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FALL 2009 — MA 227 — FINAL

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1. Part I

There are 10 problems in Part I, each worth 4 points. Place your answer on the line below the question. In Part I, there is no need to show your work, since only your answer on the answer line will be graded.

(1) Find the dot product (or scalar product) of the vectors $\langle 1, -3, 2\rangle$ and $\langle -2, 1, 3\rangle$.

$\langle 1,-3,2\rangle \cdot \langle -2,1,3\rangle =$

$1(-2)+(-3)(1)+(2)(3) = -2-3+6 = 1$

Answer: 1

(2) Express the length of the curve $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + 2\sin(t)\boldsymbol{j} + 3t\boldsymbol{k}$, $0 \le t \le 6\pi$ as an integral. (Do not evaluate the integral!)

arc length: $\int_0^{6\pi} \sqrt{\frac{\partial x}{\partial t}^2 + \frac{\partial y}{\partial t}^2 + \frac{\partial z}{\partial t}^2}\, dt$

Answer: $\int_0^{6\pi} (\sin^2(t) + 2\cos^2(t) + 9) dt$

(3) Find the gradient of the function $f(x, y) = \sqrt{xy^2 + y}$.

$\nabla f = \langle \frac{\partial}{\partial x}(xy^2+y)^{\frac{1}{2}}, \frac{\partial}{\partial y}(xy^2+y)^{\frac{1}{2}} \rangle$

$= \langle \frac{y^2}{2}(xy^2+y)^{-\frac{1}{2}}, \frac{2x+1}{2}(xy^2+y)^{-\frac{1}{2}} \rangle$

Answer: $\nabla f = \langle \frac{y^2}{2\sqrt{xy^2+y}}, \frac{2x+1}{2\sqrt{xy^2+y}} \rangle$

(4) Compute the Jacobian of the transformation $x = u^2 - 2v$, $y = 2u + v^2$.

$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2 \\ 2 & 2v \end{vmatrix} = 4uv + 4$

Answer: $J = 4uv + 4$

(5) Find rectangular coordinates of the point with spherical coordinates $\rho = 3$, $\phi = \pi$, $\theta = \pi/4$.

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$x = 3\sin(\pi)\cos(\frac{\pi}{4}) = 3(-1)(\frac{\sqrt{2}}{2})$

$y = 3\sin(\pi)\sin(\frac{\pi}{4}) = 3(-1)(\frac{\sqrt{2}}{2})$

$z = 3\cos(\pi) = 3(-1) = -3$

Answer: $(x,y,z) = (-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, -3)$

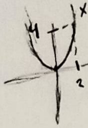
(6) Find the linearization $L(x, y)$ of $f(x, y) = \ln(x + y^2)$ at the point $(1, 2)$.

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_x(x_0,y_0)(y-y_0)$

$f_x = \frac{1}{x+y^2}$ $f_y = \frac{2y}{x+y^2}$

$L(x,y) =$ $\ln(5) + \frac{1}{5}(x-1) + \frac{4}{5}(y-2)$

(7) Reverse the order of integration in the iterated integral $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx$. (Do not evaluate the integral!)



$0 \le y \le 4$

$x = \sqrt{y}$

$0 \le x \le \sqrt{y}$

$\int_0^4 \int_0^{\sqrt{y}} f(x,y)\, dx\, dy$

(8) Find a potential function of the conservative vector field $\mathbf{F} = (1 + ye^{xy})\boldsymbol{i} + xe^{xy}\boldsymbol{j}$.

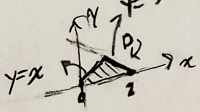
no need to check

$\int f_x dx = x + e^{xy} + g(y)$ $f_y = xe^{xy} + g'(y) = xe^{xy}$

so $g'(y) = 0$ and $g(y) = 0$

$f = x + e^{xy}$

(9) Write down the iterated integral for $\iint_D 3y\,dA$ where D is the triangular region with vertices $(0, 0)$, $(1, 1)$ and $(2, 0)$. Do not evaluate the integral.

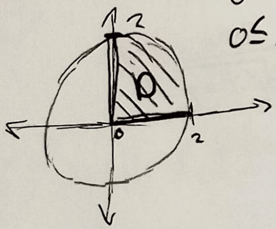


$3\int_0^2 \int_x^{2-x} y\,dy\,dx$

(10) Evaluate $\iint_D dA$ where $D = \{(x, y) : 0 \le x^2 + y^2 \le 4, 0 \le y, 0 \le x\}$.

π

$\left.\begin{matrix} 0 \le y \\ 0 \le x \end{matrix}\right\}$ only first quadrant



$\iint_D dA = \int_0^2 \int_0^{\frac{\pi}{2}} r\,d\theta\,dr = \frac{\pi}{2}\int_0^2 r\,dr = \frac{\pi}{4}\left[r^2\right]_0^2 = \frac{4\pi}{4} = \pi$

(2) A ball is thrown from ground level at an angle of $\pi/4$ radians to the ground at a speed of v_0 meters per second. The path of the ball for time $t \geq 0$ is described by the vector function $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle$, where the first component is along the horizontal while the second is along the vertical direction.

(a) What is the initial velocity vector $\boldsymbol{r}'(0)$?

(b) Assuming that gravity is the only force acting on the ball, and that the gravitational acceleration g is equal to 10 meters per second per second, find the acceleration vector $\boldsymbol{a}(t) = \boldsymbol{r}''(t)$ for the ball.

(c) Find the component functions $x(t)$ and $y(t)$ of the position vector $\boldsymbol{r}(t)$ in terms of the initial speed v_0.

(d) If the ball lands 20 meters away, find the time t_0 when the ball hits the ground, and the initial velocity v_0.

