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Date: 09/26/2021

MATH-141

CALCULUS-2

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1. Ans

$r = 2 - 2\cos\theta$

$r = 4$

$\Rightarrow 4 = 2 - 2\cos\theta$

$-2\cos\theta = 2$

$\cos\theta = -1$

θ lies between $-\pi$ to π.

$\theta = -\pi$, $\theta = \pi$

Required area:-

$$dA = \left(\frac{r^2}{2}d\theta - \frac{2(1-\cos\theta)}{2}d\theta\right)$$

$$dA = \left(\frac{r^2}{2}d\theta - (1-\cos\theta)d\theta\right)$$

$r = 4$, we get :-

$$dA = \left(\frac{4^2}{2}d\theta - (1-\cos\theta)d\theta\right)$$

$$dA = \left(8\,d\theta - (1-\cos\theta)d\theta\right)$$

$$dA = (8 - 1 + \cos\theta)d\theta$$

$$dA = (7 + \cos\theta)d\theta$$

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<u>Required Area (A)</u> :-

$$\int_{-\pi}^{\pi} dA = \int_{-\pi}^{\pi} (7+\cos\theta)\, d\theta$$

$$A = \int_{-\pi}^{\pi} (7+\cos\theta)\, d\theta$$

$$A = [7\theta + \sin\theta]_{-\pi}^{\pi}$$

$$A = (7\pi + \sin\pi) - [7(-\pi) + \sin(-\pi)]$$

$$\therefore A = (7\pi + 0) - [7(-\pi) + 0]$$

$$A = 7\pi + 7\pi$$

$$A = 14\pi$$

$\therefore$ Area $= 14\pi$

2. Ans. Given $x = \cos\theta + \theta\sin\theta$

$y = \sin\theta - \theta\cos\theta$, $0 \le \theta \le \pi$

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta$$

$$\frac{dx}{d\theta} = \theta\cos\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \theta\cos\theta)$$

$$= \cos\theta - \cos\theta + \theta\sin\theta$$

$$\frac{dy}{d\theta} = \theta\sin\theta$$

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The exact length of the curve,

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{(\theta \cos\theta)^2 + (\theta \sin\theta)^2}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{\theta^2(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{\theta^2}\, d\theta$$

$$L = \int_0^{\pi} \theta\, d\theta$$

$$L = \left[\frac{\theta^2}{2}\right]_0^{\pi}$$

$$L = \frac{\pi^2}{2}$$

$\therefore$ The exact length of the curve, $L = \frac{\pi^2}{2}$

3.(a) Ans. $x = 1 - \cos t$

$y = \sin t$

$\Rightarrow \cos t = 1 - x$

$\sin t = y$

We know that, $\sin^2 t + \cos^2 t = 1$

$\therefore y^2 + (1-x)^2 = 1$

$\therefore$ The cartesian equation is: $y^2 + (1-x^2) = 1$

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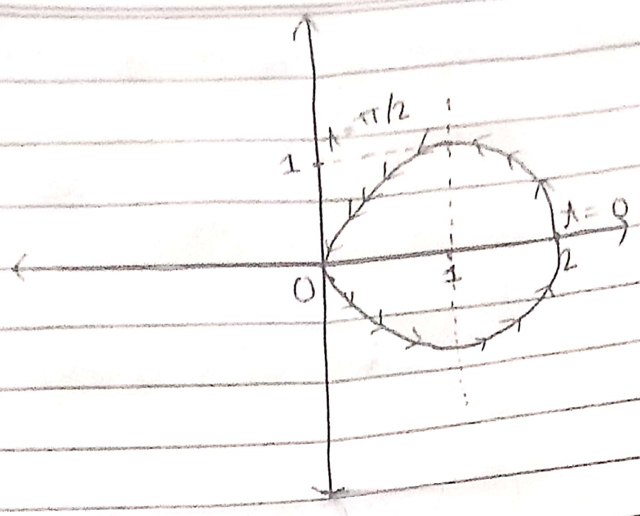
(ii)



This is the equation of the circle centered at $(1,0)$.

