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Section 7.1

1. $\int xe^{2x}\,dx$; $u = x$, $dv = e^{2x}\,dx$

Let $u = x$, $dv = e^{2x}\,dx$

$\int u\,dv = uv - \int v\,du$ --- (1)

$\frac{du}{dx} = 1$

$du = dx$

And

$dv = e^{2x}\,dx$

$\int dv = \int e^{2x}\,dx$

$v = \frac{1}{2}e^{2x}$

Substitute in (1)

$$\int xe^{2x}\,dx = x\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{1}{2}e^{2x}\right)dx$$

$$= x\left(\frac{1}{2}e^{2x}\right) - \frac{1}{2}\int(e^{2x})\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C$$

$$= \frac{xe^{2x}}{2} - \frac{1}{4}e^{2x} + C$$

3. $\int x \cos 5x \, dx$

Using integration by parts, $\int u\,dv = uv - \int v\,du$,

Assume $u = x$, $dv = \cos 5x \, dx$

Then $du = dx$, $v = \int dv$

$v = \int \cos 5x \, dx$

$v = \frac{\sin 5x}{5}$

$$\int x \cos 5x \, dx = \int x \cos 5x \, dx$$

$$= x \cdot \frac{\sin 5x}{5} - \int \frac{\sin 5x}{5} \, dx$$

$$= x \cdot \frac{\sin 5x}{5} - \left(- \frac{\cos 5x}{25} \right) + C$$

$$= x \cdot \frac{\sin 5x}{5} + \frac{\cos 5x}{25} + C$$

5. $\int t e^{-3t} \, dt$

Using integration by parts, $\int u\,dv = uv - \int v\,du$; ... ①

$u = t$ $\quad$ $dv = e^{-3t} dt$

$du = dt$ $\quad$ $v = \frac{e^{-3t}}{3}$

Substitute in ①

$$\int te^{-3t}\,dt = t\left(\frac{e^{-3t}}{3}\right) - \int\left(\frac{e^{-3t}}{-3}\right)dt$$

$$= t\left(\frac{e^{-3t}}{-3}\right) - \frac{e^{-3t}}{3^2} + C$$

7. $\int (x^2+2x)\cos x\,dx$

Using int. by parts $\int u\,dv = uv - \int v\,du$ ...... (1)

Let $u = x^2+2x$ and $dv = \cos x$ then,
$du = (2x+2)dx$ and $v = \sin x$

Sub. in (1)

$$\int (x^2+2x)\cos x\,dx = (x^2+2x)\sin x - \int \sin x(2x+2)\,dx$$

$$= (x^2+2x)\sin x - \int (2x+2)\sin x\,dx \quad \text{---} \ (2)$$

Let $u_1 = 2x+2$ and $dv_1 = \sin x$ then,
$du_1 = 2dx$ and $dv_1 = -\cos x$

Using integration by parts,

$$\int (2x+2)\sin x\,dx = (2x+2)(-\cos x) - \int(-\cos x)\cdot 2\,dx$$

$= -(2x+2)\cos x + 2\int \cos x\,dx = -(2x+2)\cos x +$
$2\sin x + C_1$ ..... (3)

Use equation ③ in ②

$$\int (x^2+2x)\cos x\,dx = (x^2+2x)\sin x -$$

$$[-2(2x+2)\cos x + 2\sin x + C_1]$$

$$= (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x - C_1$$

$$= (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C$$

11. $\int t^4 \ln t\,dt$

~~Using~~ Let $u = \ln t$, $dv = t^5\,dt$

That is,

$$u = \ln t \Rightarrow u' = 1/t$$

$$dv = t^4 \Rightarrow v = \int dv = \int t^4\,dt = t^5/5$$

Using integration by parts,

$$\int t^4 \ln t\,dt = \ln t\left(t^5/5\right) - \int t^5/5\,(1/t)\,dt$$

$$= \frac{t^5 \ln 5}{5} - \int t^4/5\,dt = \frac{t^5 \ln t}{5} - 1/5 \int t^4\,dt$$

$$= \frac{t^5 \ln t}{5} - 1/5\left[t^5/5\right] + C$$

$$= \frac{t^5 \ln t}{5} - \frac{1}{25}t^5 + C$$

$= \frac{t^5 \ln t}{5} - \frac{t^5}{25} + C$

12. $\int \tan^{-1}(2y)\,dy$

Let $u = \tan^{-1}(2y)$ and $v = 1$. Then,

$du = \frac{2}{1+4y^2}\,dy$

$du = \frac{2}{1+4\left(\frac{\tan(u)}{2}\right)}\,dy$

$du = \frac{2}{(1+\tan^2(u))}\,dy$

$du = \frac{2}{\sec^2(u)}\,dy$

$dy = \left(\frac{\sec^2(u)}{2}\right)du$

Substitute the values of $u = \tan^{-1}(2y)$, $v = 1$ and, $v = 1$ and $dy = \left(\frac{\sec^2(u)}{2}\right)du$ and use integration by parts

$\int \tan^{-1}(2y)\cdot 1)\,dy = \tan^{-1}(2y)\int 1\cdot dy - \int\left[\left(\frac{d(\tan^{-1}(2}{dy}\right.\right.$

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$= y\tan^{-1}(2y) - \int\left[\left(\frac{2}{1+4y^2}\right)(y)\right]dy + C$

$= y\tan^{-1}(2y) - \frac{1}{4}\int\left(\frac{8y}{1+4y^2}\right)dy + C$

$\int \tan^{-1}(2y)\,dy = y\tan^{-1}(2y) - \frac{1}{4}\int\left(\frac{8y}{1+4y^2}\right)dy$

$+ C \quad \ldots\ldots$ (2)

Substitute $1+4y^2 = t$ and $8y\,dy = dt$ in the integral $\int\left(\frac{8y}{1+4y^2}\right)dy$

$\int\left(\frac{8y}{1+4y^2}\right)dy = \int\frac{dt}{t} = \ln(t) + C$

$= \ln(1+4y^2) + C$

Put $\int\left(\frac{8y}{1+4y^2}\right) = \ln(1+4y^2) + C$ in (2)

$\int \tan^{-1}(2y)\,dy = y\tan^{-1}(2y) - \frac{1}{4}\ln(1+4y^2) + C$

$\left[\int 1\,dy\right]dy$

17. Let $I = \int e^{2\theta}\sin(3\theta)d\theta$ ...... (1)

Assume $u = e^{2\theta}$ and $dv = \sin 3\theta$

Then,

$du = 2e^{2\theta}d\theta$ and $v = \int \sin 3\theta\, d\theta$

$v = \int \sin 3\theta\, d\theta = -\frac{\cos 3\theta}{3}$

Using int. by parts and substituting in (1)

$I = \int e^{2\theta}\sin(3\theta)d\theta$

$= e^{2\theta}\left(\frac{-\cos 3\theta}{3}\right) - \int\left(\frac{-\cos 3\theta}{3}\right)2e^{2\theta}d\theta$

$= -\frac{1}{3}e^{2\theta}\cos 3\theta + \frac{2}{3}\int e^{2\theta}\cos 3\theta\, d\theta$ -- (2)

Consider (2)

Assume $u_1 = e^{2\theta}$ and $dv_1 = \cos 3\theta$

Then $du_1 = 2e^{2\theta}d\theta$ and $v_1 = \int \cos 3\theta\, d\theta$

$v_1 = \int \cos 3\theta\, d\theta = \frac{\sin 3\theta}{3}$

Subbing in ①

$$\int e^{2\theta}\cos 3\theta\, d\theta = \tfrac{1}{3}e^{2\theta}\sin 3\theta - \int\left(\frac{\sin 3\theta}{3}\right)2e^{2\theta}\, d\theta$$

$$= \tfrac{1}{3}e^{2\theta}\sin 3\theta - \tfrac{2}{3}\int e^{2\theta}\sin 3\theta\, d\theta$$

$$= \tfrac{1}{3}e^{2\theta}\sin 3\theta - \tfrac{2}{3}I \quad \text{(from ①)}$$

Use ③ ~~and~~ in ②

$$I = -\tfrac{1}{3}e^{2\theta}\cos 3\theta + \tfrac{2}{3}\left[\tfrac{1}{3}e^{2\theta}\sin 3\theta - \tfrac{2}{3}I\right]$$

$$= -\tfrac{1}{3}e^{2\theta}\cos 3\theta + \tfrac{2}{9}e^{2\theta}\sin 3\theta - \tfrac{4}{9}I$$

Add $\frac{4}{9}I$ on both sides

$$I + \tfrac{4}{9}I = -\tfrac{1}{3}e^{2\theta}\cos 3\theta + \tfrac{2}{9}e^{2\theta}\sin 3\theta$$

$$\tfrac{13}{9}I = \tfrac{1}{9}\left(-3e^{2\theta}\cos 3\theta + 2e^{2\theta}\sin 3\theta\right)$$

$$I = \tfrac{1}{13}\left[-3e^{2\theta}\cos 3\theta + 2e^{2\theta}\sin 3\theta\right]$$

$$= \tfrac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$$

23. $\int_0^{1/2} x\cos \pi x\, dx$

Let $u = x$ and $du = dx$
$dv = \cos \pi x \, dx$ and $v = \frac{\sin \pi x}{\pi}$

Using integration by parts

$$\int x \cos \pi x \, dx = x\left(\frac{\sin \pi x}{\pi}\right) - \int \left(\frac{\sin \pi x}{\pi}\right) dx$$

$$= x\left(\frac{\sin \pi x}{\pi}\right) + \frac{\cos \pi x}{\pi^2}$$

Now,

$$\int_0^{1/2} x \cos \pi x \, dx = \left[x\left(\frac{\sin \pi x}{\pi}\right) + \frac{\cos \pi x}{\pi^2}\right]_0^{1/2}$$

$$= \left[\frac{1}{2\pi} - \frac{1}{\pi^2}\right] = \frac{\pi - 2}{2\pi^2}$$

27. $\int_1^5 \frac{\ln R}{R^2} dR$

Using integration by parts,

Let $f(x) = \ln R$ and $g'(x) = 1/R^2 \, dR$

Then,

$f'(x) = d(\ln R) = 1/R \, dR$

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$g'(x) = 1/R^2 \, dR$

$\int g'(x) = \int 1/R^2 \, dR$

$g(x) = -1/R$

Substitute all values in

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)dx$$

~~Ans~~

$$\int_1^5 \frac{\ln R}{R^2} dR = \left[\ln R\left(-1/R\right)\right]_1^5 - \int_1^5 \left[-1/R\left(1/R\right)\right] dR$$

$$= -\left(1/5 \ln 5 - \ln 1\right) + \int_1^5 \left[-1/R\left(1/R\right)\right] dR$$

$$= -1/5 \ln 5 + \left(-1/R\right)_1^5$$

$$= -1/5 \ln 5 - \left(1/5 - 1\right) = -1/5 \ln 5 - \left(-4/5\right)$$

$$= 4/5 - 1/5 \ln 5$$

29. $\int_0^\pi x \sin x \cos x \, dx = \int_0^\pi x \cdot 1/2 \cdot 2 \sin x \cos x \, dx$ (Using $\sin 2A = 2 \sin A \cos A$)

$$= 1/2 \int_0^\pi x \cdot \sin 2x \, dx$$

Let $u = x$ and $dv = \sin 2x\,dx$
Then $du = 1\,dx$ and $v = -\dfrac{\cos 2x}{2}$

Use integration by parts

$$\int_0^{\pi} x\sin 2x\,dx = \left[x\left(-\frac{\cos 2x}{2}\right) - \int\left(-\frac{\cos 2x}{2}\right)dx\right]_0^{\pi}$$

$$= \left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx\right]_0^{\pi}$$

$$= \left[-\frac{x\cos 2x}{2} + \frac{1}{2}\cdot\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \left[-\frac{\pi\cos 2\pi}{2} + \frac{\sin 2\pi}{4}\right] - \left[-\frac{0\cos 0}{2} + \frac{\sin 0}{4}\right]$$

$$= \left[-\frac{\pi}{2} + \frac{0}{4}\right] - \left[-\frac{0}{2} + \frac{0}{4}\right] = -\frac{\pi}{2} \quad \text{--- (2)}$$

Use (2) in (1)

$$\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{4}$$

33. $\displaystyle\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

Let $u = \ln(\cos x)$, $dv = \sin x\,dx$

$\Rightarrow du = -\dfrac{\sin x}{\cos x}\,dx$

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$\int dv = \int \sin x \, dx$

$\Rightarrow v = -\cos x$

Using integration by parts,

$$\int_0^{\pi/3} \sin x \ln(\cos x)\, dx = \left[\ln|(\cos x)|(-\cos x)\right]_0^{\pi/3}$$

$$- \int_0^{\pi/3} (-\cos x)\left(-\frac{\sin x}{\cos x}\right) dx$$

$$= \left[\ln|(\cos x)|(-\cos x)\right]_0^{\pi/3} - \int_0^{\pi/3} \sin x \, dx$$

$$= \left[-\cos x \ln|(\cos x)|\right]_0^{\pi/3} - \left[-\cos x\right]_0^{\pi/3}$$

$$= \left[-\cos x \ln|(\cos x)|\right]_0^{\pi/3} + \left[\cos x\right]_0^{\pi/3}$$

$$= \left[-\cos \pi/3 \ln\left|\left(\cos \pi/3\right)\right| + \cos 0 \ln(\cos 0)\right] +$$

$$\left[\cos \pi/3 - \cos 0\right] = \left[-\tfrac{1}{2}\ln\left(\tfrac{1}{2}\right) + 1\ln(1)\right] +$$

$$\left[\tfrac{1}{2} - 1\right] = -\tfrac{1}{2}\left[\ln 1 - \ln 2\right] + 0 + \frac{1-2}{2}$$

$$= -\tfrac{1}{2}(0 - \ln 2) - \tfrac{1}{2} = \tfrac{1}{2}\ln 2 - \tfrac{1}{2}$$

Let $u = x$ and $dv = \sin 2x\,dx$
Then $du = dx$ and $v = -\frac{\cos 2x}{2}$

Use integration by parts

$$\int_0^{\pi} x \sin 2x\,dx = \left[x\left(-\frac{\cos 2x}{2}\right) - \int\left(-\frac{\cos 2x}{2}\right)dx\right]$$

$$= \left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx\right]_0^{\pi}$$

$$= \left[-\frac{x\cos 2x}{2} + \frac{1}{2}\cdot\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \left[-\frac{\pi\cos 2\pi}{2} + \frac{\sin 2\pi}{4}\right] - \left[-\frac{0\cos 0}{2} + \frac{\sin 0}{4}\right]$$

$$= \left[-\frac{\pi}{2} + \frac{0}{4}\right] - \left[-\frac{0}{2} + \frac{0}{4}\right] = -\frac{\pi}{2} \quad (2)$$

Use (2) in (1)

$$\int_0^{\pi} x \sin x \cos x\,dx = \frac{1}{2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{4}$$

33. $\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

Let $u = \ln(\cos x)$, $dv = \sin x\,dx$

$\Rightarrow du = -\frac{\sin x}{\cos x}\,dx$

$$\int dv = \int \sin x\, dx$$

$$\Rightarrow v = -\cos x$$

Using integration by parts,

$$\int_0^{\pi/3} \sin x \ln(\cos x)\, dx = \left[\ln|(\cos x)|(-\cos x)\right]_0^{\pi/3}$$

$$- \int_0^{\pi/3} (-\cos x)\left(-\frac{\sin x}{\cos x}\right) dx$$

$$= \left[\ln|(\cos x)|(-\cos x)\right]_0^{\pi/3} - \int_0^{\pi/3} \sin x\, dx$$

$$= \left[-\cos x \ln|(\cos x)|\right]_0^{\pi/3} - \left[-\cos x\right]_0^{\pi/3}$$

$$= \left[-\cos x \ln|(\cos x)|\right]_0^{\pi/3} + \left[\cos x\right]_0^{\pi/3}$$

$$= \left[-\cos \pi/3 \ln\left|\left(\cos \pi/3\right)\right| + \cos 0 \ln(\cos 0)\right] +$$

$$\left[\cos \pi/3 - \cos 0\right] = \left[-1/2 \ln(1/2) + 1 \ln(1)\right] +$$

$$\left[1/2 - 1\right] = -1/2\left[\ln 1 - \ln 2\right] + 0 + \frac{1-2}{2}$$

$$= -1/2(0 - \ln 2) - 1/2 = 1/2 \ln 2 - 1/2$$

39. $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta$

Let $u = \theta^2$

Then, $du = 2\theta\,d\theta$

$\theta\,d\theta = \frac{1}{2}du$

For the limits:

When, $\theta = \sqrt{\pi}$, the value of u will be

$u = (\sqrt{\pi})^2 = \pi$

When, $\theta = \sqrt{\pi/2}$, the value of u will be

$u = \left(\sqrt{\pi/2}\right)^2 = \pi/2$

Thus,

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2 \cos(\theta^2)(\theta\,d\theta)$$

$$= \int_{\pi/2}^{\pi} u\cdot\cos(u)\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_{\pi/2}^{\pi} u\cdot\cos(u)\,du$$

Using integration by parts,

$u = u$ $\qquad dv = \cos(u)\,du$

$du = du$ $\qquad v = \sin(u)$

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} u\cdot\cos(u)\,du$$

$$= \frac{1}{2}\left\{\left[u\cdot\sin(u)\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin(u)\,du\right\}$$

$$= \frac{1}{2}\left\{\left[u\cdot\sin(u)\right]_{\pi/2}^{\pi} - \left[-\cos(u)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{2}\left\{\left[\pi\cdot\sin(\pi) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)\right] - \left[-\cos(\pi) + \cos\left(\frac{\pi}{2}\right)\right]\right\}$$

$$= \frac{1}{2}\left[\pi\sin(\pi) - \frac{\pi}{2}\cdot\sin\left(\frac{\pi}{2}\right) + \cos(\pi) - \cos\left(\frac{\pi}{2}\right)\right] = \frac{1}{2}\left[\pi(0) - \frac{\pi}{2}(1) - 1 - 0\right]$$

$$= \frac{1}{2}\left(-\frac{\pi}{2} - 1\right) = -\frac{\pi}{4} - \frac{1}{2}$$

Exercise 7.2

1. $\int \sin^2 x \cos^3 x \, dx$

$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \, (1-\sin^2 x) \cos x \, dx$

Take $\sin x = t \Rightarrow \cos x \, dx = dt$

$\cancel{\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \, (1-\sin^2 x)}$

$= \int (t^2 - t^4) \, dt = t^3/3 - t^5/5 + C$

$= \dfrac{\sin^3 x}{3} - \dfrac{\sin^5 x}{5} + C$

3. $\int \sin^7\theta \cos^5\theta \, d\theta = \int (1-\cos^2\theta)^3 \cos^5\theta \sin\theta \, d\theta$

Take $\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$\cancel{I} = -\int (1-t^2)^3 t^5 \, dt$

$= -\int (1 - t^6 + 3t^4 - 3t^2) t^5 \, dt$

$= -\int (-t^{11} + 3t^9 - 3t^7 + t^5) \, dt$

$= \left[\dfrac{t^{12}}{12} - \dfrac{3t^{10}}{10} + \dfrac{3t^8}{8} - \dfrac{t^6}{6} \right]$

$$= \frac{(\cos\theta)^{12}}{12} - \frac{3(\cos\theta)^{10}}{10} + \frac{3(\cos\theta)^{8}}{8} - \frac{(\cos\theta)^{6}}{6}$$

Now,

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \left[\frac{(\cos\theta)^{12}}{12} - \frac{3(\cos\theta)^{10}}{10} + \frac{3(\cos\theta)^{8}}{8} - \frac{(\cos\theta)^{6}}{6} \right]_0^{\pi/2} = 0 - \frac{1}{12} - \left(0 - \frac{3}{10}\right) + 0 - \frac{3}{8}$$

$$- \left(0 - \frac{1}{6}\right) = \frac{-10 + 36 - 45 + 20}{120} = \frac{1}{120}$$

7. $\int_0^{\pi/2} \cos^2\theta \, d\theta$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\int_0^{\pi/2} \cos^2(\theta) \, d\theta = \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta = \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right)_0^{\pi/2}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} + \frac{\sin\left(2 \times \frac{\pi}{2}\right)}{2} \right) = \frac{1}{2} \left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right)$$

$$= \frac{1}{2} \left(\frac{\pi}{2} + \frac{0}{2} \right) = \frac{\pi}{4}$$

9. $\int_0^{\pi} \cos^4(2t)\,dt$

$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$\int_0^{\pi} \cos^4(2t)\,dt = \frac{1}{4}\int_0^{\pi}\left(1 + \frac{1+\cos(8t)}{2} + 2\cos(4t)\right)$

$= \left[t/4 + \frac{1}{8}\sin 4t + \frac{1}{8}\left(t + \frac{\sin 8t}{8}\right)\right]_0^{\pi}$

$= \pi/4 + \frac{1}{8}\sin 4\pi + \frac{1}{8}\left(\pi + \frac{\sin 8\pi}{8}\right) - 0$

$= \pi/4 + \pi/8 = \frac{3\pi}{8}$

15. $\int \cot x \cos^2 x\,dx = \int \frac{\cos^3 x}{\sin x}\,dx = \int \frac{\cos^2 x \cdot \cos x\,dx}{\sin x}$

$= \int \frac{1-\sin^2 x}{\sin x} \cdot \cos x\,dx$

Let $t = \sin x \rightarrow dt = \cos x\,dx$

$\int \frac{1-\sin^2 x}{\sin x} \cdot \cos x\,dx = \int \frac{1-t^2}{t} \cdot dt$

$= \int \left(1/t - t\right)dt = \ln|t| - t^2/2 + C$

$= \ln|\sin x| - \frac{\sin^2 x}{2} + C$

19. $\int t \sin^2 t \, dt$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$\int t \sin^2 t \, dt = \int t \left(\frac{1-\cos 2t}{2} \right) dt$

$= \frac{1}{2} \int t (1-\cos 2t) \, dt = \frac{1}{2} \int (t - t\cos 2t) \, dt$

$= \frac{1}{2} \left(\int t \, dt - \int t \cos 2t \, dt \right) \rightarrow (1)$

Find the integrals in (1)

$\int t \, dt = t^2/2 \longrightarrow (2)$

Find $\int t \cos 2t$ using integration by parts.

