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MATH - 141
EXAM 1

NOTE :- Not in order. Solving easy ones first & then getting to the tougher ones.

3. Evaluate $\int e^x \cos x \, dx$

Ans :- $\int e^x \cos x \, dx = \int e^{1.x} . \cos(1.x) \, dx$

Using $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

$= \frac{e^{1.x}}{1^2+1^2}(1.\cos(1.x) + 1.\sin(1.x)) + C$

$= \frac{e^x}{1+1}(\cos x + \sin x) + C$

$= \frac{e^x}{2}(\cos x + \sin x) + C$

$\therefore \int e^x \cos x \, dx = \frac{e^x}{2}(\cos x + \sin x) + C$

2. Evaluate $\int_1^2 \frac{(\ln x)^2}{x^3} dx$

Ans :- Let $\ln x = u$

$\frac{1}{x} dx = du$

$\therefore \int_{\ln(1)}^{\ln(2)} \frac{u^2}{e^{2u}} . du$

P.T.O

$$\Rightarrow \int_{\ln(1)}^{\ln(2)} \underbrace{u^2}_{I}\ \underbrace{e^{-2u}}_{II} \cdot du$$

$$\Rightarrow \left[u^2 \frac{e^{-2u}}{(-2)} - \int \frac{e^{-2u}}{(-\cancel{2})} \cdot (\cancel{2}u)\, du \right]_{\ln(1)}^{\ln(2)}$$

$$\Rightarrow \left[u^2 \frac{e^{-2u}}{(-2)} + \int u e^{-2u}\, du \right]_{\ln(1)}^{\ln(2)}$$

$$\Rightarrow \left[u^2 \frac{e^{-2u}}{(-2)} + \frac{u e^{-2u}}{(-2)} - \int \frac{e^{-2u}}{(-2)}\, du \right]_{\ln(1)}^{\ln(2)}$$

$$\Rightarrow \left[u^2 \frac{e^{-2u}}{(-2)} - \frac{u e^{-2u}}{2} + \frac{1}{2} \frac{e^{-2u}}{(-2)} \right]_{\ln(1)}^{\ln(2)}$$

$$\Rightarrow \left[-\frac{u^2 e^{-2u}}{2} - \frac{u e^{-2u}}{2} - \frac{e^{-2u}}{4} \right]_{\ln(1)}^{\ln(2)}$$

$$\Rightarrow -\frac{(\ln(2))^2}{2e^{2\ln 2}} - \frac{\ln(2)}{2e^{2(\ln 2)}} - \frac{1}{4e^{2\ln 2}} - \left(-\frac{(\ln 1)^2}{2e^{2\ln 1}} - \frac{\ln 1}{2e^{2\ln 1}} - \frac{1}{4e^{2\ln 1}} \right)$$

$$\Rightarrow -\frac{(\ln 2)^2}{2\cdot(2)^2} - \frac{\ln 2}{2(2)^2} - \frac{1}{4(2)^2} + \frac{(\ln 1)^2}{2} + \frac{\ln 1}{2} + \frac{1}{4}$$

$$\Rightarrow -\frac{(\ln 2)^2}{8} - \frac{\ln 2}{8} - \frac{1}{16} + \frac{(\ln 1)^2}{2} + \frac{\ln 1}{2} + \frac{1}{4}$$

$$\Rightarrow -0.049 - \frac{0.086}{(2)} - \frac{1}{16} + 0 + 0 + \frac{1}{4}$$

P.T.O

$\Rightarrow -0.145 - \frac{1+4}{16}$

$\Rightarrow -0.145 + \frac{3}{16}$

$\Rightarrow 0.042$

4. Evaluate $\int \cos^5 x \sin^4 x \, dx$

Ans:- $I = \int \cos^5 x \sin^4 x \, dx$

$= \int (1-\sin^2 x)^2 \sin^4 x \cos x \, dx$

Set $\sin x = u$

$\cos x \, dx = du$

$I = \int (1-u^2)^2 . u^4 \, du$

$= \int (1+u^4-2u^2) u^4 \, du$

$= \int (u^4+u^8-2u^6) \, du$

$= \int (u^8 - 2u^6 + u^4) \, du$

$= \frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5} + C$

$I = \frac{1}{9} \sin^9 x - \frac{2}{7} \sin^7 x + \frac{1}{5} \sin^5 x + C$

P.T.O

5. Evaluate $\int \frac{dx}{x^2\sqrt{16x^2-9}}$

Ans: $I = \frac{dx}{x^2\sqrt{16x^2-9}}$

Let $16x^2 = 9\sec^2\theta$

$4x = 3\sec\theta$

$x = \frac{3}{4}\sec\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$

$\therefore I = \int \frac{\frac{3}{4}\sec\theta\,.\tan\theta\, d\theta}{\frac{9\sec^2\theta}{16}\,.\sqrt{9\sec^2\theta-9}}$

