03/02/2023

CS - 355 - 2C

HW 5

Problem 1 :- Ans:- Here, given that $g(x) = \begin{cases} 1 & \text{is } x \le \frac{1}{3} \\ 2 & \text{is } x > \frac{1}{3} \end{cases}$

So, i.e. $x \to U[a=0, b=1]$

Let $f(x) = \frac{1}{b-a} = \frac{1}{1-0}$ $\because a=0, b=1$

$\therefore f(x) = 1$

When $P(x \le \frac{1}{3}) = \int_0^{\frac{1}{3}} f(x)dx$ $(\because f(x)=1)$

then $= \int_0^{\frac{1}{3}} dx = [x]_0^{\frac{1}{3}}$

$= (\frac{1}{3} - 0) = \frac{1}{3}$

and when $p(x > \frac{1}{3}) = \int_{1/3}^{1} f(x)dx$ $\because [f(x)=1]$

$= [x]_{\frac{1}{3}}^{1} = 1 - \frac{1}{3} = \frac{2}{3}$

Hence given by :- $P(y) = \begin{cases} \frac{1}{3} & y=1 \\ \frac{2}{3} & y=2 \\ 0 & o.w \end{cases}$

Then we find the formula:-

$E(y) = \sum_y y \times p(y) = 1 \times \frac{1}{3} + 2 \times \frac{2}{3}$

$= \frac{5}{3}$

$\therefore y = g(x)$

then $E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot g(x)\, dx$

$$= \int_0^{\frac{1}{3}} dx + 2\int_{\frac{1}{3}}^{1} dx$$

$$= [x]_0^{\frac{1}{3}} + 2[x]_{\frac{1}{3}}^{1}$$

$$=> \left(\frac{1}{3} - 0\right) + 2\left(1 - \frac{1}{3}\right)$$

$$=> \frac{1}{3} + 2\left(\frac{3-1}{3}\right)$$

$$=> \frac{1}{3} + 2 \times \frac{2}{3}$$

$$=> \frac{1}{3} + \frac{4}{3}$$

$$=> \frac{5}{3}$$

Hence, proved

Problem 2:- Ans:- We have the joint distribution $P(x, y)$ as follows:-

$P(x,y)$	x		
	0	1	
y	0	0.60	0.10
	1	0.10	0.20

(a) If two events X and Y are independent in a joint distribution then $P_{X,Y}(X=x, Y=y) = P_X(X=x) P_Y(Y=y)$.

Let us take the probability $P_{X,Y}(X=1, Y=1)$. Its value from the above table is 0.20.

$P_X(X=1) = 0.1 + 0.2 = 0.3$

$P_X(Y=1) = 0.10 + 0.20 = 0.30$

So, $P_X(X=x) P_Y(Y=y) = 0.3 \times 0.3 = 0.09$

$P_{X,Y}(X=x, Y=y) \neq P_X(X=x) P_Y(Y=y)$.

So, the events X and Y are not independent, i.e. hardware and software failures are dependent.

(b) Let $Z = X+Y$. Then the possible values of Z are 0, 1 and 2.

$$P(Z=0) = P(X=0, Y=0)$$
$$= \underline{0.60}$$

$$P(Z=1) = P(X=0, Y=1) + P(X=1, Y=0)$$
$$= 0.10 + 0.10$$
$$= 0.20$$

$$P(Z=2) = P(X=1, Y=1)$$
$$= \underline{0.20}$$

The probability distribution of Z is :-

z	0	1	2
$P(Z=z)$	0.60	0.20	0.20

The distribution is valid as $\sum P(Z=z) = 0.60 + 0.20 + 0.20$

$= \underline{\underline{1}}$

The value of $E(X+Y)$ is given as $E(Z) = \sum z \cdot P(Z=z)$

$$\sum z \cdot P(Z=z) = (0\times 0.60) + (1\times 0.20) + (2\times 0.20)$$
$$= 0 + 0.20 + 0.40$$
$$= \underline{\underline{0.60}}$$

The expected total number of failures during 1 day is $\underline{\underline{0.60}}$

<u>Problem 3</u> :- Ans:- Let X be the number of candy bars that one should eat to find a ticket.

Now, X can take values 1, 2, 3, 4, ... ∞

It can be easily observed that X follows geometric distribution with probability p.

$$\therefore \text{pmf of } X = p(x) = \begin{cases} p(1-p)^{x-1} & ; x = 1, 2, 3, 4, \ldots \\ 0 & ; \text{otherwise} \end{cases}$$

$$\therefore \text{mean } E(X) = \sum_{x=\cancel{0}}^{\infty} x\, p(x)$$
$$= \sum_{x=1}^{\infty} \cancel{p}\, p x (1-p)^{x-1} = p \sum_{x=1}^{\infty} x (q)^{x-1},$$

let $1-p = q$

$$\therefore E(X) = p[1 + 2q + 3q^2 + 4q^3 + \ldots\ldots \infty]$$
$$= p \cdot \frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\therefore E(X) = \underline{\underline{\frac{1}{p}}} \text{ mean}$$

