^4 = 36t^2(1+t^2)$

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 6t\sqrt{1+t^2}$

$L = \int_{0}^{1} 6t\sqrt{1+t^2}\, dt = 6\int_{0}^{1} \frac{\sqrt{u}\, du}{2}$

$= 3\int_{0}^{1} \sqrt{u}\, du = 3\left[\frac{u^{3/2}}{3/2}\right]$

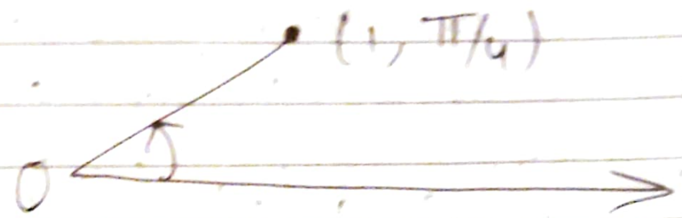
$= 2\left[(1+t^2)^{3/2}\right]_0^1 = 2(2^{3/2}-1)$

$= 4\sqrt{2} - 2$

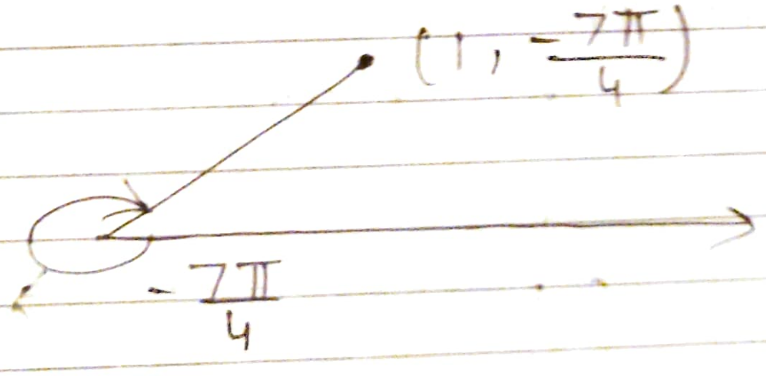
EDG3

Exercise 10.3

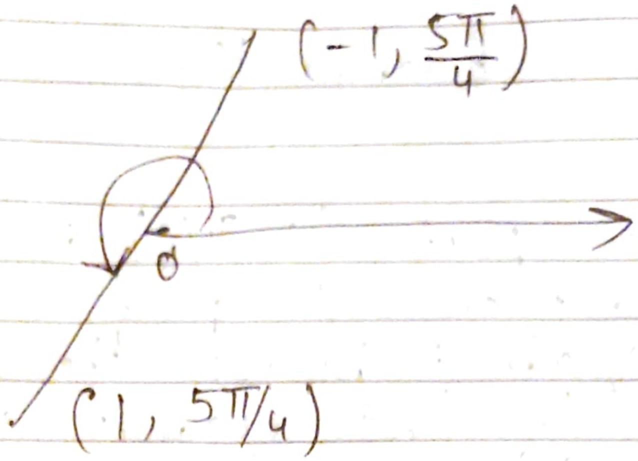
1. (a) The point $(1, \pi/4)$ is located one unit from the pole in the first quadrant because the angle $\pi/4$ is in the first quadrant and $r=1$ is positive.



If the angle is measured in clockwise direction, then the corresponding point is $\left(1, -\frac{7\pi}{4}\right)$



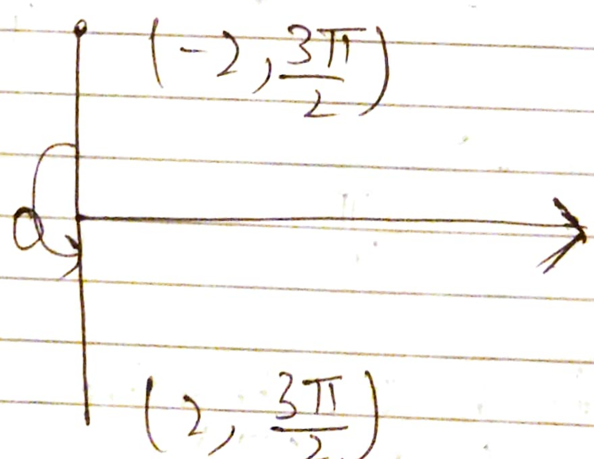
If $r<0$, then the corresponding point is $\left(-1, \frac{5\pi}{4}\right)$



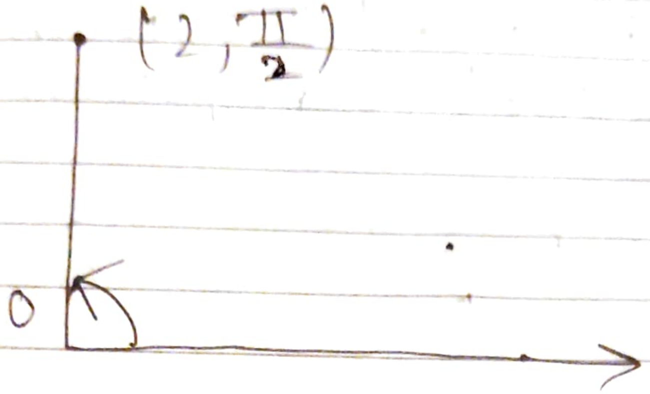
Therefore, the other pairs of the polar coordinates $(1, \pi/4)$ are

$$\left(1 - \frac{7\pi}{4}\right) \text{ and } \left(-1, \frac{5\pi}{4}\right)$$

(b) The point $\left(-2, \frac{3\pi}{2}\right)$ is located two units from the pole in first quadrant because the angle $\frac{3\pi}{2}$ is in third quadrant and $r = -2$ is negative.

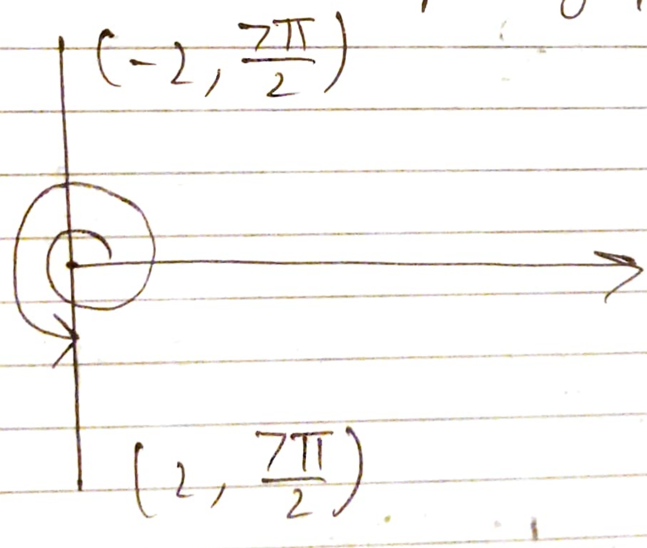


If $r > 0$ then the corresponding point is $\left(2, \frac{\pi}{2}\right)$ because if $r > 0$, then the ray lies in the first quadrant



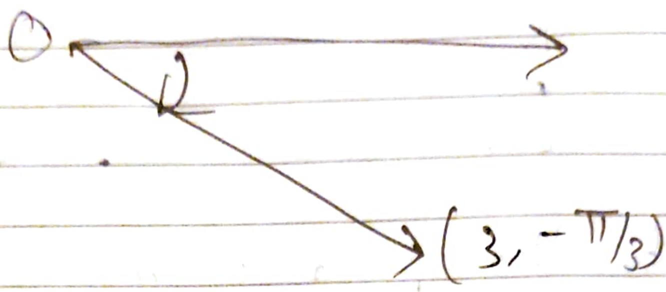
If the angle is measured in clockwise direction, then the corresponding point is $(2, \frac{\pi}{2})$

If $r<0$ then corresponding point is $(-2, \frac{7\pi}{2})$

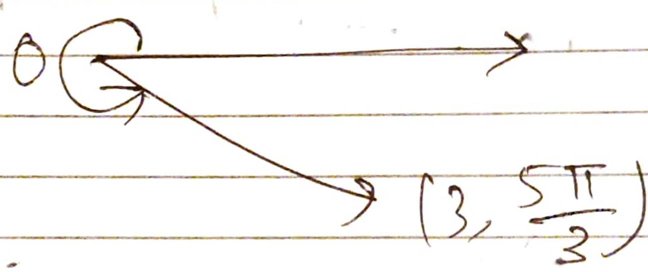


Therefore, the other pair of polar coordinates of $(-2, \frac{3\pi}{2})$ are $(2, \pi/2)$ and $(-2, \frac{7\pi}{2})$

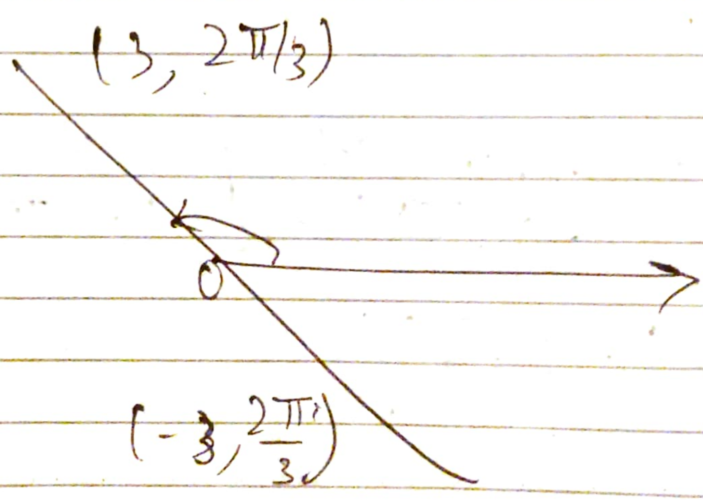
(c) The point $(3, -\pi/3)$ is located three units from the pole in fourth quadrant because the angle $-\pi/3$ is in fourth quadrant and $r=3$ is positive



If the angle is measured in anticlockwise direction, then the corresponding point is
$\left(3, \frac{5\pi}{3}\right)$

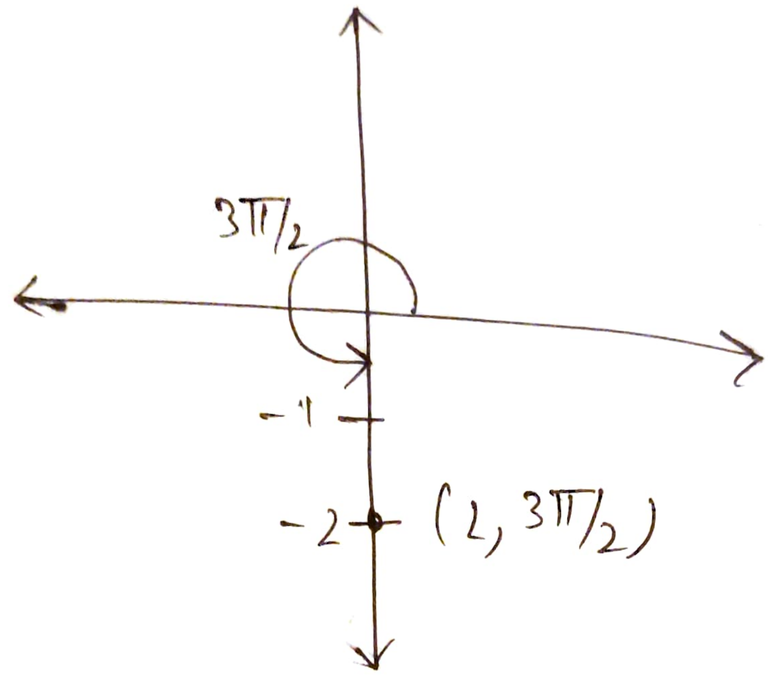


If $r<0$, then the corresponding point is
$\left(-3, \frac{2\pi}{3}\right)$



Therefore, the other pairs of coordinates of $(3, -\pi/3)$ are $(3, \frac{5\pi}{3})$ and $(-3, \frac{2\pi}{3})$

3. (a)