$I = \int \frac{4}{3}\frac{\cancel{\sec\theta}\,\cancel{\tan\theta}\, d\theta}{\sec^{\cancel{2}}\theta\,.\,3\cancel{\tan\theta}}$ $\left[\because \sec^2\theta-1=\tan^2\theta\right]$

$I = \frac{4}{9}\int \cos\theta\, d\theta$

$I = \frac{4}{9}\sin\theta + C$

Since, $x = \frac{3}{4}\sec\theta$

$\cos\theta = \frac{3}{4x}$

$\sin\theta = \sqrt{1-\frac{9}{(4x)^2}} = \frac{\sqrt{16x^2-9}}{4x}$

$I = \frac{\cancel{4}}{9}\times\frac{\sqrt{16x^2-9}}{\cancel{4}x} + C = \frac{\sqrt{16x^2-9}}{9x}+C$

P.T.O

1. First make a substitution & then use integration by parts to evaluate the integral.

$$\int \frac{\arcsin(\ln x)}{x}\,dx$$

Ans.- Set $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\therefore \int \arcsin u\,du = u \arcsin u - \int \frac{u}{\sqrt{1-u^2}}\,du$$

$\Rightarrow$ Set $w = \arcsin u$

$dw = \frac{1}{1-u^2}\,du$

$dV = du$

$V = u + C$

$$\left\{\begin{aligned} t &= 1-u^2 \\ \frac{dt}{-2} &= \frac{+\cancel{2}u\,du}{+\cancel{2}} \\ -\frac{1}{2}\,dt &= u\,du \end{aligned}\right\}$$

~~$\Rightarrow$ ~~

$$\therefore \int \arcsin u\,du = u \arcsin u + \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$\Rightarrow u \arcsin u + \frac{1}{2}\int t^{-1/2}\,dt$$

$$\Rightarrow u \arcsin u + \frac{1}{\cancel{2}}\,\frac{t^{1/2}}{\cancel{1/2}} + C$$

$$\left\{\begin{aligned} t &= 1-u^2 \\ t &= 1-(\ln x)^2 \end{aligned}\right.$$

$$\therefore \ln x \cdot \arcsin(\ln x) + \sqrt{1-(\ln x)^2} + C$$

P.T.O

6. Evaluate $\displaystyle\int \frac{x^2}{(a^2-x^2)^{3/2}}\,dx$

Ans: Let $x = a\cos\theta$

$\theta = \cos^{-1}\left(\frac{x}{a}\right)$

$\left\{\begin{array}{l}\sin^2\theta + \cos^2\theta = 1 \\ \sin^2\theta = 1-\cos^2\theta\end{array}\right.$

$dx = -a\sin\theta\, d\theta$

$$\int \frac{x^2}{(a^2-x^2)^{3/2}}\,dx = \int \frac{(a\cos\theta)^2}{(a^2-a^2\cos^2\theta)^{3/2}}\cdot(-a)\sin\theta\, d\theta$$

$$= \int \frac{a^2\cos^2\theta\,(-a)\sin\theta\, d\theta}{(a^2\sin^2\theta)^{3/2}}$$

$$= -a\int \frac{a^2\cos^2\theta\sin\theta\, d\theta}{(a\sin\theta)^3}$$

$$= -a\int \frac{a^2\cos^2\theta\sin\theta\, d\theta}{a^3\sin^3\theta}$$

$$= -\int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= -\int \frac{1-\sin^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \frac{\sin^2\theta - 1}{\sin^2\theta}\, d\theta$$

$$= \int\left(\frac{\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}} - \frac{1}{\sin^2\theta}\right)d\theta$$

P.T.O

$$\Rightarrow \int (1 - \operatorname{cosec}^2\theta)\, d\theta$$

$$\Rightarrow \int d\theta - \int \operatorname{cosec}^2\theta \, d\theta$$

$$\Rightarrow \theta + \cot\theta + C$$

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right) + \frac{\cos\theta}{\sin\theta} + C$$

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right) + \frac{x}{a} \cdot \frac{1}{\sqrt{1 - \frac{x}{a^2}}} + C$$

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right) + x \cdot \frac{1}{\sqrt{a^2 - x^2}} + C$$

