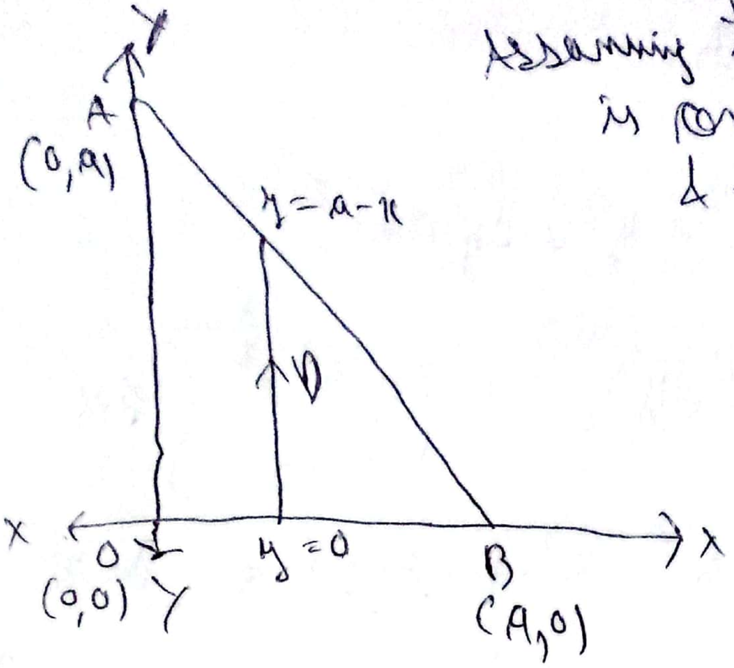
Ans:

Assuming that the triangle is on the positive axis & the vertex is (0,0).



Equation of line AB: $y - a = \frac{0-a}{a-0}(x-0)$

$\Rightarrow y - a = \frac{-\cancel{a}^{1}}{\cancel{a}_{1}} x$

$y - a = -x$

$y = a - x$

$\therefore D = \{(x,y) \mid 0 \le x \le a,\ 0 \le y \le a-x\}$

The vertex of the triangle opposite to the hypotenuse is O (0,0).

$\rho(x,y) \propto \left(\sqrt{x^2+y^2}\right)^2$ $\quad\because z = \sqrt{x^2+y^2}$

z = distance from vertex (0,0) to hypotenuse.

$\therefore \rho(x,y) \propto x^2+y^2$

$\rho(x,y) = k(x^2+y^2)$

$\therefore$ Center of mass $= (\bar{x}, \bar{y})$

$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$

$$m = \iint_D \rho(x,y)\, dA$$

$$= \int_0^a \int_0^{a-x} k(x^2 + y^2)\, dy\, dx$$

$$= \int_0^a k \left[x^2 (y)_0^{a-x} + \left(\frac{y^3}{3}\right)_0^{a-x} \right] dx$$

$$= \int_0^a k \left[x^2 a - x^3 + \frac{(a-x)^3}{3} \right] dx$$

$$= \int_0^a k \left[\frac{x^2 a - x^3}{1} + \frac{a^3 - 3a^2 x + 3ax^2 - x^3}{3} \right] dx$$

$$= k \int_0^a \left(\frac{3x^2 a - 3x^3 + a^3 - 3a^2 x + 3ax^2 - x^3}{3} \right) dx$$

$$= \frac{k}{3} \int_0^a (3x^2 a - 3x^3 + a^3 - 3a^2 x + 3ax^2 - x^3)\, dx$$

$$= \frac{k}{3} \left[\frac{\cancel{3} x^3 a}{\cancel{3}} - \frac{3x^4}{4} + x a^3 - \frac{3a^2 x^2}{2} + \frac{\cancel{3} a x^3}{\cancel{3}} - \frac{x^4}{4} \right]_0^a$$

$$= \frac{k}{3} \left[a^4 - \frac{3a^4}{4} + a^4 - \frac{3a^4}{2} + a^4 - \frac{a^4}{4} \right]$$

$$= \frac{k}{3} \left(\frac{3a^4}{1} - \frac{3a^4}{4} - \frac{3a^4}{2} - \frac{a^4}{4} \right)$$

$$= \frac{k}{3} \left(\frac{12a^4 - 3a^4 - 6a^4 - a^4}{4} \right) = \frac{k}{3} \left(\frac{2a^4}{4} \right)$$

$$= \frac{\cancel{2}^{1} k a^4}{\cancel{12}\, 6}$$

$$m = \frac{k a^4}{6}$$

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$$\bar{x} = \frac{M_y}{m}$$

$$M_y = \iint_D x \rho(x,y)\, dA$$

$$= \int_0^a \int_0^{a-x} x\, k(x^2+y^2)\, dy\, dx$$

$$= k\int_0^a \left[\int_0^{a-x} x^3\, dy + \int_0^{a-x} x y^2\, dy \right] dx$$

$$= k \int_0^a \left[x^3 (y)_0^{a-x} + x\left(\frac{y^3}{3}\right)_0^{a-x} \right] dx$$

$$= k\int_0^a \left[x^3(a-x) + x\frac{(a-x)^3}{3} \right] dx$$

$$= k\int_0^a \left[x^3 a - x^4 + \frac{x(a^3 - 3a^2x + 3ax^2 - x^3)}{3} \right] dx$$

$$= k\int_0^a \left[\frac{x^3 a - x^4}{1} + \frac{xa^3 - 3a^2x^2 + 3ax^3 - x^4}{3} \right] dx$$