$x = r\cos\theta$ and $y = r\sin\theta$

$r = 2$ and $\theta = 3\pi/2$

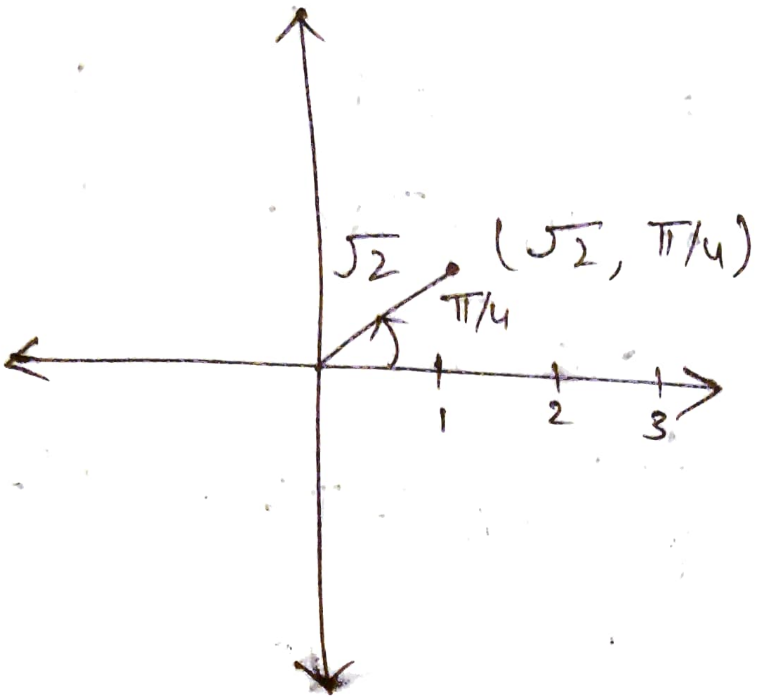
$x = 2\cos 3\pi/2 = 2\cdot(0) = 0$

And

$y = 2\sin \frac{3\pi}{2} = 2(-1) = -2$

Therefore the Cartesian coordinate is $(0, -2)$

(b)



$x = r\cos\theta$ and $y = r\sin\theta$

$r = \sqrt{2}$ and $\theta = \pi/4$

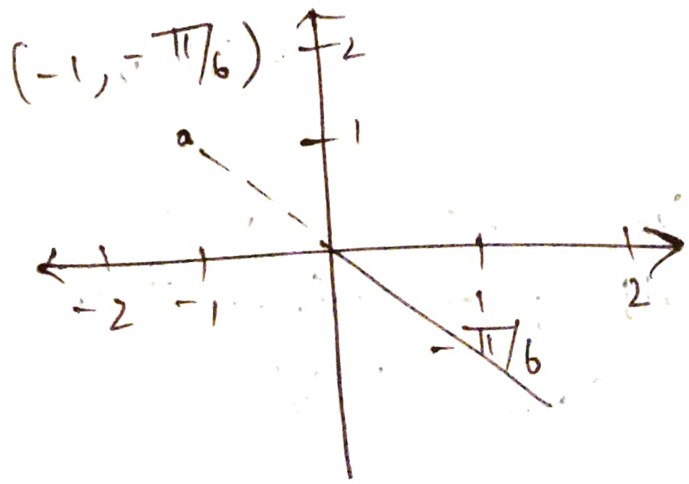
$x = \sqrt{2}\cos \pi/4 = \sqrt{2}(1/\sqrt{2}) = 1$

And

$y = \sqrt{2}\sin(\pi/4) = \sqrt{2}(1/\sqrt{2}) = 1$

Thus, the Cartesian coordinate is $(1,1)$

(c)



$x = r\cos\theta$ and $y = r\sin\theta$

$r = -1$ and $\theta = -\pi/6$

$x = -1\cos(-\pi/6) = -1\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{\sqrt{3}}{2}$

And,

$y = -1\sin(-\pi/6) = -1(-1/2)$

$= 1/2$

Thus, the Cartesian coordinate is

$\left(-\frac{\sqrt{3}}{2}, 1/2\right)$

5. (a) $r = \sqrt{x^2+y^2}$

$= \sqrt{(-4)^2+(4)^2} = \sqrt{32} = 4\sqrt{2}$

$\theta = \tan^{-1}(y/x) = \tan^{-1}(4/-4)$

$= \tan^{-1}(-1) = \pi - \pi/4$

$= 3\pi/4$

(i) When $r > 0$ and $0 \leq \theta \leq 2\pi$ the polar coordinates are,

$(4\sqrt{2}, 3\pi/4)$

(ii) When $r<0$ and $0 \leq \theta \leq 2\pi$
the polar coordinates are,

$$\left(-4\sqrt{2}, \frac{3\pi}{4} + \pi\right) = \left(-4\sqrt{2}, \frac{7\pi}{4}\right)$$

(b) $r = \sqrt{x^2+y^2}$

$$= \sqrt{(3)^2 + (3\sqrt{3})^2} = \sqrt{9+27} = 6$$

$\theta = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{3\sqrt{3}}{3}\right)$

$= \tan^{-1}(\sqrt{3}) = \pi/3$

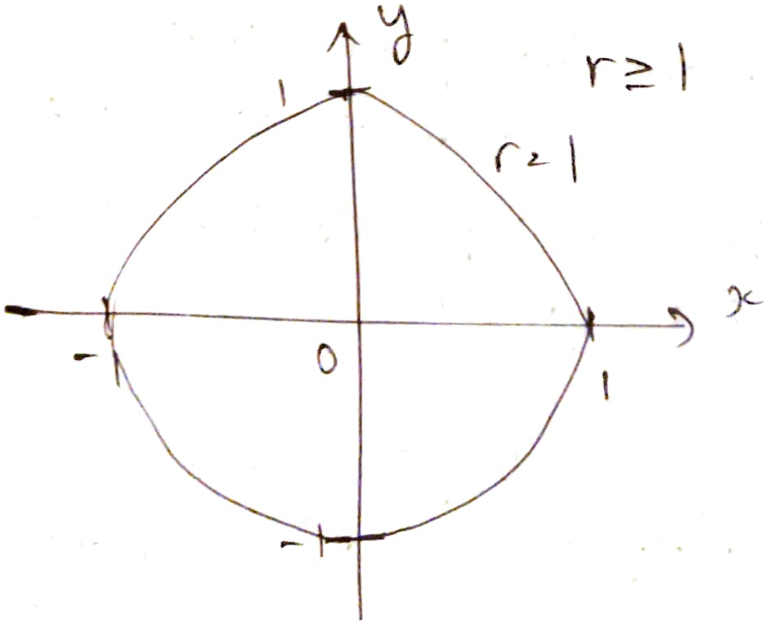
(i) When $r>0$ and $0 \leq \theta \leq 2\pi$,

$(6, \pi/3)$

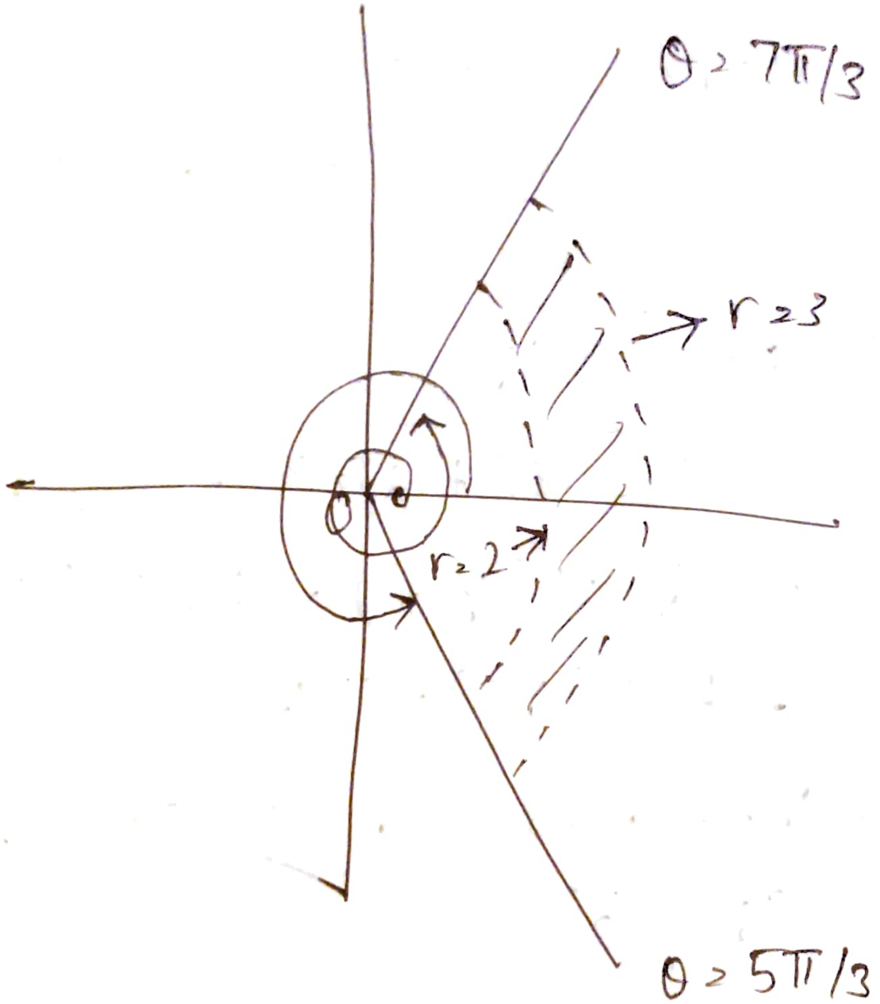
(ii) When $r<0$ and $0 \leq \theta \leq 2\pi$

$$(-6, \pi/3 + \pi) = (-6, 4\pi/3)$$

7. $r \geq 1$



11.



15. $r^2 = 5$

$x^2 + y^2 = 5$

Therefore the given curve in Cartesian coordinates is $\Rightarrow x^2+y^2=5$

The curve represents the circle $(0,0)$ and radius $\sqrt{5}$

17. $x = r\cos\theta, y = r\sin\theta \quad \ldots (1)$

$x^2+y^2 = r^2$ and $\tan\theta = y/x \quad \ldots (2)$

Here $x = r\cos\theta$ then $\cos\theta = x/r$

$r = 5\cos\theta$

$r = 5(x/r) = 5x/r$

$5x = r^2$

Sub. in (2)

$x^2 + y^2 = r^2 = 5x$

$x^2 + y^2 - 5x = 0$

$x^2 - 5x + y^2 = 0$

$x^2 - 5x + (5/2)^2 + y^2 = (5/2)^2$

(Add 5/2 on both sides)

$(x - 5/2)^2 + y^2 = \frac{25}{4}$

Therefore, the cartesian equation of a curve represents circle with center $(5/2, 0)$ and radius $5/2$

21. $y = 2$

$r\sin\theta = 2$

$r = 2/\sin\theta = 2\csc\theta$

Hence, the required equation in the polar system is $r = 2\csc\theta$

23. $y = r\sin\theta \,,\, x = r\cos\theta$

$y = 1 + 3x$

$\Rightarrow r\sin\theta = 1 + 3r\cos\theta$

$\Rightarrow r(\sin\theta - 3\cos\theta) = 1$

$\Rightarrow r = \dfrac{1}{\sin\theta - 3\cos\theta}$

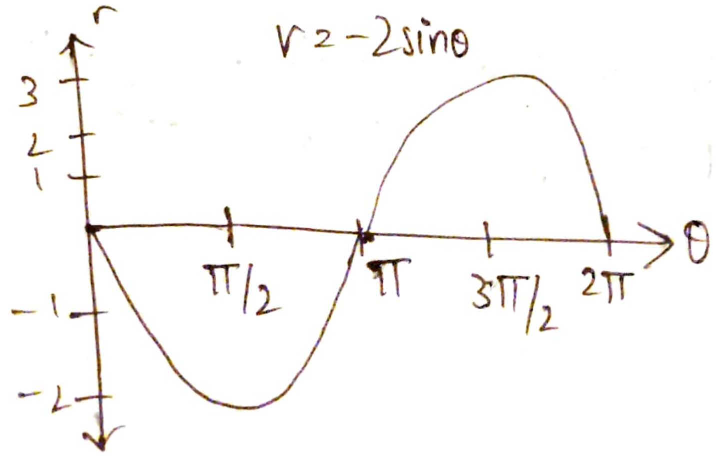
28. (A) $(x-2)^2 + (y-3)^2 = 5^2$

(B) $r = 4$

$x^2 + y^2 = 4^2 = 16$

29.