8. Evaluate $\int \frac{x^2 + x + 1}{(x^2+1)^2}\, dx$

Ans.:- Using Partial Fraction Decomposition :-

$$\int \left(\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

$$\Rightarrow \underbrace{\int \frac{1}{(x^2+1)}\, dx}_{I_1} + \underbrace{\int \frac{x}{(x^2+1)^2}\, dx}_{I_2}$$

$$I_1 = \int \frac{1}{(x^2+1)}\, dx$$

$$I_1 = \tan^{-1} x$$

P.T.O

$$I_2 = \int \frac{x}{(x^2+1)^2}\, dx$$

Let $u = x^2 + 1$

$du = 2x\, dx$

$dx = \frac{1}{2x}\, du$

$$I_2 = \int \frac{1}{2u^2}\, du$$

Using $\int u^n\, du = \frac{u^{n+1}}{n+1}$

$$I_2 = \frac{1}{2} \int \frac{1}{u^2}\, du$$

$$= \frac{-1}{2u}$$

$$\therefore\ I_2 = \frac{-1}{2(x^2+1)}$$

$$I = I_1 + I_2 + C$$

$$I = \tan^{-1} x - \frac{1}{2(x^2+1)} + C$$

9. Evaluate $\int_1^4 \frac{e^{\sqrt{w}}}{\sqrt{w}}\, dw$

(x)

Ans:- Let $u = \sqrt{w}$

$du = \frac{1}{2\sqrt{w}} dw$

$2\,du = \frac{dw}{\sqrt{w}}$

When $w = 1$, $u = 1$,
When $w = 4$, $u = 2$

$\therefore \int_1^2 e^u \cdot 2\,du$

$\Rightarrow 2\int_1^2 e^u\,du$

$\Rightarrow 2\left[e^t\right]_1^2$

$\Rightarrow \underline{\underline{2\left[e^2 - 1\right]}}$

7. Evaluate $\int_0^1 \sqrt{x^2+1}\,dx$

Ans:- Let $x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

When $x = 0$, $\theta = 0$,
When $x = 1$, $\theta = \frac{\pi}{4}$

$\therefore \int_0^{\frac{\pi}{4}} \sqrt{\tan^2\theta + 1} \cdot \sec^2\theta\,d\theta$

$\Rightarrow \int_0^{\frac{\pi}{4}} \sec\theta \cdot \sec^2\theta\,d\theta$
(x)

P.T.O

$$\Rightarrow \int_0^{\frac{\pi}{4}} \sec^3\theta \, . \, d\theta$$

Let $u = \sec\theta$

$du = \sec\theta \tan\theta$

$dV = \sec^2\theta$

$V = \tan\theta$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \sec^2\theta \, . \, \sec\theta \, d\theta$$

$$= \left[\sec\theta \tan\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan\theta \, . \, \sec\theta \tan\theta \, d\theta$$

$$= \left(\sec\frac{\pi}{4} \tan\frac{\pi}{4} - \sec 0 \tan 0\right) - \int_0^{\frac{\pi}{4}} \sec\theta \, . \tan^2\theta \, d\theta$$

$$= (\sqrt{2} - 0) - \int_0^{\frac{\pi}{4}} \sec\theta \, . (\sec^2\theta - 1) \, d\theta$$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} (\sec^3\theta - \sec\theta) \, d\theta$$

$$\therefore I = \sqrt{2} - \int_0^{\frac{\pi}{2}} \sec^2\theta \, . \sec\theta + \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$\Rightarrow I + I = \sqrt{2} + \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$2I = \sqrt{2} + \left[\log|\sec\theta + \tan\theta|\right]_0^{\frac{\pi}{4}}$$

$$I = \frac{\sqrt{2}}{2} + \frac{1}{2}\left[\log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log|\sec 0 + \tan 0|\right]$$

$$\Rightarrow \frac{1}{\sqrt{2}} + \frac{1}{2}\left(\log|\sqrt{2}+1| + 0\right) \quad [\because \log 1 = 0]$$

$$\therefore I = \frac{1}{\sqrt{2}} + \frac{1}{2}\log|\sqrt{2}+1|$$

10. Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin B \cot B}{\sec B}\, dB$

Ans:
$$\int_{\pi/6}^{\pi/3} \frac{\cancel{\sin B} \cdot \frac{\cos B}{\cancel{\sin B}}}{\frac{1}{\cos B}}\, dB$$

$$\Rightarrow \int_{\pi/6}^{\pi/3} \cos^2 B\, dB$$

$$\Rightarrow \int_{\pi/6}^{\pi/3} \frac{1}{2}[1 + \cos 2B]\, dB$$

$$\Rightarrow \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \left(\frac{\sin 2B}{2}\right)_{\pi/6}^{\pi/3}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{2} \times 0\right]$$

$$= \frac{1}{2} \times \frac{4\pi}{6} = \frac{\pi}{3}$$