When $t = 0$, the coordinates are $(2,0)$ & when $t = \pi/2$, the coordinates are $(1,1)$. This shows the orientation of the curve with increasing values of t.

4. Ans:

$f(x) = e^x$

$a = 3$

Taylor series is given by :-

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f^2 a + \ldots\ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots\ldots$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^n a}{n!} \cdot (x-a)^n \longrightarrow (1)$$

Since, $f(x) = e^x$,

$f(3) = e^3$,

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Differentiating with respect to 'x', we get:-

$f'(x) = e^x$,
$f'(3) = e^3$

$f''(x) = e^x$
$f''(3) = e^3$

$f'''(x) = e^x$
$f'''(3) = e^3$

$f''''(x) = e^x$
$f''''(3) = e^3$

From equation (1), we get:-

$$e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!} (x-3)^n \longrightarrow (2)$$

Using ratio test to find the radius of conver-
-gence.

Let $a_n = \dfrac{e^3}{n!}$,

then $a_{n+1} = \dfrac{e^3}{(n+1)!}$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{e^3}{(n+1)!} \cdot \frac{n!}{e^3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{n!}}{(n+1) \cdot \cancel{n!}} \right|$$

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$$\Rightarrow \lim_{n\to\infty} \left|\frac{1}{n+1}\right|$$

$$\Rightarrow \frac{1}{\infty} = 0$$

$$\therefore \frac{1}{R} = 0$$

$$R = \infty$$

$\therefore$ The radius of convergence is infinite.

5. Ans:- By root test,

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$$

Let $u_n = \dfrac{n^{2n}}{(1+2n^2)^n}$

$$L = \lim_{n\to\infty} (u_n)^{1/n}$$

$$= \lim_{n\to\infty} \left(\left(\frac{n^2}{1+2n^2}\right)^n\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n^2}+2\right)}$$

$$L = \frac{1}{2}$$

$\therefore$ L is a finite real number.

$\therefore$ It is convergent.

P.T.O

6 Ans. Given series,

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

The series in terms of a_n :-

$$a_n = \frac{(2n)!}{(n!)^2}$$

$$\therefore a_{n+1} = \frac{(2n+2)!}{((n+1)!)^2}$$

To define L, we use :-

$$L = \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)(2n)!}{(n+1)^2 (n!)^2} \times \frac{(n!)^2}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)}{(n+1)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2} \right|$$

$$= \frac{(2+0)(2+0)}{(1+0)^2} \quad \left[\because \lim_{n\to\infty} \frac{1}{n} = 0\right]$$

$$= 4 = 4 > 1$$

$\therefore$ By Ratio test, the given series diverges.

(P.T.O)

7 Ans: Ans Given series :-

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

Let $f(x) = \frac{1}{x\sqrt{\ln x}}$

Three reasons ~~to~~ use the Integral test here :-

(i) $f(x)$ is greater than 0 when $x > 2$.

(ii) $f(x)$ is decreasing. $f'(x) > 0$ for $x > 2$

(iii) $f(x)$ is continuous for $[2, \infty)$.

Applying Integral test :-

Comparing with $\int_2^{\infty} f(x)\,dx$.

$$\therefore \lim_{b \to \infty} \int_2^b \frac{dx}{x\sqrt{\ln x}}$$

$$= \lim_{b \to \infty} \int_{\ln(2)}^{\ln(b)} \frac{du}{\sqrt{u}}$$

$$\left[\begin{array}{l} \because \ln x = u \\ \frac{dx}{x} = du \\ \begin{array}{c|c|c} x & 2 & b \\ \hline u & \ln(2) & \ln(b) \end{array} \end{array} \right]$$

$$\therefore \lim_{b \to \infty} \left[\frac{u^{-1/2+1}}{-\frac{1}{2}+1} \right]_{\ln(2)}^{\ln(b)}$$