θ	$r = -2\sin\theta$
0	0
$\pi/2$	-2
π	0
$\frac{3\pi}{2}$	2



$r = -2\sin\theta$

Multiply with r on both sides

$r^2 = -2r\sin\theta$ .... (1)

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\sin^2\theta + \cos^2\theta) = r^2$.

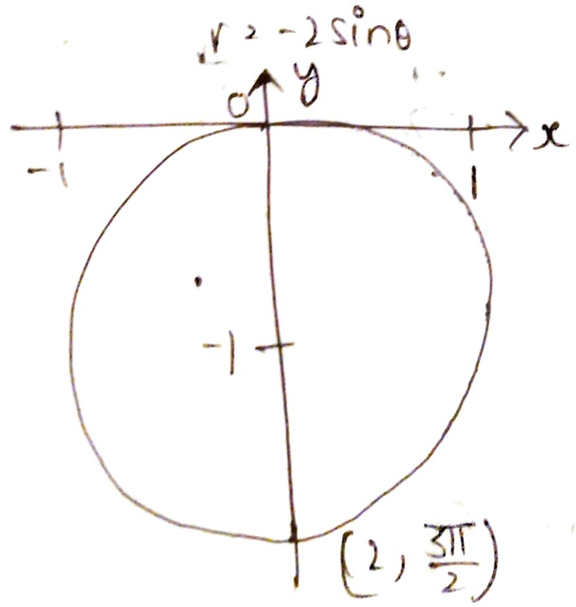
Substitute in (1)

$x^2 + y^2 = -2y$

$x^2 + y^2 + 2y = 0$

$x^2 + (y+1)^2 = 1$

This is a circle of radius 1 centered at $(0, -1)$



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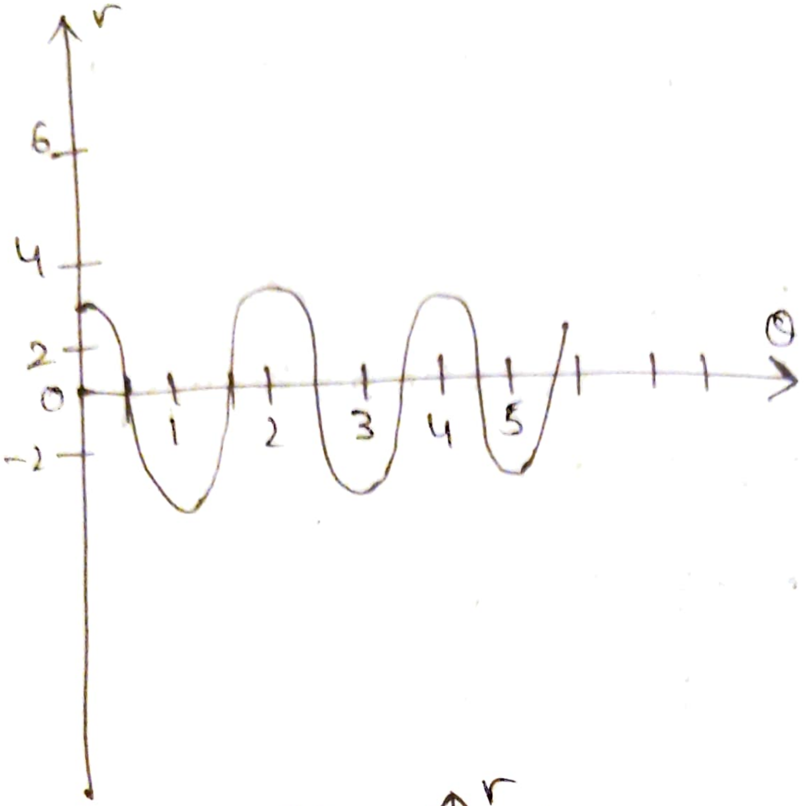

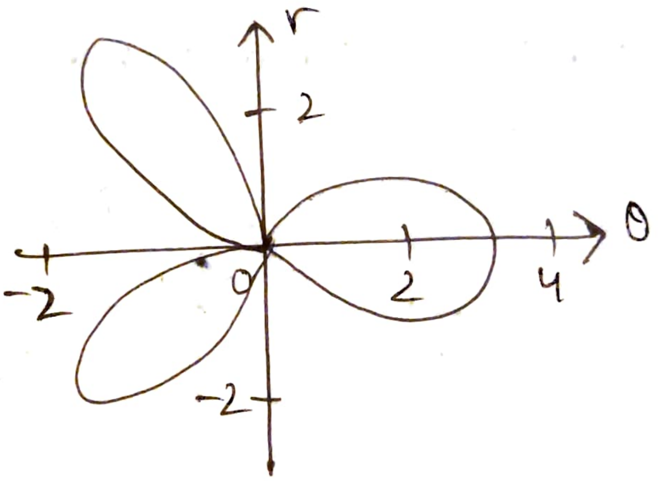

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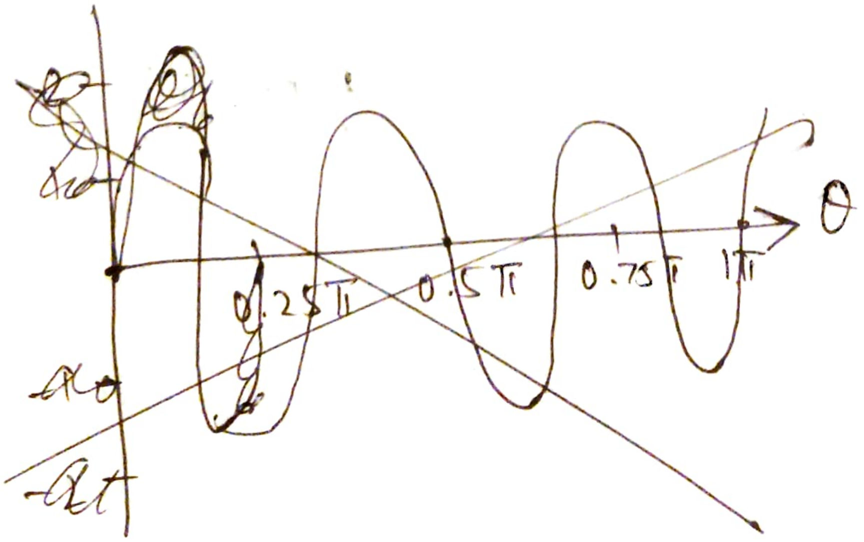
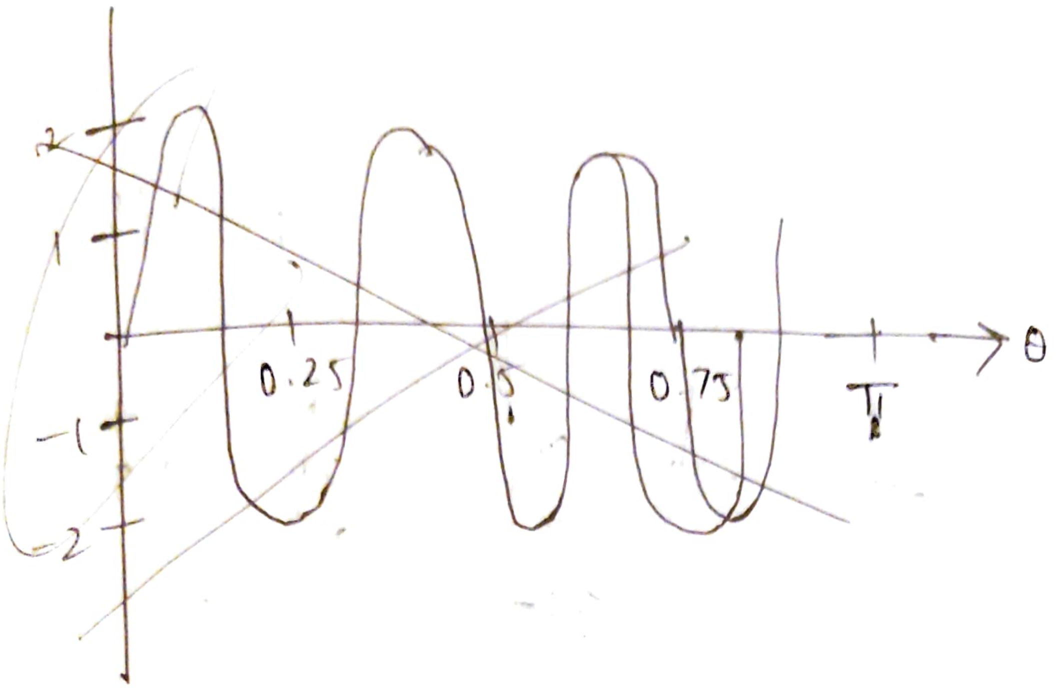
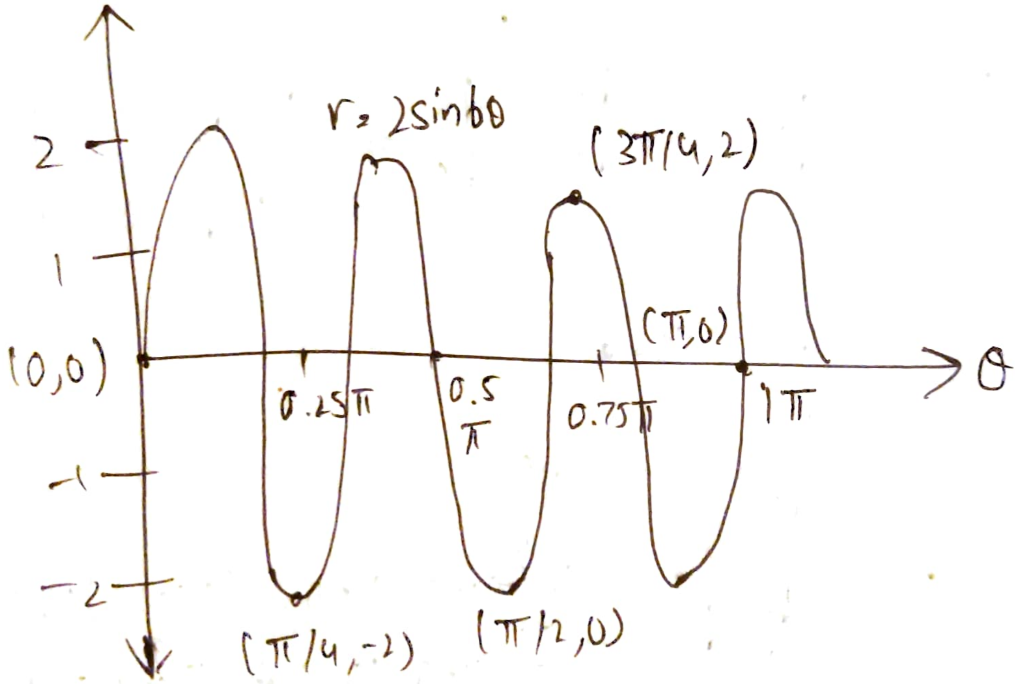
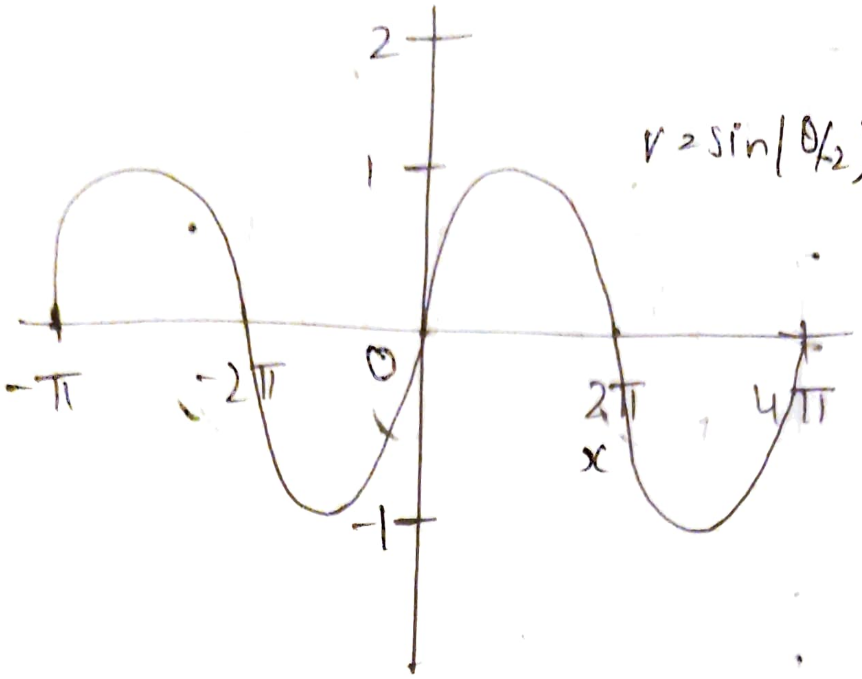






θ	$r = 2\sin 6\theta$
0	0
$\pi/4$	-2
$\pi/2$	0
$3\pi/4$	2
π	0

45.



