04/17/2023 CS 355

SPRING 2023

HW-8

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Problem 1 Ans:-

The joint PDF given by :-

$$f_{X,Y}(x,y) = \begin{cases} \boxed{R(x+y) ; 0 \le x \le 2 ; \; 0 \le y \le 2} \\ 0 ; \text{ otherwise} \end{cases}$$

(a) For the valid joint PDF it is known that :-

1

$$\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Here, $0 \le x \le 2$; $0 \le y \le 2$

Here, $\int_0^2 \int_0^2 c(x+y)\,dx\,dy = 1$

$$c\int_0^2 \left(\frac{x^2}{2} + yx\right)_0^2 dy = 1$$

$$c\int_0^2 \left(\frac{2^2}{2} + 2y\right) dy = 1$$

2; ≤ 2

$$= c\int_0^2 (2+2y)\,dy = 1$$

$$= c\left(2y + 2\frac{y^2}{2}\right)_0^2 = 1$$

$$= c\left(\left(2y + y^2\right)_0^2\right) = 1$$

$$= c\left(\left(2(2) + (2)^2\right) - \left((2\times 0) + 0^2\right)\right) = 1$$

$= k(4+4-0) = 1$

$= 8k = 1$

$k = \frac{1}{8}$

$\therefore$ The value of $k = \frac{1}{8} = 0.125$

The marginal distribution of X, $f_X(x)$ is as below:-

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \quad ; \boxed{-\infty \le x < \infty}$$

Here, $f_X(x) = \int_0^2 k(x+y)\,dy \quad ; \boxed{0 \le x \le 2}$

$= k\int_0^2 (x+y)\,dy$

3

$= \frac{1}{8}(xy$

$= \frac{1}{8}((2x$

$= \frac{1}{8}(2x$

$= \frac{1}{8}(2x$

$= \frac{1}{4}(x$

$\therefore f_X(x) =$

4

$$= \frac{1}{8}\left(xy + \frac{y^2}{2}\right)_0^2$$

$$= \frac{1}{8}\left(\left(2x + \frac{2^2}{2}\right) - \left(0x + \frac{0^2}{2}\right)\right)$$

$$= \frac{1}{8}(2x+2-0)$$

$$= \frac{1}{8}(2x+2)$$

$$= \frac{1}{4}(x+1)$$

$$\therefore f_X(x) = \frac{1}{4}(x+1) \; ; 0 \le x \le 2$$

$\boxed{; -\infty \le x < \infty}$ Now, the marginal distribution of Y, $f_Y(y)$ is as below:-

$\boxed{; 0 \le x \le 2}$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \; ; -\infty \le y < \infty$$

Here, $f_Y(y) = \int_0^2 k(x+y)\,dx$;
$0 \leq y \leq 2$

$$= \frac{1}{8}\left(\frac{x^2}{2} + yx\right)_0^2$$

$$= \frac{1}{8}\left(\left(\frac{2^2}{2} + 2y\right) - \left(0y + \frac{0^2}{2}\right)\right)$$

$$= \frac{1}{8}(2 + 2y - 0)$$

$$= \frac{1}{4}(1+y)$$

$\therefore f_Y(y) = \frac{1}{4}(y+1)$; $0 \leq y \leq 2$

The expected value of X, E(X) is as below:-

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Here, $E(X) = \int_0^2 x\left(\frac{1}{4}\right)(x+1)\, dx$

$$= \frac{1}{4}\int_0^2 (x^2+x)\, dx$$

$$= \frac{1}{4}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^2$$

$$= \frac{1}{4}\left(\left(\frac{2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{0^3}{3} + \frac{0^2}{2}\right)\right)$$

$$= \frac{1}{4}\left(\frac{8}{3} + 2 - 0\right)$$

$$= \frac{1}{4}\left(\frac{14}{3}\right)$$

$$= \frac{7}{6} = \underline{\underline{1.1667}}$$

6

The expected value of Y, $E(Y)$ is as below:-

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

Here, $E(Y) = \int_0^2 y\left(\frac{1}{4}\right)(y+1)\,dy$

$$= \frac{1}{4}\int_0^2 (y^2+y)\,dy$$

$$= \frac{1}{4}\left(\frac{y^3}{3} + \frac{y^2}{2}\right)_0^2$$

$$= \frac{1}{4}\left(\left(\frac{2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{0^3}{3} + \frac{0^2}{2}\right)\right)$$