P.T.O

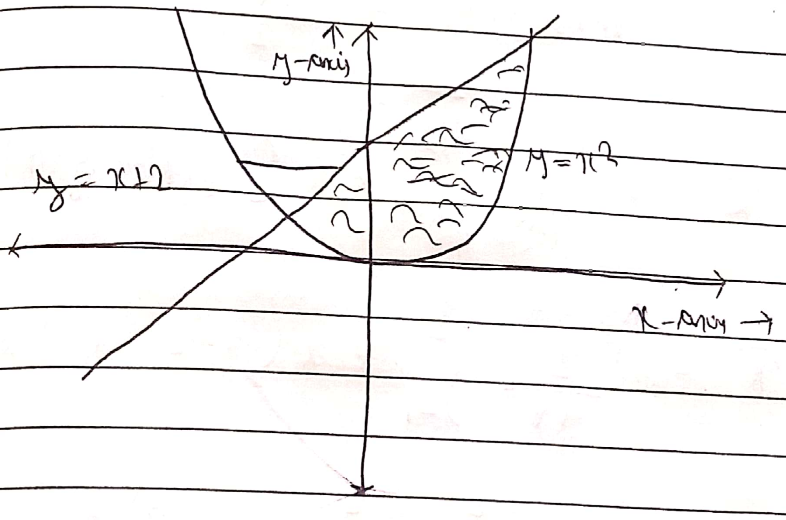
$\Rightarrow 2 \lim_{b \to \infty} \left[u^{1/2} \right]_{\ln(2)}^{\ln(b)}$

$\Rightarrow 2 \lim_{b \to \infty} \left(\sqrt{\ln b} - \sqrt{\ln 2} \right)$

$\Rightarrow \infty$

$\therefore$ According to the Integral test,
The given series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges & is divergent.

8. Ans) Given $y = x^2$ & $y = x + 2$



Point of intersection :-

$y = x^2 = x + 2$

$\Rightarrow x^2 = x + 2$
$\Rightarrow x^2 - x - 2 = 0$
$\Rightarrow (x+1)(x-2) = 0$
$\Rightarrow x + 1 = 0, \; x - 2 = 0$
$\Rightarrow x = -1, \; x = 2$

P.T.O

Let density = ρ (constant)

Mass, $m = \int_a^b \rho\, [f(x) - g(x)]\, dx$

$m = \int_{-1}^{2} \rho\, (x + 2 - x^2)\, dx$

$m = \rho \left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \right)_{-1}^{2}$

$m = \rho \left(\frac{3}{2} + 6 - 3 \right)$

$m = \frac{9}{2} \rho$

$\therefore$ Mass $(m) = \frac{9\rho}{2}$

$\bar{x} = \frac{1}{m} \int_a^b \rho x\, [f(x) - g(x)]\, dx$

$\bar{x} = \frac{2}{9\rho} \int_{-1}^{2} \rho x\, (x + 2 - x^2)\, dx$

$\bar{x} = \frac{2}{9} \int_{-1}^{2} (x^2 + 2x - x^3)\, dx$

$\bar{x} = \frac{2}{9} \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4} \right]_{-1}^{2}$

$\bar{x} = \frac{2}{9} \left(3 + 3 - \frac{15}{4} \right)$

$= \frac{\cancel{2}^{1}}{\cancel{9}} \left(\frac{\cancel{9}}{\cancel{4}_{2}} \right) = \frac{1}{2}$