55. $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{dr/d\theta \sin\theta + r\cos\theta}{dr/d\theta \cos\theta - r\sin\theta}$

$r = 2\cos\theta$

$\frac{dr}{d\theta} = -2\sin\theta$

$dy/dx = \frac{(-2\sin\theta)\sin\theta + (2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (2\cos\theta)\sin\theta}$

$= \frac{2[\cos^2\theta - \sin^2\theta]}{-4\cos\theta\sin\theta}$

$= \frac{2\cos 2\theta}{-4\cos\theta\sin\theta}$

Substitute $\theta = \pi/3$

$$\left.dy/dx\right|_{\theta=\pi/3} = \frac{2\cos\left(\frac{2\pi}{3}\right)}{-4\cos(\pi/3)\sin(\pi/3)}$$

$$= \frac{-1}{-\sqrt{3}} = 1/\sqrt{3}$$

59. $$\frac{dy}{dx} = \frac{dr/d\theta\,\sin\theta + r\cos\theta}{dr/d\theta\,\cos\theta - r\sin\theta}$$

$$= \frac{\left(d/d\theta(\cos 2\theta)\right)\sin\theta + (\cos 2\theta)\cdot\cos\theta}{\left(d/d\theta(\cos 2\theta)\right)\cos\theta - (\cos 2\theta)\sin\theta}$$

$$= \frac{(-2\sin 2\theta)\sin\theta + \cos 2\theta\cos\theta}{(-2\sin 2\theta)\cos\theta - \cos 2\theta\sin\theta}$$

$$= \frac{2\sin 2\theta\sin\theta - \cos 2\theta\cos\theta}{2\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{2\sin(2\cdot\pi/4)\sin\pi/4 - \cos(2\cdot\pi/4)\cos\pi/4}{2\sin(2\pi/4)\cos\pi/4 + \cos(2\pi/4)\sin(\pi/4)}$$

$$= \frac{2\cdot 1\cdot 1/\sqrt{2} - 0\cdot 1/\sqrt{2}}{2\cdot 1\cdot 1/\sqrt{2} + 0\cdot 1/\sqrt{2}} = 1$$

61. $x = r\cos\theta$, $y = r\sin\theta$

$r = 3\cos\theta$

$x = r\cos\theta = 3\cos\theta \cdot \cos\theta = 3\cos^2\theta$

$y = r\sin\theta = 3\cos\theta\sin\theta$

$= 3/2\,(2\cos\theta \cdot \sin\theta)$

$= 3/2\,(\sin 2\theta)$

$\frac{dx}{d\theta} = d/d\theta\,(3\cos^2\theta) = 3\,d/d\theta(\cos^2\theta)$

$= 3(2\cos\theta) \cdot d/d\theta\,(\cos\theta)$

$= 3\cancel{2}(\cos\theta)(-\sin\theta)$

$= 3(-2\cos\theta\sin\theta)$

$= -3\sin(2\theta)$

$\frac{dy}{d\theta} = d/d\theta\,(3/2\sin 2\theta)$

$= 3/2\; d/d\theta\,(\sin 2\theta)$

$= 3/2\,(2\cos 2\theta)$

$= 3\cos 2\theta$

$dy\cancel{\theta}/dx = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos 2\theta}{-3\sin 2\theta} = \frac{\cos 2\theta}{-\sin 2\theta}$

Set $dy/dx = 0$

$$\frac{-\cos 2\theta}{\sin 2\theta} = 0$$

$\cos 2\theta = 0$

$\cos 2\theta = \cos \pi/2, \cos 3\pi/2, \cos 5\pi/2$

$2\theta = \pi/2, \frac{3\pi}{2}, \dots$

$\theta = \pi/4, 3\pi/4$

Substitute these values in $r = 3\cos\theta$

At $\theta = \pi/4$,

$r = 3\cos(\pi/4) = 3(1/\sqrt{2}) = 3/\sqrt{2}$

At $\theta = 3\pi/4$

$r = 3\cos(3\pi/4) = 3(-1/\sqrt{2}) = -3/\sqrt{2}$

Thus, the points on the given curve where tangent line is at horizontal, $(3/\sqrt{2}, \pi/4)$ and $(-3/\sqrt{2}, 3\pi/4)$

Set $dx/dy = 0$

$$\frac{-\sin(2\theta)}{\cos(2\theta)} = 0$$

$\sin 2\theta = 0$

$\sin 2\theta = \sin 0, \sin \pi$

$\theta = 0$ or $\pi/2$

Substitute those values in $r = 3\cos\theta$

At $\theta = 0$

$r = 3\cos(0) = 3(1) = 3$

At $\theta = \pi/2$

$r = 3\cos(\pi/2) = 3(0) = 0$

Thus, the points on the given curve which the tangent line is vertical,

$(3, 0)$ and $(0, \pi/2)$

Exercise 10.4

1. $A = \int_a^b \frac{1}{2} r^2 d\theta \quad \ldots (1)$

$a \le \theta \le b$

$a = \pi/2,\ b = \pi$

$r = {}^{-\theta/4}$

Sub. in (1)

$A = \frac{1}{2} \int_{\pi/2}^{\pi} (e^{-\theta/4})^2 d\theta$

$= \frac{1}{2} \int_{\pi/2}^{\pi} \left(e^{-\frac{2\theta}{4}}\right) d\theta$

$= \frac{1}{2} \int_{\pi/2}^{\pi} (e^{-\theta/2}) d\theta$

$= \frac{1}{2} \left[\frac{e^{-\theta/2}}{-1/2} \right]_{\pi/2}^{\pi}$

$= 1/2 \left[-2e^{-\theta/2} \right]_{\pi/2}^{\pi}$

$= 2 \cdot 1/2 \left[e^{-\pi/2} - e^{-\pi/4} \right]$

$= -\left(e^{-\pi/2} - e^{-\pi/4} \right)$

$A = e^{-\pi/4} - e^{-\pi/2}$

5. $A = \int_a^b 1/2 \, r^2 \, d\theta$

$r^2 = \sin 2\theta$

$\sin 2\theta = 0$

$2\theta = 0$

$2\theta = 0, \pi$

$\theta = 0, \pi/2$

$A = \int_0^{\pi/2} 1/2 (\sin 2\theta) \, d\theta$

$= 1/2 \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2}$

$= -1/4 \left[\cos 2\theta \right]_0^{\pi/2}$

$= -1/4 \left[\cos 2(\pi/2) - \cos(0) \right]$

$= -1/4 \left[\cos \pi - \cos 0 \right] = -1/4 [-1-1]$

$= -1/4 [-2] = 1/2$

7. $A = \int_a^b \frac{1}{2} r^2 d\theta$

Substitute $r = 4+3\sin\theta$ and $a = -\pi/2$, $b = \pi/2$

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(4+3\sin\theta)^2 d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 9\sin^2\theta + 24\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 16\,d\theta + \frac{1}{2}\cdot 9\int_{-\pi/2}^{\pi/2} \sin^2\theta\,d\theta +$$

$$\frac{1}{2}\cdot 24\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta$$

$$= 8\int_{-\pi/2}^{\pi/2} d\theta + \frac{9}{2}\int_{-\pi/2}^{\pi/2} \sin^2\theta\,d\theta + 12\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta$$

$$= 8\int_{-\pi/2}^{\pi/2} d\theta + \frac{9}{4}\int_{-\pi/2}^{\pi/2} (1-\cos 2\theta)\,d\theta +$$

$$12\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta$$

$$= 8\int_{-\pi/2}^{\pi/2} d\theta + 9/4\int_{-\pi/2}^{\pi/2} d\theta - 9/4\int_{-\pi/2}^{\pi/2} \cos 2\theta \, d\theta +$$

$$12\int_{-\pi/2}^{\pi/2} \sin\theta \, d\theta$$

$$A = 8[\theta]_{-\pi/2}^{\pi/2} + 9/4[\theta]_{-\pi/2}^{\pi/2} - 9/4\left[\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$+ 12[-\cos\theta]_{-\pi/2}^{\pi/2}$$

$$= 8(\pi/2 + \pi/2) + 9/4(\pi/2 + \pi/2)$$

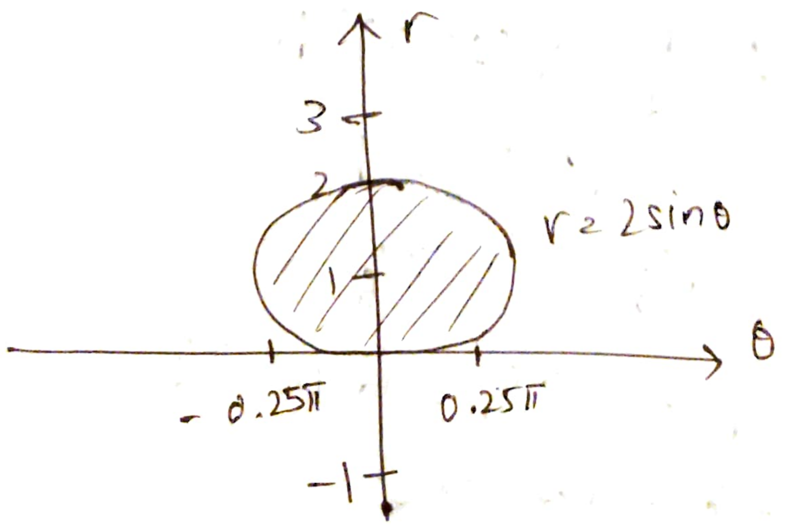
$$- 9/8\left[\sin\frac{2\pi}{2} + \sin\frac{2\pi}{2}\right] - 12\left[\cos\frac{\pi}{2} -\right.$$

$$\left.\cos \pi/2\right]$$

$$= 8\pi + 9/4\pi - 9/8(0+0) - 12(0-0)$$

$$= 41/4\pi$$

9.



$$A = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta)^2)\, d\theta \quad \ldots\ldots (1)$$

$$\alpha \leq \theta \leq \beta$$

$f(\theta) = 2\sin\theta$ and $\beta = \pi$ and

$\alpha = 0$

Substitute in (1)

$$A = \frac{1}{2}\int_{0}^{\pi} 4\sin^2\theta\, d\theta$$

$$= 2\int_{0}^{\pi}\left[\frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= 2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi}$$

$$= 2\left[\frac{(\pi - 0)}{2} - \frac{(0-0)}{4}\right]$$

$$= \pi$$

17. $$A = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2\, d\theta \quad \ldots\ldots (1)$$

$$\alpha \leq \theta \leq \beta$$

$$-\pi/6 \leq \theta \leq \pi/6$$

$\alpha = -\pi/6$ and $\beta = \pi/6$

$f(\theta) = 4\cos(3\theta)$.