$\vec{F} = m\vec{a} \rightarrow \vec{F} = \langle 0, -mg \rangle$

$\vec{a} = \langle 0, -g \rangle$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{r}'(t) = \langle x'(t), y'(t) \rangle$

$\vec{r}''(t) = \langle x'(t), y'(t) \rangle = \vec{a} = \langle 0, -g \rangle$

$\int \vec{a}\, dt = \langle c_1, -gt + c_2 \rangle = \vec{r}'(t)$

$\therefore \vec{r}(t) = \langle c_1 t, -\frac{1}{2} g t^2 + c_2 t \rangle \quad \vec{r}(0) = \langle 0, 0 \rangle$

so $c_1, c_2 = 0$

(4) Find the points (x, y, z) on the double cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 4, 0)$.

$$d = \sqrt{(x_1 - x_0)^2 + (y_2 - y_0)^2 + (z_1 - z_0)^2}$$

$$z = \pm\sqrt{x^2 + y^2}$$

$$d = \sqrt{(x-4)^2 + (y-4)^2 + (x^2+y^2)}$$

$$d^2 = (x-4)^2 + (y-4)^2 + (x^2+y^2)$$

$$\left.\begin{array}{l} f_x = 2(x-4) + 2x = 4x - 8 = 0 \quad x = 2 \\ f_y = 2(y-4) + 2y = 4y - 8 = 0 \quad y = 2 \end{array}\right\} \text{crit point at } (2,2)$$

$f_{xx} = 4$ $f_{yy} = 4$ $D = 16 > 0$, $f_{xx} > 0$ so min

$f_{yx} = 0$

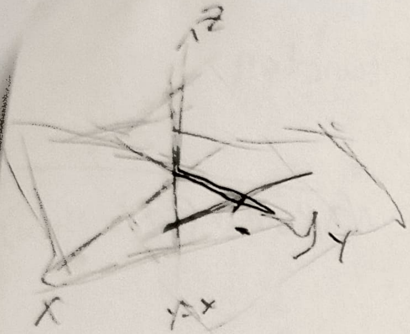
$z = \pm\sqrt{2^2 + 2^2} = \pm\sqrt{8} = \pm 2\sqrt{2}$ shortest dist at $(2, 2, \pm 2\sqrt{2})$

xy plane

(5) Find the volume of the solid lying between the planes $z = 0$, $z = x$ above the triangle bounded by the lines $x = 0$, $y = x$, and $y + x = 1$ in the x-y-plane.

y axis

$y = 1 - x$

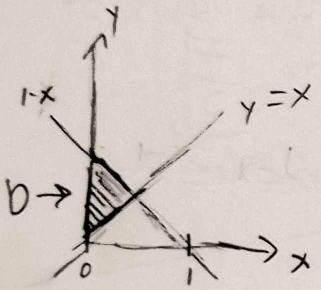


$0 \le z \le x$

$0 \le x \le \frac{1}{2}$

$x \le y \le 1-x$

$$\int_0^{\frac{1}{2}}\int_0^{x}\int_x^{1-x} dy\,dz\,dx = \int_0^{\frac{1}{2}}\int_0^{x} y\Big]_{y=x}^{y=1-x} dz\,dx$$



$$= \int_0^{\frac{1}{2}}\int_0^{x}\big((1-x)-x\big)\,dz\,dx = \int_0^{\frac{1}{2}}\int_0^{x} 1-2x\,dz\,dx$$

$$= \int_0^{\frac{1}{2}} x(1-2x)\,dx = \int_0^{\frac{1}{2}} x - 2x^2\,dx = \frac{1}{2}x^2 - \frac{2}{3}x^3\Big]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{1}{4}\right) - \frac{2}{3}\left(\frac{1}{8}\right) = \frac{1}{8} - \cancel{\frac{2}{24}}\,\frac{1}{12} = \frac{1}{24}$$

(6) Use spherical coordinates to find the mass of the solid that lies within the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{3(x^2 + y^2)}$ if the density μ of the material in the solid is given by

$$\mu(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

← integret this function

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

① $z = \sqrt{3(x^2+y^2)}$

$\rho\cos\phi = \sqrt{3(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)} = \sqrt{3\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta)}$

$= \sqrt{3\rho^2\sin^2\phi(1)} = \rho\sin\phi(\sqrt{3})$

$\rho\cos\phi = \rho\sin\phi(\sqrt{3}) \rightarrow \cos\phi = \sqrt{3}\sin\phi$ this occurs at angle $\boxed{\frac{\pi}{3}}$, so

$0 \le \phi \le \frac{\pi}{3}$

② $x^2+y^2+z^2 = 1 \rightarrow \rho^2 = 1 \quad \rho = 1$ so $0 \le \rho \le 1$

③ We know that the sphere makes a whole revolution, so θ varies from $0 \le \theta \le 2\pi$

④ $M(\theta, \phi, \rho) = \frac{1}{\rho} \quad \iiint \frac{1}{\rho}\, dV = \int_0^1 \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \frac{1}{\rho}(\rho^2\sin\phi)\, d\theta\, d\phi\, d\rho =$

$\int_0^1 \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \rho\sin\phi\, d\theta\, d\phi\, d\rho = 2\pi\int_0^1 \int_0^{\frac{\pi}{3}} \rho\sin\phi\, d\phi\, d\rho = 2\pi\int_0^1 \left[-\rho\cos\phi\right]_{\phi=0}^{\phi=\frac{\pi}{3}} d\rho$

$= 2\pi\int_0^1 \rho\left[-\frac{1}{2}+1\right] d\theta = \pi\int_0^1 \rho\, d\rho = \boxed{\pi}$