$\therefore \bar{x} = \frac{1}{2}$

P.T.O

$$\bar{y} = \frac{1}{m}\int_a^b \frac{\rho\,[(f(x))^2 - (g(x))^2]}{2}\,dx$$

$$\bar{y} = \frac{2}{9\rho}\int_{-1}^{2} \frac{\rho\,[(x+2)^2 - (x^2)^2]}{2}\,dx$$

$$\bar{y} = \frac{1}{9}\int_{-1}^{2} [x^2 + 4x + 4 - x^4]\,dx$$

$$\bar{y} = \frac{1}{9}\left[\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5}\right]_{-1}^{2}$$

$$\bar{y} = \frac{1}{9}\left(3 + 6 + 12 - \frac{31}{5}\right)$$

$$\bar{y} = \frac{1}{9}\left(\frac{72}{5}\right)$$

$$= \frac{8}{5}$$

$$\therefore \bar{y} = \frac{8}{5}$$

$\therefore$ Centroid of the region bounded by the curves :- $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{8}{5}\right)$

4. Ans: $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$

Let $\ln x = t$

$$\frac{dx}{x} = dt$$

Limits for t are 0 to ∞.

$$\Rightarrow \int_0^\infty \frac{t\,dt}{e^t} = \int_0^\infty t \cdot e^{-t}\,dt$$

Using Integration by parts, we get :-

$$\Rightarrow \left[-t e^{-t} - e^{-t}\right]_0^\infty$$

$$\Rightarrow \left[\frac{-t}{e^t} - \frac{1}{e^t}\right]_0^\infty$$

$$\Rightarrow \left[\lim_{t\to\infty} \frac{-t}{e^t} - 0\right] - [0-1]$$

$$\Rightarrow \underline{\underline{1}}$$

$$\therefore \int_1^\infty \frac{\ln x}{x^2}\,dx = 1$$

$\therefore$ The value is finite, hence, the integral is convergent.

10 Ans $I = \int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$

Let $x = \sec\theta$

$$\frac{dx}{d\theta} = \sec\theta \tan\theta$$

$$\therefore dx = \sec\theta \cdot \tan\theta \cdot d\theta$$

For limiting value,

$x = \sec\theta$

at $x = \sqrt{2}$, $\sqrt{2} = \sec\theta$

$$\frac{1}{\sqrt{2}} = \cos\theta$$

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$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \cos^{-1}\left(\cos\frac{\pi}{4}\right)$$

$$\theta = \frac{\pi}{4}$$

At $x=2$, $2 = \sec\theta$

$$\frac{1}{2} = \cos\theta$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \cos^{-1}\left(\cos\frac{\pi}{3}\right)$$

$$\theta = \frac{\pi}{3}$$

$$\therefore I = \int_{\pi/4}^{\pi/3} \frac{\sec\theta \, . \tan\theta . d\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}}$$

$$I = \int_{\pi/4}^{\pi/3} \frac{\tan\theta \, d\theta}{\sec\theta\sqrt{\tan^2\theta}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\tan\theta \, . d\theta}{\sec\theta . \tan\theta}$$

$$= \int_{\pi/4}^{\pi/3} \cos\theta \, . d\theta$$

$$= \left[\sin\theta\right]_{\pi/4}^{\pi/3}$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}$$

P.T.O

$$I = \frac{\sqrt{6} - 2}{2\sqrt{2}}$$

By rationalisation:-

$$I = \frac{\sqrt{6} - 2}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{12} - 2\sqrt{2}}{4}$$

$$= \frac{2\sqrt{3} - 2\sqrt{2}}{4}$$

$$I = \frac{\cancel{2}(\sqrt{3} - \sqrt{2})}{\cancel{4}\,2}$$

$$I = \frac{(\sqrt{3} - \sqrt{2})}{2}$$

11. Ans: $\int \ln x \, dx$

$\ln x \int 1 \, dx - \int \left[\frac{d}{dx} (\ln x) \int 1 \, dx \right] dx$

$\Rightarrow \ln x (x) - \int \left(\frac{1}{x} \int 1 \, dx \right) dx$

$\Rightarrow x \ln x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx$

$\Rightarrow x \ln x - \int 1 \, dx$

$\Rightarrow x \ln x - x + C$

$\Rightarrow x (\ln x - 1) + C$

12. Ans. (a) & (b) Note:- Can't remember right now. Sorry Professor!