Substitute in eq (1),

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (4\cos(3\theta))^2 d\theta$$

$$= 8\int_{-\pi/6}^{\pi/6} \cos^2(3\theta) d\theta$$

$$= 8\int_{-\pi/6}^{\pi/6} \left[\frac{1+\cos(6\theta)}{2}\right] d\theta$$

$$= 4\int_{-\pi/6}^{\pi/6} (1+\cos(6\theta)) d\theta$$

$$A = 4\left[\theta + \frac{\sin(6\theta)}{6}\right]_{-\pi/6}^{\pi/6}$$

$$= 4\left(\pi/6 - (-\pi/6)\right) + 2/3(\sin(\pi) - \sin(-\pi))$$

$$= 4(\pi/6 + \pi/6) + 2/3(0-0)$$

$$A = 4\left(\frac{2\pi}{6}\right) = \frac{4\pi}{3}$$

21. $A = \int_a^b \frac{1}{2} r^2 d\theta$

Take $r = 0$

$1 + 2\sin\theta = 0$

$\sin\theta = -1/2$

$\theta = 7\pi/6, 11\pi/6$ since sine function is negative in Quadrant III and IV

Inner loop interval $\left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$

$$A = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(1+2\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1+4\sin\theta + 4\sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left[1+4\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (3+4\sin\theta - 2\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[3\theta - 4\cos\theta - \sin 2\theta\right]_{7\pi/6}^{11\pi/6}$$

$$= \frac{1}{2}\left[\left(3\cdot\frac{11\pi}{6} - 4\cos\frac{11\pi}{6} - \sin\frac{11\pi}{3}\right) - \left(3\cdot\frac{7\pi}{6} - 4\cos\frac{7\pi}{6} - \sin\frac{7\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{11\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{2}\left[2\pi - 3\sqrt{3}\right] = \pi - \frac{3}{2}\sqrt{3}$$

23. $r = 4\sin\theta$, $r = 2$

Substitute 2 for r in $r = 4\sin\theta$

$4\sin\theta = 2$

$\sin\theta = 1/2$

$\theta = \pi/6, 5\pi/6$

$r_1 = f(\theta), r_2 = g(\theta) \quad a \le \theta \le b$

$$A = \int_a^b 1/2\,(r_1^2 - r_2^2)\,d\theta$$

$$= 1/2 \int_{\pi/6}^{5\pi/6} \left[(4\sin\theta)^2 - (2)^2\right] d\theta$$

$$= 1/2 \int_{\pi/6}^{5\pi/6} (16\sin^2\theta - 4)\,d\theta$$

$$= 1/2 \int_{\pi/6}^{5\pi/6} \left[16\left(\frac{1-\cos 2\theta}{2}\right) - 4\right] d\theta$$

$$= 1/2 \int_{\pi/6}^{5\pi/6} [8 - 8\cos 2\theta - 4]\,d\theta$$

$$= 1/2 \int_{\pi/6}^{5\pi/6} [4 - 8\cos 2\theta]\,d\theta$$

$$= 1/2 \left[4\theta - 8\left(\frac{\sin 2\theta}{2}\right)\right]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left[4\theta - 4\sin 2\theta\right]_{\pi/6}^{5\pi/6}$$

$$= 2\left[\theta - \sin 2\theta\right]_{\pi/6}^{5\pi/6}$$

$$= 2\left[\frac{5\pi}{6} - \sin\left(2\left(\frac{5\pi}{6}\right)\right) - \frac{\pi}{6} + \sin\left(2\left(\frac{\pi}{6}\right)\right)\right]$$

$$= 2\left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= 2\left[\frac{2\pi}{3} + \sqrt{3}\right] = \frac{4\pi}{3} + 2\sqrt{3}$$

27. $3\cos\theta = 1 + \cos\theta$

$\cos\theta = 1/2$

$\theta = \pi/3, -\pi/3$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$

$$= \int_{0}^{\pi/3} \left[9\cos^2\theta - 1 - \cos^2\theta - 2\cos\theta\right] d\theta$$

$$= \int_{0}^{\pi/3} \left[8\cos^2\theta - 2\cos\theta - 1\right] d\theta$$

$$= \int_{0}^{\pi/3} \left[8\left(\frac{1+\cos 2\theta}{2}\right) - 2\cos\theta - 1\right] d\theta$$

$$= \int_0^{\pi/3} [4 + 4\cos 2\theta - 2\cos\theta - 1]\, d\theta$$

$$= \int_0^{\pi/3} [3 + 4\cos 2\theta - 2\cos\theta]\, d\theta$$

Therefore, the area $A = \left[3\theta + \frac{4\sin 2\theta}{2} - 2\sin\theta\right]_0^{\pi/3}$

$$= \left[\pi + \frac{4\sin(2\pi/3)}{2} - 2\sin(\pi/3)\right]$$

$$= \left[\pi + 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\right] = \pi$$

29. $A = 1/2 \int_a^b ([f(\theta)]^2)\, d\theta$

Solve $3\sin\theta = 3\cos\theta$

$\frac{\sin\theta}{\cos\theta} = 1$

$\tan\theta = 1$

So, $\theta = \pi/4$

$$A = 1/2 \int_0^{\pi/4} (3\sin\theta)^2 d\theta + 1/2 \int_{\pi/4}^{\pi/2} (3\cos\theta)^2 d\theta$$

$$= \frac{9}{2}\int_0^{\pi/4} \sin^2\theta\, d\theta + \frac{9}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{9}{2}\int_0^{\pi/4}\left(\frac{1-\cos 2\theta}{2}\right)d\theta + \frac{9}{2}\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{9}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} + \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{9}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right) + \frac{9}{4}\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{9}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right) + \frac{9}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{9}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{9\pi}{8} - \frac{9}{4}$$

37. $r = \sin\theta$, $r = 1 - \sin\theta$

$$\sin\theta = 1 - \sin\theta$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

Thus the values of θ between 0 and 2π that satisfy both the equation are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Therefore, the point of intersection are $\left(\frac{1}{2}, \frac{\pi}{6}\right)$; $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$ and the pole.

41. $r = \sin\theta$, $r = \sin 2\theta$

At the points of intersection,

$\sin 2\theta = \sin\theta$

$\sin\theta - 2\sin\theta\cos\theta = 0$

$\sin\theta(1 - 2\cos\theta) = 0$

$\sin\theta = 0$ and $1 - 2\cos\theta = 0$

$\sin\theta = 0$ and $\cos\theta = 1/2$

If $\sin\theta = 0$ then $\theta = 0°$

Sub. in $r = \sin\theta$

$r = \sin 0° = 0$

So, the point of intersection is $(0, 0)$

If $\cos\theta = 1/2$ then $\theta = \pi/3$

Plug $\theta = \pi/3$ into $r = \sin\theta$

This implies that

$r = \sin \pi/3 = \frac{\sqrt{3}}{2}$

So, the point of intersection is $(\sqrt{3}/2, \pi/3)$

Plug $\theta = \frac{2\pi}{3}$ into $r = \sin\theta$

$r = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

So, the point of intersection is $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$

Thus, the point of intersection of the two curves are $\left(\frac{\sqrt{3}}{2}, \pi/3\right), (\sqrt{3}/2, 2\pi/3)$; $(0, 0)$

45. $r = f(\theta), a \le \theta \le b$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad \ldots\ldots (1)$$

Here $r = 2\cos\theta$

$\Rightarrow \frac{dr}{d\theta} = -2\sin\theta$

From eq (1)

$$L = \int_0^{\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2} d\theta$$

$$= \int_0^{\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta} d\theta$$

$$= \int_0^{\pi} 2\sqrt{\cos^2\theta + \sin^2\theta} d\theta$$

$$L = 2\int_0^{\pi} 1 d\theta$$

$$= 2[\theta]_0^{\pi} = 2(\pi - 0) = 2\pi$$

Exercise 10.5

1. The equation of the parabola with vertex $(0,0)$, focus $(0,p)$ and diretrix $y = -p$ is

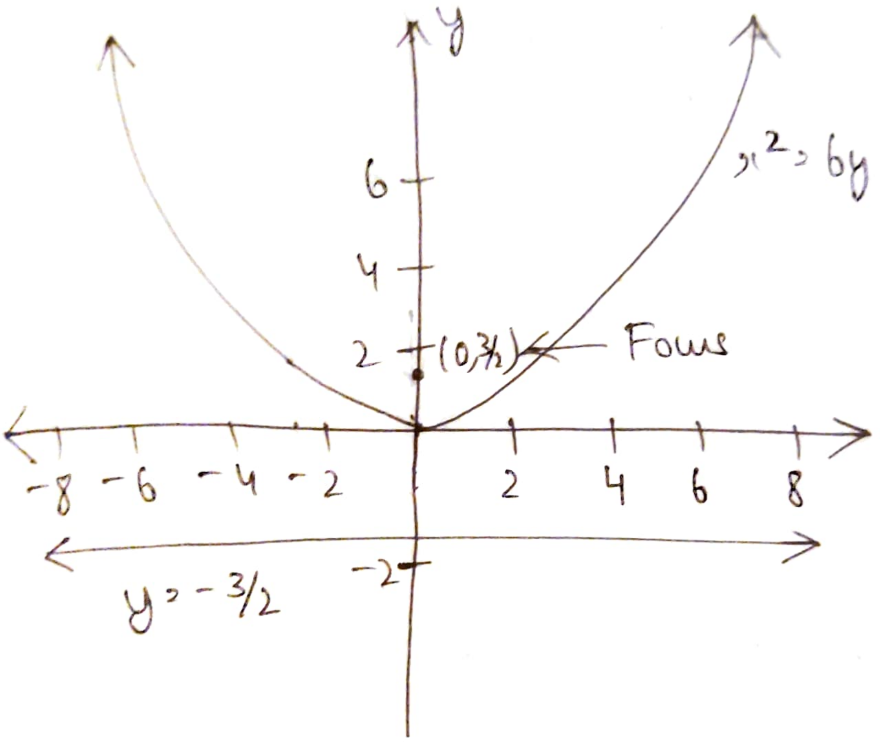
$x^2 = 4py$

Rewrite the equation $x^2 = 6y$

$x^2 = 4(3/2)y$

Comparing with $x^2 = 4py$, the value is $p = 3/2$

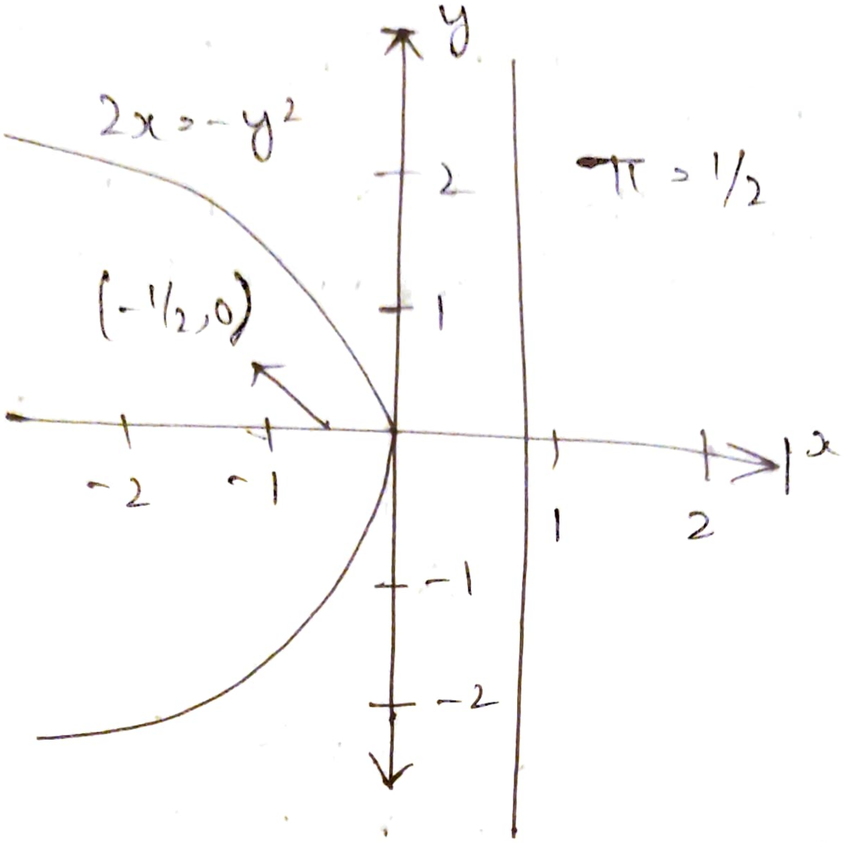
Therefore, the vertex of the parabola is $(0,0)$, focus $(0, 3/2)$ and diretrix is $y = -3/2$



3. Given parabola $y^2 = -2x$
Compare this with standard parabola
$y^2 = -4px$
Then $p = 1/2$
Hence the focus of the parabola is $(-p, 0)$
$= (-1/2, 0)$

Equation of the diretrix is
$x = p \Rightarrow x = 1/2$

Vertex is $(0,0)$



5. $(x+2)^2 = 8(y-3) \ldots\ldots (1)$

Assume, $x+2 = X$ and $y-3 = Y$

The equation becomes,

$X^2 = 8Y \ldots\ldots (2)$

By comparing with $x^2 = 4py$

$4p = 8$ [for equation (2)]

