13.8 15.4 A particle moves along line segments from the origin to the points (1,0,0), (1,2,1), (0,2,1), and back to the origin under the influence of the force field

$$\vec{F}(x,y,z) = z^2\,\hat{i} + 2xy\,\hat{j} + 4y^2\,\hat{k}$$

Find the work done.

Work done along each line segment = W_1, W_2, W_3, W_4

$$W = W_1 + W_2 + W_3 + W_4$$

$$\vec{r}_1(t) = \langle t, 0, 0 \rangle \quad ; \quad t \in [0,1]$$

$\therefore$ Work done for line segment from (0,0,0) to (1,0,0) :-

$$W_1 = \int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) = 0$$

$$= \int_0^1 \langle 0, 0, 0 \rangle \cdot \langle 1, 0, 0 \rangle$$

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Work done for line segment from $(1,0,0)$ to $(1,2,1)$:-

$$\frac{\cancel{56}^{28}}{\cancel{6}_3}$$

$$\vec{r}_2(t) = \langle 1, 2t, t \rangle$$

$$\vec{r}_2\,'(t) = \langle 0, 2, 1 \rangle$$

$$W_2 = \int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2\,'(t)$$

$$= \int_0^1 \langle t^2, 4t, 16t^2 \rangle \cdot \langle 0,2,1 \rangle$$

$$= \int_0^1 \langle 0, 8t, 16t^2 \rangle$$

$$\cancel{= (8+16) = 24} = \left(\frac{8t^2}{2} + \frac{16t^3}{3} \right)_0^1$$

$$= 4 + \frac{16}{3} = \boxed{\frac{28}{3}}$$

Work done for line segment from $(1,2,1)$ to $(0,2,1)$:-

$$\vec{r}_3(t) = \langle 1-t, 2, 1 \rangle$$

$$\vec{r}_3\,'(t) = \langle -1, 0, 0 \rangle$$

$$W_3 = \int_0^1 \vec{F}(\vec{r}_3(t)) \cdot \vec{r}_3\,'(t)$$

$$= \int_0^1 \langle 1, 4-4t, 16 \rangle \cdot \langle -1,0,0 \rangle$$

$$= \int_0^1 \langle -1, 0, 0 \rangle$$

$$= -1$$

Work done for line segment from $(0,2,1)$ to $(0,0,0)$:-

$\vec{r}_4(t) = \langle 0, 2-2t, 1-t \rangle$

$\vec{r}_4\,'(t) = \langle 0, -2, -1 \rangle$

$$W_4 = \int_0^1 \vec{F}(\vec{r}_4(t)) \cdot \vec{r}_4\,'(t)$$

$$= \int_0^1 \langle (1-t)^2, 0, 4(2-2t)^2 \rangle \cdot \langle 0, -2, -1 \rangle$$

$$= \int_0^1 \langle 0, 0, -4(2-2t)^2 \rangle$$

$$= \int_0^1 -4(4 - 2\times 2\times 2t + 4t^2)\,dt$$

$$= \int_0^1 (-16 + 32t - 16t^2)\,dt$$

$$= -16 + \left(\frac{\cancel{32}^{16} t^2}{\cancel{2}_1}\right)_0^1 - \left(\frac{16t^3}{3}\right)_0^1$$

$$= -\cancel{16} + \cancel{16} - \frac{16}{3}$$

$$= -\frac{16}{3}$$

$$\therefore W = 0 + \frac{28}{3} - \frac{1}{1}^{\times 3} - \frac{16}{3}$$

$$= \frac{28 - 3 - 16}{3} = \cancel{\frac{\ \ }{1}}$$

$$= \frac{28-19}{3} = \frac{\cancel{9}^3}{\cancel{3}_1} = \underline{\underline{3}}$$