Let $u = t$ and $dv = \cos 2t \, dt$ so that $du = dt$ and $v = \frac{\sin 2t}{2}$

$\int t \cos 2t \, dt = t \cdot \frac{\sin 2t}{2} - \int \frac{\sin 2t}{2} \, dt$

$= \frac{1}{2} t \sin 2t - \frac{1}{2} \int \sin 2t \, dt = \frac{1}{2} t \sin 2t - \left[-\frac{\cos 2t}{2} \right]$

$= \frac{t \sin 2t}{2} + \frac{\cos 2t}{4} \rightarrow (3)$

Substitute (2) and (3) in (1)

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$\int t\sin^2 t\,dt = \frac{1}{2}\left(\int t\,dt - \int t\cos 2t\,dt\right)$

$= \frac{1}{2}\left(\frac{t^2}{2} - \left(\frac{t\sin 2t}{2} + \frac{\cos 2t}{4}\right)\right) + C$

$= \frac{t^2}{4} - \frac{t\sin 2t}{4} - \frac{\cos 2t}{8} + C$

21. $\int (\tan x)\sec^3 x\,dx = \int \sec^2 x(\tan x\sec x)\,dx$

Take $\sec x = t \Rightarrow (\tan x\sec x)dx = dt$

$\int (\tan x)\sec^3 x\,dx = \int t^2\,dt = t^3/3 + C$

$= \frac{\sec^3 x}{3} + C$

23. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int dx$
(Using $\tan^2 x = \sec^2 x - 1$)
~~Using Integration by parts,~~

$= \tan x - x + C$

27. $\int \tan^3 x\sec x\,dx = \int (\tan^2 x\tan x\sec x\,dx) =$

$\int (\tan^2 x(\sec x\tan x))\,dx = \int (\sec^2 x - 1)(\sec x\tan x)\,dx$
Since
$(\tan^2 x = \sec^2 x - 1)$

Let $u = \sec x$

$du = \sec x \tan x$

$\int \tan^3 x \sec x \, dx = \int (u^2-1) du$

$= u^3/3 - u + C$

$= \frac{(\sec x)^3}{3} - \sec x + C = \frac{\sec^3 x}{3} - \sec x + C$

31. $\int \tan^5 x \, dx = \int \tan^4 x \cdot \tan x \, dx$

$= \int (\tan^2 x)^2 \cdot \tan x \, dx = \int (\sec^2 x - 1)^2 \cdot \tan x \, dx$

Let $u = \sec x$

$du = \sec x \tan x \, dx$

$\tan x \, dx = \frac{du}{\sec x} = \frac{du}{u}$

$\int \tan^5 x \, dx = \int (u^2-1)^2 \cdot \left(\frac{du}{u}\right) = \int \left(\frac{u^4 - 2u^2 + 1}{u}\right) du$

$= \int \left(u^3 - 2u + 1/u\right) du$

$= \frac{u^4}{4} - \frac{2u^2}{2} + \ln|u| + C_1$

$= \frac{u^4}{4} - u^2 + \ln|u| + C_1$

$= \frac{\sec^4 x}{4} - \sec^2 x + \ln|\sec x| + C_1$

$= 1/4 \sec^4 x - (1 + \tan^2 x) + \ln|\sec x| + C_1$

$= \frac{1}{4}\sec^4 x - 1 - \tan^2 x + \ln|\sec x| + C_1$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$

Where $C = C_1 - 1$

49. $\int x(\tan^2 x)\,dx$

Use integration by parts,

$\int x(\tan^2 x)\,dx = x\int \tan^2 x\,dx - \int \frac{d}{dx}(x)\int \tan^2 x\,dx$

$= x(-x + \tan(x)) - \int(-x + \tan(x))\,dx$

$= x(-x + \tan(x)) - \left(\ln|\sec(x)| - x^2/2\right) + C$

$= -x^2 + x\tan(x) - \ln|\sec(x)| + x^2/2 + C$

This implies,

$\int x(\tan^2 x)\,dx = -x^2 + x\tan(x) - \ln|\sec(x)| + x^2/2 + C = x\tan(x) - \ln|\sec(x)| - \frac{x^2}{2} + C$

56(A) Substitute $u = \cos x \Rightarrow du = -\sin x\,dx$

Thus, $\int \sin x \cos x\,dx = \int u(-du)$

$= -\int u\,du = -u^2/2 + C = -\frac{1}{2}\cos^2 x + C_1$

(B) Substitute $u = \sin x \Rightarrow du = \cos x\,dx$

$\int \sin x \cos x\,dx = \int u\,du = u^2/2 = \frac{1}{2}\sin^2 x + C_2$

(c) Use $\sin 2x = 2\sin x \cos x$

$\int \sin x \cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = \frac{1}{2}\int \sin 2x\,dx$

$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + C = -\frac{1}{4}\cos 2x + C_3$

(D) Integrate by parts.

Take $u = \sin x$ $dv = \cos x\,dx$
$du = \cos x\,dx$ $v = \sin x$

$\int \sin x \cos x\,dx = \sin x \cdot \sin x - \int \sin x \cos x\,dx$

$2\int \sin x \cos x\,dx = \sin^2 x + C_4'$

$\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x + C_4$ [where $C_4 = C_4'/2$]

Since $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$-\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2 x - \frac{1}{4} = -\frac{1}{2}\cos^2 x + \frac{1}{4}$

Thus the answers differ from each other by constants.

URBAN EDGE

Exercise 7.3

1. $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}}$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4(1-\sin^2\theta)}}$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4\cos^2\theta}} = \int \frac{2\cos\theta\, d\theta}{(2\sin^2\theta^2)\sqrt{(2\cos\theta)^2}}$$

$$= \int \frac{d\theta}{4\sin^2\theta} = \frac{1}{4}\int \csc^2\theta\, d\theta = \frac{1}{4}(-\cot\theta) + C$$

$$\int \frac{dx}{x\sqrt{4-x^2}} = -\frac{1}{4}\cot\theta + C$$

Consider

$x = 2\sin\theta$

$\sin\theta = x/2$ $\left\{ \begin{array}{l} \sin\theta = \frac{\text{Perpendicular}}{\text{hypotenuse}} = x/2 \\ \text{Hyp} = 2 \quad \text{Perp} = x \end{array} \right.$

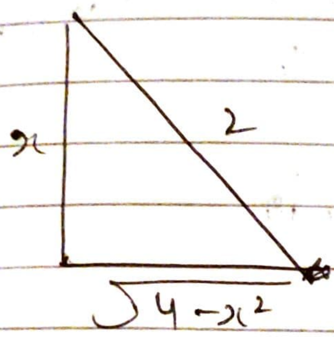
Since Pythagoras Theorem says:

$(\text{Hypotenuse})^2 = (\text{Perpendicular})^2 + (\text{Base})^2$

Sub. Perpendicular $= x$, Hypotenuse $= 2$

$(2)^2 = (\text{Base})^2 + (x)^2$

$(\text{Base}) = \sqrt{4-x^2}$



From the diagram,

$$\cot\theta = \frac{\text{Base}}{\text{Perpendicular}}$$

$$= \frac{\sqrt{4-x^2}}{x}$$

Substitute: $\frac{\sqrt{4-x^2}}{x}$ for $\cot\theta$ in integral

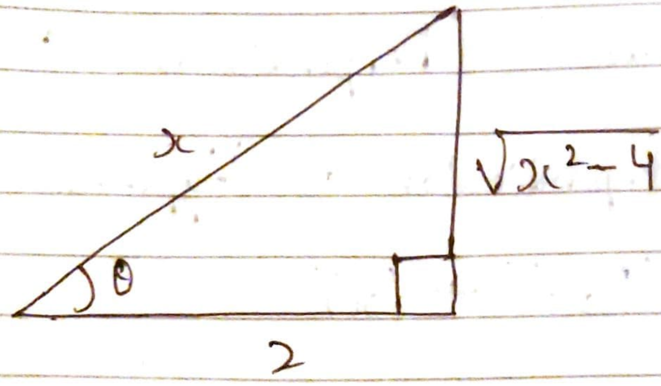
$$\int \frac{dx}{x\sqrt{4-x^2}} = -\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{x}\right) + C$$

3. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

$x = 2\sec\theta$

$\sec\theta = x/2 \Rightarrow x = \dfrac{2}{\cos\theta} \Rightarrow \cos\theta = 2/x$

$dx = 2\sec\theta \tan\theta \, d\theta$



$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{(2\sec\theta)^2-4}}{2\sec\theta}\,(2\sec\theta\tan\theta\,d\theta)$$

$$= \int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= \int \sqrt{4\tan^2\theta}\,\tan\theta\,d\theta = \int 2\tan\theta\tan\theta\,d\theta$$

$$= \int 2\tan^2\theta\,d\theta = \int 2(\sec^2\theta-1)\,d\theta =$$

$$2\tan\theta - 2\theta + C$$

$$= 2\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\left(\sec^{-1}\left(\frac{x}{2}\right)\right) + C$$

(From fig. $\tan\theta = \frac{\sqrt{x^2-4}}{2}$)

$$= \sqrt{x^2-4} - 2\sec^{-1}(x/2) + C$$

URBAN EDGE

5. $\int \frac{\sqrt{x^2-1}}{x^4} dx$

Let $x = \sec\theta$, where $0 < \theta < \pi/2$ or $\pi/2 < \theta < \frac{3\pi}{2}$

Then $dx = \sec\theta \tan\theta \, d\theta$

So, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = |\tan\theta| = \tan\theta$

$\theta d\theta)$ Thus, $\int \frac{\sqrt{x^2-1}}{x^4} dx = \int \frac{\tan\theta}{\sec^4\theta} \cdot \sec\theta \cdot \tan\theta \, d\theta$

$= \int \tan^2\theta \cdot \frac{1}{\sec^3\theta} d\theta = \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^3\theta \, d\theta$

$= \int \sin^2\theta \cos\theta \, d\theta$

Now substitute $u = \sin\theta$ so that $du = \cos\theta \, d\theta$

So, $\int \frac{\sqrt{x^2-1}}{x^4} dx = \int \sin^2\theta \cos\theta \, d\theta = \int u^2 du$

$= 1/3 \, u^3 + C = 1/3 \sin^3\theta + C$

Substitute $u = \sin\theta$

Thus, $\int \frac{\sqrt{x^2-1}}{x^4} dx = 1/3 \sin^3\theta + C \quad \ldots (1)$

If $x = \sec\theta$, then

URBAN EDGE

$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{x}$

Use $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + 1/x^2 = 1$

$\sin\theta = \sqrt{1 - \frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{x}$

(i) is as,

$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = 1/3\sin^3\theta + C$

$= 1/3\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = 1/3\frac{\left(\sqrt{x^2-1}\right)^3}{x^3} + C$

$= 1/3\frac{(x^2-1)^{3/2}}{x^3} + C$

7. $\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

Take $x = a\tan\theta$ so that

$dx = a\sec^2\theta\,d\theta$

Limits:

When $x = 0$

$0 = a\tan\theta$

$0 = \tan\theta$

$0 = \theta$

When $x = a$

$a = a\tan\theta$

$1 = \tan\theta$

$\theta = \pi/4$

$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^{3/2}}$$

$$= \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{(a^2)^{3/2}(1+\tan^2\theta)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{a^3(\sec^2\theta)^{3/2}}$$

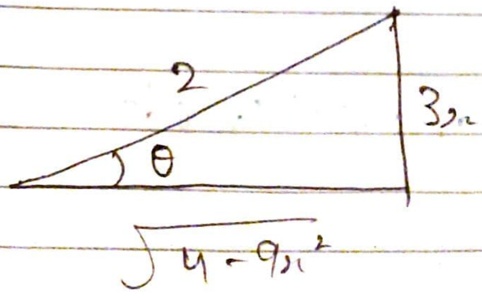
$$= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{a^2\sec^3\theta} = 1/a^2 \int_0^{\pi/4} \frac{1}{\sec\theta}\, d\theta =$$

$$1/a^2 \int_0^{\pi/4} \cos\theta\, d\theta = 1/a^2\,[\sin\theta]_0^{\pi/4}\, d\theta$$

$$= 1/a^2\,(\sin\pi/4 - \sin 0) = 1/a^2\,(1/\sqrt{2} - 0)$$

$$= 1/\sqrt{2}a^2$$

10. $$\int_0^{2/3} \sqrt{4-9x^2}\, dx = \int_0^{2/3} \sqrt{2^2-(3x)^2}\, dx$$



Let $3x = 2\sin\theta \Rightarrow x = \frac{2}{3}\sin\theta$

(Since $x = a\sin\theta$), where $-\pi/2 \le \theta \le \pi/2$

Then $dx = \frac{2}{3}\cos\theta\, d\theta$

When $x = 0$, then $\theta = 0$

When $x = 2/3$, then

$2/3 = 2/3 \sin\theta$

$\sin\theta = 1$

$\sin\theta = \sin\pi/2 \Rightarrow \theta = \pi/2$

$$\int_0^{2/3} \sqrt{4-9x^2}\,dx = \int_0^{\pi/2} \sqrt{4-9\left(\frac{2}{3}\sin\theta\right)^2}$$

$$\frac{2}{3}\cos\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2} \sqrt{4-9\left(\frac{4}{9}\sin^2\theta\right)}\cos\theta\, d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \sqrt{4-4\sin^2\theta}\,\cos\theta\, d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \sqrt{4(1-\sin^2\theta)}\cos\theta\, d\theta$$

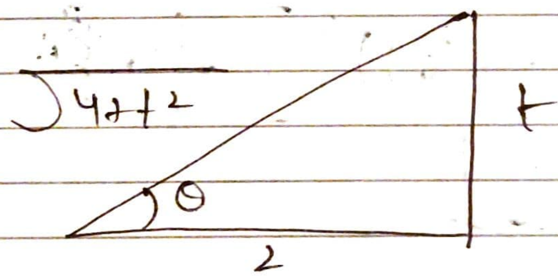
$$= \frac{4}{3}\int_0^{\pi/2} \sqrt{(1-\sin^2\theta)}\cos\theta\, d\theta$$

$= \frac{4}{3}\int_0^{\pi/2}\sqrt{\cos^2\theta}\,\cos\theta\,d\theta = \frac{4}{3}\int_0^{\pi/2}\cos^2\theta\,d\theta$

$= \frac{4}{3}\int_0^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)d\theta = \frac{4}{6}\int_0^{\pi/2}(1+\cos2\theta)\,d\theta$

$= \frac{2}{3}\left[\theta + \frac{\sin2\theta}{2}\right]_0^{\pi/2} = \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3}$

12. $\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \int_0^2 \frac{dt}{\sqrt{2^2+t^2}}$



Let $t = 2\tan\theta$

Then,

$\sqrt{2^2+t^2} = \sqrt{2^2+2^2\tan^2\theta} = \sqrt{2^2(1+\tan^2\theta)}$

$= \sqrt{4(\sec^2\theta)} = 2\sec\theta$

Diff. t w.r.t θ

$dt = 2\sec^2\theta \, d\theta$

When $t = 0$, then $\theta = 0$
When $t = 2$, then $\theta = \pi/4$

$$\int_0^2 \frac{dt}{\sqrt{2^2+t^2}} = \int_0^{\pi/4} \frac{2\sec^2\theta \, d\theta}{2\sec\theta} = \int_0^{\pi/4} \sec\theta \, d\theta$$

$$= \left[\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} + C$$

$$= \left[\ln\left|\sec(\pi/4) + \tan(\pi/4)\right| - \ln|\sec(0) + \tan(0)|\right] = \left[\ln(\sqrt{2}+1) - \ln(1)\right]$$

$$= \ln(\sqrt{2}+1)$$

14. $$\int_0^1 \frac{dx}{(x^2+1)^2}$$

Assume $x = \tan\theta$
$dx = \sec^2\theta \, d\theta$

The limits:

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$
When $x = 1$, $\tan\theta = 1$, so $\theta = \pi/4$

Substitute,

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{(\tan^2\theta+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{\sec^4\theta}$$

$$= \int_0^{\pi/4} \cos^2\theta \, d\theta = \int_0^{\pi/4} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{1}{2}\left[\frac{\pi}{4} + \frac{\sin \frac{\pi}{2}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{\pi}{8} + \frac{1}{4}$$

15. $\int_0^a x^2 \sqrt{a^2 - x^2}$

Let $x = a\sin\theta \Rightarrow dx = a\cos\theta \, d\theta$

For $x = 0$ | For $x = a$
$a\sin\theta = 0$ | $a\sin\theta = a$
$\theta = 0$ | $\theta = \pi/2$

Substitute

$$\int_0^a x^2 \sqrt{a^2 - x^2} \, dx = \int_0^{\pi/2} (a\sin\theta) \sqrt{a^2 - (a\sin\theta)^2} \, (a\cos\theta \, d\theta)$$

$$= \int_0^{\pi/2} a^2\sin^2\theta \sqrt{a^2 - a^2\sin^2\theta} \, (a\cos\theta \, d\theta)$$

$$\cancel{= \int_0^{\pi/2} a^2\sin^2\theta \sqrt{a^2 - a^2\sin^2\theta} \, (a\cos\theta \, d\theta)}$$

$$= \int_0^{\pi/2} a^3 \sin^2\theta \cos\theta \sqrt{a^2(1-\sin^2\theta)}\, d\theta$$

$$= \int_0^{\pi/2} a^3 \sin^2\theta \cos\theta \sqrt{a^2 \cos^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} a^3 \sin^2\theta \cos\theta \sqrt{(a\cos\theta)^2}\, d\theta$$

$$= \int_0^{\pi/2} a^3 \sin^2\theta \cos\theta (a\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} a^4 \sin^2\theta \cos^2\theta\, d\theta = \frac{a^4}{4}\int_0^{\pi/2} 4\sin^2\theta\cos^2\theta\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} (2\sin\theta\cos\theta)^2 d\theta = a^4/4 \int_0^{\pi/2} (\sin 2\theta)^2 d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} (\sin^2 2\theta)\, d\theta$$

Use $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$= \frac{a^4}{4}\int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{a^4}{8}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta$$

$$= \frac{a^4}{8}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2} (\cos 4\theta)\, d\theta\right]$$

$$= \frac{a^4}{8}\left[[\theta]_0^{\pi/2} - \left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}\right]$$

$$= \frac{a^4}{8}\left[\left[\pi/_2 - 0\right] - 1/_4\left[\sin\left(4\left(\pi/_2\right)\right) - \sin(0)\right]\right]$$

$$= \frac{a^4}{8}\left[\left[\pi/_2\right] - 1/_4\left[\sin 2\pi - \sin 0\right]\right]$$

$$= a^4/_8\left[\pi/_2 - 1/_4\left[0-0\right]\right] = \frac{a^4}{8}\left[\frac{\pi}{2}\right]$$

$$= \frac{\pi a^4}{16}$$

19. $$\int \frac{\sqrt{1+x^2}}{x}\,dx$$

Let $x = \tan\theta$
Then $dx = \sec^2\theta$

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta\,d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec^3\theta\,d\theta}{\tan\theta}$$

$$= \int \frac{1}{\cos^3\theta\left(\frac{\sin\theta}{\cos\theta}\right)}\,d\theta = \int \frac{1}{\cos^2\theta\sin\theta}\,d\theta$$

$$= \int \csc\theta\sec^2\theta\,d\theta$$

$= \int \csc\theta(1+\tan^2\theta)\,d\theta = \int \csc\theta\,d\theta + \int \csc\theta\tan^2\theta\,d\theta$

$= \int \csc\theta\,d\theta + \int \frac{1}{\sin\theta}\,\frac{\sin^2\theta}{\cos^2\theta}\,d\theta$

$= \int \csc\theta\,d\theta + \int \frac{\sin\theta}{\cos^2\theta}\,d\theta$

$= \int \csc\theta\,d\theta + \int \tan\theta\sec\theta\,d\theta$

$= \ln|\csc\theta - \cot\theta| + \sec\theta + C$

Here $x = \tan\theta$
And, $1/x = \cot\theta$

Use $1+\cot^2\theta = \csc^2\theta$

$\csc^2\theta = 1 + (1/x)^2$

$\csc^2\theta = \frac{x^2+1}{x^2}$

$\csc\theta = \sqrt{\frac{x^2+1}{x^2}} = \frac{\sqrt{x^2+1}}{x}$

So $1+x^2 = 1+\tan^2\theta = \sec^2\theta$

Then $\sec\theta = \sqrt{1+x^2}$

Put all these values into result of prev. step

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \ln|\operatorname{cosec}\theta - \cot\theta| + \sec\theta + C$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \sqrt{1+x^2} + C$$

$$= \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

21. $$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx$$

Let $5x = 3\sin\theta$ so that $5dx = 3\cos\theta\,d\theta$

When $x=0$ so $\theta=0$ and when $x=0.6$ so $\theta=\pi/2$

So,

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx = 3/5 \int_0^{\pi/2} \frac{\left(\frac{3\sin\theta}{5}\right)^2}{\sqrt{9-(3\sin\theta)^2}} \cos\theta\,d\theta$$

$$= 3/5 \cdot 9/25 \int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{9-9\sin\theta}} \cos\theta\,d\theta$$

$$= 3/5 \cdot 9/25 \int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cos\theta\,d\theta$$

$$= 3/5 \cdot 9/25 \int_0^{\pi/2} \frac{\sin^2\theta}{3\sqrt{1-\sin^2\theta}} \cos\theta\,d\theta$$

$$\Rightarrow \frac{a}{125}\int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cdot\cos\theta\,d\theta$$

$$= \frac{a}{125}\int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta$$

$$= \frac{a}{125}\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= \frac{a}{250}\int_0^{\pi/2} (1-\cos 2\theta)\,d\theta$$

$$= \frac{a}{250}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a}{250}\left[\frac{\pi}{2}\right] = a/500\,\pi$$

31. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}}$

(A) Substitute $x = a\tan\theta$

~~x = a tan~~ $dx = a\sec^2\theta\,d\theta$

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2}$$
$$= a\sqrt{\tan^2\theta + 1} = a\sec\theta$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int \sec\theta\,d\theta$$

$= \ln|\sec\theta + \tan\theta| + C_1$

Undo the substitution

$\tan\theta = x/a$, $\sec\theta = \dfrac{\sqrt{x^2+y^2}}{a}$

$= \ln\left|\dfrac{\sqrt{x^2+a^2}}{a^2} + \dfrac{x}{a}\right| + C_1$

$= \ln\left|\dfrac{\sqrt{x^2+a^2} + x}{a}\right| + C_1$

$= \ln\left(\sqrt{x^2+a^2} + x\right) - \ln(a) + C_1$

$= \ln\left(\sqrt{x^2+a^2} + x\right) + C_1$

URBAN EDGE

Exercise 7.4

1. a) $$\frac{4+x}{(1+2x)(3-x)} = \frac{A}{(1+2x)} + \frac{B}{(3-x)}$$

b) $$\frac{1-x}{x^3+x^4} = \frac{1-x}{x^3(1+x)}$$

$$\frac{1-x}{x^3(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{1+x}$$

3. (a) $$\frac{1}{x^2+x^4} = \frac{1}{x^2(1+x^2)}$$

$$\frac{1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}$$

(b)
$$\begin{array}{r|l} & 1 \\ x^3-3x^2+2x & x^3+1 \\ & x^3-3x^2+2x \\ \hline & 3x^2-2x+1 \end{array}$$

$$\frac{x^3+1}{x^3-3x^2+2x} = 1 + \frac{3x^2-2x+1}{x^3-3x^2+2x}$$

URBAN EDGE

$$x^3 - 3x^2 + 2x = x(x^2 - 3x + 2)$$

$$= x(x-1)(x-2)$$

$$\frac{x^3+1}{x^3-3x^2+2x} = 1 + \frac{3x^2-2x+1}{x(x-1)(x-2)}$$

$$= 1 + \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

5. a)

$$\begin{array}{r} x^4 + 4x^2 + 16 \\ x^2 - 4 \,\big)\, x^6 \qquad\qquad\quad \\ \underline{x^6 - 4x^2} \qquad\quad \\ 4x^4 \qquad \\ \underline{4x^4 - 16x^2} \\ 16x^2 \\ \underline{16x^2 - 64} \\ 64 \end{array}$$

$$\frac{x^6}{x^2-4} = (x^4 + 4x^2 + 16) + \frac{64}{(x-2)(x+2)}$$

$$= (x^4 + 4x^2 + 16) + \frac{A}{x-2} + \frac{B}{x+2}$$

b) $$\frac{x^4}{(x^2-x+1)(x^2+2)^2} = \frac{Ax+B}{(x^2-x+1)} + \frac{Cx+D}{(x^2+2)}$$

$$+ \frac{Ex+F}{(x^2+2)^2}$$

7. $\int \frac{x^4}{x-1}\,dx$

By long division

$$\frac{x^4}{x-1} = (x^3 + x^2 + x + 1) + \frac{1}{x-1}$$

$$\int \frac{x^4}{x-1}\,dx = \int (x^3 + x^2 + x + 1)\,dx + \int \frac{1}{x-1}\,dx = \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x\right)$$

$$+ \ln|x-1| + C$$

11. Factor the denominator:

$$2x^2 + 3x + 1 = 2x^2 + 2x + x + 1$$
$$= 2x(x+1) + 1(x+1)$$
$$= (2x+1)(x+1)$$

$$\frac{2}{(2x+1)(x+1)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)}$$

$$= \frac{A(x+1) + B(2x+1)}{(2x+1)(x+1)}$$

$$= \frac{(A+2B)x + A + B}{(2x+1)(x+1)}$$

Equate the coefficients

$A + 2B = 0$
$A + B = 0$

$$\begin{array}{r} A + 2B = 0 \\ \underline{(-)A +_{(-)} B = 2} \\ B = -2 \end{array}$$