$p = 2$

The focus is $(0, P)$

$\Rightarrow X = 0$ and $Y = 2$

so $x+2=0$ and $y-3=2$

$\Rightarrow x=-2$ and $y=5$

The focus of the parabola is $(-2,5)$

The vertex is at $(0,0)$ [for (2)

$\Rightarrow X=0$ and $Y=0$

$\Rightarrow x+2=0$ and $y-3=0$

$\Rightarrow x=-2$ and $y=3$

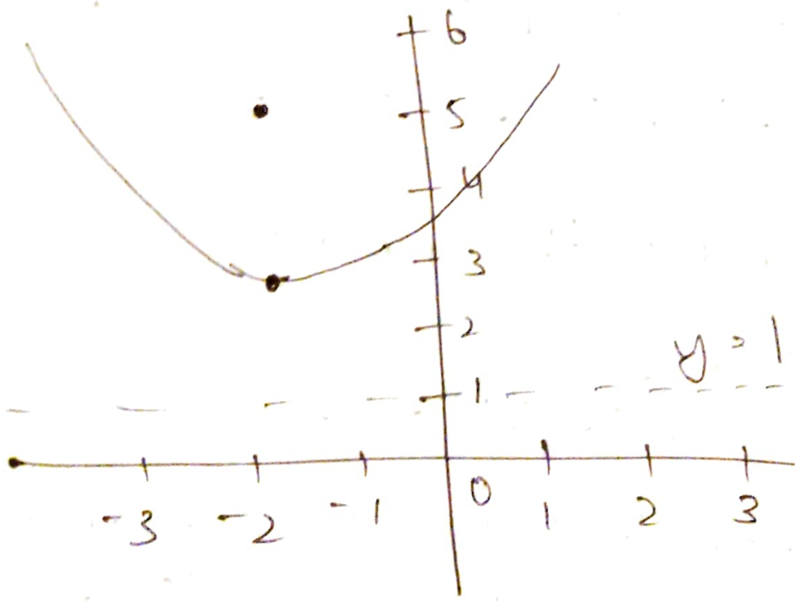
The vertex of the parabola is at $(-2,3)$

The diretrix for (2) is $Y=-P$

$\Rightarrow y-3=-2$

$\Rightarrow y=1$

The diretrix of the parabola is at $y=1$



7. $y^2+6y+2x+1=0$

$$y^2+6y=-2x-1$$

$$y^2+6y+9=-2x-1+9$$

$$(y+3)^2=-2x+8$$

$$(y+3)^2=-2(x-4)$$

$$(y-(-3))^2=-2(x-4)$$

$$(y-(-3))^2=4(-1/2)(x-4)$$

By comparing $(y-(-3))^2=4(-1/2)(x-4)$ with $(y-k)^2=4p(x-h)$, get $h=4$, $k=-3$ and $p=-1/2$

The vertex of the parabola is $(h,k)=(4,-3)$

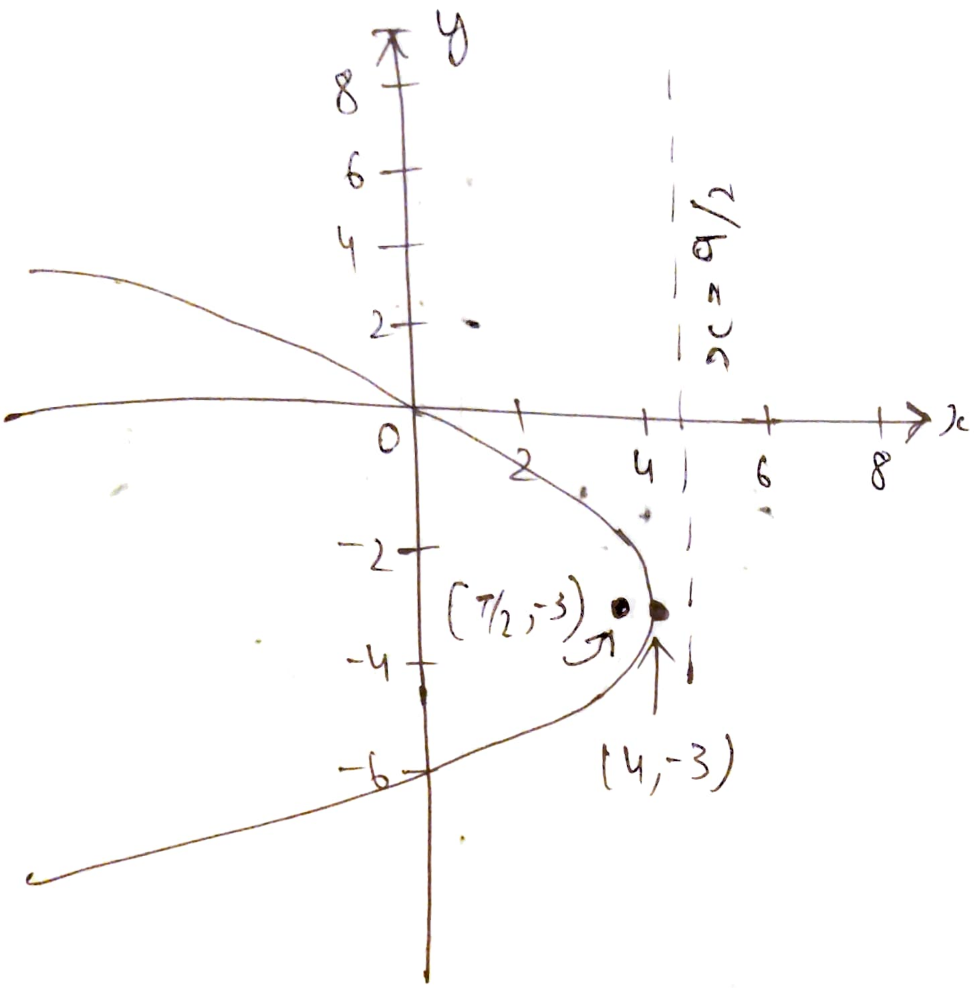
Focus of the parabola is $(h+p,k)=$

$$(4-1/2,-3)=(7/2,-3)$$

Diretrix of the parabola is,

$$x=h-p$$
$$=4-(-1/2)=9/2$$

Therefore, the vertex, focus and diretrix of the parabola are $(4,-3)$, $(7/2,-3)$, $x=9/2$



11. Given $x^2/2 + y^2/4 = 1$

Compare with the standard form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$,

where $a > b > 0$

$\Rightarrow a = 2, b = \sqrt{2}$

Length of the major axis is $2a = 4$

Length of the minor axis is $2b = 2\sqrt{2}$

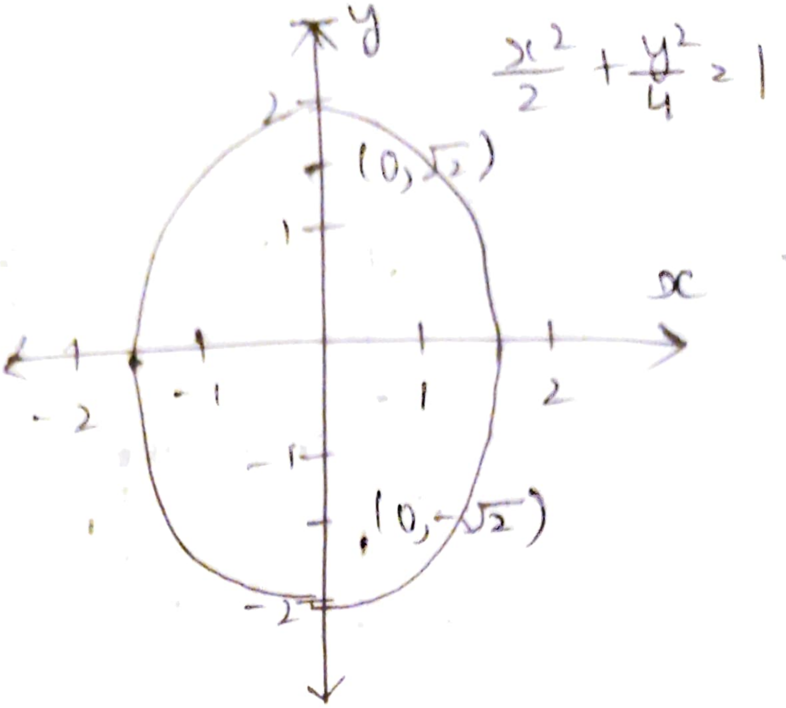
Therefore vertices are $(0, \pm a) = (0, \pm 2)$

We know that $c^2 = a^2 - b^2$

$\Rightarrow c^2 = 2$

$\Rightarrow c = \pm\sqrt{2}$

Therefore the foci are $(0, \pm c) = (0, \pm\sqrt{2})$



15. $9x^2 - 18x + 4y^2 = 27$

$9x^2 - 18x + 9 + 4y^2 = 27 + 9$

$9x^2 - 2 \cdot 3x \cdot 3 + 9 + 4y^2 = 36$

$(3x - 3)^2 + 4y^2 = 36$

$9(x-1)^2 + 4y^2 = 36$

$$\frac{9(x-1)^2}{36} + \frac{4y^2}{36} = 1$$

$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{(x-1)^2}{2^2} + \frac{y^2}{3^2} = 1$$

