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SUMMER 2022 — MA 227 — FINAL

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1. Part I

There are 6 problems in Part I, each worth 3 points. Place your answer on the line below the question. In Part I, there is no need to show your work, since only your answer on the answer line will be graded.

(1) Find the dot product (or scalar product) of the vectors $\langle 1, -2, 0\rangle$ and $\langle -2, 1, 3\rangle$.

$\langle 1,-2,0\rangle \cdot \langle -2,1,3\rangle =$

$1(-2)+(-2)(1)+0(3) = -4$ $\quad$ -4

(2) Compute the curl of the vector field $\langle y, 2z, 3x\rangle$.

$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & 3x \end{vmatrix} = \langle -2, -3, -1\rangle$ $\quad$ $\langle -2, -3, -1\rangle$

(3) Find a parametrization for a circle with radius 3 centered at the point with coordinates $(2, 1)$ in the x-y-plane.

$x = 4\cos(\theta)+2$

$y = 4\sin(\theta)+1$ $\quad$ $\vec{r}(\theta) = \langle 4\cos(\theta)+2,\ 4\sin(\theta)+1$

(4) Compute the Jacobian of the transformation $x = e^{s-t}$, $y = e^{s+t}$.

$x_s = e^{s-t}$ $\quad$ $x_t = -e^{s-t}$

$y_s = e^{s+t}$ $\quad$ $y_t = e^{s+t}$

$J = \begin{vmatrix} e^{s-t} & -e^{s-t} \\ e^{s+t} & e^{s+t} \end{vmatrix} =$ $\quad$ $e^{2s} + e^{2s} = 2e^{2s}$

☆ (5) The temperature on a metal plate is given by $T(x, y) = 2x + y^2$. In which direction does it increase fastest at the point $(3, 1)$?

Fastest change in temp: $\frac{\nabla T(x,y)}{|\nabla T(x,y)|}$

$\nabla T = \langle 2, 2y\rangle$ $\quad$ Fastest increase at $\frac{\langle 2, 2\rangle}{\sqrt{8}}$

$|\nabla T| = \sqrt{2^2+(2y)^2} = \sqrt{4+4y^2}$

$|\nabla T(3,1)| = \sqrt{4+4} = 2\sqrt{2}$

(6) Find the linearization $L(x, y)$ of $f(x, y) = \sin(\pi xy)$ at the point $(1, 1)$.

☆

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ $\quad$ $L(x,y) = -\pi x - \pi y + 2\pi$

$f_x = \pi y\cos(\pi xy)$ $\quad$ $f_y = \pi x\cos(\pi xy)$

$L(x,y) = \underbrace{\sin(\pi)}_{0} + \pi\underbrace{\cos(\pi)}_{-1}(x-1) + \pi\underbrace{\cos(\pi)}_{-1}(y-1) = 0 - \pi(x-1) - \pi(y-1) = -\pi x - \pi y + 2\pi$

2. Part II

There are 6 problems in Part II with the weight indicated. On Part II problems show all your work! Your work, as well as the answer, will be graded. Your solution must include enough detail to justify any conclusions you reach in answering the question.

(1) [14 points] Find the critical points of

$$f(x, y) = 2x^3 + 2xy^2 + 2x^2 + y^2$$

and determine which are local maxima, local minima, and saddle points.

$f_x = 6x^2 + 2y^2 + 4x = 0$

$f_y = 4xy + 2y = 0 \rightarrow \overset{(a)}{2}y(2x+1) = 0$ plug $y=0$ into f_x: $6x^2 + 4x = 0$

either $y=0$ or $x = \frac{1}{2}$ $\quad 2x(3x+2) \quad x=0$ or $x = -\frac{2}{3}$

points: $(0, 0)$ OR $(-\frac{2}{3}, 0)$

(b) $x = \frac{1}{2}$ $\quad 6(\frac{1}{4}) + 2y^2 + 4(\frac{1}{2}) = 2y^2 + 2 + \frac{3}{2} = 2y^2 + \frac{7}{2}$

$2y^2 + \frac{7}{2} = 0 \rightarrow y^2 = \frac{1}{4} \rightarrow y = \pm\frac{1}{2}$

points: $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$

$D: f_{xx}f_{yy} - f_{xy}^2 \qquad f_{xx} = 12x + 4 \quad f_{yy} = 4x + 2 \quad f_{xy} = 4y$

$(12x+4)(4x+2) - 16y^2 = 48x + 12x + 16x + 8 = 4(12x + 3x + 4x + 2) =$

$4(19x + 2)$

If $D > 0$, $f_{xx} > 0$, min

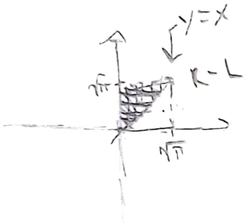
$D < 0$, saddle point

$D > 0$, $f_{xx} < 0$, max

(2) [13 points] Evaluate the integral

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy$$

by reversing the order of integration.