Sub. the value of B:

$A + 2(-2) = 0$
$A - 4 = 0$
$A = 4$

Thus,

$$\frac{2}{(2x+1)(x+1)} = \frac{4}{(2x+1)} - \frac{2}{(x+1)}$$

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \frac{4}{(2x+1} - \frac{2}{(x+1)}\,dx$$

$$= 2\int_0^1 \left(\frac{2}{2x+1}\right)dx - 2\int_0^1 \frac{1}{x+1}\,dx$$

$$= \left[2\ln(2x+1) - 2\ln(x+1)\right]_0^1$$

$= 2\ln(2+1) - 2\ln(1+1) - (2\ln(0+1) - 2\ln(0+1))$
$= 2\ln 3 - 2\ln 2 - (2\ln 1 - 2\ln 1)$
$= 2\ln 3 - 2\ln 2 - 0$

$= 2(\ln 3 - \ln 2) = 2\ln 3/2$

15.

$$\begin{array}{r} x+3 \\ x^3-3x+2 \;\overline{)\; x^3-4x+1} \\ \underline{x^3-3x^2+2x} \\ 3x^2-6x+1 \\ \underline{3x^2-9x+6} \\ 3x-5 \end{array}$$

$$\frac{x^3-4x+1}{x^2-3x+2} = (x+3) + \frac{3x-5}{x^2-3x+2}$$

$$= (x+3) + \frac{3x-5}{(x-2)(x-1)} = (x+3) + \frac{A}{(x-2)} + \frac{B}{(x-1)}$$

$$\frac{x^3-4x+1}{x^2-3x+2} = \frac{(x+3)(x-2)(x-1) + A(x-1) + B}{(x-1)(x-2)}$$

$x^3-4x+1 = (x+3)(x-2)(x-1) + A(x-1) + B(x-2)$
$\rightarrow$ (1)

Let $x=1$ in (1)

$-2 = 0 + A(0) + B(-1) \rightarrow B = 2$

URBAN EDGE

Let $x = 2$ in (1)

$1 = 0 + A(1) + B(0) \rightarrow A = 1$

$$\int_{-1}^{0} \frac{x^3 - 4x + 1}{x^2 - 3x + 2} = \int_{-1}^{0} \left[(x+3) + \frac{1}{x-2} + \frac{2}{x-1} \right] dx$$

$$= \left[\frac{x^2}{2} + 3x + \ln|x-2| + 2\ln|x-1| \right]_{-1}^{0}$$

$$= 0 + 0 + \ln 2 + 2\ln 1 - 1/2 + 3 - \ln 3 - 2\ln 2$$

$$= 5/2 - \ln 3 - \ln 2$$

19. $$\frac{x^2 + x + 1}{(x+1)^2 (x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$$

$$\frac{x^2 + x + 1}{(x+1)^2 (x+2)} = \frac{A(x+1)(x+2) + B(x+2) + C(x+1)^2}{(x+1)^2 (x+2)}$$

$(x-2)$

$$x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2 \quad \ldots (1)$$

Let $x = -1$ in (1)

$(-1)^2 + (-1) + 1 = A(-1+1)(-1+2) + B(-1+2) + C(-1+1)^2$

$1 - 1 + 1 = A(0)(1) + B(1) + C(0)^2$

$1 = B(1)$

$B = 1$

Let $x = -2$ in (1)

$x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

$(-2)^2 + (-2) + 1 = A(-2+1)(-2+2) + B(-2+2) + C(-2+1)^2$

$4 - 2 + 1 = A(-1)(0) + B(0) + C(-1)^2$

$3 = C(1)^2$
$C = 3$

Equate the coeffiencients of x^2 on both sides in (1)

$x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

$x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 2x + 1)$

$x^2 + x + 1 = (A + C)x^2 + (3A + B + 2C)x + (2A + 2B + C)$
$1 = A + C$
$1 = A + 3$
$A = -2$

EDGE

$$\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = \int_0^1 \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{(x+2)}\right] dx$$

$$= -2\int_0^1 \left(\frac{1}{x+1}\right) dx + \int_0^1 \frac{1}{(x+1)^2}\,dx + 3\int_0^1 \frac{1}{x+2}\,dx$$

$$= \left[-2\ln|x+1| - \frac{1}{x+1} + 3\ln|x+2|\right]_0^1$$

$$= \left(-2\ln 2 - \tfrac{1}{2} + 3\ln 3\right) - \left(-2\ln|1| - 1 + 3\ln 2\right)$$

$$= \left(-2\ln 2 - \tfrac{1}{2} + 3\ln 3\right) - \left(0 - 1 + 3\ln 2\right)$$

$$= -2\ln 2 - \tfrac{1}{2} + 3\ln 3 + 1 - 3\ln 2$$

$$= -5\ln 2 + \tfrac{1}{2} + 3\ln 3$$

21. $\dfrac{1}{(t^2-1)^2} = \dfrac{1}{(t+1)^2(t-1)^2}$

$$\frac{1}{(t^2-1)^2} = \frac{A}{(t+1)} + \frac{B}{(t+1)^2} + \frac{C}{(t-1)} + \frac{D}{(t-1)^2}$$

$1 = A(t+1)(t-1)^2 + B(t-1)^2 + C(t-1)(t+1)^2 +$
$D(t+1)^2 \ldots \quad (1)$

Let $t = 1$ in (1)

$1 = D(2)^2 \rightarrow D = 1/4$

Let $t = -1$ in (1)

$1 = B(-2)^2 \to B(1/4)$

Equate the coefficients of t^3 in (1)

$0 = A + C$

$1 = A + B - C + D$
$1 = A - C + 1/2$
$1/2 = A - C$

From $0 = A + C$, $A - C = 1/2$,

$A + C = 0$

$A - C = 1/2$

$2A = 1/2 \to A = 1/4$

$C = -A \to C = -1/4$

$$\int \frac{dt}{(t^2-1)^2} = \int \left(\frac{1/4}{(t+1)} + \frac{1/4}{(t+1)^2} - \frac{1/4}{(t-1)} + \frac{1/4}{(t-1)^2} \right) dt$$

$= 1/4 \ln|t+1| - \frac{1}{4(t+1)} - 1/4 \ln|t-1| -$

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$$\frac{1}{4(t-1)} + C$$

23. $$\frac{10}{(x-1)(x^2+9)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+9} \quad \ldots\ldots (1)$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$
$$10 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$$
$$10 = (A+B)x^2 + (C-B)x + 9A - C \quad \ldots\ldots (2)$$

Sub $x = 1$

$$10 = A((1)^2+9) + (B(1)+C)(1-1)$$

$10 = 10A$

$A = 10/10 \qquad A = 1$

Now equate the coefficient of x^2

$A + B = 0$
$B = -A$
$B = -1$

Now equate the coefficient of x

$C - B = 0$
$C = B$
$C = -1$

Sub. all values

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{(x-1)} + \frac{-x-1}{(x^2+9)}\,dx$$

$$= \int \frac{1}{(x-1)}\,dx + \int \frac{-x}{(x^2+9)}\,dx + \int \frac{-1}{(x^2+9)}\,dx$$

$$= \int \frac{1}{(x-1)}\,dx - \int \frac{x}{(x^2+9)}\,dx - \int \frac{1}{(x^2+9)}\,dx$$

...... (2)

Since $\int \frac{1}{(x-1)}\,dx = \ln|x-1| + c$

and $\int \frac{1}{(x^2+9)}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$

Assume $x^2+9 = t$

$2x\,dx + 0 = dt$

$x\,dx = \frac{1}{2}dt$

Then,

$$\int \frac{x}{(x^2+9)}\,dx = \int \frac{\frac{1}{2}dt}{t} = \frac{1}{2}\ln|t| + c$$

$$= \frac{1}{2}\ln|x^2+9| + c$$

From (2)

URBAN EDGE

$$\int \frac{10}{(x-1)(x^2+9)}dx = \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}(x/3) + C$$

34.

$$\begin{array}{r|l} & x^2 \\ x^3+1 & x^5+x-1 \\ & x^5+x^2 \\ \hline & x-1-x^2 \end{array}$$

$$\frac{x^5+x-1}{x^3+1} = x^2 + \frac{x-1-x^2}{x^3+1}$$

$$= x^2 + \frac{x-1-x^2}{(x+1)(x^2-x+1)} = x^2 - \frac{1}{x+1}$$

$$\int \frac{x^5+x-1}{x^3+1}dx = \int \left(x^2 - \frac{1}{x+1}\right)dx$$

$$= x^3/3 - \ln|x+1| + C$$

41.

$$\int \frac{dx}{x^2+x\sqrt{x}}$$

Take $x = t^2 \Rightarrow dx = 2t\,dt$

$$\int \frac{dx}{x^2+x\sqrt{x}} = \int \frac{(2t\,dt)}{t^4+t^3} = 2\int \frac{dt}{t^3+t^2}$$

$$\frac{1}{t^2(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

$$= \frac{At(1+t) + B(1+t) + Ct^2}{t^2(1+t)}$$

$$\Rightarrow At(1+t) + B(1+t) + Ct^2 = 1$$

Equating the like powers of 't' we get

$A + C = 0$, $A + B = 0$, $B = 1$

Solving, we get $A = -1$, $B = 1$, $C = 1$

Therefore $\frac{1}{t^2(1+t)} = \frac{-1}{t} + \frac{1}{t^2} + \frac{1}{t+1}$

$$\int \frac{dt}{t^3+t^2} = \int \frac{-1}{t}\,dt + \int \frac{1}{t^2}\,dt + \int \frac{1}{t+1}\,dt$$

$$= -\ln|t| - \frac{1}{t} + \ln|t+1| + C$$

$$= -2\ln|\sqrt{x}| - \frac{2}{\sqrt{x}} + 2\ln|\sqrt{x}+1| + C$$

47. $\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}\,dx = \int \frac{e^{x} \cdot e^{x}\,dx}{e^{2x}+3e^{x}+2}$

Substitute $e^{x} = t \Rightarrow e^{x}dx = dt$

$$\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}\,dx = \int \frac{t \cdot dt}{t^2+3t+2}$$

$$\frac{t}{t^2+3t+2} = \frac{t}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1}$$

$$\Rightarrow t = A(t+1) + B(t+2)$$

Substitute $t = -1$ Substitute $t = -2$

$B = -1$ $A = 2$

$$\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}\,dx = \int \left(\frac{2}{t+2} + \frac{-1}{t+1} \right) dt$$

$$= 2\ln|t+2| - \ln|t+1| + C$$

$$= 2\ln(e^{x}+2) - \ln(e^{x}+1) + C$$

$$= \ln\left[(e^{x}+2)^2/(e^{x}+1)\right] + C$$

448. $$\int \frac{\cos x}{\sin^2 x + \sin x}\,dx$$

Substitute $\sin x = t \Rightarrow$ ~~cos t~~ $\cos x\,dx = dt$

$$\int \frac{\cos x}{\sin^2 x+\sin x}\,dx = \int \frac{dt}{t^2+t} = \int \frac{dt}{t(t+1)}$$

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

$1 = A(t+1) + Bt$

Substitute $t = 0$: $A = 1$

$t = -1$ $B = -1$

$$\int \frac{\cos x}{\sin^2 x + \sin x}\,dx = \int \frac{dt}{t(t+1)} = \int\left(\frac{1}{t} - \frac{1}{t+1}\right)$$

$= \ln|t| - \ln|t+1| + C$

$= \ln\left|\frac{t}{t+1}\right| + C = \ln\left|\frac{\sin x}{\sin x + 1}\right| + C$

65. $(3x - x^2) = x(3-x)$

$$-x^2 + 3x \,)\, x^2 + 1 \,(\, -1$$
$$\underline{x^2 - 3x}$$
$$3x + 1$$

$$\frac{x^2+1}{x(3-x)} = -1 + \frac{3x+1}{3x - x^2} \quad \ldots (2)$$

$$\frac{3x+1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x}$$

$$\frac{3x+1}{x(3-x)} = \frac{A(3-x) + Bx}{x(3-x)}$$

$3x+1 = A(3-x) + Bx$

$3x+1 = (B-A)x + 3A$

Equate coefficients of x and constants

$3A = 1 \Rightarrow A = 1/3$

And,

$B - A = 3 \Rightarrow B - 1/3 = 3 \Rightarrow B = 10/3$

Thus,

$$\frac{3x+1}{x(3-x)} = \frac{1}{3x} + \frac{10}{3(3-x)} \quad \ldots\ldots (3)$$

Sub (3) in (2)

$$\frac{x^2+1}{x(3-x)} = -1 + \left(\frac{1}{3x} + \frac{10}{3(3-x)}\right)$$

$$\int_1^2 \frac{x^2+1}{x(3-x)}\,dx = \int_1^2 (-1)\,dx + \int_1^2 (1/3x)\,dx$$

$$+ \int_1^2 \left(\frac{10}{3(3-x)}\right) dx$$

$$= (-x)_1^2 + \left(1/3 \ln|x|\right)_1^2 - \left(\frac{10}{3} \ln|3-x|\right)_1^2$$

URBAN EDGE

$= -1 + \frac{1}{3}\ln|2| - \frac{1}{3}\ln|1| - \frac{10}{3}\ln|1| +$
$\frac{10}{3}\ln|2|$

$= -1 + \frac{1}{3}\ln|2| + \frac{10}{3}\ln|2|$

$= \frac{11}{3}\ln|2| - 1$

Exercise 7.5

1. $\int \frac{\cos x}{1-\sin x}\,dx$

Substitute $1-\sin x = y$

Differentiate:

$0 - \cos x\,dx = dy$

$\cos x\,dx = -dy$

$\int \frac{\cos x}{1-\sin x}\,dx = -\int \frac{1}{y}\,dy = -\ln y + C$

$= -\ln(1-\sin x) + C$

5. $I = \int \frac{t}{t^4+2}\,dt$

URBAN EDGE

Let $u = t^2$

$du = 2t\,dt$

$$I = \int \frac{du/2}{u^2+2} = 1/2 \int \frac{1}{u^2+(\sqrt{2})^2}\,du$$

Using $\int \frac{dx}{x^2+a^2} = 1/a \tan^{-1}(x/a)$

$$1/2 \int \frac{1}{u^2+(\sqrt{2})^2}\,du = 1/2\left(1/\sqrt{2} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right) + C$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

Substitute $u = t^2$

$$\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right) + C$$

7. $$\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy$$

$\tan^{-1} y = u$

$$\frac{1}{1+y^2}\frac{dy}{du} = 1 \Rightarrow \frac{1}{1+y^2}\,dy = du$$

$y = 1, \quad u = \tan^{-1}(1) = \pi/4$

$y = -1, \quad u = \tan^{-1}(-1) = -\pi/4$

$$\Rightarrow \int_{-1}^{1} e^{\arctan y} \frac{1}{1+y^2}\, dy$$

$$= \int_{-\pi/4}^{\pi/4} e^u\, du = [e^u]_{-\pi/4}^{\pi/4}$$

$$= e^{\pi/4} - e^{-\pi/4}$$

9. $$\frac{x+2}{x^2+3x-4} = \frac{x+2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

$$\frac{x+2}{x^2+3x-4} = \frac{A(x-1) + B(x+4)}{(x+4)(x-1)}$$

$$x + 2 = A(x-1) + B(x+4) \ldots (1)$$

Let $x = 1$ in (1)

$3 = B(5) \rightarrow B = 3/5$

Let $x = -4$ in (1)

$-2 = A(-5) = A = 2/5$

URBAN EDGE

$$\int_2^4 \frac{x+2}{x^2+3x-4} = \int_2^4 \left[\frac{2}{5(x+4)} + \frac{3}{5(x-1)}\right] dx$$

$$= \left[\tfrac{2}{5}\ln(x+4) + \tfrac{3}{5}\ln(x-1)\right]_2^4$$

$$= \tfrac{2}{5}\ln 8 + \tfrac{3}{5}\ln 3 - \tfrac{2}{5}\ln 6 - \tfrac{3}{5}\ln 1$$

$$= \tfrac{2}{5}\ln(2^3) + \tfrac{3}{5}\ln 3 - \tfrac{2}{5}\ln(2\cdot 3) - \tfrac{3}{5}\ln 1$$

$$= 3\times\tfrac{2}{5}\ln(2) + \tfrac{3}{5}\ln 3 - \tfrac{2}{3}[\ln(2) + \ln(3)]$$

$$- \tfrac{3}{5}\ln 1$$

$$= \tfrac{6}{5}\ln 2 + \tfrac{3}{5}\ln 3 - \tfrac{2}{5}\ln 2 - \tfrac{2}{5}\ln 3 - \tfrac{3}{5}(0)$$

$$= \left(\tfrac{6}{5}\ln 2 - \tfrac{2}{5}\ln 2\right) + \left(\tfrac{3}{5}\ln 3 - \tfrac{2}{5}\ln 3\right)$$

$$= \tfrac{4}{5}\ln 2 + \tfrac{1}{5}\ln 3 = \tfrac{1}{5}\ln(2^4) + \tfrac{1}{5}\ln 3$$

$$= \tfrac{1}{5}\ln(16) + \tfrac{1}{5}\ln 3 = \tfrac{1}{5}(\ln 16 + \ln 3)$$

$$= \tfrac{1}{5}\ln(16\cdot 3) = \tfrac{1}{5}\ln(48)$$

15. $\int x\sec x\tan x\,dx$

Apply integration by parts

Let $u = x$, $dv = \sec x\tan x\,dx$

URBAN EDGE

$u = x \rightarrow du = dx$
$dv = \sec x \tan x \, dx \rightarrow v = \sec x$

$$\int x \sec x \tan x \, dx = x \sec x - \int \sec x \, dx$$

$$= x \sec x - \ln|\sec x + \tan x| + C$$

30. $\int_{-1}^{2} |e^x - 1| \, dx$

$$|e^x - 1| = \begin{cases} -e^x + 1, & x < 0 \\ e^x - 1, & x > 0 \end{cases}$$

$$\int_{-1}^{2} |e^x - 1| \, dx = \int_{-1}^{0} (1 - e^x) \, dx + \int_{0}^{2} (e^x - 1) \, dx$$

$$= [x - e^x]_{-1}^{0} + [e^x - x]_0^2$$

$$= -1 - (-1 - 1/e) + (e^2 - 2 - 1)$$

$$= 1/e + e^2 - 3$$

41. $\int \theta \tan^2 \theta \, d\theta = \int \theta (\sec^2\theta - 1) \, d\theta = \int (\theta \sec^2\theta - \theta) \, d\theta$

$$= \int \theta \sec^2\theta \, d\theta - \int \theta \, d\theta$$

$I_1 = \int \theta \sec^2\theta \, d\theta \qquad I_2 = \int \theta \, d\theta$

$$\int \theta \tan^2\theta \, d\theta = I_1 - I_2$$

$I_2 = \int \theta d\theta = \left[\frac{\theta^2}{2}\right] + C_2$

I_1: Evaluate I_1

Let $u = \theta$ and $dv = \sec^2\theta d\theta$

$\frac{du}{d\theta} = \frac{d}{d\theta}(\theta)$ $dv = \sec^2\theta d\theta$

$\frac{du}{d\theta} = 1$ $v = \int \sec^2\theta d\theta$

$du = d\theta$ $v = \tan\theta$

Using integration ~~po~~ by parts

$I_1 = \int \theta \sec^2\theta d\theta = \theta\tan\theta - \int \tan\theta d\theta$

$= \theta\tan\theta - \ln|\sec\theta| + C_1$

Substitute back I_1 and I_2

$\int \theta \tan^2\theta d\theta = I_1 - I_2$

$= \theta\tan\theta - \ln|\sec\theta| + C_1 - \left[\theta^2/2 + C_2\right)$

$= \theta\tan\theta - \ln|\sec\theta| + C_1 - \frac{\theta^2}{2} - C_2$

$= \theta\tan\theta - \ln|\sec\theta| - \theta^2/2 + C$ $[C = C_1 - C_2]$

URBAN EDGE

46 $\int \frac{(x-1)e^x}{x^2} dx = \int \left(\frac{x}{x^2} - \frac{1}{x^2} \right) e^x dx$

$= \int (1/x - 1/x^2) e^x dx$

$\Rightarrow \int \frac{(x-1)}{x^2} e^x dx = \int \frac{1}{x} e^x dx - \int \frac{1}{x^2} e^x dx$

$I = \int 1/x \, e^x dx \qquad f(x) = 1/x \quad f'(x) = -1/x^2$

$\qquad g'(x) = e^x \cdot g(x) = e^x$

Using integration by parts

$= (1/x)(e^x) - \int e^x (- 1/x^2) dx + C$

$= \int 1/x \cdot e^x dx = 1/x \, e^x + \int 1/x^2 e^x dx + C$

Substitute back I

$\int \frac{(x-1)}{x} \cdot e^x dx = 1/x \, e^x + \int \frac{1}{x^2} e^x dx + C$

$- \int 1/x^2 \cdot e^x dx = 1/x \, e^x$

$\int \frac{(x-1) e^x dx}{x^2} = 1/x \, e^x + C$

URBAN EDGE

Exercise 7.6

1. By the table of integrals,

$$\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C \quad \cdots (1)$$

Take $a=5,\ b=2$ in (1) we get

$$\int_0^{\pi/2} \cos 5x \cos 2x\, dx = \left[\frac{\sin 3x}{6} + \frac{\sin 7x}{14}\right]_0^{\pi/2}$$

$$= -1/6 - 1/14 = -5/21$$

3. By the table of integrals

$$\int \sqrt{u^2-a^2}\, du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C \quad \cdots (1)$$

Take $a = \frac{\sqrt{3}}{2}$ in (1)

$$\int_1^2 \sqrt{4x^2-3}\, dx = 2\int_1^2 \sqrt{x^2-\left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-\left(\frac{\sqrt{3}}{2}\right)^2} - 3/8 \ln\left|x+\sqrt{x^2-3/4}\right|\right]_1^2$$

$$= 2\left[\sqrt{\frac{13}{4}} - \frac{3}{8}\ln\left|2 + \frac{\sqrt{13}}{2}\right| - \frac{1}{4} + \frac{3}{8}\ln\left|\frac{3}{2}\right|\right]$$

$$= \sqrt{13} - \frac{3}{4}\ln\left|2 + \frac{\sqrt{13}}{2}\right| - \frac{1}{2} + \frac{3}{4}\ln\left|\frac{3}{2}\right|$$

6. By the table of integrals

$$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\left(\frac{u}{a}\right) + C \quad \ldots\ldots (1)$$

Take $a=2$ in (1) we get

$$\int_0^2 x^2\sqrt{4-x^2}\,dx = \left[\frac{x}{8}(2x^2-4)\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \pi$$

7. Take $\sin x = u \Rightarrow \cos x\,dx = du$

$$\int \frac{\cos x}{\sin^2 x - 9}\,dx = \int \frac{du}{u^2-3^2}$$

$$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C \quad \ldots (1)$$

URBAN EDGE

Take $a = 3$ in (1) we get

$$\int \frac{du}{u^2 - 3^2} = \frac{1}{6} \ln \left| \frac{u-3}{u+3} \right| + C$$

$$\int \frac{\cos x}{\sin^2 x - 9}\, dx = \int \frac{du}{u^2 - 3^2} = \frac{1}{6} \ln \left| \frac{u-3}{u+3} \right| + C$$

$$= \frac{1}{6} \ln \left| \frac{\sin x - 3}{\sin x + 3} \right| + C$$

18. $$\int \frac{dx}{2x^3 - 3x^2} = \int \frac{dx}{x^2(2x-3)}$$

From the table of integarations,

$$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} - \frac{b}{a^2} \ln \left| \frac{a+bu}{u} \right| + C \quad \ldots (1)$$

