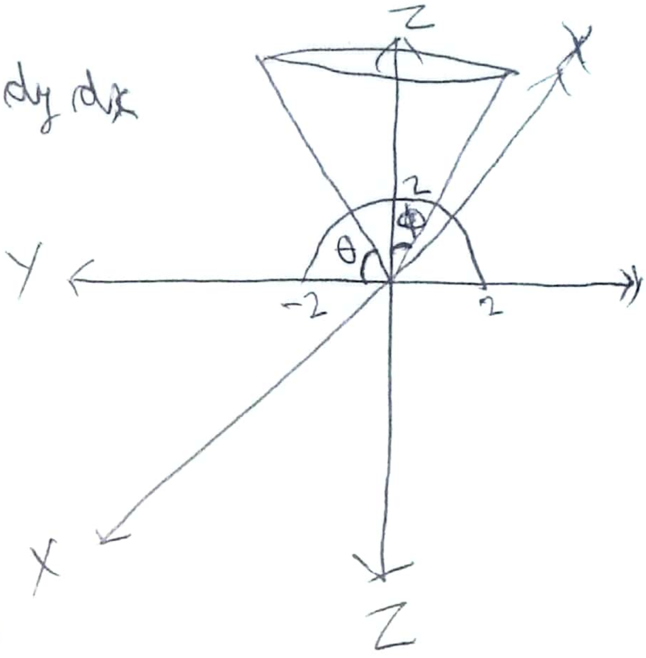
37. An Evaluate the integral by changing to spherical coordinates,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$$



In spherical coordinates,

let $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$dz\,dy\,dx = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$z = \sqrt{x^2+y^2} = \text{Cone}$

$z = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$

$\rho \cos\phi = \sqrt{\rho^2 \sin^2\phi (\cos^2\theta + \sin^2\theta)}$

$\cancel{\rho^2} \cos^2\phi = \cancel{\rho^2} \sin^2\phi$

$1 = \dfrac{\sin^{\cancel{2}}\phi}{\cos^{\cancel{2}}\phi}$

$1 = \tan\phi$

$\phi = \tan^{-1}(1)$

$\phi = \dfrac{\pi}{4}$

$\dfrac{3\cancel{\pi}}{4_1} \times \dfrac{\overset{45}{\cancel{180}}}{\pi}$

∴

$\boxed{\cancel{6.825 \times 10^4} \quad \cancel{2.278 \times 10^4} \quad 22}$

(57) $z = \sqrt{2-x^2-y^2}$ = hemisphere

$\rho\cos\phi = \sqrt{2-\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta}$

$\rho\cos\phi = \sqrt{2-\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)}$

$\rho^2\cos^2\phi = 2-\rho^2\sin^2\phi$

$\rho^2(\cos^2\phi+\sin^2\phi) = 2$

$\rho = \pm\sqrt{2}$

Our lower limit is gonna be 0,

$\therefore \rho = +\sqrt{2}$

$z = \sqrt{1-x^2}$ = hemisphere on y-axis

$\rho\cos\phi = \sqrt{1-\rho^2\sin^2\phi\cos^2\theta}$

$\cancel{\sqrt{2}}\cdot\frac{1}{\cancel{\sqrt{2}}} = \sqrt{1-\cancel{2}\times\frac{1}{\cancel{2}}\cos^2\theta}$

$1 = \cancel{\sqrt{}}\sin^2\theta$

$1 = \sin\theta$

$\theta = \sin^{-1}(1)$

$\theta = \frac{\pi}{2}$

$\therefore$ Region E in spherical coordinates:-

$\{(\theta,\phi,\rho) : 0 \le \theta \le \frac{\pi}{2}; 0 \le \phi \le \frac{\pi}{4}; 0 \le \rho \le \sqrt{2}\}$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \rho \sin\phi \cos\theta \cdot \rho \sin\phi \sin\theta \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \rho^4 \sin^3\phi \sin\theta \cos\theta \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \left(\frac{\rho^5}{5}\right)_0^{\sqrt{2}} \sin^3\phi \sin\theta \cos\theta \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{5} \int_0^{\frac{\pi}{4}} 4\sqrt{2} \sin^3\phi \sin\theta \cos\theta \, d\phi \, d\theta$$

$$= \frac{4\sqrt{2}}{5} \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{4}} \frac{3 \sin\phi}{4} \, d\phi - \int_0^{\frac{\pi}{4}} \frac{\sin 3\phi}{4} \, d\phi \right] \sin\theta \cos\theta \, d\theta$$

$$\sin 3\phi = 3\sin\phi - 4\sin^3\phi$$

$$\sin^3\phi = \frac{3\sin\phi - \sin 3\phi}{4}$$

$$= \frac{4\sqrt{2}}{5} \int_0^{\frac{\pi}{2}} \left[\frac{3}{4} (-\cos\phi)_0^{\frac{\pi}{4}} - \frac{1}{4} \left(-\frac{\cos 3\phi}{3}\right)_0^{\frac{\pi}{4}} \right] \sin\theta \cos\theta \, d\theta$$

$$= \frac{4\sqrt{2}}{5} \int_0^{\frac{\pi}{2}} -\frac{3}{4}\left(\frac{1}{\sqrt{2}} - \frac{1}{1}\right) + \frac{1}{12}\left(-\frac{1}{\sqrt{2}} - \frac{1}{1}\right) \sin\theta \cos\theta \, d\theta$$

$$= \frac{4\sqrt{2}}{5} \int_0^{\frac{\pi}{2}} -\frac{3}{4}\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right) + \frac{1}{12}\left(\frac{-1-\sqrt{2}}{\sqrt{2}}\right) \sin\theta \cos\theta \, d\theta$$

$$= \frac{4\sqrt{2}}{5} \int_0^{\frac{\pi}{2}} \left(\frac{(-3+3\sqrt{2})^{\times 3}}{4\sqrt{2}} - \left(\frac{1+\sqrt{2}}{12\sqrt{2}}\right)\right) \sin\theta \cos\theta \, d\theta$$

$$= \frac{4\sqrt{2}}{5} \int_0^{\frac{\pi}{2}} \left(\frac{-9+9\sqrt{2}-1-\sqrt{2}}{12\sqrt{2}}\right) \sin\theta \cos\theta \, d\theta$$

$$= \frac{4\cancel{\sqrt{2}}}{5} \cdot \left(\frac{8\sqrt{2}-10}{\underset{3}{\cancel{12\sqrt{2}}}}\right) \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2\theta \, d\theta$$

$$= \frac{\cancel{2}(4\sqrt{2}-5)}{15} \cdot \frac{1}{\cancel{2}} \left(\frac{-\cos 2\theta}{2}\right)_0^{\frac{\pi}{2}}$$

$$\begin{array}{r|l} 4 & 4,\ 12 \\ 3 & 1,\ 3 \\ & 1,\ 1 \end{array}$$

$$= \frac{4\sqrt{2}-5}{30} \cdot (1 + 1)$$

$$= \frac{4\sqrt{2}-5}{15}$$