$0 \leq y \leq \sqrt{\pi}$

$y \leq x \leq \sqrt{\pi}$

new lims; $0 \leq x \leq \sqrt{\pi}$, $0 \leq y \leq x$

$$\int_0^{\sqrt{\pi}} \int_0^{x} \cos(x^2)\, dy\, dx = \int_0^{\sqrt{\pi}} \left[y\cos(x^2) \right]_{y=0}^{y=x} dx = \int_0^{\sqrt{\pi}} x\cos(x^2)\, dx$$

$u = x^2 \quad du = 2x\,dx$

new lims: $u = \sqrt{\pi}^2$, $u = 0^2$

$$\frac{1}{2}\int_0^{\pi} \cos(u)\,du = \frac{1}{2}\left[\sin(u)\right]_0^{\pi} = \frac{1}{2}(\sin(\pi) - 0)$$

$$= \frac{1}{2}(0) = 0$$

$f_{xyz} = 2y \quad f_{yxz} = 2y$

$f_{zyx} = 2y$

conservative ✓

(3) [13 points] Consider the vector field

$$\boldsymbol{F}(x, y, z) = \langle \overset{f_x}{y^2 z}, \overset{f_y}{2xyz + 2yz}, \overset{f_z}{y^2 + xy^2 + 3z^2} \rangle.$$

① [Find a function f such that $\nabla f = \boldsymbol{F}$ and use it to compute the integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$
② [when C is circular arc on the hemisphere $x^2 + y^2 + z^2 = 4$, $x \geq 0$ and the plane $x + y = 0$ joining the points $(0, 0, 2)$ and $(1, -1, \sqrt{2})$.

①

$$\int f_x \, dx = \int y^2 z \, dx = x \cdot y^2 z + g(y,z) \qquad f_y = 2xyz + g'(y,z)$$

$$2xyz + g'(y,z) = 2xyz + 2yz \qquad g'(y,z) = 2yz \qquad \int g'(y,z)\,dy = y^2 z + h(z)$$

$$f = xy^2 z + y^2 z + \cancel{g(z)} \quad f_z = xy^2 + y^2 + g'(z) = y^2 + xy^2 + 3z^2 \qquad g'(z) = 3z^2$$

$$\int g'(z)\,dz = z^3 + c \qquad \boxed{f = xy^2 z + y^2 z + 3z^2 + c}$$

② $$\int_C F \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(t))\Big]_a^b$$

$$= f(\vec{r}(t)) - f(\vec{r}(a))$$

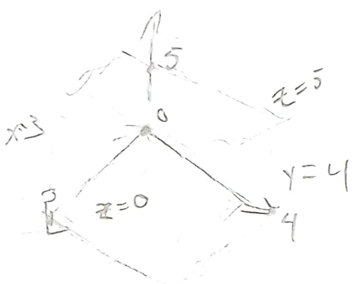
We can stick in the points b/c F is conservative

$$= f(1,1,\sqrt{2}) - f(0,0,2) = \sqrt{2} + \sqrt{2} + 2\sqrt{2} + c - (8 + c) = 4\sqrt{2} - 8$$

(4) [13 points] Let

$$\boldsymbol{F}(x, y, z) = \langle x^2yz, xy^2z, xyz^2\rangle$$

and let S be the surface of the box bounded by the planes $x = y = z = 0$, $x = 3$, $y = 4$, and $z = 5$. Find the flux of $\boldsymbol{F}$ across S, that is, the integral $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$.