$(h, k) = (1, 0)$, $a = 3$ and $b = 2$

$c^2 = a^2 - b^2$

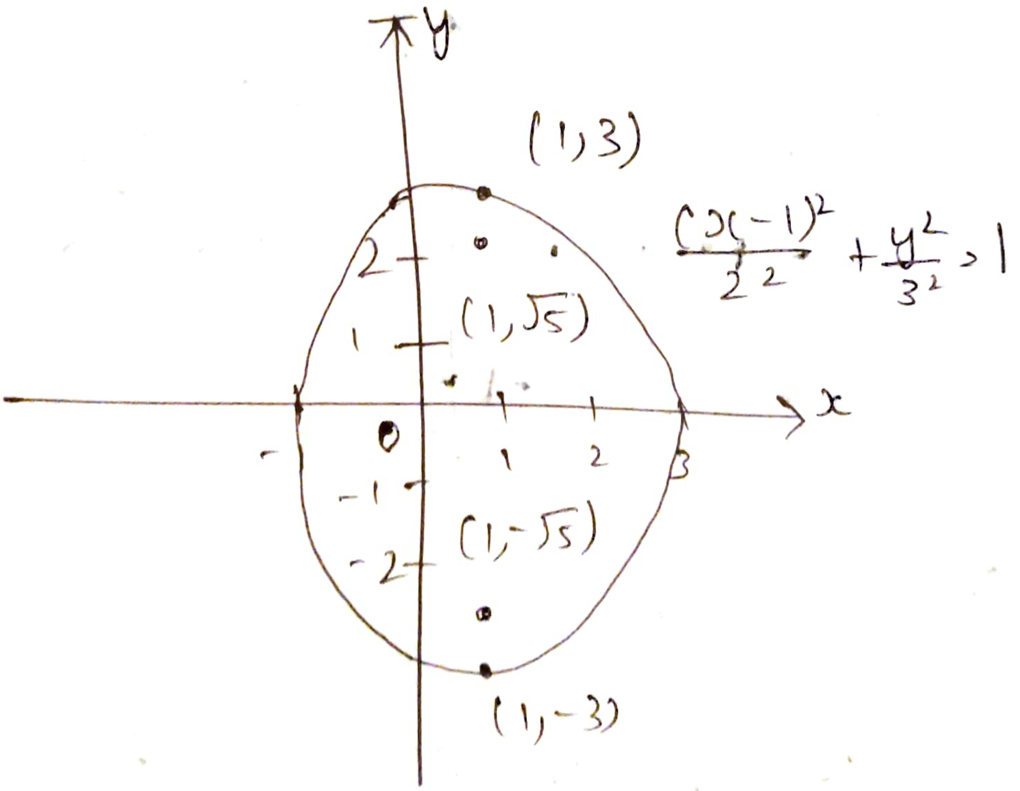
$= 3^2 - 2^2 = 9 - 4 = 5$

$c = \pm\sqrt{5}$

$(h, k \pm a) = (1, \pm 3)$ and

$(h, k \pm c) = (1, \pm\sqrt{5})$

The vertices are $(1,-3), (1,3)$ and the foci are $(1,-\sqrt{5}), (1,\sqrt{5})$



13. Given $\frac{x^2}{9} + \frac{y^2}{1} = 1$

Compare with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$

Therefore $a = 3, b = 1$

$2a = 6$

$2b = 2$

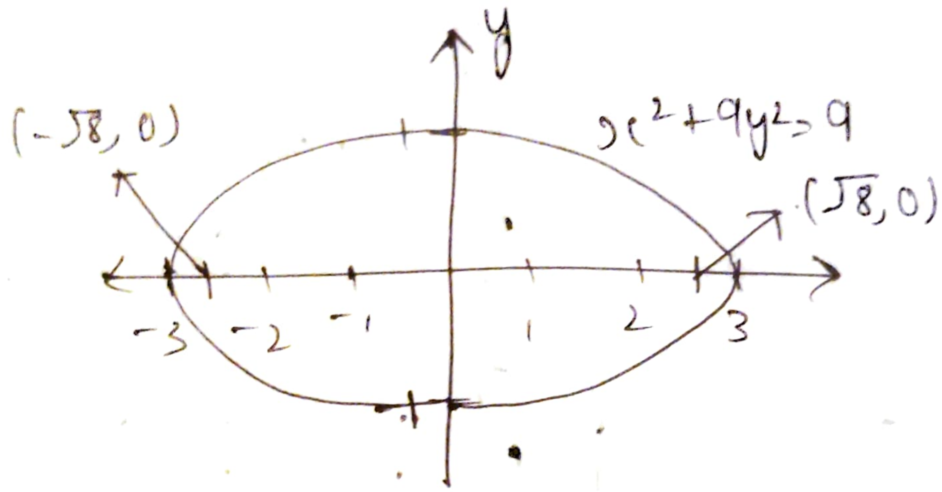
Therefore vertices are $(\pm a, 0) = (\pm 3, 0)$

We know that $c^2 = a^2 - b^2$

$\Rightarrow c^2 = 8$

$\Rightarrow c = \pm\sqrt{8}$

Therefore the foci are $(\pm c, 0) = (\pm\sqrt{8}, 0)$



19. Given $\frac{y^2}{25} - \frac{x^2}{9} = 1$

Compare with the standard form

$y^2/a^2 - x^2/b^2 = 1$

Therefore $a = 5, b = 3$

Hence its vertices are $(0, \pm a) = (0, \pm 5)$

$c^2 = a^2 + b^2$

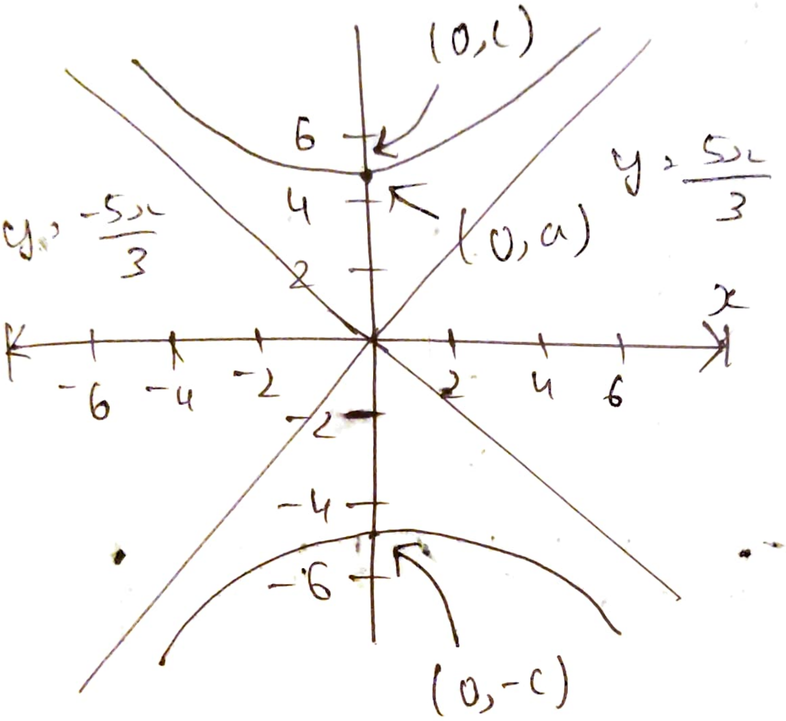
$c^2 = 34$

$c = \pm\sqrt{34}$

Therefore the foci are $(0, \pm c) =$
$(0, \pm\sqrt{34})$

Asymptotes are $y = \pm a/b\, x$

$\Rightarrow y = \pm \frac{5x}{3}$



21. Given $\frac{x^2}{100} - \frac{y^2}{100} = 1$

Compare with the standard form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Therefore $a = 10, b = 10$

Hence the vertices are $(\pm a, 0) = (\pm 10, 0)$

We know that $c^2 = a^2 + b^2$

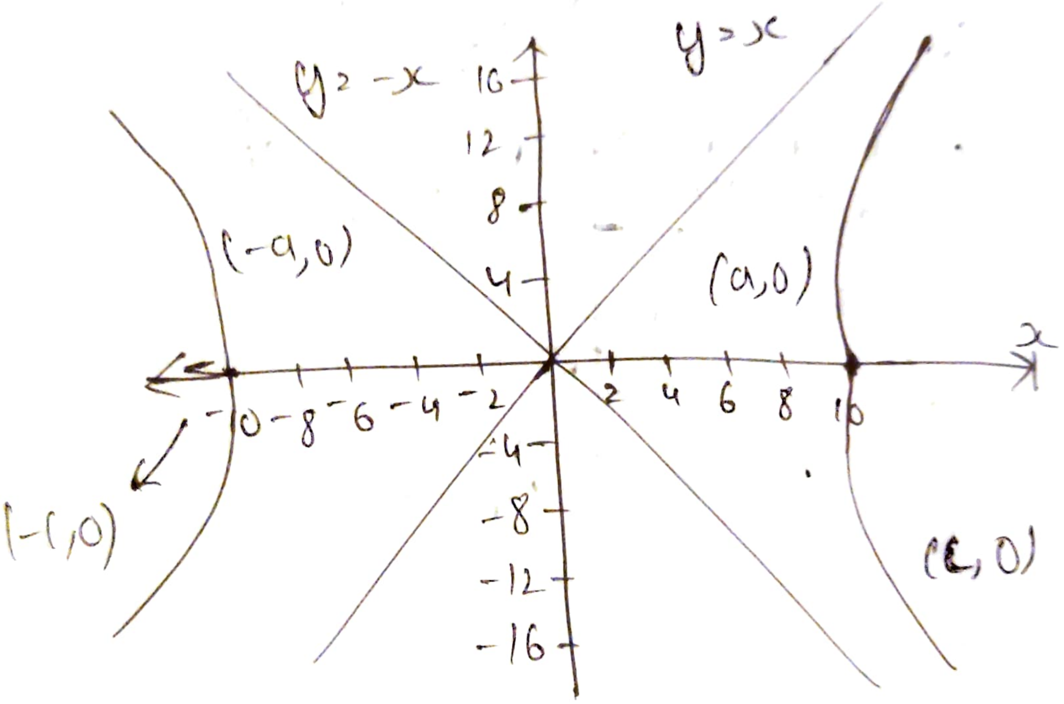
$\Rightarrow c^2 = 200$

$\Rightarrow c = \pm 10\sqrt{2}$

Therefore the foci are $(\pm c, 0) = (\pm 10\sqrt{2}, 0)$

Asymptotes are $y = \pm b/a\, x$

$\Rightarrow y = \pm x$



23. $x^2 - y^2 + 2y = 2$

$x^2 - y^2 + 2y - 1 = 2 - 1$

$x^2 - (y-1)^2 = 1$

$\frac{(x-0)^2}{1^2} - \frac{(y-1)^2}{1^2} = 1$

Compare with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$,

get $h=0$, $k=1$, $a^2=1$ and $b^2=1$

Vertices of the hyperbola are $(h \pm a, k)$

$= (0 \pm 1, 1) = (\pm 1, 1)$

$c^2 = 1^2 + 1^2 = 2$

$\Rightarrow c = \pm\sqrt{2}$

So the foci of the hyperbola are

$(h \pm c, k) = (0 \pm \sqrt{2}, 1) = (\pm\sqrt{2}, 1)$

Asymptotes of the hyperbola are,

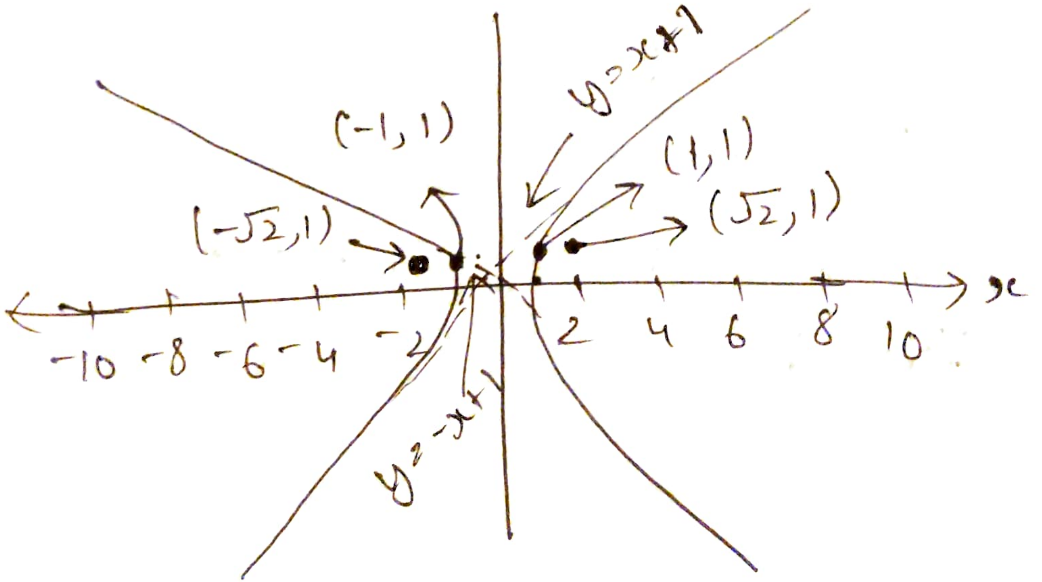
$y - k = \pm (a/b)(x-h)$

$y - 1 = \pm (1/1)(x-0)$

$y - 1 = \pm x$

Therefore, the vertices, foci and asymptotes of the hyperbola are

$(\pm 1, 1)$, $(\pm\sqrt{2}, 1)$, $y = \pm x + 1$



Excercise 10.6

1. The diretrix $x = 4 = d$ is to the right of the focus hence

$$r = \frac{ed}{1 + e\cos\theta}$$

Here $d = 4$ and $e = 1/2$

Therefore required equation of the ellipse is

$$r = \frac{1/2 \cdot 4}{1 + 1/2 \cos\theta}$$

$$r = \frac{4}{2(1 + 1/2 \cos\theta)} \Rightarrow r = \frac{4}{2 + \cos\theta}$$

2. ~~The~~ $x = -3 = -d$ is left of the focus hence

$$r = \frac{ed}{1 - e\cos\theta}$$

The conic is a parabola if $e = 1$

Here $d = 3$

Therefore

$$r = \frac{1(3)}{1-1\cdot\cos\theta} \Rightarrow r = \frac{3}{1-\cos\theta}$$

3. $y=2 \Rightarrow d$ is parallel to the polar axis hence

$$r = \frac{ed}{1+e\sin\theta}$$

Here $d=2$ and $e=1.5$

Therefore

$$r = \frac{(1.5)(2)}{1+0.5\sin\theta} = \frac{3}{1+(3/2)\sin\theta}$$

$$r = \frac{6}{2+3\sin\theta}$$

6. $r = 4\csc\theta$

$y = r\sin\theta$

$y = \left(4\frac{1}{\sin\theta}\right)\sin\theta$

$y = 4$

The polar eq. is in the form of,

$$r = \frac{ed}{1\pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1\pm e\sin\theta}$$