Compare the function $x^2(2x-3)$ with $u^2(a+bu)$ to get $u^2 = x^2$, $u = x$, $a = -3$, and $b = 2$

Substitute these values in eq. (1)

$$\int \frac{dx}{x^2(2x-3)} = \frac{1}{-3x} + \frac{2}{(-3)^2} \ln \left| \frac{-3+2x}{x} \right| + C$$

$$= -\frac{1}{-3x} + \frac{2}{9} \ln \left| \frac{-3+2x}{x} \right| + C$$

$= \frac{1}{3x} + \frac{2}{9} \ln\left|\frac{3+2x}{x}\right| + C$

19. Let $\sin x = t \Rightarrow \cos x\,dx = dt$

$\int \sin^2 x \cos x \ln(\sin x)\,dx = \int t^2 \ln t\,dt$

Use the formula $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}$

$[(n+1)\ln u - 1] + C$

Where $n = 2$, then

$\int t^2 \ln t\,dt = t^3/9\,(3\ln t - 1) + C$

$= \frac{\sin^3 x}{9}\,[3\ln(\sin x) - 1] + C$

20. Take $\sin\theta = u \Rightarrow \cos\theta\,d\theta = du$

$\int \frac{2\sin\theta\cos\theta}{\sqrt{5-\sin\theta}}\,d\theta = 2\int \frac{u\,du}{\sqrt{5-u}}$

By the table of integrals

$\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$ ....(1)

Take $a = 5$, $b = -1$ in (1) we get

$$\int \frac{u\,du}{\sqrt{5-u}} = -\frac{2}{3}(u+10)\sqrt{5-u} + C$$

$$\int \frac{2\sin\theta\cos\theta}{\sqrt{5-\sin\theta}}\,d\theta = 2\int \frac{u\,du}{\sqrt{5-u}}$$

$$= -\frac{4}{3}(u+10)\sqrt{5-u} + C$$

$$= -\frac{4}{3}(\sin\theta+10)\sqrt{5-\sin\theta} + C$$

21. Let $e^x = t \Rightarrow e^x dx = dt$

$$\int \frac{e^x}{3-e^{2x}}\,dx = \int \frac{dt}{3-t^2} = \int \frac{dt}{(\sqrt{3})^2 - t^2}$$

Use the formula $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

$$\int \frac{dt}{(\sqrt{3})^2-t^2} = \frac{1}{2\sqrt{3}}\ln\left|\frac{t+\sqrt{3}}{t-\sqrt{3}}\right| + C$$

$$= \frac{1}{2\sqrt{3}}\cdot\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right| + C$$

URBAN EDGE

23. Use the formula

$\int \sin^n u\,du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u\,du \ldots (1)$

$n = 5$

$\int \sec^5 x\,dx = \frac{1}{4} \tan x \sec^3 x + \frac{3}{4} \int \sec^3 x\,dx$

Again use (1)

$\int \sec^5 x\,dx = \frac{1}{4} \tan x \sec^3 x + \frac{3}{4} \left[\frac{1}{2} \tan x \sec x + \frac{1}{2} \int \sec x\,dx \right]$

Use $\int \sec u\,du = \ln|\sec u + \tan u| + C$

$\int \sec^5 x\,dx = \frac{1}{4} \tan x \sec^3 x + \frac{3}{8} \tan x \sec x + \frac{3}{8} \ln|\sec x + \tan x| + C$

32. Let $\tan\theta = x \Rightarrow \sec^2\theta\,d\theta = dx$

$\int \frac{\sec^2\theta \tan^2\theta}{\sqrt{9 - \tan^2\theta}}\,d\theta = \int \frac{x}{\sqrt{9 - x^2}}\,dx$

Use the formula $\int \frac{u^2}{\sqrt{a^2 - u^2}}\,du =$

$- \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{-x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} +$$

$$= -\frac{\tan\theta}{2}\sqrt{9-\tan^2\theta} + \frac{9}{2}\sin^{-1}\left(\frac{\tan\theta}{3}\right)$$

Exercise 7.7

5\. (b) $\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big]$ where n is even and $\Delta x = \frac{b-a}{n}$ .... (3)

So,

$\Delta x = \frac{b-a}{n} = \frac{2-0}{10} = 1/5 = 0.2$

~~$x_0 = a + i\Delta x = 0 +$~~

$x_0 = a + i\Delta x = 0 + 0(0.2) = 0$

$x_1 = a + i\Delta x = 0 + 1(0.2) = 0.2$

$x_2 = a + i\Delta x = 0 + 2(0.2) = 0.4$

Simililarly we get, $x_3 = 0.6$, $x_4 = 0.8$,

$x_5 = 1$, $x_6 = 1.2$, $x_7 = 1.4$, $x_8 = 1.6$

$x_9 = 1.8$, $x_{10} = 2$

Now substitute these values in (3), we have

$$\int_0^2 \frac{x}{1+x^2}\,dx \approx S_{10}$$

$$= \frac{0.2}{3}\Big[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1) + 2f(1.3) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)\Big]$$

$$\approx \frac{0.2}{3}\Big[0 + 4(0.192307692) + 2(0.344827586) + 4(0.441176471) + 2(0.487804878) + 4(0.5) + 2(0.491803279) + 4(0.472972973) + 2(0.449438202) + 4(0.424528302) + 0.4\Big]$$

$$\approx \frac{0.2}{3}\Big[0 + 0.769230769 + 0.689655172 + 1.764805882 + 0.975609756 + 2 + 0.983608557 + 1.891891892 + 0.898876404 + 1.698113208 + 0.4\Big]$$

$$\approx \frac{0.2}{3}(12.07168964) \approx \underline{0.804779309}$$

$$\int_0^2 \frac{x}{1+x^2}\,dx = \Big[\tfrac{1}{2}\ln(1+x^2)\Big]_0^2 = \tfrac{1}{2}\ln(5)$$

$$\approx 0.8047189560$$

Therefore, the error is

$E_S = 0.804718956 0 - 0.804779309$
$\approx -0.000060353 0$
≈ -0.000060

7. (a) $\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + f(x_n) + 2\left(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right)\right]$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ .....(1)

For the given problem, $a = 1$, $b = 2$, $n = 10$,

$f(x) = \sqrt{x^3 - 1}$

So,

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = 0.1$$

$x_0 = a + i\Delta x = 1 + 0(1/10) = 1$

$x_1 = a + i\Delta x = 1 + 1(1/10) = 11/10 = 1.1$

$x_2 = a + i\Delta x = 1 + 2(1/10) = \frac{12}{10} = 1.2$

Similarly we get $x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$, $x_6 = 1.6$, $x_7 = 1.7$, $x_8 = 1.8$, $x_9 = 1.9$, $x_{10} = 2$

Now substitute these values in (1), we have

$$\int_1^2 \sqrt{x^3-1}\,dx \approx T_{10}$$

$$= \frac{(0.1)}{2}\left[f(1)+f(2)+2\left(f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)+f(1.6)+f(1.7)+f(1.8)+f(1.9)\right)\right]$$

$$\approx \frac{1}{20}\left[0+2.645751311+2\left(0.575325995+0.853229916+1.094074952+1.320605922+1.541103501+1.759545396+1.97813043+2.198181066+2.42053713\right)\right]$$

$$\approx \frac{1}{20}\left[0.2.645751311+2(13.74073355)\right]$$

$$\approx \frac{1}{20}\left[0.+2.645751311+27.4814671\right]$$

$$\approx \frac{1}{20}\left[30.12721841\right] \approx 1.506360921$$

(b) $\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x}_1)+f(\bar{x}_2)+ \dots f(\bar{x}_n)\right]$ where $\Delta x = \frac{b-a}{n}$ and

$\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$ = midpoint of $[x_{i-1}, x_i]$ .....(2)

URBAN EDGE

For the given problem, $a = 1, b = 2, n = 1$
$f(x) = \sqrt{x^3 - 1}$. So,

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = 0.1$$

The 10 intervals are

$[1, 1.1], [1.1, 1.2], [1.2, 1.3], [1.3, 1.4$
$[1.4, 1.5], [1.5, 1.6], [1.6, 1.7], [1.7, 1.8$
$[1.8, 1.9], [1.9, 2]$

The midpoint of 10 subintervals are

$\bar{x}_1 = \frac{1}{2}(x_0 + x_1) = \frac{1}{2}(1 + 1.1) = 1.05$
$\bar{x}_2 = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}(1.1 + 1.2) = 1.15$

Similarly, we get

$\bar{x}_3 = 1.25, \bar{x}_4 = 1.35, \bar{x}_5 = 1.45, \bar{x}_6 = 1.55,$
$\bar{x}_7 = 1.65, \bar{x}_8 = 1.75, \bar{x}_9 = 1.85, x_{10} = 1.95$

Now substitute these values in (2), we have

$$\int_1^2 \sqrt{x^3 - 1}\, dx \approx M_{10} =$$

$$= \frac{1}{10}\Big[f(1.05) + f(1.15) + f(1.25) + f(1.35$$
$$+ f(1.45) + f(1.55) + f(1.65) +$$
$$f(1.75) + f(1.85) + f(1.95)\Big]$$

$$\approx \frac{1}{10}\left[\begin{array}{l} 0.397020151 + 0.721716703 + 0.976281209 \\ + 1.208459764 + 1.431301855 + 1.650416614 \\ + 1.868722826 + 2.087911639 + 2.309031182 \\ + 2.532760352 \end{array}\right]$$

$$\approx \frac{1}{10}[15.18362229] \approx 1.518362229$$

(c) $\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$ where n is even and

$$\Delta x = \frac{b-a}{n} \quad \dots (3)$$

For the given problem, $a = 1$, $b = 2$, $n = 10$, $f(x) = \sqrt{x^3 - 1}$

So,

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = 0.1$$

$x_0 = a + i\Delta x = 1 + 0(1/10) = 1$

$x_1 = a + i\Delta x = 1 + 1(1/10) = \frac{11}{10} = 1.1$

$x_2 = a + i\Delta x = 1 + 2(1/10) = 12/10 = 1.2$

Similarly we get, $x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$, $x_6 = 1.6$, $x_7 = 1.7$, $x_8 = 1.8$, $x_9 = 1.9$, $x_{10} = 2$

Substitute those values in (3)

$$\int_1^2 \sqrt{x^3-1}\, dx \approx S_{10}$$

$$\approx \frac{1}{30}\begin{bmatrix} 0 + 2.301303 + 1.706458 + \\ 4.376299 + 2.641211 + 6.164414 \\ + 3.519090 + 7.912521 + 4.396392 \\ + 9.682148 + 2.645751 \end{bmatrix}$$

$$\approx \frac{1}{30}[45.34556] \approx 1.1511519$$

11. (a) $\int_a^b f(x)\,dx \approx T_n =$

$$\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots \right.$$

$$\left. 2f(x_{n-1}) + f(x_n) \right]$$ where

$\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Here, $f(x) = x^3 \sin x$, $a = 0$, $b = 4$ and $n = 8$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{8} = 1/2 = 0.5$$

Find $x_1, x_2, x_3, \dots x_n$ using the result $x_i = a + i\Delta x$

$x_0 = 0 + 0 \times 0.5 = 0$

$x_1 = 0 + 1 \times 0.5 = 0.5$

$x_2 = 0 + 2 \times 0.5 = 1$

$x_3 = 0 + 3 \times 0.5 \; 1.5$

$x_4 = 0 + 4 \times 0.5 = 2$

$x_5 = 0 + 5 \times 0.5 = 2.5$

$x_6 = 0 + 6 \times 0.5 = 3$

$x_7 = 0 + 7 \times 0.5 = 3.5$

$x_8 = 0 + 8 \times 0.5 = 4$

URBAN EDGE

Then,

$f(0) = 0^3 \sin(0) = 0$

$f(0.5) = 0.5^3 \sin(0.5) = 0.059928$

$f(1) = 1^3 \sin(1) = 0.841470$

$f(1.5) = 1.5^3 \sin(1.5) = 3.36655$

$f(2) = 2^3 \sin(2) = 7.274379$

$f(2.5) = 2.5^3 \sin(2.5) = 9.35113$

$f(3) = 3^3 \sin(3) = 3.81024$

$f(3.5) = 3.5^3 \sin(3.5) = -15.0398$

$f(4) = 4^3 \sin(4) = -48.435535$

$$\int_0^4 x^3 \sin x \, dx \approx T_8 =$$

$$0.25 \Big[0 + 2(0.059928) + 2(0.841470) + 2(3.36655) + 2(7.274379) + 2(9.35113) + 2(3.810240) + 2(-15.0398) + (-48.435535) \Big]$$

$$= -7.27689055$$

(b) $$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

$$= \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \dots + (\bar{x}_n) \right]$$

Where $\Delta x = \dfrac{b-a}{n}$ and $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$

Find mid points, using the result

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2}$$

For $i = 1$,

$$\bar{x}_i = \frac{0 + 0.5}{2} = 0.25$$

For $i = 2$,

$$\bar{x}_2 = \frac{0.5 + 1}{2} = 0.75$$

Similarly the remaining mid points

$\bar{x}_3 = 1.25$, $\bar{x}_4 = 1.75$, $\bar{x}_5 = 2.25$,

$\bar{x}_6 = 2.75$, $\bar{x}_7 = 3.25$, $\bar{x}_8 = 3.75$

$f(x) = x^3 \sin x$, $a = 0$, $b = 4$ and $n = 8$

$$\int_0^4 x^3 \sin x \, dx \approx M_8$$

$$= \frac{1}{2} [0.003865 + 0.287566 + 1.85348 + 5.273549 + 8.862740 + 7.937355 - 3.714136 - 30.14092$$

$$= 0.5 [-9.636501] = -4.818251$$

URBAN EDGE

(c) $\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) +$

$2f(x_2) + 4f(x_3) + \dots 2f(x_{n-2}) +$

$4f(x_{n-1}) + f(x_n)]$ Where n is even,

$\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

From the part (a) use the values of

$x_1, x_2, x_3 \dots$ and x_n.

$\int_0^4 x^3 \sin x\,dx \approx S_8$

$= \frac{1}{6}[0.023971 + 1.682941 + 13.466182 + 14.548758 + 37.404509 + 7.620480 - 60.159323 - 48.435359]$

$= \frac{1}{6}[-33.632097] = -5.605349$

16. (a) $\int_a^b f(x)\,dx = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2)$

$+ \dots 2f_{n-1} + 2f(x_n)] \dots (1)$

Here $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

Here $a = 1$, $b = 3$

$$\Delta x = \frac{3-1}{4} = \frac{1}{2} \qquad x_0 = 1$$

Then,

$x_1 = 1 + 1(1/2) = 3/2$

$x_2 = 1 + 2(1/2) = 2$

And,

$x_3 = 1 + 3(1/2) = 5/2$

$x_4 = 1 + 4(1/2) = 3$

$$\int_1^3 \frac{\sin t}{t}\,dt = 1/4\left[\frac{\sin 1}{1} + 2\left(\frac{\sin(3/2)}{(3/2)}\right) + 2\left(\frac{\sin 2}{2}\right) + 2\left(\frac{\sin(5/2)}{(5/2)}\right) + \left(\frac{\sin 3}{3}\right)\right] \approx 0.9016$$

(b) $\int_a^b f(x)\,dx = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \dots$
$\dots + f(\bar{x}_n)]$ .....(2)

URBAN EDGE

$\Delta x = \frac{b-a}{n}$

$\bar{x_i} = \frac{1}{2}(x_{i-1} + x_i)$

Here $a = 1$, $b = 3$

$\Delta x = \frac{3-1}{4}$

$\Delta x = \frac{1}{2}$

$x_0 = 1$

Intervals are $[1, 3/2]$, $\left[\frac{3}{2}, \frac{4}{2}\right]$, $\left[\frac{4}{2}, \frac{5}{2}\right]$

, $\left[\frac{5}{2}, 3\right]$

$\bar{x}_1 = 5/4$ $\bar{x}_2 = 7/4$ $\bar{x}_3 = 9/4$ $\bar{x}_4 = 11/4$

Substitute in (2)

$$\int_1^3 \frac{\sin t}{t}\,dt = \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f(7/4) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] = \frac{1}{2}\left[\frac{\sin(5/4)}{5/4} + \frac{\sin(7/4)}{7/4} + \frac{\sin(9/4)}{(9/4)} + \frac{\sin(11/4)}{11/4}\right] \approx 0.90303$$

(c) $\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$

$\dots (3)$

Here $\Delta x = \frac{b-a}{2}$, n is even

Use values from (a) and substitute in (3)

$$\int_1^3 \frac{\sin t}{t}\,dt = \frac{1}{6}\left[\frac{\sin 1}{1} + 4\left(\frac{\sin(3/2)}{3/2}\right) + 2\left(\frac{\sin(2)}{2}\right) + \cancel{4}\frac{\sin(5/2)}{5/2} + \frac{\sin 3}{3} \right]$$

≈ 0.9025

22. Let $f(x) = e^{x^2}$

First, find the fourth derivative

Differentiate both sides

$f'(x) = d/dx\, e^{x^2} = 2xe^{x^2}$

Again, diff. both sides

$f''(x) = 2xe^{x^2}(2x) + 2e^{x^2}$

$= 4x^2e^{x^2} + 2e^{x^2}$

URBAN EDGE

$f''(x) = (4x^2 + 2)e^{x^2}$

Diff. both sides

$f'''(x) = (4x^2+2)(2x)e^{x^2} + 8xe^{x^2}$

$= (8x^3 + 4x + 8x)e^{x^2}$

$= (8x^3 + 12x)e^{x^2}$

Diff. again

$f^{(4)}(x) = (8x^3+12x)(2x)e^{x^2} + (24x^2+12)e^{x^2}$

$= (16x^4 + 24x^2 + 24x^2 + 12)e^{x^2} =$

$(16x^4 + 48x^2 + 12)e^{x^2}$

$|f^4(x)| = (16x^4 + 48x^2 + 12)e^{x^2} \leq (16+48+12)e$

$|f^4(x)| \leq 76e$ for $0 \leq x \leq 1$

Take $K = 76e$, $a = 0$, $b = 1$

$\frac{K(b-a)^5}{180n^4} < 0.00001$

$\frac{76e}{180n^4} < 0.00001$

$$n^4 > \frac{76e}{0.0018}$$

$$n > \sqrt[4]{\frac{76e}{0.0018}} \approx 18.4$$

Therefore, $n = 20$ (n must be even) gives the desired accuracy

29. Here $a = 0$, $b = 6$ and $n = 10$.

$$\Delta x = \frac{b-a}{n} = \frac{6-0}{6}$$

$$\Delta x = 1$$

(a) $T_6 = \frac{\Delta x}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)\right]$

$$= \frac{1}{2}\left[2 + 2\cdot(1) + 2\cdot(3) + 2\cdot(5) + 2\cdot(4) + 2\cdot(3) + 4\right]$$

$$T_6 = 19$$

(b) $M_6 = \Delta x\left[f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)\right]$

$$= 1\left[1.3 + 1.5 + 4.5 + 4.7 + 3.3 + 3.3\right.$$

$M_6 = 18.6$

Thus, $M_6 = 18.6$

(c) $S_6 = \frac{\Delta x}{3}[f(0) + 4f(1) + 2f(2) + 4f(3)$

$+ 2f(4) + 4f(5) + f(6)]$

$= 1/3 [2 + 4\cdot(1) + 2(3) + 4(5) + 2(4) +$

$4(3) + 4] = \frac{56}{3}$

$S_6 = 18.\bar{6}$

~~Thus~~

30. Divide the interval $[0, 16]$ into $n = 8$ subintervals, $\Delta w = \frac{16}{8} = 2$

The subintervals are $[0,2]$, $[2,4]$, $[4,6]$, $[6,8]$, $[8,10]$, $[10,12]$, $[12,14]$, and $[14,16]$

$A \approx \frac{\Delta w}{3}[W(0) + 4W(2) + 2W(4) +$

$4W(6) + 2W(8) + 4W(10) + 2W(12) +$

$4W(14) + W(16)]$

$= 2/3 [24.8 + 14.4 + 27.2 + 11.2 + 20 + 9.6$

$+ 19.2] = 84.2\bar{6}\ m^2$

Exercise 6.1

1. The area A of the region bounded by curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f, g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

From the diagram, $\sqrt[3]{x} \geq 1/x$ for $x \geq 1$

Let $f(x) = \sqrt[3]{x}$, $g(x) = 1/x$

$$A = \int_1^8 \left(\sqrt[3]{x} - 1/x\right) dx =$$

$$\left[\frac{x^{1/3+1}}{1/3+1} - \ln x\right]_1^8 = \left[\frac{3}{4} x^{4/3} - \ln x\right]_1^8$$

$$= \left[\frac{3}{4}(8)^{4/3} - \ln 8\right] - \left[\frac{3}{4} - \ln 1\right]$$

$$= [12 - \ln 8] - [3/4 - \ln 1] = 12 - \ln 8 - 3/4$$

$$= 45/4 - \ln 8$$

3. The desired area is given by

$$A = \int_{y=-1}^{1} (e^y - (y^2 - 2))\,dy =$$

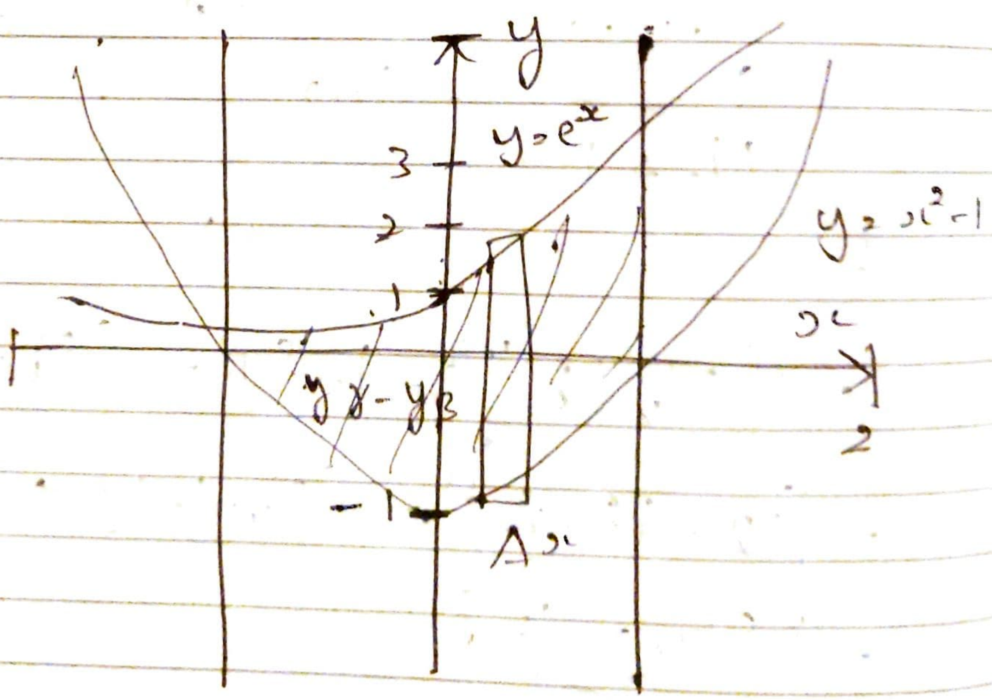
$$\int_{y=-1}^{1} (e^y - y^2 + 2)\,dy = \left[e^y - y^3/3 + 2y\right]_{y=-1}^{1}$$

$$= \left(e - 1/3 + 2\right) - \left(e^{-1} + 1/3 - 2\right)$$

$$= e - e^{-1} - 2/3 + 4$$

$$= e - 1/e + 10/3$$

5.