Now let us find $E(x^2)$:-

$$E(x^2) = E[x(x-1)+x] \quad [\because x^2 = x(x-1)+x]$$

$$= E[x(x-1)+E(x)] \quad — (1)$$

So, let us find :-

$$E[x(x-1)] = \sum_{x=1}^{\infty} x(x-1)\, p(x)$$

$$= \sum_{x=1}^{\infty} x(x-1)\, p(1-p)^{x-1}$$

Let $1-p = q$

$$E[x(x-1)] = p \sum_{x=1}^{\infty} x(x-1)\, q^{x-1}$$

$$= p \sum_{x=1}^{\infty} (x-1)\, \underbrace{x q^{x-1}}_{\frac{d}{dq} q^x}$$

$$\therefore E[x(x-1)] = p \sum_{x=1}^{\infty} (x-1) \frac{d}{dq} q^x$$

$$= p \frac{d}{dq} \sum_{x=1}^{\infty} (x-1)\, q^x$$

$$E[x(x-1)] = p \frac{d}{dq} \Big(q^2 \cdot \underbrace{\sum_{x=1}^{\infty} (x-1) q^{x-2}}_{\frac{1}{(1-q)^2}}\Big)$$

$$E[x(x-1)] = p \frac{d}{dq} \frac{q^2}{(1-q)^2} = \frac{2q}{p^2}$$

Substituting in 1 :-

$$var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2(1-p)}{p^2} - \frac{1}{p^2} + \frac{1}{p}$$

$$var(X) = \frac{1-2p+p}{p^2}$$

$$\therefore var(X) = \frac{1-p}{p^2}$$

Problem 4 :- Ans:- Given PDF is :- $f(x) = \begin{cases} cx^2, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(a) $f(x)$ is said to be a valid density function of X if :-

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{|x|} cx^2\,dx = 1$$

$$\int_{-1}^{1} cx^2\,dx = 1$$

$$c\left[\frac{x^3}{3}\right]_{-1}^{1} = 1$$

$$c\left[1^3 - (-1)^3\right] = 3$$

$$c(1+1) = 3$$

$$c = \frac{3}{2}$$

The range of x is -1 to 1 since $|x| <= 1$ implies,
$-1 \leq x \leq 1$

$\therefore$ The PDF is :- $f(x) = \begin{cases} \frac{3}{2}x^2, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(b) The expected value of X is :-

$$E(X) = \int_{-1}^{1} x\, f(x)\, dx$$

$$= \int_{-1}^{1} x \frac{3}{2} x^2\, dx$$

$$= \frac{3}{2} \left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{3}{8}\left[1^4 - (-1)^4\right]$$

$$= 0$$

Since $\frac{3}{2x^2}$ is a curve symmetric to $x=0$, its expected value is 0.

$$E(X^2) = \int_{-1}^{1} x^2 \frac{3}{2} x^2\, dx$$

$$= \frac{3}{2}\int_{-1}^{1} x^4\, dx$$

$$= \frac{3}{2}\left[\frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{3}{10}\left[1^5 - (-1)^5\right]$$

$$= \frac{3}{10} \times 2 = \frac{3}{5}$$

$\therefore$ The variance of X is :- $Var(x) = \frac{3}{5}$

Since, $Var(x) = E(X^2) - [E(X)]^2$

(c) $P\left(x \geq \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} \frac{3}{2} x^2 \, dx$

$= \frac{3}{2}\left[\frac{x^3}{3}\right]_{\frac{1}{2}}^{1}$

$= \frac{3}{6}\left[1^3 - \left(\frac{1}{2}\right)^2\right]$

$= \frac{3}{6}\left[1 - \frac{1}{4}\right]$

$= \frac{3}{6} \times \frac{3}{4}$

$= \frac{9}{24}$

$= \underline{\underline{0.375}}$

Problem 5 :- Ans:- The probability density function of X is:- $f(x) = 4x^3$

The required probability is:-

$$P\left(X \leq \frac{2}{3} \mid X > \frac{1}{3}\right) = \frac{P\left(X \leq \frac{2}{3} \text{ and } X > \frac{1}{3}\right)}{P\left(X > \frac{1}{3}\right)}$$

$$= \frac{P\left(\frac{1}{3} < X \leq \frac{2}{3}\right)}{P\left(X > \frac{1}{3}\right)}$$

$$= \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} 4x^3 \, dx}{\int_{\frac{1}{3}}^{1} 4x^3 \, dx}$$

$$= \frac{4 \times \left(\frac{x^4}{4}\right)\Big|_{\frac{1}{3}}^{\frac{2}{3}}}{4 \times \left(\frac{x^4}{4}\right)\Big|_{\frac{1}{3}}^{1}}$$

$$= \frac{\left(\frac{2}{3}\right)^4 - \left(\frac{1}{3}\right)^4}{1 - \left(\frac{1}{3}\right)^4}$$

$$= \frac{\frac{16}{81} - \frac{1}{81}}{1 - \frac{1}{81}}$$

$$= \frac{\frac{15}{81}}{\frac{80}{81}}$$

$$= \frac{15}{80} = \underline{\underline{0.1875}}$$

The conditional probability is $\underline{0.1875}$

Problem 6

Exp. No.	Experiment/Subject		Date	
Name	Lab Partner		Locker/ Desk No.	Course & Section No.

Ans: Here, the probability of obtaining a 1 = $\frac{1}{6}$

Since there are 6 equally possible outcomes.

Probability of obtaining a 2 = $\frac{1}{6}$

$\therefore$ From the multinomial distribution below, the joint PMF of X and Y is given by :-

$$P(X=x\,;\,Y=y) = \frac{4!}{x!\times y!\times(4-x-y)!}\times\left(\frac{1}{6}\right)^x\cdot\left(\frac{1}{6}\right)^y\cdot\left(\frac{4}{6}\right)^{4-x-y}$$

From the expression above, the table for the joint PMF of X and Y, is as below :-

x	y	P(x, y)
0	0	0.1975
0	1	0.1975
0	2	0.0741
0	3	0.0123
0	4	0.0008
1	0	0.1975
1	1	0.1481
1	2	0.0370
1	3	0.0031

Signature	Date	Witness/TA	Date

x	y	$p(x, y)$
2	0	0,0741
2	1	0,0370
2	2	0,0046
3	0	0,0123
3	1	0,0031
4	0	0,0008