$$= \frac{1}{4}\left(\frac{8}{3} + 2 - 0\right)$$

$$= \frac{1}{4}\left(\frac{14}{3}\right)$$

$$= \frac{7}{6} = 1.1667$$

$$E[2X+4Y] = 2E[X] + 4E[Y]$$

$$= \cancel{2}^{1} \times \frac{7}{\cancel{6}_{3}} + \cancel{4}^{2} \times \frac{7}{\cancel{6}_{3}}$$

$$= \frac{7}{3} + \frac{14}{3}$$

$$= \frac{\cancel{21}^{7}}{\cancel{3}_{1}}$$

$$= \underline{\underline{7}}$$

(b) $Cov(X,Y) = E[XY] - E[X] \times E[Y]$

Then, $E[XY] = \int_0^2 \int_0^2 XY . f(X,Y) \, dx \, dy$

$$= \int_0^2 \int_0^2 XY \cdot \frac{1}{8}(x+y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \int_0^2 XY \cdot (x+y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \int_0^2 (x^2y + xy^2)\,dx\,dy$$

$$= \frac{2}{3} + \frac{2}{3}$$

$$= \frac{4}{3}$$

We know that $E[X]$ and $E[Y] = \frac{7}{6}$ & $\frac{7}{6}$.

$\therefore$ $Cov(X,Y) = E[XY] - E[X] \times E[Y]$

$$= \frac{4}{3} - \frac{7}{6} \times \frac{7}{6}$$

$$= \frac{4^{\times 12}}{3} - \frac{49}{36}$$

$$\begin{array}{r|l} 3 & 3, 36 \\ 3 & 1, 12 \\ 4 & 1, 4 \\ & 1, 1 \end{array}$$

$$= \frac{48 - 49}{36}$$

$$= -\frac{1}{36}$$

$$= -0.028$$

Problem 2 Ans:-

A factory produces X_n gadgets on day 'n', where X_n are independently and identically distributed random variables R.V.

Mean = 5

Variance = 9

(a) Let $W_N = \sum_{n=1}^{N} X_n$

$$P\left[\sum_{n=1}^{100} X_n \leq 440\right] =$$

$$P[W_{100} \leq 400]$$

$\{\because$ By De Moivre-Laplace approximation to Binomial $\} \approx \phi\left(\dfrac{440-100 \times 5}{\sqrt{9 \times 100}}\right)$

$$= \phi\left(\frac{440-500}{\sqrt{900}}\right)$$

$$= \phi\left(\frac{-60}{30}\right)$$

$$= \phi(-2)$$

$$= 0.02275$$

∴ The probability is 0.02275

(b) Finding the largest value of n such that

$$P(X_1 + \dots + X_n > 200 + 5n) \le 0.05$$

Largest value o to n

$\boxed{x5)}$

$$P(X_1 + \dots + X_n \geq 200 + 5n) \le 0.05$$

$$P(W_n \geq 200 + 5n) =$$

$$1 - \Phi\left(\frac{200 + 5n - 5n}{3\sqrt{n}}\right)$$

$$= 1 - \Phi\left(\frac{200}{3\sqrt{n}}\right) \le 0.05$$

$$= \phi\left(\frac{200}{3\sqrt{n}}\right) \geq 0.95$$

$$= \frac{200}{3\sqrt{n}} \geq 1.65$$

{ ∴ Z value for 0.95 = 1.65 }

$$= \frac{200}{3\sqrt{n}} = 1.65$$

$$= \frac{200}{3} = 1.65\sqrt{n}$$

$$= n = \left(\frac{200}{3 \times 165}\right)^2$$

$$n = \left(\frac{40000}{4.95}\right)^2$$

$n = 1632.4865$

$\therefore$ Probability $= 1632$

(c) Here,

$P[N \geq 220] = 1 - P$[you have -ve produced 100 by $N = 219$]

$= 1 - P[219 \rightarrow 1000]$

$$= \phi\left[\frac{1000 - 219(5)}{3\sqrt{219}}\right]$$

$= \phi(-2.1398)$

$= 0.016$

$P[N \geq 220] = 0.016$