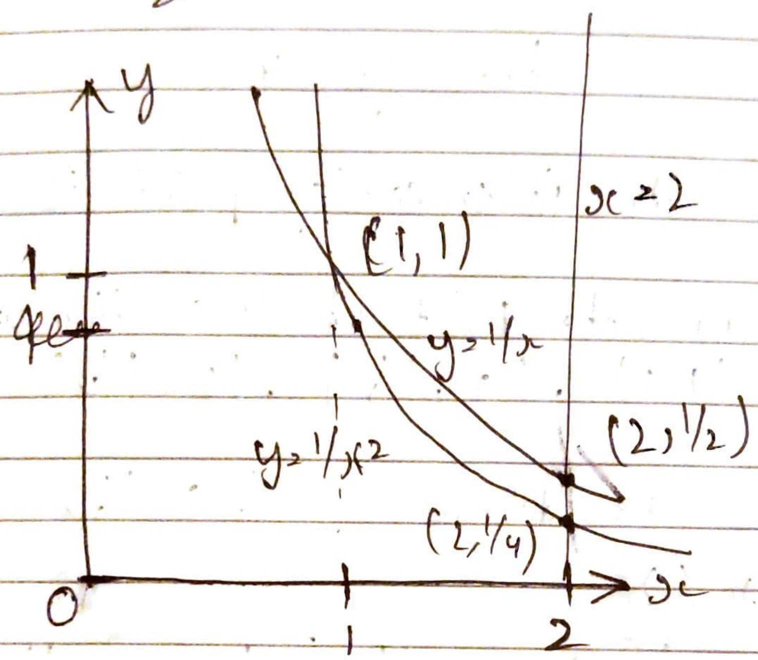


$$A = \int_{-1}^{1} (y_\gamma - y_\beta) = \int_{-1}^{1} (e^x - (x^2 - 1))\,dx$$

$$= \left[e^{x} - \frac{x^3}{3} + x\right]_{-1}^{1} = (e - e^{-1}) - 2/3 + 2$$

$$= 4/3 + (e - e^{-1})$$

9.



Find the intersection points of the curves $y = 1/x$, $y = 1/x^2$

$1/x = 1/x^2$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x = 1$

If $x = 0$ then $y = 1/0 = \infty$

If $x = 1$ then $y = 1/1 = 1$

Thus the two curves intersect at point $(1,1)$

Find the intersection points of curves $y = 1/x$, $x = 2$

If $x = 2$ then $y = 1/2$

Hence, the intersection point is $(2, 1/2)$

Find the intersection points of the curves
$y = 1/x^2, x = 2$

If $x = 2$ then $y = 1/4$

Hence the intersecting point is $(2, 1/4)$

Therefore, the point of intersection of the curves is $(1,1), (2, 1/2), (2, 1/4)$

From the diagram, $1/x \geq 1/x^2$ and $1 \leq x \leq 2$.

$$A = \int_1^2 \left[1/x - 1/x^2\right] dx = \int_1^2 1/x \, dx - \int_1^2 1/x^2$$

$$= [\ln x]_1^2 - [-1/x]_1^2 = [\ln x]_1^2 + [1/x]_1^2$$

$$= \ln 2 - \ln 1 + [1/2 - 1/1]$$

$$= \ln - 0 - 1/2 = \ln 2 - 1/2$$

11. Find the intersection points of the curves $x = 1 - y^2, x = y^2 - 1$

$$1 - y^2 = y^2 - 1$$

$$2y^2 = 2$$

$$y^2 = 1 \Rightarrow y = \sqrt{1} = \pm 1$$

Substitute y values in any one of the two curve equations to get the corresponding x values.

When $y = \pm 1$, then

$x = (1)^2 - 1 = 1 - 1 = 0$

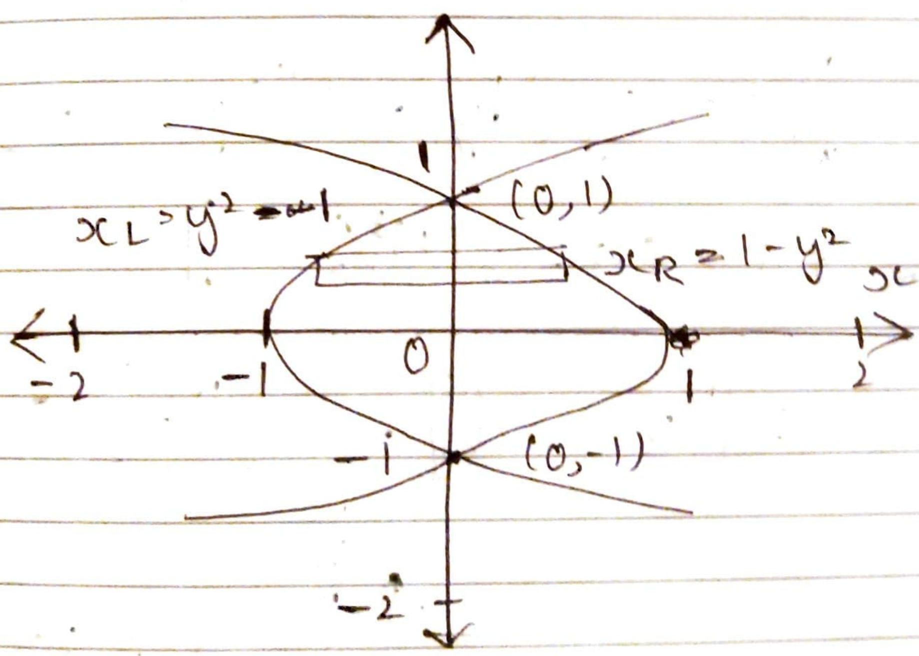
$x = 0$

When $y = -1$, then

$x = (-1)^2 - 1 = 1 - 1 = 0$

$x = 0$

Then the intersecting points of the two curves are $(0, 1), (0, -1)$.



From the above figure, the left and right boundary of the curves are

$x_R = 1 - y^2$ and $x_L = y^2 - 1$

Integrate between the appropriate y values $y = -1$ and $y = 1$

Then, the area of the shaded region is as follows:

$$A = \int_{-1}^{1} [x_R - x_L] dy$$

$$= \int_{-1}^{1} [(1-y^2) - (y^2-1)] dy$$

$$= \int_{-1}^{1} [1-y^2-y^2+1] dy$$

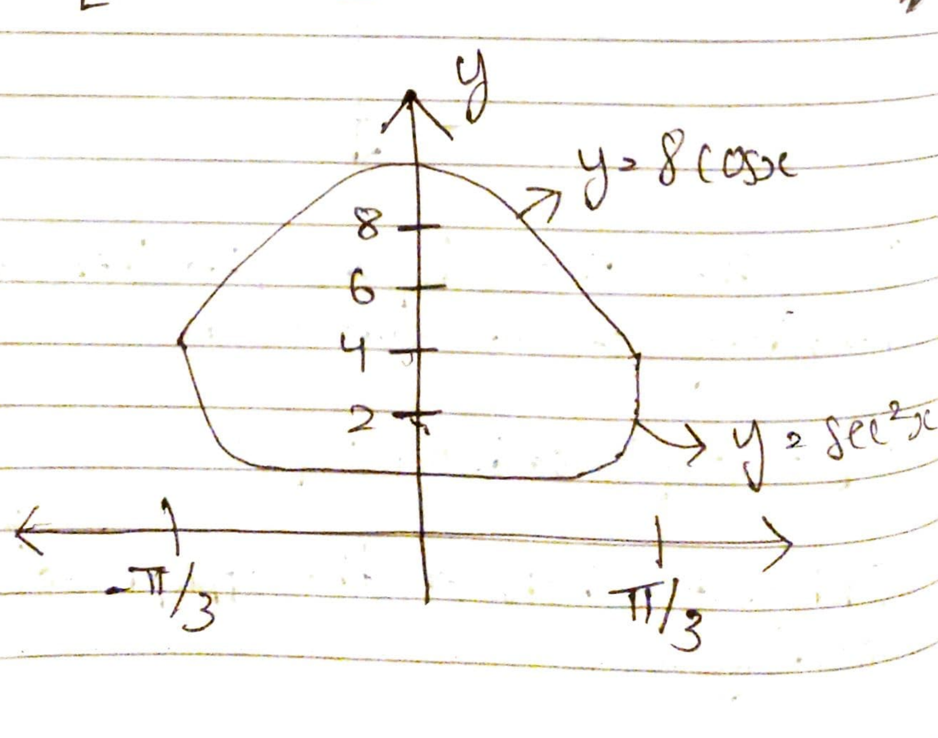
$$= \int_{-1}^{1} [2-2y^2] dy = 2\int_{0}^{1} [2-2y^2] dy$$

$$= 2\int_{0}^{1} [2-2y^2] dy = 2\left[2y - 2\frac{y^3}{3}\right]_0^1$$

$$= 2\left[2(1) - 2(1)^3/3 - 0\right]$$

$$= 2\left[2 - 2/3\right] = 2\left[4/3\right] = 8/3$$

15.



From the ~~de~~diagram, $8\cos x \geq \sec^2 x$ for $-\frac{\pi}{3} \leq x \leq \pi/3$

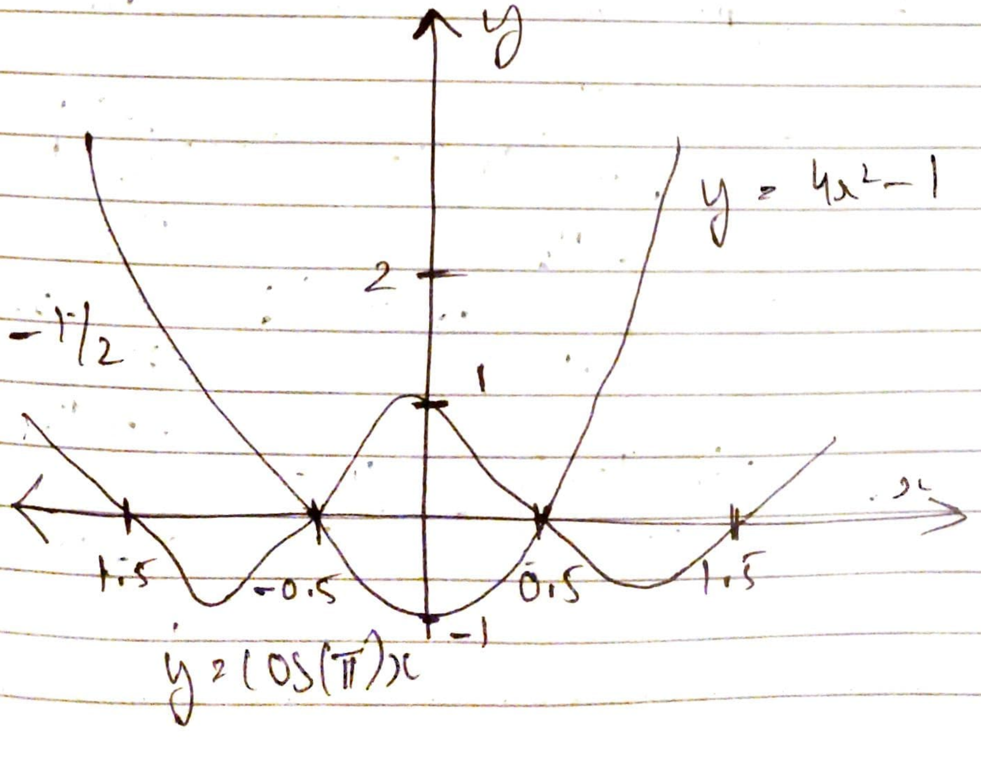
Area of the shaded region is $\int_{-\pi/3}^{\pi/3} (8\cos x - \sec^2 x)\,dx$

$= \int_{-\pi/3}^{\pi/3} (8\cos x - \sec^2 x)\,dx = \int_{-\pi/3}^{\pi/3} 8\cos x\,dx - \int_{-\pi/3}^{\pi/3} \sec^2 x\,dx = 8[\sin x]_{-\pi/3}^{\pi/3} - [\tan x]_{-\pi/3}^{\pi/3}$

$= 8[\sin \pi/3 - \sin(-\pi/3)] - [\tan \pi/3 - \tan(-\pi/3)] = 8\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - (\sqrt{3} + \sqrt{3})$

$= 8\sqrt{3} - 2\sqrt{3} = 6\sqrt{3}$

19.



Find the point of intersection as follows:

From the figure, two curves intersect at the points $x = -1/2$ and $x = 1/2$

Hence the region of the integration is

$-1/2 \le x \le 1/2$

$$A = \int_{-1/2}^{1/2} (y_{upper} - y_{lower})\, dx$$

$$= \int_{-1/2}^{1/2} (y_T - y_B)\, dx$$

From the figure, the upper curve is $y = 4x^2 - 1$ and lower curve is $y = \cos(\pi x)$

$$A = \int_{-1/2}^{1/2} (\cos \pi x - (4x^2 - 1))\, dx$$

$$= 2\int_0^{1/2} (\cos \pi x - (4x^2 - 1))\, dx$$

$$= 2\left[\int_0^{1/2} (\cos \pi x)\, dx - \int_0^{1/2} (4x^2 - 1)\, dx\right]$$

$$= \left[\frac{\sin \pi x}{\pi}\right] - \left(\frac{4x^3}{3} - x\right)\Bigg]_0^{1/2}$$

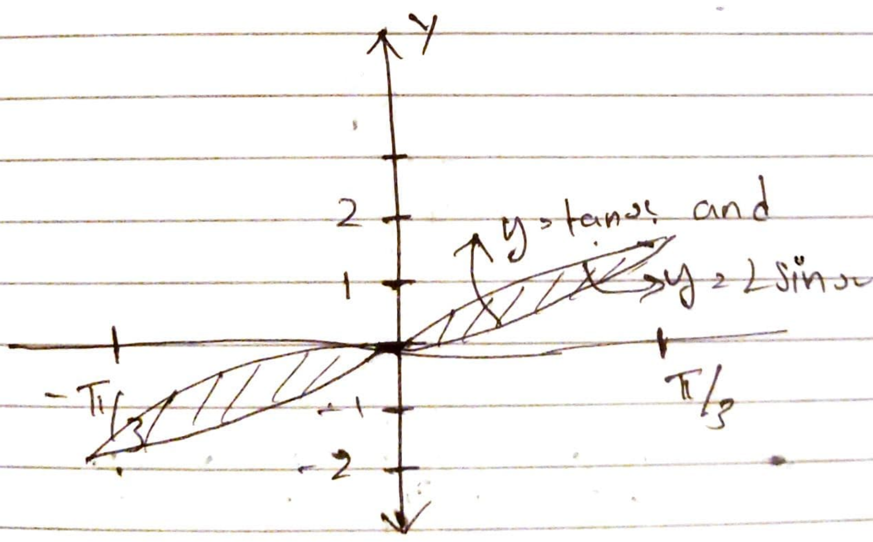
$$= 2\left[\frac{\sin \pi x}{\pi} - \frac{4x^3}{3} + x\right]_0^{1/2}$$

$$= 2\left[\frac{1}{\pi}\left(\sin\left(\pi/2\right) - \sin(0)\right) - 4/3\left((1/2)^3 - 0\right) + (1/2 - 0)\right] = 2\left[1/\pi\,(1-0) - 4/3\,(1/8 - 0) + 1/2\right] = 2\left[1/\pi - 4/3\,(1/8) + (1/2)\right]$$

$$= 2\left[1/\pi - 1/6 + 1/2\right] = 2\left(1/\pi + 2/6\right)$$

$$= 2/\pi + 2/3$$

21.



From the above graph you can see that in interval $[-\pi/3, 0]$ the curve $y = \tan x$ is above ~~y~~ $y = 2\sin x$ and in interval $\left[0, \pi/3\right]$ the curve $y = \tan x$ is below the $y = 2\sin x$

$$A = \int_{-\pi/3}^{0} (\tan x - 2\sin x)\,dx + \int_{0}^{\pi/3} (2\sin x - \tan x)\,dx$$

$$= \Big[\ln|\sec x| + 2\cos x\Big]_{-\pi/3}^{0} + \Big[-2\cos x - \ln|\sec x|\Big]_{0}^{\pi/3}$$

$$= (\ln|\sec 0| + 2\cos 0) - (\ln|\sec -\pi/3| + 2\cos -\pi/3)$$

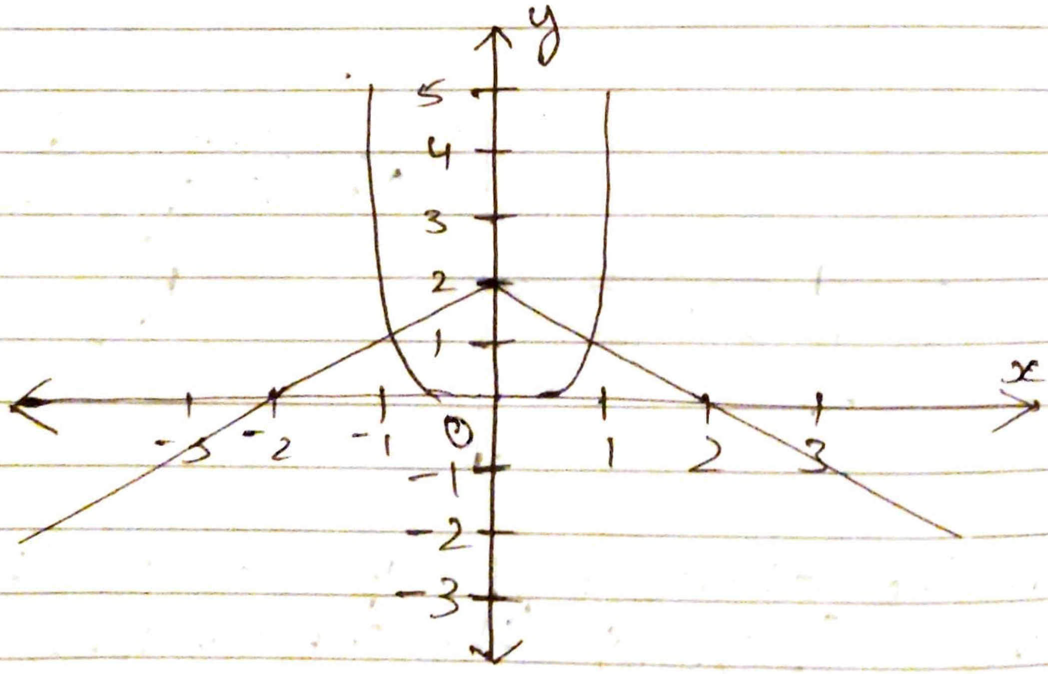
$$+ (-2\cos \pi/3 - \ln|\sec \pi/3|) - (-2\cos 0 - \ln|\sec 0|)$$

$$= (0+2) - (\ln 2 + 1) + (-1 - \ln 2) - (-2 - 0)$$

$$= 2 - \ln 2 - 1 - 1 - \ln 2 + 2$$

$$= 2 - 2\ln 2$$

25.



From the above figure, the intersecting points of the curve is -1 to 1.

Observe that, $2-|x|-x^4 = \begin{cases} 2+x-x^4 & \text{when } -1\leq x<0 \\ 2-x-x^4 & \text{when } 0\leq x\leq 1 \end{cases}$

$$A = \int_{-1}^{1} [2-|x|-x^4]\,dx = A_1 + A_2$$

$$= \int_{-1}^{0} [2+x-x^4]\,dx + \int_{0}^{1} [2-x-x^4]\,dx$$

Here, $A_1 = \int_{-1}^{0} [2+x-x^4]\,dx$ and

$A_2 = \int_{0}^{1} [2-x-x^4]\,dx$

Find $A_1 = \int_{-1}^{0} [2+x-x^4]\,dx$

$$A_1 = \int_{-1}^{0} [2+x-x^4]\,dx = \left[2x + \frac{x^2}{2} - \frac{x^5}{5}\right]_{-1}^{0}$$

$$= [0+0-0] - \left[-2 + 1/2 + 1/5\right] = \frac{13}{10}$$

Thus, $A_1 = 13/10$

$$A_2 = \int_{0}^{1} [2-x-x^4]\,dx = \left[2x - \frac{x^2}{2} - \frac{x^5}{5}\right]_{0}^{1}$$

$$= \left[2 - 1/2 - 1/5\right] - [0-0-0] = \frac{13}{10}$$

Thus, $A_2 = \frac{13}{10}$

$A = A_1 + A_2 = \frac{26}{10} = \frac{13}{5}$ //

29\. (a) The objective is to find the total area between the curves for $0 \le x \le 5$.

The area between the two curves for $0 \le x \le 5$ is the sum of the area of the two shaded regions.

That is,

$12 + 27 = 39$.

Thus the total area of the region is 39

(b) The objective is to find the value of the integral $\int_0^5 [f(x) - g(x)]\,dx$

From the diagram it is clear that, the area of the region from $x=0$ to $x=2$ is,

$\int_0^2 [g(x) - f(x)]\,dx = 12$

And, the area of the region from $x=2$ to $x=5$ is,

$\int_2^5 [f(x) - g(x)]\,dx = 27$.

Then,

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$\int_0^5 [f(x) - g(x)]\,dx = \int_0^2 [f(x) - g(x)]\,dx +$

$\int_2^5 [f(x) - g(x)]\,dx = \int_0^2 - [g(x) - f(x)]\,dx$

$+ \int_2^5 [f(x) - g(x)]\,dx = -\int_0^2 [g(x) - f(x)]\,dx$

$+ \int_2^5 [f(x) - g(x)]\,dx = -12 + 27 = 15$

34. The equation to the line passes through the points $(x_1, y_1), (x_2, y_2)$ is

$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

The equation to the line passes through the points $(2,0), (0,2)$ is

$(y - 0) = \frac{2-0}{0-2}(x - 2) \Rightarrow y = -x + 2$

$\Rightarrow y = 2 - x$

The equation to the line passes through the points $(2,0), (-1,1)$ is

$(y - 0) = \frac{1-0}{-1-2}(x - 2)$

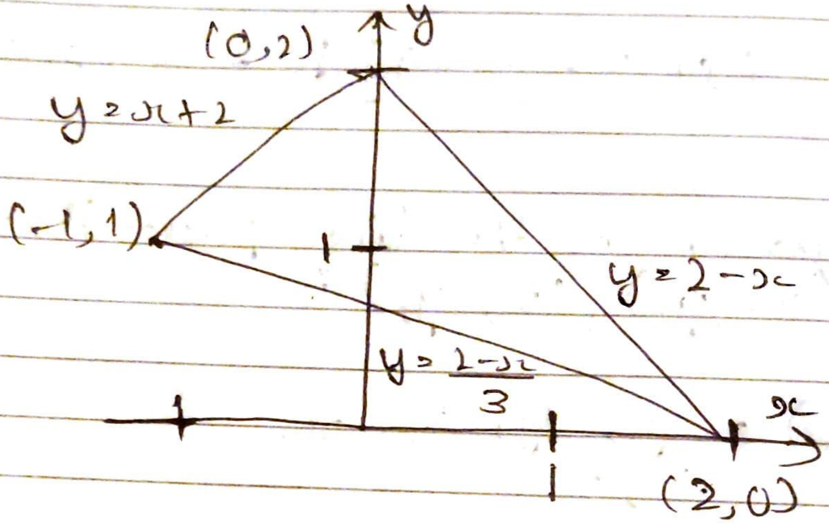
$\Rightarrow -3y = x - 2$

$\Rightarrow y = \frac{2-x}{3}$

The equation to the line passes through the points $(0,2),(-1,1)$ is

$$(y-2) = \frac{1-2}{-1-0}(x-0) \Rightarrow y-2 = x$$

$$\Rightarrow y = x+2$$



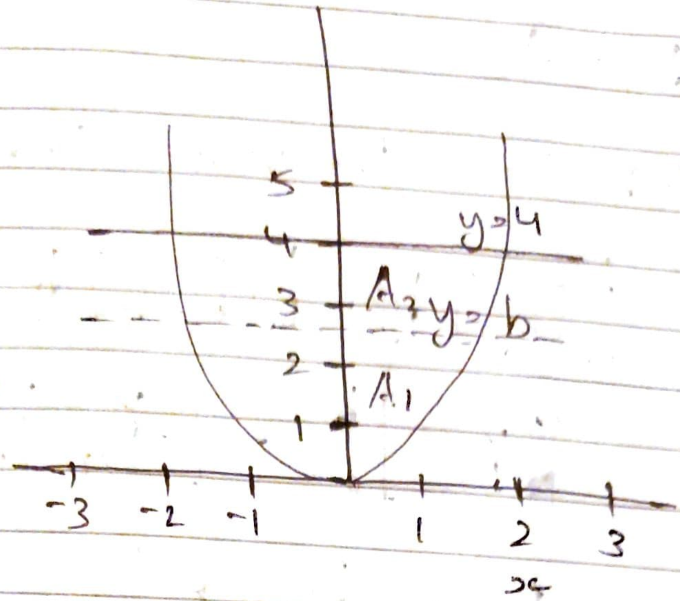
Equations of the sides passing through the given vertices is shown in the above graph

$$A = \int_{-1}^{0} \left((x+2) - \left(\frac{2-x}{3}\right)\right) dx + \int_{0}^{2} \left((2-x) - \left(\frac{2-x}{3}\right)\right) dx$$

$$= \left[\frac{4}{3}\,\frac{(x+1)^2}{2}\right]_{-1}^{0} + \left[\frac{4x}{3} - \frac{x^2}{3}\right]_{0}^{2}$$

$$= \frac{2}{3} + \frac{4}{3} = 2$$

57.