★ $\iiint_E \operatorname{div}\vec{F}\, dV = \iint_S \cdot\, d\vec{S}$ Gauss' Law

$\operatorname{div}\vec{F} = \nabla\cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle x^2yz, xy^2z, xyz^2\rangle$

$= 2xyz + 2xyz + 2xyz = \boxed{6xyz = \nabla\cdot\vec{F}}$

We know from the box given that lims of integration will be:

$x \in [0,3] \quad y\in[0,4] \quad z\in[0,5]$

$6\int_0^5\int_0^4\int_0^3 xyz\,dx\,dy\,dz = \frac{6}{2}\int_0^5\int_0^4 \left[x^2yz\right]_{x=0}^{x=3} dy\,dz = 21\int_0^5\int_0^4 yz\,dy\,dz$

$= \frac{21}{2}\int_0^5 \left[y^2z\right]_{y=0}^{y=4} dz = 21(8)\int_0^5 z\,dz = 21(4)\left[z^2\right]_0^5 = 21(25)(4)$

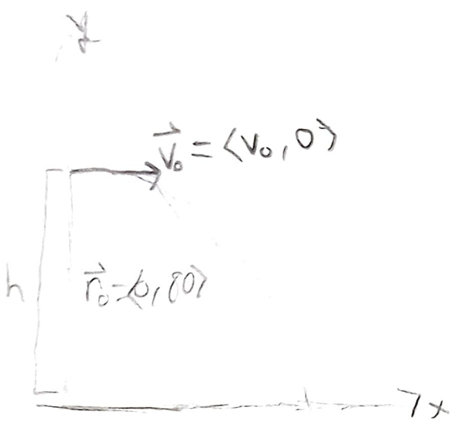
(5) [14 points] A ball is thrown horizontally from a tower of height h meters with initial speed of v_0 meters per second. Choose coordinates to describe the path of the ball until it hits ground, assuming that gravity is the only force acting on the ball and that the gravitational acceleration g is equal to 10 meters per second squared. If $h = 80$ meters and the ball lands 100 meters from the tower, find the time t_0 when the ball hits the ground and the initial speed v_0.

$F = m\vec{a} = \langle 0, -mg \rangle$

$\Downarrow$

$\vec{a} = \langle 0, -g \rangle$

$\vec{v}(t) = \langle 0, -gt \rangle + \vec{v}_0 =$



(6) [15 points] Find the maximum and minimum values of the function $f(x, y, z) = 2x+2y$ subject to the constraints $\underbrace{x + 2y + z = 1}_{1}$ and $\underbrace{y^2 + z^2 = 2}_{2}$.

$\nabla f = \lambda \nabla g + \mu \nabla h$

$\nabla f = \langle 2, 2, 0\rangle$ $\lambda \nabla g = \lambda\langle 1, 2, 1\rangle$ $\mu \nabla h = \langle 0, 2y, 2z\rangle$

$2 = \lambda + 0 \to \boxed{2 = \lambda}$ plug λ into other eq's

$2 = 2\lambda + 2\mu y \to 2 = 4 + 2\mu y \to -1 = \mu y^{*} \to \boxed{-\frac{1}{\mu} = y}$ 3

$0 = \lambda + 2\mu z \to 0 = 2 + 2\mu z \to -1 = \mu z \to \boxed{-\frac{1}{\mu} = z}$ 4

plug these into eq. 2

1a) $x + \left(-\frac{2}{\mu}\right) + \left(-\frac{1}{\mu}\right) = 1 \to x - \frac{3}{\mu} = 1 \to x = 1 + \frac{3}{\mu}$

2 $\left(-\frac{1}{\mu}\right)^2 + \left(-\frac{1}{\mu}\right)^2 = 2 \to \frac{1}{\mu^2} + \frac{1}{\mu^2} = 2 \to \frac{2}{\mu^2} = 2 \to \frac{1}{\mu^2} = 1 \to \mu^2 = 1$

$\mu = \pm\sqrt{1} = \pm 1$

Using 1a) $x = 1 + \frac{3}{\mu} \to x = 1 + \frac{3}{\pm 1}$ $x = 4, -2$ ($\mu = +1$, $\mu = -1$)

Using 3 and our vals for μ: $y = -1, +1$

Using 4 and our vals for μ: $z = -1, +1$

So our points are,
$(4, -1, -1) = P_1$
and
$(-2, 1, 1) = P_2$

now plug in P_1 and P_2 to f:

$f(4, -1, -1) = 2(4) + (-1)(2) = 8 - 2 = 6$ max

$f(-2, 1, 1) = 2(-2) + (2)(1) = -4 + 2 = -2$ min

μ cannot be 0, otherwise we would have $-1 = (0)y$ or $-1 = 0$