So, the polar equation of the ellipse is,

$$r = \frac{ed}{1+e\sin\theta} = \frac{0.6\times4}{1+0.6\sin\theta} = \frac{12}{5+3\sin\theta}$$

7. $r = \frac{ed}{1+e\sin\theta}$

$y=d$ is the directrix

If the conic is a parabola, then $e=1$

So, $r = \frac{d}{1+\sin\theta}$

Substitute $(r,\theta) = (3, \pi/2)$ in the equation to find d

$$3 = \frac{d}{1+\sin\pi/2} \Rightarrow 3 = \frac{d}{2}$$

$\Rightarrow d = 6$

Substitute $d=6$ in $r = \frac{d}{1+\sin\theta}$

$$r = \frac{6}{1+\sin\theta}$$

8. $r = -2\sec\theta$

Multiply both sides by $\cos\theta$

$r\cos\theta = -2\sec\theta \cdot \cos\theta$

$r\cos\theta = -2$

$x = -2$

Therefore, $d = 2$

The conic equation in polar coordinates

$$r = \frac{ed}{1 - e\cos\theta}$$

Substitute the values of e and d

$$r = \frac{2 \cdot 2}{1 - 2\cos\theta} = \frac{4}{1 - 2\cos\theta}$$

10.(a) $r = \dfrac{1}{2 + \sin\theta} = \dfrac{1/2}{1 + 1/2 \sin\theta} = \dfrac{ed}{1 + e\sin\theta}$

So, the value of eccentricity is $e = 1/2$

(b) The conic is an ellipse

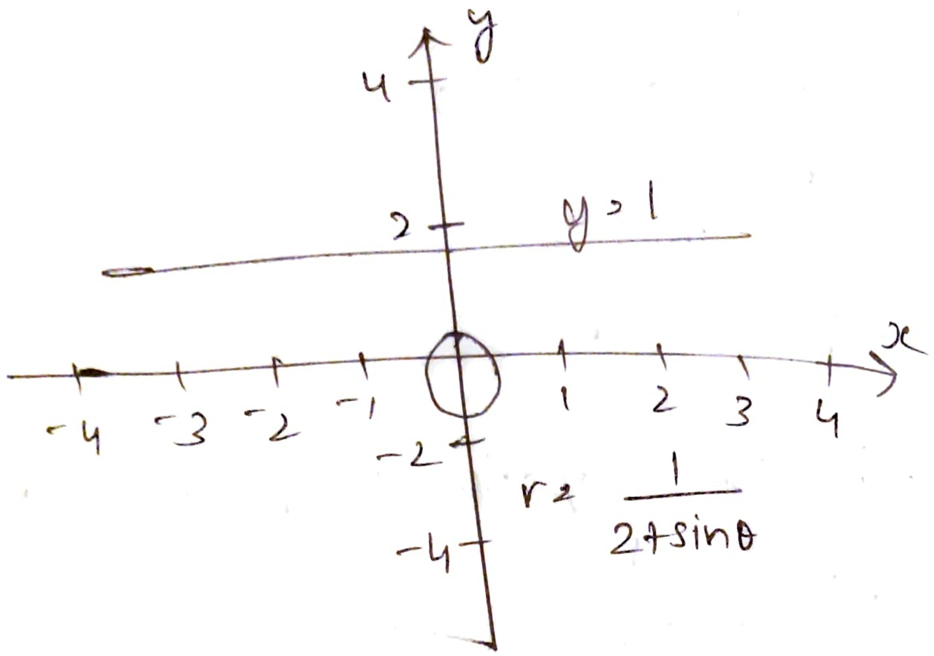
(c) The eq. of the diretrix is

$ed = 1/2$

$d = 1$

$y = 1$

(d)



$$r = \frac{2}{3+3\sin\theta}$$

Divide numerator and denominator by 3,

$$r = \frac{2}{3+3\sin\theta} = \frac{2/3}{1+\sin\theta}$$

This is in the form $r = \frac{ed}{1+e\sin\theta}$

On comparison we obtain,

$e = 1$

$ed = 2/3$

$\Rightarrow 1 \cdot d = 2/3$

$\Rightarrow d = 2/3$

(a) The eccentricity is $e = 1$

(b) Since $e = 1$, we have the given conic as a parabola

(c) $y = 2/3$

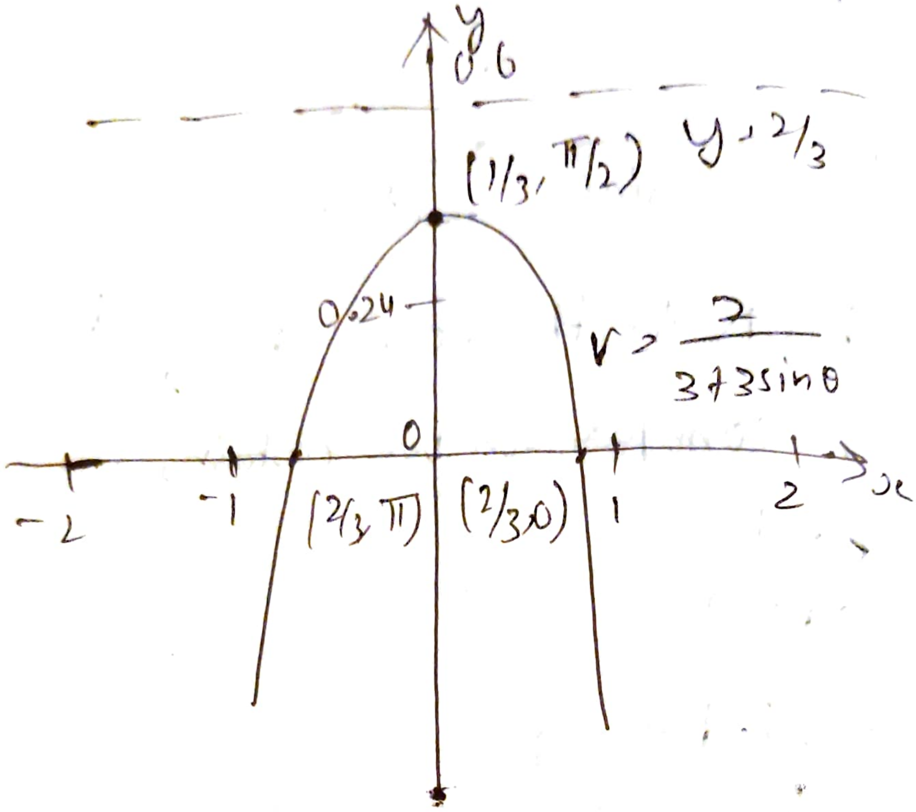
(d) $e = 1$, $y = 2/3$

When $\theta = 0$, $r = 2/3$

When $\theta = \pi$, $r = 2/3$

When $\theta = \pi/2$, $r = 1/3$

So, the vertices have the polar coordinates $(2/3, 0)$, $(2/3, \pi)$ and $(1/3, \pi/2)$



13. $r = 9/6+2\cos\theta$

Divide the numerator and denominator by 6

$$r = \frac{3/2}{1+1/3\cos\theta}$$

Compare with $r = \dfrac{ed}{1 \pm e\cos\theta}$

(a) $e = 1/3$

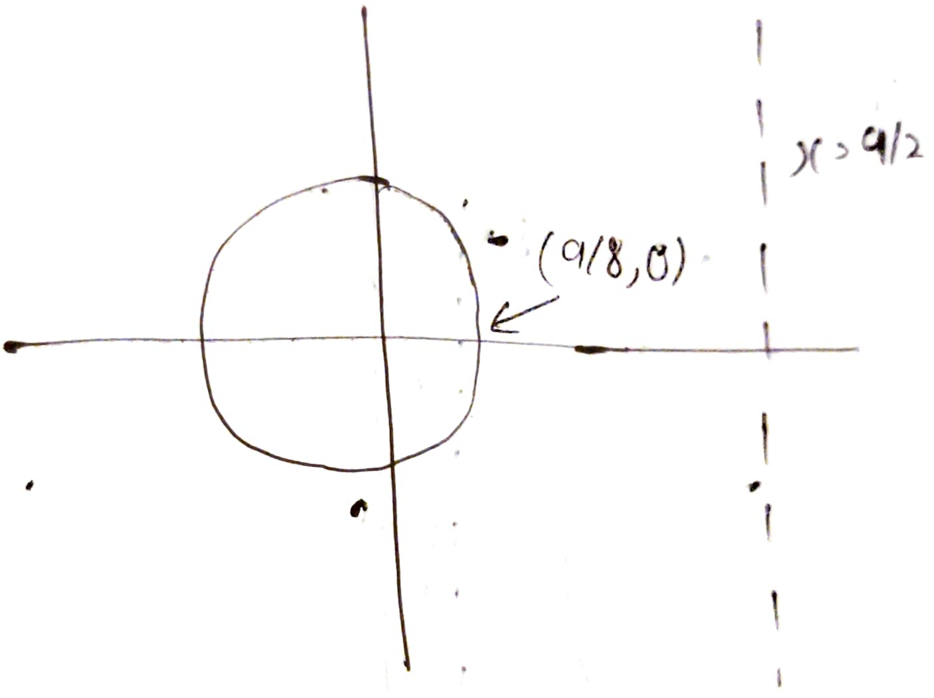
(b) Since $e = 1/3 < 1$, the conic is an ellipse

(c) $ed = 3/2 \Rightarrow d = 3/(2e)$

$\Rightarrow d = \frac{3}{2/3} \Rightarrow d = 9/2$

$x = 9/2$

(D)



15. $r = \frac{3}{4 - 8\cos\theta}$

Divide the numerator and denominator by 4.

$\Rightarrow r = \frac{3/4}{1 - 2\cos\theta}$

Compare with $r = \frac{ed}{1 \pm e\cos\theta}$

(A) $e = 2$

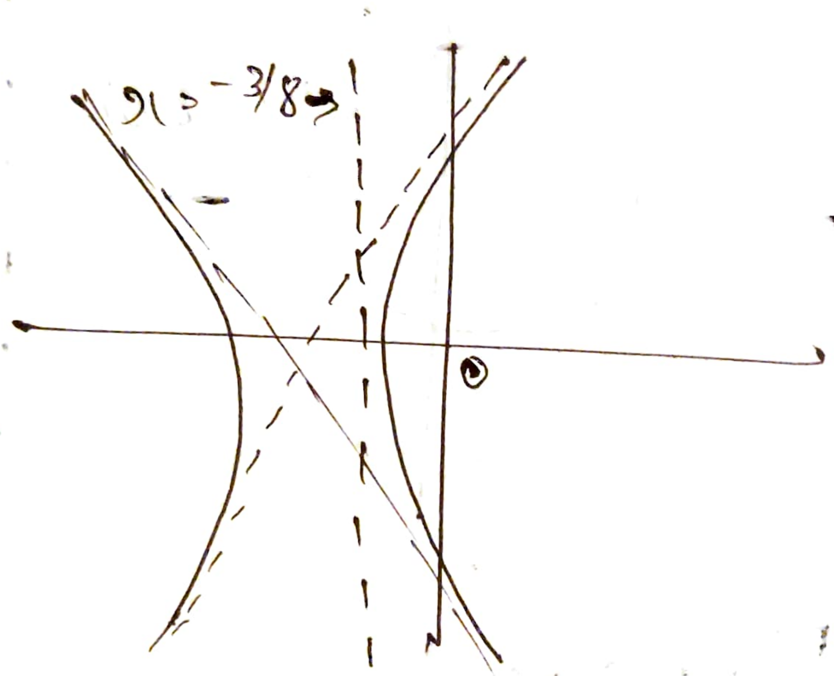
(B) $e = 2 > 1$ so the conic is a hyperbola

(C) $ed = 3/4 \Rightarrow d = 3/(4e)$

$\Rightarrow d = 3/8$

$x = \cancel{8} - 3/8$

(D)



16. $r = \dfrac{4}{2+3\cos\theta}$

Divide the numerator and denominator by 2.

$r = \dfrac{4}{2+3\cos\theta}$

$$r = \frac{2}{1 + 3/2 \cos\theta}$$

Compare with $r = \dfrac{ed}{1 + e\cos\theta}$

(a) So, the value of eccentricity is

$e = 3/2$

(b) The conic is a hyperbola.

(c) $ed = 2$

$d = 2/3/2$

$d = 4/3$

The equation of the directrix is

$y = 4/3$

d)