If a region bounded by the curves $x = f(y)$, $x = g(y)$ and the lines $x = c$ and $x = d$, where f and g are continuous and $f(x) \geq g(x)$, then its area is given by

$$A = \int_c^d [f(y) - g(y)]\, dy$$

Thus, the area A_1 is given by,

$$A_1 = \int_0^b [\sqrt{y} - 0]\, dy$$

$$= \int_0^b [\sqrt{y}]\, dy$$

Similarly, the area A_2 is given by,

$$A_2 = \int_b^4 [\sqrt{y} - 0]\, dy = \int_b^4 [\sqrt{y}]\, dy$$

Since the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two equal areas, so

$A_1 = A_2$

Substitute the values and simplify,

$$\int_0^b [\sqrt{y}]\,dy = \int_b^4 [\sqrt{y}]\,dy$$

$$\left[\frac{y^{3/2}}{3/2}\right]_0^b = \left[\frac{y^{3/2}}{3/2}\right]_b^4$$

$$b^{3/2} = 4^{3/2} - b^{3/2}$$

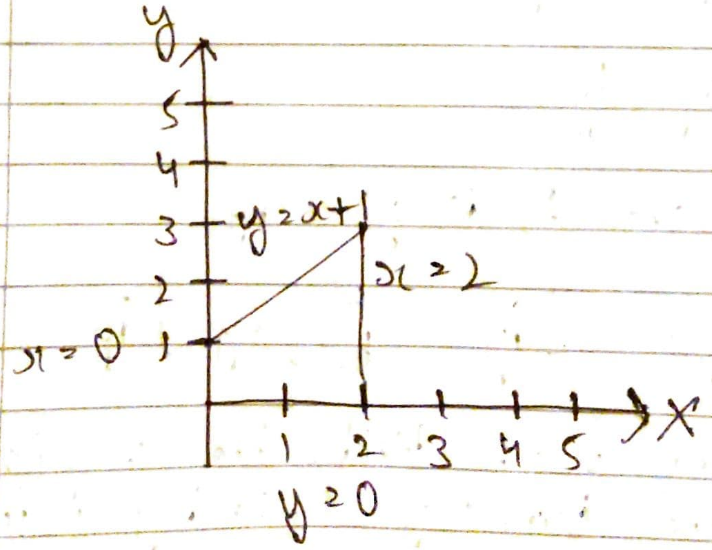
$$2b^{3/2} = 8$$

Solve the above expression for b,

$$b^{3/2} = 4$$

$$b = 2.52$$

Exercise 6.2



Cross section is in the shape of a disk with radius $(x+1)$

The area of the cross section is,

$A(x) = \pi r^2 = \pi (x+1)^2$

Volume of the solid is $\int_0^2 A(x)\,dx$

$\int_0^2 A(x)\,dx = \int_0^2 \pi (x+1)^2\,dx \quad \ldots\ldots (1)$

Let $x+1 = t$

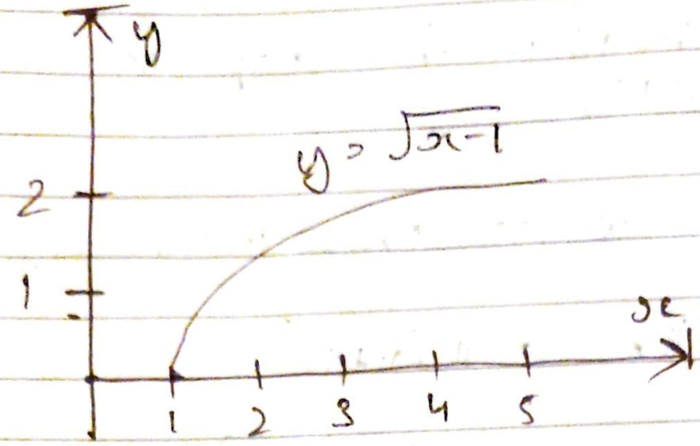
Diff. both sides
$dx = dt$
The limits
If $x = 0$ then $t = 1$
If $x = 2$ then $t = 3$

From equation (1)

$\int_0^2 A(x)\,dx = \int_0^2 \pi (x+1)^2\,dx = \int_1^3 \pi t^2\,dt$

$= \pi \left[\frac{t^3}{3} \right]_1^3 = \pi \left[\frac{3^3}{3} - \frac{1^3}{3} \right] = \frac{26\pi}{3}$

3.



Cross sectional area is $A(x) = \pi(\sqrt{x-1})^2$

Volume of the solid is

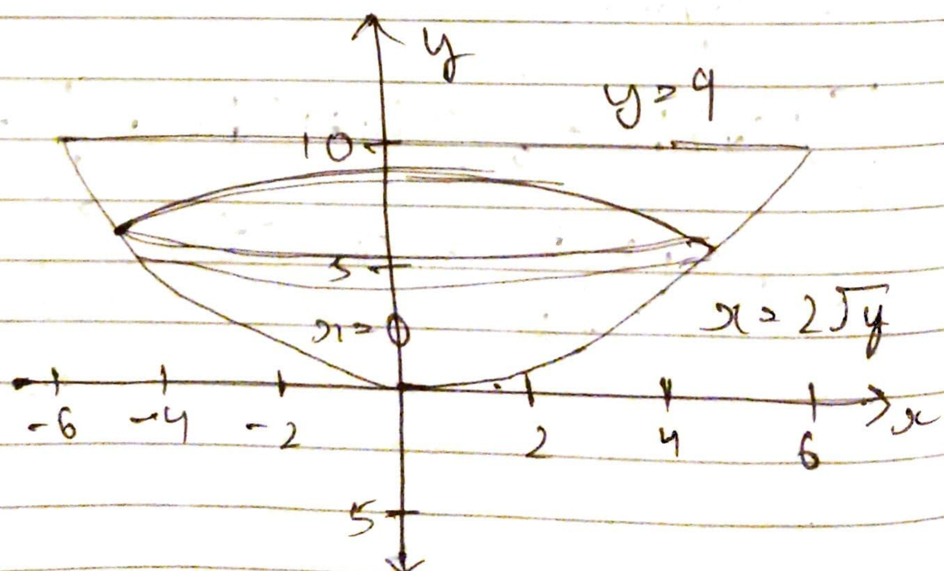
$$V = \int_1^5 A(x)\,dx = \int_1^5 \pi(\sqrt{x-1})^2\,dx$$

$$= \int_1^5 \pi(x-)\,dx = \pi\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \pi\left[\left(\frac{25}{2} - 5\right) - \left(\tfrac{1}{2} - 1\right)\right]$$

$$= \pi\left[\tfrac{15}{2} + \tfrac{1}{2}\right] = 8\pi$$

5.



Slicing at height y, a circular disk with radius x is obtained where $x = 2\sqrt{y}$

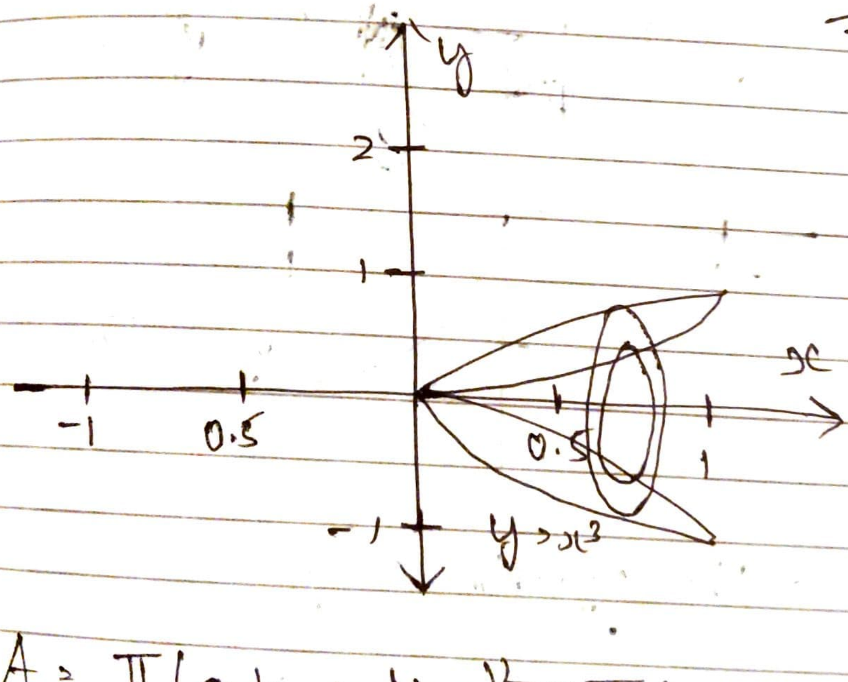
$$A = \pi x^2 = \pi(2\sqrt{y})^2 = \pi 4y$$

Since the solid lies between $y=0$ and $y=9$, its volume is

$$V = \int_0^9 A\,dy = \pi \int_0^9 |4y|\,dy = \pi \left|2y^2\right|_0^9$$

$$= \pi\left(\left(2(9)^2\right) - \left(2(0)^2\right)\right) = \underline{\underline{162\pi}}$$

7.



$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$
$A = \pi x^2 - \pi x^6$
$A = \pi(x^2 - x^6)$

Since the solid lies between $x=0$ and $x=1$, its volume is

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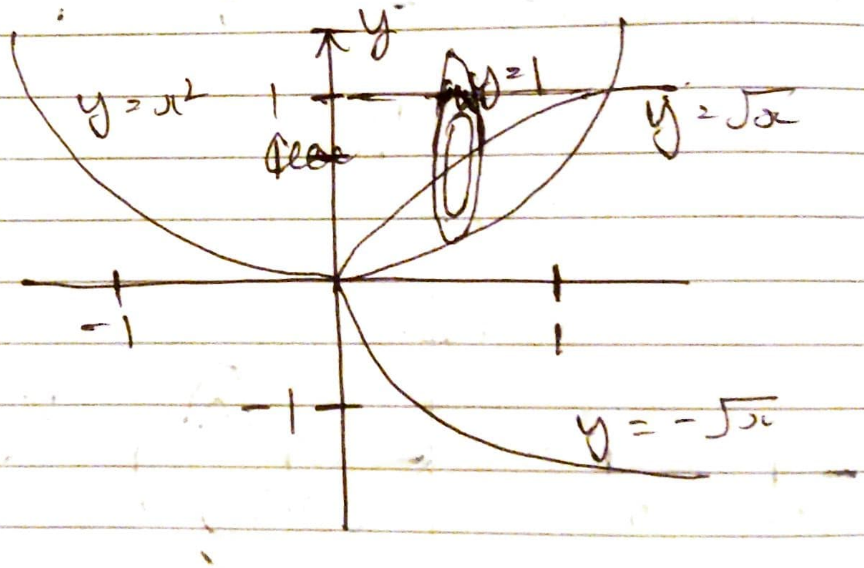
$$V = \int_0^1 A\,dx = \pi \int_0^1 |x^2 - x^6|\,dx$$

$$= \pi \left| \frac{x^3}{3} - \frac{x^7}{7} \right|_0^1$$

$$= \pi \left[\left(\frac{1^3}{3} - \frac{1^7}{7} \right) - \left(\frac{0^3}{3} - \frac{0^7}{7} \right) \right]$$

$$= \frac{4\pi}{21}$$

11.



Rewrite the equation $x = y^2$

$x = y^2 \Rightarrow y = \pm\sqrt{x} \Rightarrow y = \sqrt{x}$

The outer radius of the disk is the distance of the curve $y = x^2$ with the line $y = 1$. Therefore, outer radius is given by $R = 1 - x^2$

Similarly, the inner radius is the distance of the curve $y=\sqrt{x}$ with the line $y=1$
Therefore, inner radius is given by $r = 1-\sqrt{x}$

$$A(x) = \pi (R)^2 - \pi (r)^2 = \pi (1-x^2)^2 - \pi(1-\sqrt{x})^2$$

$$V = \int_0^1 A(x)\,dx$$

$$= \pi \int_0^1 \left[(1-x^2)^2 - (1-\sqrt{x})^2\right] dx$$

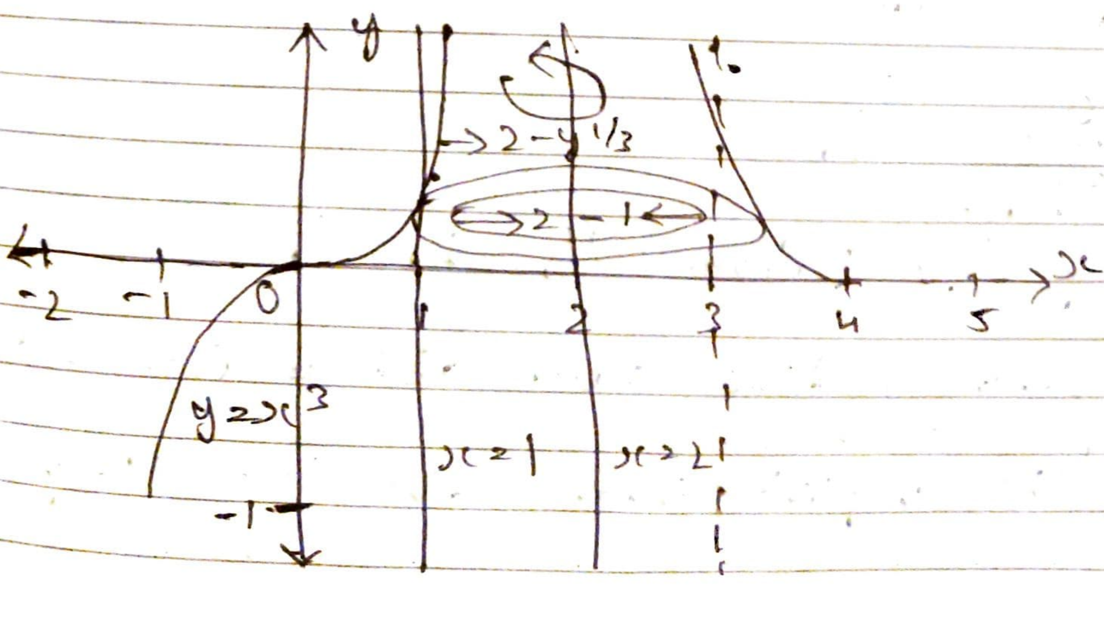
$$= \pi \int_0^1 \left[1-2x^2+x^4 - (1-2\sqrt{x}+x)\right] dx$$

$$= \pi \int_0^1 \left[-2x^2+x^4+2\sqrt{x}-x\right] dx$$

$$= \pi \left[-\frac{2x^3}{3}+\frac{x^5}{5}+\frac{4}{3}x^{3/2}-\frac{x^2}{2}\right]_0^1$$

$$= \frac{11\pi}{30}$$

15.



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$\Rightarrow y = x^3$
$x = y^{1/3}$

The curves are $x = y^{1/3}$, $x = 1$

$V = \int_a^b A(y)\,dy$

$A(y) = \pi\,(\text{outer radius})^2 - \pi\,(\text{inner radius})^2$

$A(y) = \pi\left(2 - y^{1/3}\right)^2 - \pi(2-1)^2$

$= \pi\left(4 + y^{2/3} - 4y^{1/3}\right) - \pi$

$= \pi\left(4 + y^{2/3} - 4y^{1/3} - 1\right)$

$= \pi\left(3 + y^{2/3} - 4y^{1/3}\right)$

17. $y^2 = 1 - y^2$
$2y^2 = 1$
$y^2 = 1/2$
$y = \pm 1/\sqrt{2}$
Therefore,
y varies from $-1/\sqrt{2}$ to $1/\sqrt{2}$

Inner radius our distance from $x = 3$ to $x = 1 - y^2$
This implies; inner radius is $3 - (1 - y^2)^2$

$2+y^2$

Similarly, outer radius is $3-y^2$

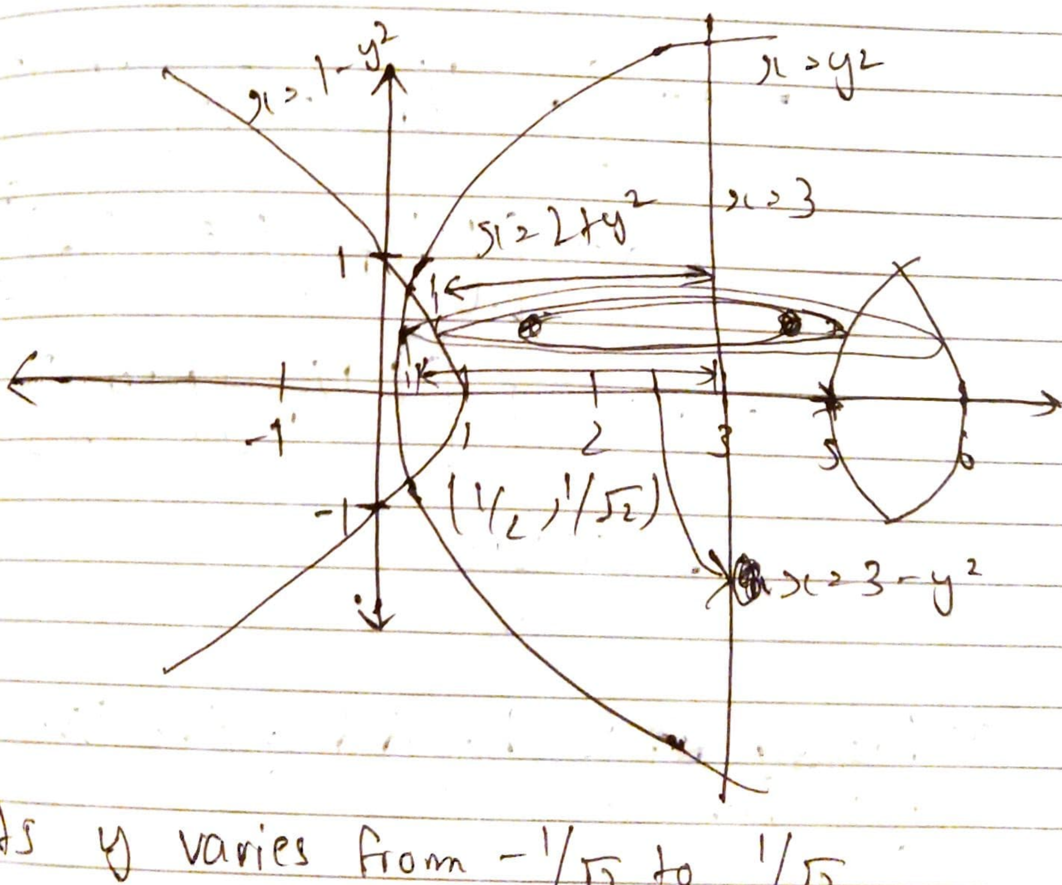
$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

$= \pi(3-y^2)^2 - \pi(2+y^2)^2$

$= \pi((3-y^2)^2 - (2+y^2)^2)$

$= \pi(9+y^4-6y^2-(4+y^4+4y^2))$

$= \pi(5-10y^2)$



As y varies from $-1/\sqrt{2}$ to $1/\sqrt{2}$

$V = \pi\int_{-1/\sqrt{2}}^{1/\sqrt{2}} A(y)\,dy = \pi\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (5-10y^2)\,dy$

$$= 2\pi \int_0^{1/\sqrt{2}} (5 - 10y^2)\, dy$$

$$= \cancel{3}\ 2\pi \left[5y - \frac{10y^3}{3} \right]_0^{1/\sqrt{2}}$$

$$= 2\pi \left[5\left(\frac{1}{\sqrt{2}}\right) - \frac{10}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - \left(5(0) - \frac{10}{3}(0)^3\right) \right]$$

$$= 2\pi \left[5\left(\frac{1}{\sqrt{2}}\right) - \frac{10}{3}\left(\frac{1}{2\sqrt{2}}\right) \right]$$

$$= 2\pi \left[5\left(1/\sqrt{2}\right) - 5/3\left(1/\sqrt{2}\right) \right]$$

$$= 2\pi \left[\left(1/\sqrt{2}\right)\left(5 - 5/3\right) \right]$$

$$= \sqrt{2}\sqrt{2}\,\pi \left[\left(1/\sqrt{2}\right)\left(\frac{10}{3}\right) \right]$$

$$= \frac{10\pi\sqrt{2}}{3}$$

19. Region R_1 is bounded by the line $y = x$ from $0 \le x \le 1$

Now required volume is.

URBAN EDGE

$$V = \int_0^1 (\pi x^2)\,dx = \pi\left[\frac{x^3}{3}\right]_0^1 = \pi/3$$

21. The equation of the line AB is $x=1$
The region R_1 is bounded by the line OB is $x=y$. So the area of a cross-section through y is

$$A(y) = \pi(\text{radius})^2 = \pi(1-y)^2$$

The solid lies between $y=0$ and $y=1$

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi(1-y)^2\,dy$$

$$= \pi\left[-\frac{(1-y)^3}{3}\right]_{y=0}^{y=1}$$

$$= \pi\left[-\frac{(1-1)^3}{3} + \frac{(1-0)^3}{3}\right]$$

$$= \pi\left[-0 + 1/3\right] = \pi/3$$

25. Here line AB is $x=1$

Region R_2 is bounded by the line $y=1$, and the curve $y = \sqrt[4]{x}$ from $0 \le x \le 1$
$\Rightarrow 0 \le y \le 1$

Cross sectional area is

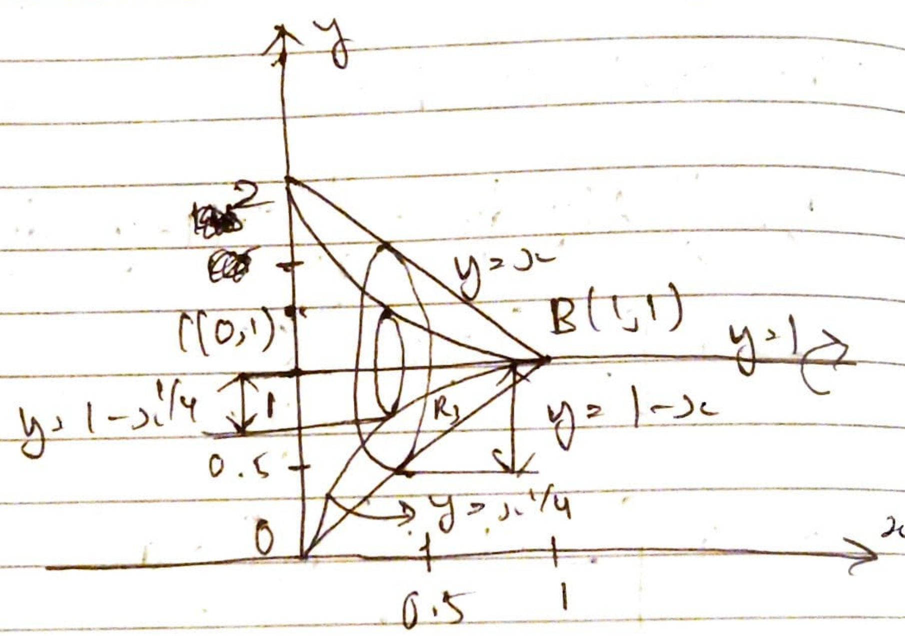
$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

$= \pi(1)^2 - \pi(1-y^4)^2$

$V = \int_0^1 \left(\pi(1)^2 - \pi(1-y^4)^2\right) dy$

$= \pi\left[\frac{2y^5}{5} - \frac{y^9}{9}\right]_0^1 = \frac{13\pi}{45}$

30.