$$= k\int_0^a \left(\frac{3x^3a - 3x^4 + xa^3 - 3a^2x^2 + 3ax^3 - x^4}{3} \right) dx$$

$$= \frac{k}{3}\left[\frac{3x^4 a}{4} - \frac{3x^5}{5} + \frac{x^2a^3}{2} - \frac{3a^2x^3}{3} + \frac{3ax^4}{4} - \frac{x^5}{5} \right]_0^a$$

$$= \frac{k}{3}\left[\frac{3a^5}{4} - \frac{3a^5}{5} + \frac{a^5}{2} - \frac{3a^5}{3} + \frac{3a^5}{4} - \frac{a^5}{5} \right]$$

$$= \frac{k}{3}\left(\frac{45a^5 - 36a^5 + 30a^5 - 60a^5 + 45a^5 - 12a^5}{60}\right)$$

$$\begin{array}{r|l} 2 & 2,3,4,5 \\ 2 & 1,3,2,5 \\ 3 & 1,3,1,5 \\ 5 & 1,1,1,5 \\ & 1,1,1,1 \end{array}$$

$$= \frac{k}{3}\left(\frac{120a^5 - 108a^5}{60}\right)$$

$$= \frac{k}{3}\left(\frac{\cancel{12}\,a^5}{\cancel{60}_{5}}\right) = \frac{ka^5}{15} = M_y$$

$$\therefore \bar{x} = \frac{M_y}{m} = \frac{\frac{ka^5}{15}}{\frac{ka^4}{6}} = \frac{\cancel{k}a^{\cancel{5}}}{\cancel{15}\,5} \times \frac{\cancel{6}^{\,2}}{\cancel{ka^4}}$$

$$\bar{x} = \frac{2a}{5}$$

$$\bar{y} = \frac{M_x}{m}.$$

$$M_x = \iint_D y\,\rho(x,y)\,dA$$

$$= \int_0^a \int_0^{a-x} y\,k(x^2+y^2)\,dy\,dx$$

$$= k\int_0^a \left[\int_0^{a-x} yx^2\,dy + \int_0^{a-x} y^3\,dy\right] dx$$

$$= k\int_0^a \left[\frac{y^2x^2}{2} + \frac{y^4}{4}\right]_0^{a-x} dx$$

$$= k\int_0^a \left[\frac{(a-x)^2x^2}{2} + \frac{(a-x)^4}{4}\right] dx$$

$$= k \int_0^a \left[\frac{(a^2 - 2ax + x^2)x^2}{2} + \frac{(a-x)^2(a-x)^2}{4} \right] dx$$

$$= k \int_0^a \frac{x^2a^2 - 2ax^3 + x^4}{2} + \frac{(a^2 - 2ax + x^2).(a^2 - 2ax + x^2)}{4} \, dx$$

$$= k \int_0^a \left[\frac{x^2a^2 - 2ax^3 + x^4}{2} + \frac{a^4 - 2a^3x + x^2a^2 - 2a^3x + 4a^2x^2 - 2ax^3 + x^2a^2 - 2ax^3 + x^4}{4} \right] dx$$

$$= k \int_0^a \left(\frac{x^2a^2 - 2ax^3 + x^4}{2} + \frac{a^4 - 4a^3x + 2x^2a^2 + 4a^2x^2 - 4ax^3 + x^4}{4} \right) dx$$

$$= k \int_0^a \left(\frac{2x^2a^2 - 4ax^3 + 2x^4 + a^4 - 4a^3x + 2x^2a^2 + 4a^2x^2 - 4ax^3 + x^4}{4} \right) dx$$

$$= \frac{k}{4} \left[\frac{2x^3a^2}{3} - \frac{4ax^4}{4} + \frac{2x^5}{5} + a^4x - \frac{4a^3x^2}{2} + \frac{2x^3a^2}{3} + \frac{4a^2x^3}{3} - \frac{4ax^4}{4} + \frac{x^5}{5} \right]_0^a$$

$$= \frac{k}{4} \left[\frac{2a^5}{3} - \frac{4a^5}{4} + \frac{2a^5}{5} + a^5 - \frac{4a^5}{2} + \frac{2a^5}{3} + \frac{4a^5}{3} - \frac{4a^5}{4} + \frac{a^5}{5} \right]$$

$= \frac{k}{4}\left[\frac{40a^5 - \cancel{60a^5} + 24a^5 + \cancel{160a^5} - 120a^5 + 40a^5 + 80a^5 - 60a^5 + 12a^5}{60}\right]$

$= \frac{k}{4}\left(\frac{196a^5 - 180a^5}{60}\right) = \frac{k}{4}\left(\frac{16a^5}{60}\right) = \frac{ka^5}{15}$

$\therefore \bar{y} = \frac{M_x}{m} = \frac{\frac{ka^5}{15}}{\frac{ka^4}{6}} = \frac{ka^5}{15} \times \frac{6}{ka^4}$

$\bar{y} = \frac{2a}{5}$

$\therefore$ Center of mass of the lamina $= \left(\frac{2a}{5}, \frac{2a}{5}\right)$

17. Ans Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 7.

D is bounded by $y = 1 - x^2$ and $y = 0$; $\rho(x,y) = ky$

$0 \le y \le 1 - x^2$

since $y = 0$:- $0 = 1 - x^2$

$x^2 = 1$

$x = \pm\sqrt{1}$

$x = +1$ or -1

$\therefore -1 \le x \le 1$

$\therefore I_x = \iint y^2 \rho(x,y)\, dA$

$I_y = \iint_D x^2 \rho(x,y)\, dA$ & $I_0 = I_x + I_y$