$A(y) = \pi(1-x)^2 - \pi(1-x^{1/4})^2$

$= \pi\left[(1-x)^2 - (1-x^{1/4})^2\right]$

The two curves intersect at $x = 0$ and $x = 1$

And so x varies from 0 to 1

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi \left[(1-x)^2 - (1-x^{1/4})^2 \right) dx$$

$$= \int_0^1 \pi \left(-2x - x^2 + 2x^{1/4} - x^{1/2} \right) dx$$

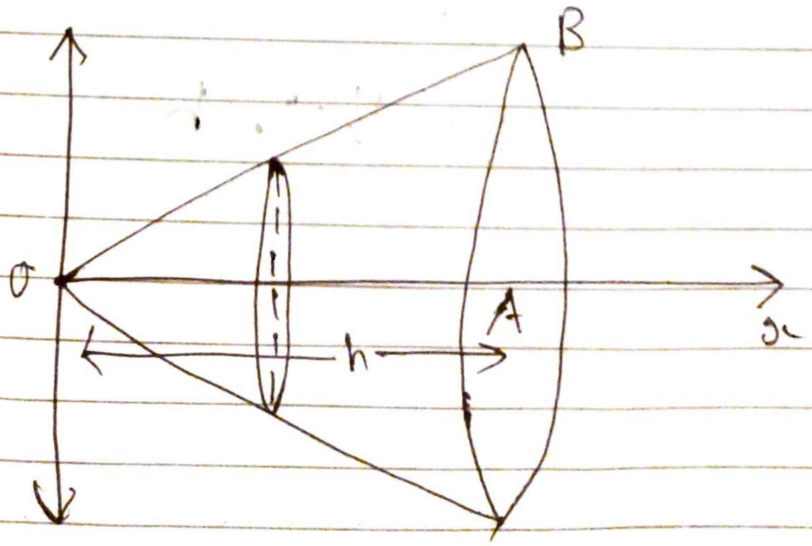
$$= \pi \left(-\frac{2x^2}{2} + \frac{x^3}{3} + \frac{2x^{1/4+1}}{1/4+1} - \frac{x^{1/2+1}}{1/2+1} \right)_0^1$$

$$= \pi \left(-\frac{2x^2}{2} + \frac{x^3}{3} + \frac{2x^{5/4}}{5/4} - \frac{x^{3/2}}{3/2} \right)_0^1$$

$$= \pi \left(-\frac{2(1)^2}{2} + \frac{(1)^3}{3} + \frac{8(1)^{5/4}}{5} - \frac{2(1)^{3/2}}{3} \right)$$

$$= \frac{4}{15}\pi$$

47.



Equation of line OB => $y = \frac{r}{h} x$

$A(x) = \pi(\text{radius})^2$

Or $A(x) = \pi \left(\frac{r}{h}x\right)^2$

$= \frac{\pi r^2}{h^2} x^2$

$$V = \int_0^h A(x)\,dx$$

$$= \int_0^h \pi \frac{r^2}{h^2} x^2\,dx$$

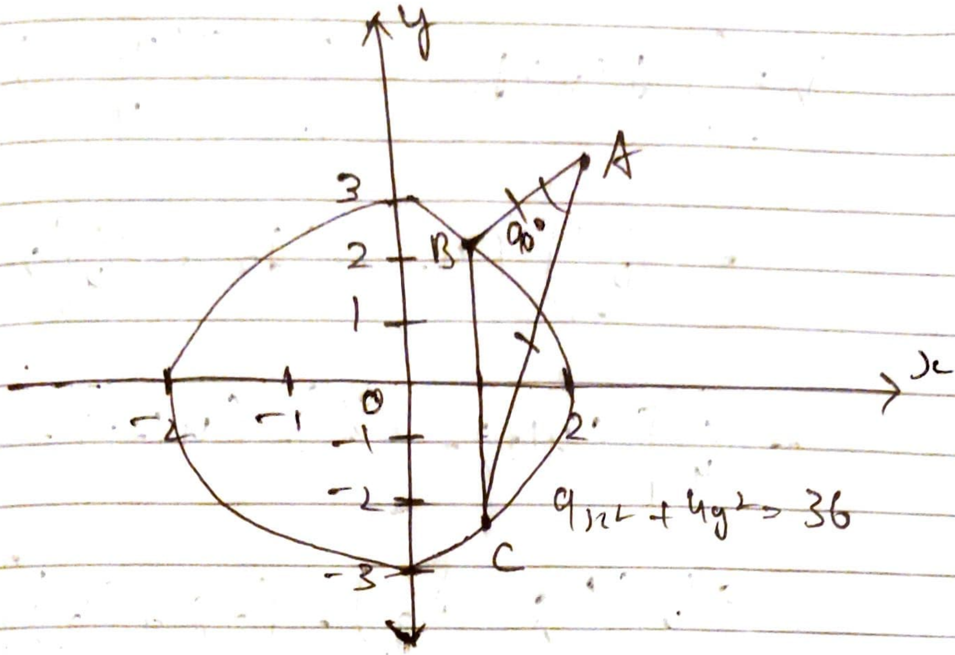
$$= \frac{\pi r^2}{h^2} \int_0^h x^2\,dx$$

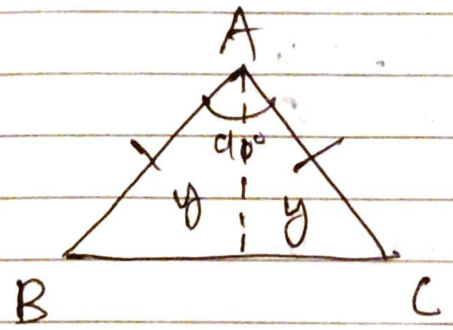
$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3}$$

Or $V = \frac{1}{3}\pi r^2 h$

55.





$$9x^2+4y^2=36$$

$$4y^2=36-9x^2$$

$$y^2=\frac{36-9x^2}{4}$$

$$y=\frac{3}{2}\sqrt{4-x^2}$$

$$L=2y=2\left(\frac{3}{2}\sqrt{4-x^2}\right)=3\sqrt{4-x^2}$$

$$A=\frac{L^2}{4}=\frac{(3\sqrt{4-x^2})^2}{4}=\frac{9(4-x^2)}{4}$$

x varies from -2 to 2

$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} \frac{9(4-x^2)}{4}\,dx$$

$$= \frac{9}{4}\int_{-2}^{2}(4-x^2)\,dx = \frac{9}{4}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{9}{4}\left[\left(4(2) - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)\right]$$

$$= \frac{9}{4}\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right]$$

$$= \frac{9}{4}\left[\frac{32}{3}\right] = 24$$

Exercise 6.3

3. $V = \int_a^b 2\pi x f(x)\,dx$, where $0 \le a \le b$

x varies from $x = 0$ to $x = 1$

That is, $a = 0$ and $b = 1$

The volume bounded by $y = \sqrt[3]{x}$, $y = 0$ and $x = 1$ about y-axis is

$$V = \int_a^b 2\pi x f(x)\,dx = \int_0^1 2\pi x (x^{1/3})\,dx$$

$$= \int_0^1 2\pi x^{1+1/3}\,dx = \int_0^1 2\pi x^{1+1/3}\,dx$$

$$= 2\pi \int_0^1 x^{4/3}\,dx$$

$= 2\pi \left(\frac{x^{4/3+1}}{4/3+1} \right)_0^1 = 2\pi \left(\frac{x^{7/3}}{7/3} \right) =$

$2\pi \left(\frac{(1)^{7/3} - 0}{7/3} \right) = 2\pi \left(\frac{(1)^{7/3} - 0}{7/3} \right)$

$= \frac{6\pi}{7}$ cubic units

5. $V = \int_a^b 2\pi x f(x)\,dx \; ; \; 0 \leq a \leq b$

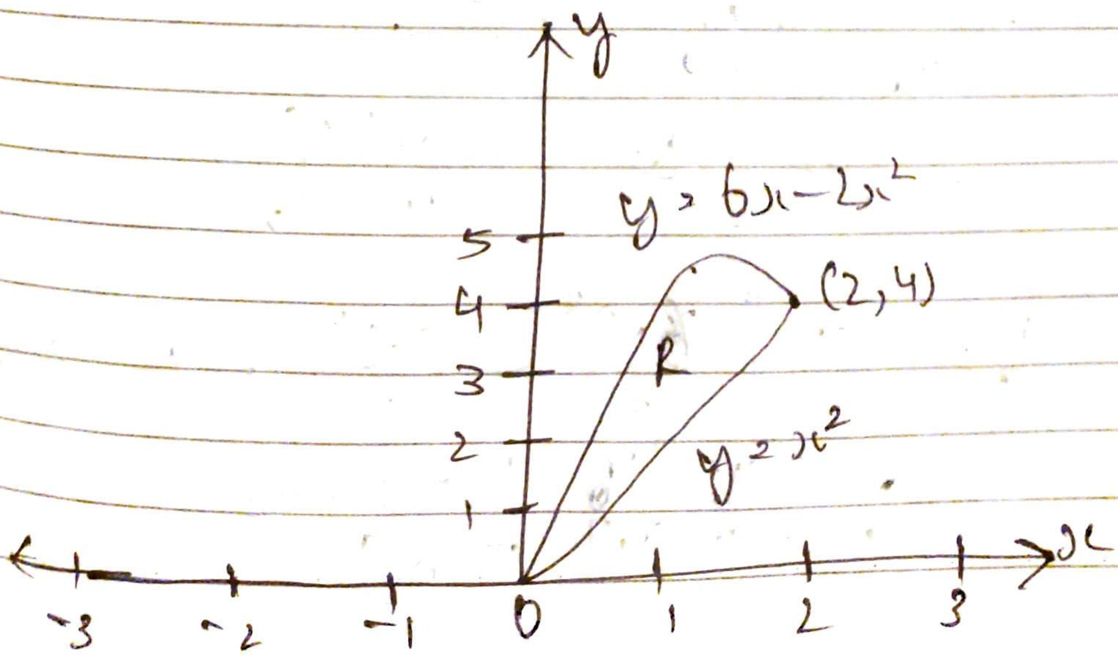
$= \int_0^1 2\pi x e^{-x^2}\,dx$

Let $u = x^2$; $du = 2x\,dx$

$= \pi \int_0^1 e^{-u}\,du = \pi \left[-e^{-u} \right]_0^1 =$

$\pi [1 - 1/e]$

7.



The point of intersection of the curves is (2, 4). It has radius x, circumference is $2\pi x$

Height of the shell $= f(x) = 6x^2 - 2x^2 - x^2$
$= 6x - 3x^2$

Point of intersection of the curves

$6x - 2x^2 = x^2$

$6x - 3x^2 = 0$

$3x(2-x) = 0$

$x = 0, 2$

$$V = \int_a^b 2\pi \, [\text{Shell Radius}][\text{Shell Height}]\,dx$$

$$= \int_a^b 2\pi [x][f(x)]\,dx$$

Here x varies from 0 to 2

Sub $f(x) = 6x - 3x^2$

$$V = \int_{x=0}^{2} 2\pi x (6x - 3x^2)\,dx$$

$$= \int_0^2 2\pi (6x^2 - 3x^3)\,dx$$

$$= 2\pi \left[\frac{6x^3}{3} - \frac{3x^4}{4} \right]_0^2$$

$$V = 2\pi \left[\frac{6 \cdot 2^3}{3} - \frac{3 \cdot 2^4}{4} - \frac{6 \cdot 0^3}{3} + \frac{3 \cdot 0^4}{4} \right]$$

$$= 2\pi [16 - 12 - 0 + 0]$$

$$= 2\pi [4] = 8\pi$$

9. It has radius y, circumference $2\pi y$ and height $f(y) = 1/y$

By shell method $V = \int_c^d 2\pi y f(y) dy$

And y varies from $c = 1$ to $d = 3$

$$V = \int_1^3 2\pi y f(y) dy = \int_1^3 2\pi y (1/y) dy$$

$$= 2\pi \int_1^3 dy = 2\pi [y]_1^3 = 2\pi [3-1] = 4\pi$$

11. Volume of the solid obtained by rotating about x-axis the region between the curves $x = f(y)$, $x = g(y)$, $a \leq y \leq b$ is $\int_a^b 2\pi y (f(y) - g(y)) dy$

Here height of the shell is $x = y^{2/3}$

$$= \int_0^8 2\pi y (y^{2/3} - 0) dy = \int_0^8 2\pi y^{5/3} dy$$

$$= 2\pi \left[\frac{y^{8/3}}{8/3} \right]_0^8 = 2\pi (3/8) \left[y^{8/3} \right]_0^8$$

$$= 2\pi (3/8) \left[(8)^{8/3} - (0)^{8/3} \right] = \left(\frac{3\pi}{4} \right) \left[(2^3)^{8/3} \right]$$

$$= \frac{3\pi}{4} (2^8) = 192\pi$$

13. $V = \int_a^b 2\pi y f(y) dy$ where $0 \leq a \leq b$

For rotation about the x-axis, the solid has radius y.

The height of the solid $f(y)$ is $x_2 - x_1 =$
$2-(1+(y-2)^2)$

The limits of the solid is,

$$1+(y-2)^2=2$$
$$1+y^2-4y+4=2$$
$$y^2-4y+3=0$$
$$(y-1)(y-3)=0$$
$$y=1,3$$

$$V=\int_1^3 2\pi y\left(2-(1+(y-2)^2)\right)dy$$
$$=2\pi\int_1^3 y\left(2-(1+y^2-4y+4)\right)dy$$
$$=2\pi\int_1^3 y(2-1-y^2+4y-4)dy$$
$$=2\pi\int_1^3 y(-3-y^2+4y)dy$$
$$=2\pi\int_1^3 (-y^3+4y^2-3y)dy$$
$$=2\pi\left[\frac{-(3^4-1^4)}{4}+\frac{4(3^3-1^3)}{3}-\frac{3(3^2-1^2)}{2}\right]$$
$$=2\pi\left[\frac{-80}{4}+\frac{104}{3}-\frac{24}{2}\right]=$$

$$2\pi\left[\frac{-80.37+104.4-24.6}{12}\right] = 2\pi\left[\frac{32}{12}\right]$$

$$= 2\pi\left(8/3\right) = 16\pi/3$$

15. $y = f(x)$ $a \leq x \leq b$ is $\int_a^b$

Volume of the solid obtained about y-axis the region below the curve

$y = f(x)$ $a \leq x \leq b$ is $\int_a^b 2\pi x\,[f(x)]\,dx$

Curves $y = x^3$; $y = 8$ intersect for $x = 2$

Limits are $0 \leq x \leq 2$

Here radius of the shell is $3 - x$ and height is $8 - x^3$

$$V = \int_0^2 2\pi(3-x)(8-x^3)\,dx$$

$$= 2\pi\int_0^2 (24 - 8x - 3x^3 + x^4)\,dx$$

$$= 2\pi\left[24x - 4x^2 - \frac{3x^4}{4} + \frac{x^5}{5}\right]_0^2$$

$$= 2\pi\left[24(2-0) - 4(2^2-0) - 3/4(2^4-0) + 1/5(2^5-0)\right] = \frac{264\pi}{5}$$

URBAN EDGE

17. $y = 4x - x^2, \quad y = 3$

Determine the limits of integration

$$3 = 4x - x^2$$
$$x^2 - 4x + 3 = 0$$
$$x^2 - 3x - x + 3 = 0$$
$$x(x-3) - 1(x-3) = 0$$
$$(x-1)(x-3) = 0$$
$$x = 1, 3$$

A typical shell has radius $(x-1)$, circumference $2\pi(x-1)$ and height $(4x - x^2 - 3)$

$$V = \int_1^3 2\pi(x-1)(4x - x^2 - 3)\,dx$$

$$= \int_1^3 2\pi(4x^2 - x^3 - 3x - 4x + x^2 + 3)\,dx$$

$$= \int_1^3 2\pi(-x^3 + 5x^2 - 7x + 3)\,dx$$

$$= 2\pi\left[-\frac{x^4}{4} + \frac{5x^3}{3} - \frac{7x^2}{2} + 3x\right]_1^3$$

$$= 2\pi\left(\left[-\frac{3^4}{4} + \frac{5(3)^3}{3} - \frac{7(3)^2}{2} + 3(3)\right] - \left[\frac{1^4}{4} + \frac{5(1)^3}{3} - \frac{7(1)^2}{2} + 3(1)\right]\right)$$

$= 2\pi\left(\left[-\frac{81}{4} + \frac{135}{3} - \frac{63}{2} + 9\right] - \left[-\frac{1}{4} + \frac{5}{3} - \frac{7}{2} + 3\right]\right) = 2\pi\left(\frac{9}{4} - \frac{11}{12}\right) = 2\pi\left(\frac{4}{3}\right)$

$= \frac{8\pi}{3}$

19. Volume of the solid obtained by rotating about x-axis the region between the curves

$x = f(y),\ x = g(y);\ a \le y \le b$ is $\int_a^b 2\pi y\,[f(y) - g(y)]\,dy$

$2y^2 = 2$

$y^2 = 1$

$y = 1$ (since $y \geq 0$)

Limits are $0 \le y \le 1$

Here radius of shell is $(2-y)$ and height is $(2-2y^2)$

$V = \int_0^1 2\pi(2-y)(2-2y^2)\,dy$

$= 2\pi\int_0^1 (4 - 4y^2 - 2y + 2y^3)\,dy$

$= 2\pi\left[4y - \frac{4y^3}{3} - \frac{2y^2}{2} + \frac{2y^4}{4}\right]_0^1$

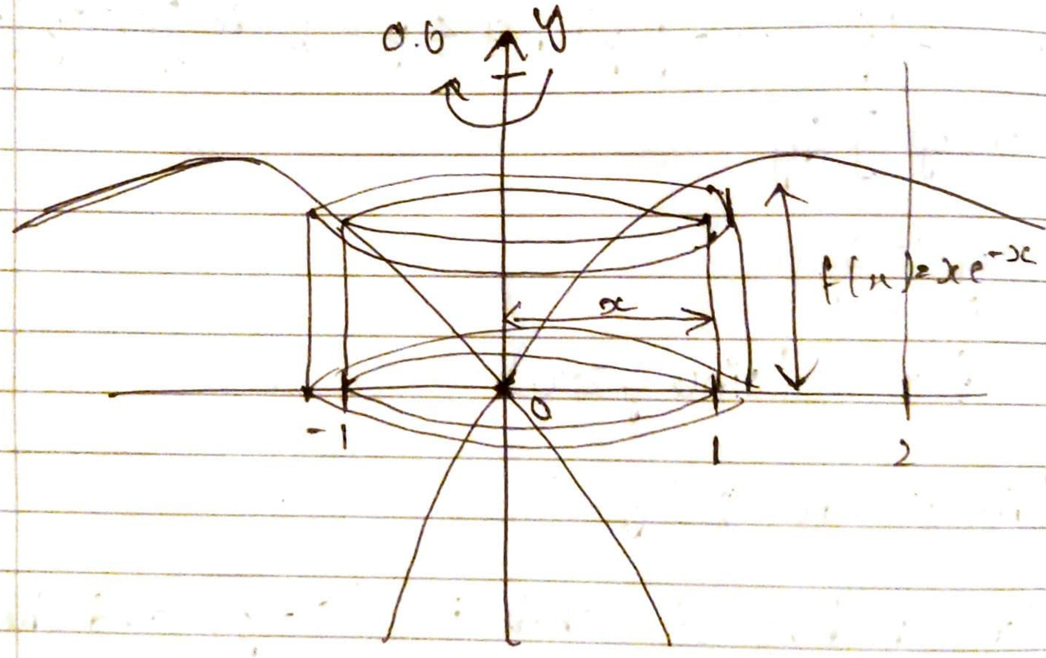
$= 2\pi\left[4 - \frac{4}{3} - 1 + \frac{2}{4}\right] = \frac{13\pi}{3}$

21. (a) It has radius x, circumference $2\pi x$, and height $f(x) = y = xe^{-x}$

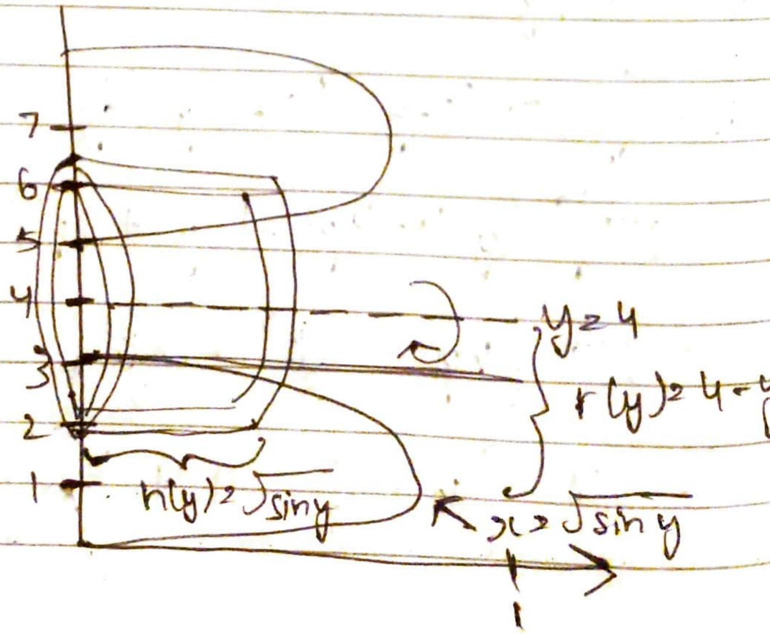
By shell method $V = \int_a^b 2\pi x\,(f(x))\,dx$
And x varies from $a=0$ to $b=2$

$$V = \int_0^2 2\pi(x)\, f(x)\, dx$$

$$= \int_0^2 2\pi(x)(xe^{-x})\,dx$$



25.



URBAN EDGE

$$V = 2\pi \int_c^d r(y)h(y)\,dy$$

$$= 2\pi \int_0^{\pi} (4-y)\sqrt{\sin y}\,dy$$

37. Notice that a typical shell has radius x, circumference $2\pi x$, and height given by the following function:

$$f(x) = -x^2 + 6x - 8$$

$$V = \int_2^4 2\pi x f(x)\,dx$$

$$= \int_2^4 2\pi x(-x^2 + 6x - 8)\,dx$$

$$= 2\pi \int_2^4 (-x^3 + 6x^2 - 8x)\,dx$$

$$= 2\pi \left(-\frac{x^4}{4} + 2x^3 - 4x^2\right)\Big]_2^4$$

$$= 2\pi\left(-\frac{(4)^4}{4} + 2(4)^3 - 4(4)^2\right) -$$

$$2\pi\left(-\frac{(2)^4}{4} + 2(2)^3 - 4(2)^2\right)$$

$$= 2\pi(-64 + 128 - 64) - 2\pi(-4 + 16 - 16)$$

$$= 2\pi(0) - 2\pi(-4) = 8\pi \text{ unit}^3$$

39. Find the point of intersection of the curves.

$y^2 - x^2 = 1, \; y = 2$

$2^2 - x^2 = 1$

$x^2 = 4 - 1$

$x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

Two curves intersect at $x = \pm\sqrt{3}$

Cross section is is the shape of a washer with outer radius 2 and inner radius $\sqrt{1+x^2}$

$A(x) = \pi\left[(\text{Inner radius})^2 - (\text{Outer radius})^2\right]$

$= \pi\left(2^2 - \left(\sqrt{1+x^2}\right)^2\right) = \pi(4-1-x^2)$

$= \pi(3-x^2)$

$V = \int_{-\sqrt{3}}^{\sqrt{3}} A(x)\,dx = \int_{-\sqrt{3}}^{\sqrt{3}} \pi(3-x^2)\,dx$

$= 2\pi\int_0^{\sqrt{3}} (3-x^2)\,dx = 2\pi\left[3x - \frac{x^3}{3}\right]_0^{\sqrt{3}}$

$= 2\pi\left(3\sqrt{3} - \frac{3\sqrt{3}}{3}\right)$

$= 2\pi \times 3\sqrt{3}\left(2/3\right) = 4\sqrt{3}\pi$

41. $x^2+(y-1)^2=1$

$(y-1)^2=1-x^2$

$y-1=\pm\sqrt{1-x^2}$

$y=1\pm\sqrt{1-x^2}$

$y=1+\sqrt{1-x^2}$ or $y=1-\sqrt{1-x^2}$

$y=2\sqrt{1-x^2}$ $\quad x^2+(y-1)^2=1$

$V=\int_a^b 2\pi y f(x)\,dx$

The limits of x vary from 0 to 1

$V=\int_0^1 2\pi x\left(2\sqrt{1-x^2}\right)\cancel{dy}\,dx$

$=\int_0^1 4\pi\left(x\sqrt{1-x^2}\right)dx$

$=4\pi\int_0^1\left(x\sqrt{1-x^2}\right)dx$

$=4\pi\int_0^1\left(x\sqrt{1-x^2}\right)dx$

Let $t=1-x^2\rightarrow dt=-2x\,dx$

When $x=0$, then $t=1$

When $x=1$, then $t=0$

$=4\pi\int_1^0\sqrt{t}\left(-\frac{dt}{2}\right)=-2\pi\int_1^0\left(\sqrt{t}\right)dt$

$=-2\pi\int_1^0 t^{1/2}\,dt=-2\pi\left[\frac{2t^{3/2}}{3}\right]_1^0$

$$= \frac{-4\pi}{3}[0-1] = \frac{4\pi}{3}$$

Exercise 6.4

1. (a) From the data,

The weight of the gorilla is,

$m = 360 lb$

The height or distance of the tree is,
$d = 20 ft$

Work done is,

$W = Fd$

Since the force is 360 lb and distance is 20ft,

$W = (360 lb)(20 ft) = 7200$ ft-lb

(b) The amount of time it takes the gorilla to climb the tree doesn't change the amount of work done. Therefore the work done is still 7200 ft-lb

URBAN EDGE

3. Work done in moving is

$W = \int_a^b f(x)\,dx = \int_1^{10} 5x^{-2}\,dx = \left[\frac{5x^{-1}}{-1}\right]_1^{10}$

$= -5(10^{-1}-1) = 9/2 = 4.5$ lb-ft

Hence the work done is 4.5 lb-ft

7. Since 4 in $= 4/12 = 1/3$ ft
And 6 in $= 1/2$ ft

Now the required force to hold a spring stretched 4 in $= 1/3$ ft, is given $F = 10$ lb

By Hooke's law, we have

$F(x) = kx$

When $x = 1/3$ ft $\quad f(x) = 10$ lb (given)

So $\frac{1}{3}k = 10$

or $k = 30$

Then the equation of force is $F(x) = 30x$

The work done in stretching the spring $x = 0$ to $x = 1/2$ ft is

$W = \int_0^{1/2} 30x\,dx = 30\int_0^{1/2} x\,dx = 30\left[\frac{x^2}{2}\right]_0^{1/2}$ ft-lb

$= 30\left[\frac{1}{8}\right]$ ft-lb

Or $W = \frac{15}{4}$ ft-lb

13. (a) In a coordinate system, take the top of the building to be $x = 50$. The end of the rope is originally at $x = 0$ (the ground is down at 60 feet hight, that is half part of the ~~p~~ building height).

Now, slicing the rope up into thin pieces of uniform thickness dx

This amount to taking a partition

$0 = x_0 < x_1 < x_2 < \ldots < x_n = 50$ where $\Delta x = 50/n$ and $x_i = i\Delta x$

$F_i = (0.5\ \text{lb/ft})(\Delta x\ \text{feet})$
$= 0.5\,\Delta x\ \text{lb}$

The distance for the ith piece must be lifted is,

$D_i = 50 - x_i$

So, the work done on the ith slice of the rope is,

$W_i = 0.5(50 - x_i)\Delta x$ ft-lb

The total work is the sum.

It follows that

$$\sum_{i=1}^{n} 0.5(50 - x_i)\Delta x$$

Letting $n \to \infty$

$$W = \int_0^{50} 0.5(5 - x)\,dx = 0.5\int_0^{50}(50 - x)\,dx$$

$$= 0.5\left[50x - \frac{x^2}{2}\right]_0^{50} = 0.5\left[50(50) - \frac{50^2}{2}\right]$$

$= 625$ ft-lb

(b) $W = F \cdot d$

$F = 0.5x$, $d = dx$

$$W = \int_{20}^{50} 0.5x\,dx = 0.5\left(\frac{x^2}{2}\right]_{25}^{50}$$

$$= 0.5\left(\frac{50^2}{2} - \frac{25^2}{2}\right) = \frac{1875}{4} =$$

468.75 ft-lb

15. Weight of cable = 2 lb/ft

Weight of coal = 800 lb

Distance = 500 ft

Let ith part of the cable has Δx
So the weight of ith part of cable is = $2\Delta x$ and the distance from mine shaft of the ith part of the cable is = x_i.

So, work done on the ith part of the cable is = $2x_i\Delta x$

Total work done,

$$W_{ca} = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i \Delta x = \int_0^{500} 2x\,dx$$

$$= [x^2]_0^{5000} = 250000 \text{ ft-lb}$$

Since weight of coal is = 800 lb
And distance is = 500 ft
So work done to lift the coal up to the top of mine shaft is

$$W_{co} = 800 \times 500 = 40,0000 \text{ ft-lb}$$

Total work done

$W = W_{ca} + W_{co} = 250000 + 400000$

$W = 650,000$ ft-lb

19. Divide the chain into small parts with length Δx.

If x_i^* is a point in the i-th sub interval, then all the points in the interval are lifted by approximately in the same amount namely x_i^*.

The chain weights $\frac{25}{10} = 2.5$ pounds per foot, so the weight of the i-th part is $2.5\Delta x$.

Thus, the work done on the i-th part, in foot pounds, is $(2.5\Delta x) \times x_i^* = 2.5x_i^*\Delta x$

Get the total work done by adding all these approximations and let the number of parts becomes large (so $\Delta x \to 0$)

Therefore, $W = \lim_{n\to\infty} \sum_{i=1}^{n} 2.5x_i^*\Delta x$

$$W = \frac{1}{2} \int_0^{20} 2.5x_i \, \Delta x$$

$$= \frac{1}{2}(2.5)\left[\frac{x^2}{2}\right]_0^{10}$$

$$= \frac{2.5}{2} \times \frac{100}{2} = 62.5$$

Hence, the required work done is 62.5 ft.

20. Consider Δx is a thickness of horizontal slice of the water

Then, the volume of the slice is,

$$V_i = \pi\,(\text{radius})^2\,(\text{height}) = \pi(12)^2 \Delta x$$
$$= 144\pi \Delta x \ ft^3$$

$$F = 144 \cdot \pi \Delta x\, ft^3 \cdot 62.5 \frac{lb}{ft^3} = 9000\pi \Delta x \, lb$$

Let x be the height from the bottom of the pool, then each side of water has to move a distance of,

$$d = 5 - x$$

Because, the side of the pool is 5 ft high.

The slices become infinitely thin (Δx becomes dx). The water depth is 4 ft, so the limits are from 0 to 4

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$W = \int_0^4 (9000\pi)(5-x)\,dx$

$= 9000\pi \left[5x - \frac{x^2}{2}\right]_0^4$

$= 9000\pi\,[20-8-0] = 108000\pi$

$\approx$ 339292 ft-lb

21. Let us say that the water is Δx metres thick and at a depth of x_i

So this piece has a volume of $2 \times 1 \times \Delta x$

$m =$ density of water $\times$ volume
$= d \times V = 1000\,kg/m^3 \times 2\Delta x\,m^3$
$= 2000\,\Delta x\,kg$

$F = mg$
$= (2000\,\Delta x\,kg)(9.8\,m/s^2)$
$= 19600\,\Delta x\,N$

Work done in lifting one slice is,

$F \cdot x = (19600\,\Delta x\,N)\cdot(x\,m)$
$= 19600\,x\,\Delta x\,N\cdot m$
$= 19600\,x\,\Delta x\,J$

So, the portion of the chain from x to $x + \Delta x$ metres to be the following

$$\lim_{n \to \infty} \sum_{i=1}^{n} 19600 x_i^* \Delta x$$

Therefore, the total work done to pump half of the water out of the aquarium is,

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 19600 x_i^* \Delta x$$

$$= 19600 \int_0^{1/2} x\,dx$$

$$= 19600 \left[\frac{x^2}{2}\right]_0^{1/2}$$

$$= 9800 \left[x^2\right]_0^{1/2}$$

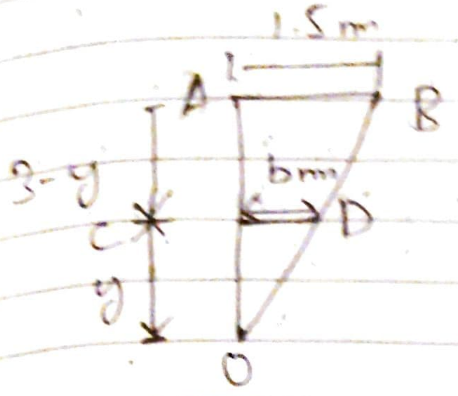
$$= 9800 \left[1/4\right] - 0$$

$$= 2450 \text{ J}$$

23. The height of the tank is 3 m, so divide the interval $[0, 3]$ ~~for~~ into N layers and let y be the height of the ith layer

And each layer is in the shape of a rectangle with length 8 m and width 26 m.

Consider the similar triangles OAB, OCD.



Use the property of similar triangles

$$\frac{OA}{AB} = \frac{OC}{CD} \Rightarrow \frac{3}{1.5} = \frac{y}{b}$$

$$b = \frac{1.5y}{3}$$

Area of the rectangular ith layer is

$A = (\text{length})(\text{width}) = 8(2b) = 16b$

$= 16\left(\frac{1.5y}{3}\right) = 8y$

The volume of the ith layer is,

$V = $ (Area of the rectangle)(Thickness of the rectangle) $= A\Delta y = 8y\Delta y$

Density of water is 1000 kg/m^3

Force on the ith layer is,

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$F = g\,(\text{density})V = 9.8(1000)(8y)\Delta y$

The ith layer has to be lifted a vertical distance of $(3-y)$ m and plus 2 m (along the spout), so the layer has to be lifted a vertical distance of $(3-y+2)$ m or $(5-y)$ m

The work done to raise this ith layer to the top is the product of F and the distance $(5-y)$ m is,

$W_i = 9.8(1000)(8y)(5-y)\Delta y$

To find the total work done, add all the work done to raise each of the N layers in the interval $[0, 3]$ to the top and then take the limit as $n \to \infty$

$$W = \lim_{n\to\infty} \sum_{i=1}^{N} W_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{N} 9.8(1000)(8y)(5-y)\Delta y$$

$$= \int_0^3 9.8(1000)(8y)(5-y)\,dy$$

$$= 9.8(1000)(8)\int_0^3 (y)(5-y)\,dy$$

$$= 78400\int_0^3 (5y-y^2)\,dy$$

$$= 78400\left[\frac{5y^2}{2} - \frac{y^3}{3}\right]_0^3 = 78400\left[\frac{5(3)^2}{2} - \frac{3^3}{3}\right]$$

$$= 78400\left[\frac{45}{2} - 9\right] = 78400\left[\frac{27}{2}\right]$$

$$= 1058400 \approx 1.058400 \times 10^6 \approx 1.06 \times 10^6 \text{ J}$$

33 (a) $\text{Force} = \dfrac{GMm}{r^2}$

Work = Force * Distance

$$\text{Work} = \int_a^b \frac{GMm}{r^2}\,dr = GMm\int_a^b r^{-2}\,dr$$

$$\text{Work} = GMm\left[\frac{r^{-1}}{-1}\right]_a^b = -GMm\left[1/r\right]_a^b$$

$$\text{Work} = -GMm\left[1/b - 1/a\right] = GMm\left(1/a - 1/b\right)$$

$$\text{Work} = \frac{(b-a)\,GMm}{ab}$$

Exercise 6.5

1. $$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2}(3x^2+8x)\,dx$$

$$= \frac{1}{3}\left[\frac{3x^3}{3} + \frac{8x^2}{2}\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[x^3 + 4x^2\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[2^3 - (-1)^3 + 4(2^2 - (-1)^2)\right]$$

$$= \frac{1}{3}\left[8 + 1 + 4(3)\right]$$

$$= \frac{1}{3}\left[9+12\right] = 7$$

2. $$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-0}\int_0^4 \sqrt{x}\,dx$$

$$= \frac{1}{4}\left[\frac{x^{3/2}}{3/2}\right]_0^4 = \frac{1}{6}\left[4^{3/2} - 0\right] = \frac{4}{3}$$

5. $$f_{ave} = \frac{1}{(\pi/2 - 0)}\int_0^{\pi/4} e^{\sin t}\cos t\,dt$$

$$= \frac{2}{\pi} \int_0^{\pi/2} e^{\sin t} \cos t\, dt$$

Substitute: $u = \sin t$, then $du = \cos t\, dt$ and

$$\int e^{\sin t} \cos t\, dt = \int e^t dt = e^t = e^{\sin t}$$

So,

$$f_{ave} = 2/\pi \int_0^{\pi/2} e^{\sin t} \cos t\, dt$$

$$= 2/\pi \left[e^{\sin t} \right]_0^{\pi/2}$$

$$= 2/\pi \left[e^{\sin \pi/2} - e^0 \right] = 2/\pi\, (e-1)$$

7. $h_{ave} = \frac{1}{b-a} \int_a^b h(x)\, dx = \frac{1}{\pi - 0} \int_0^{\pi} \cos^4 x \cdot \sin x$

$dx = \frac{1}{\pi} \int_0^{\pi} \cos^4 x \sin x\, dx \quad \ldots \ldots (1)$

Assume $\cos x = t$

Thus, $-\sin x\, dx = dt \Rightarrow \sin x\, dx = -dt$

When $x = 0$, $t = 1$ and $x = \pi$, $t = -1$

Substitute in (1)

$$h_{ave} = 1/\pi \int_1^{-1} -t^4 dt = -1/\pi \int_1^{-1} t^4 dt$$

$$= 1/\pi \int_{-1}^{1} t^4 dt = 1/\pi \left[t^5/5 \right]_{-1}^{1}$$

$= \frac{1}{5\pi}[1-(-1)] = \frac{2}{5\pi}$

9. (a) $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$f(x) = (x-3)^2$ and $a=2, b=5$

$= \frac{1}{5-2}\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left[\frac{(x-3)^3}{3}\right]_2^5$

$= \frac{1}{9}\left[(5-3)^3 - (2-3)^3\right]$

$= \frac{1}{9}\left[2^3 - (-1)^3\right] = \frac{1}{9}(8+1) = \frac{9}{9}$

$= 1$

(b) $f(c) = (c-3)^2$

$f_{ave} = 1$

$f_{ave} = f(c)$

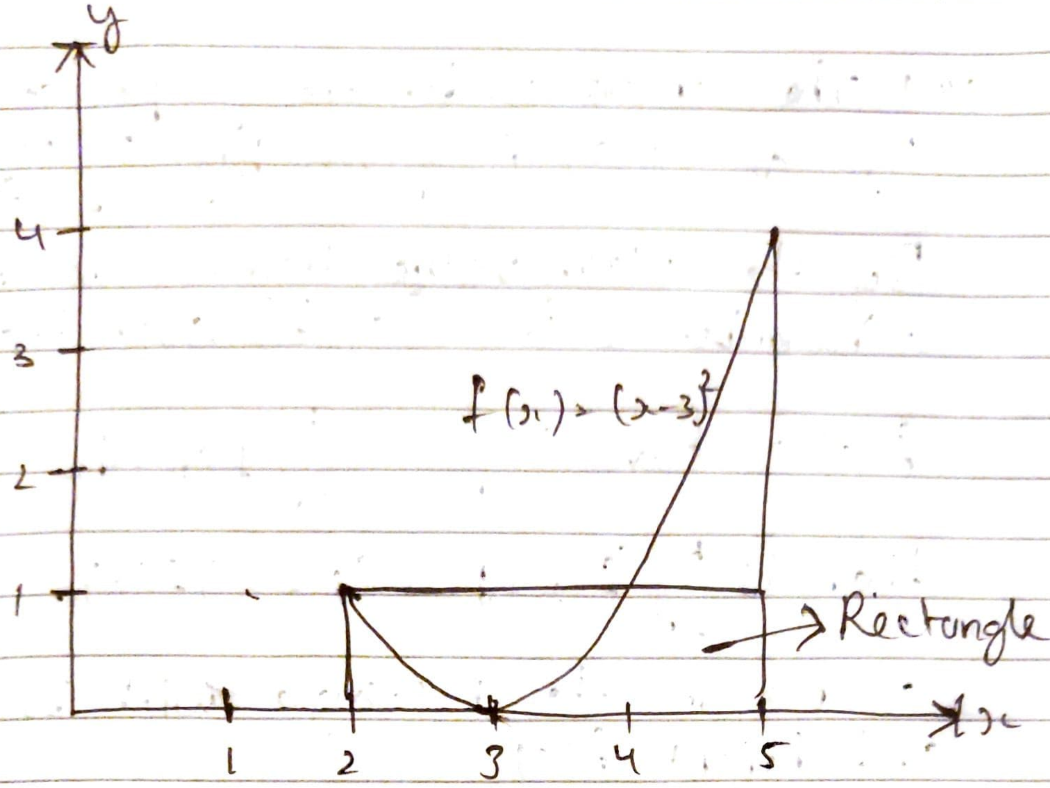
$1 = (c-3)^2$

$(c-3) = \sqrt{1}$

$c-3 = \pm 1$

$c = 3 \pm 1 = 2, 4$

(c)



10. (a) $f(x) = 1/x$, $b = 3$, $a = 1$

$$f_{ave} = \frac{1}{3-1}\int_1^3 1/x \, dx = 1/2 \int_1^3 1/x \, dx$$

$$= 1/2 \, [\ln x]_1^3 = 1/2 \, [\ln 3 - \ln 1]$$

$$= 1/2 \, (\ln 3 - 0) = 1/2 \ln 3.$$

Hence $f_{ave} = 1/2 \ln 3$

(b) $f_{ave} = f(c)$

$$\Rightarrow 1/2 \ln 3 = 1/c \Rightarrow c = \frac{2}{\ln 3}$$

(c) The area under the graph of $f(x) = 1/x$ on $[1, 3]$ is given by $\int_1^3 1/x \, dx$.

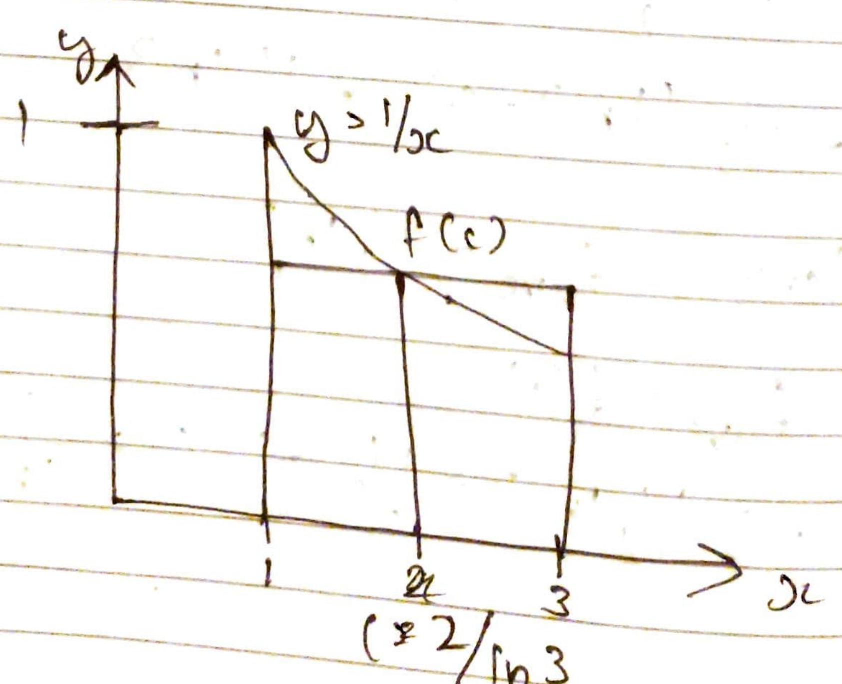
By the Mean value theorem, for f there is a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

that is $\int_a^b f(x)\,dx = (b-a) f(c)$.

= area of the rectangle with base $(b-a)$ and height $f(c)$

In order to sketch a rectangle whose area is same as the area under the curve f we choose $c = \frac{2}{\ln 3}$ in $[1, 3]$ then $f(c)$ is the height of the rectangle and its width is $3-1=2$



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13. We have $\int_1^3 f(x)\,dx = 8$

Since f is continuous so the mean value theorem of integral we have

$$\int_1^3 f(x)\,dx = f(c)(3-1)$$

Or $8 = 2f(c)$ Or $f(c) = 4$

This means the function $f(x)$ takes on the value 4 at least once on the interval $[1,3]$

17. $T(t) = 50 + 14\sin\frac{\pi t}{12}$

$a = 0,\ b = 12$

$$T_{ave} = \frac{1}{b-a}\int_a^b T(t)\,dt$$

$$= \frac{1}{12}\int_0^{12}\left(50 + 14\sin\frac{\pi t}{12}\right)dt$$

$$= \frac{1}{12}\left[50t + 14\times\frac{12}{\pi}\left(-\cos\frac{\pi t}{12}\right)\right]_0^{12}$$

$$= \frac{1}{12}\left[\left(50\times12\right) - 14\times\left(\frac{12}{\pi}\right)\cdot\cos\pi\right) -$$

$$\left(\left(50\times0\right) - 14\times\left(\frac{12}{\pi}\right)\cdot\cos 0\right)\right]$$

$$= \frac{1}{12}\left[600 + \frac{168}{\pi} + \frac{168}{\pi}\right] = 1/12\left[600 + \frac{336}{\pi}\right]$$

$T_{ave} = 50 + \frac{28}{T_1} \,^{\circ}F$ or $\approx 59^{\circ}F$