half of the sphere $x^2+y^2+z^2=1$.

[Hint :- Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2+y^2\le 1$, oriented downward, and $S_2 = S \cup S_1$.]

It is an open surface, so we can attach a disk to the bottom of the hemisphere and close it. The closed surface can be S_2 & the disk S_1.

$$\iint_S \vec{F}.d\vec{S} = \iint_{S_2} \vec{F}.d\vec{S} - \iint_{S_1} \vec{F}.d\vec{S}$$

$$\therefore \iint_{S_2} \vec{F}.d\vec{S} = \iint_{S_2} \text{div}\,\vec{F}.dV$$

$$\text{div}\,\vec{F} = \nabla.\vec{F}$$

$$= \frac{d}{dx}(z^2 x) + \frac{d}{dy}\left(\frac{y^3}{3} + \tan z\right) + \frac{d}{dz}(x^2 z + y^2)$$

$$= z^2 + y^2 + x^2$$

$$\therefore \iint_{S_2} \vec{F}.d\vec{S} = \iint_{S_2} x^2+y^2+z^2\, dV$$

Using spherical coordinates :-

$x = \rho \sin\phi \cos\theta$; $y = \rho \sin\phi \sin\theta$; $z = \rho \cos\phi$

$\therefore x^2+y^2+z^2 = \rho^2$; $\rho^2 = 1$; $\therefore \rho = \pm 1$

$\therefore \quad 0 \leq \rho \leq 1 \; ; \; 0 \leq \phi \leq \frac{\pi}{2} \; ; \; 0 \leq \theta \leq 2\pi$

θ is from 0 to 2π because we do one revolution around the hemisphere.

ϕ is from 0 to $\frac{\pi}{2}$ because ϕ can only go from 0 to π in spherical coordinates, but since we only have the top half of the hemisphere here $\therefore$ ϕ will only go to $\frac{\pi}{2}$.

$$\therefore \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho^2 \, dV$$

$dV = \rho^2 \sin\phi$ from Jacobian of the transformation equations.

$$\therefore \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\frac{\rho^5}{5}\right)_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{5}\left(-\cos\phi\right)_0^{\frac{\pi}{2}} \cdot (2\pi - 0)$$

$$= \frac{1}{5}\left(0 - (-1)\right) \cdot 2\pi$$

$$= \frac{2\pi}{5}$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot n \, d\vec{S}$$

S_1 is part of the plane $z = 0$ which is oriented downwards.

n is the normal & $n = 0\hat{i} + 0\hat{j} - 1\hat{k}$

no idea why the normal is $\langle 0, 0, -1 \rangle$.

Probably because S_1 is part of the plane $z = 0$ & it is oriented downwards.

For dS, we can use the surface area formula.

$$\therefore dS = \sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1} \; dA$$

z is 0;

$$\therefore dS = \sqrt{0^2 + 0^2 + 1} = \underline{\underline{1}} \, dA$$

$$\therefore \iint_{S_1} \left\{ z^2 x \hat{i} + \left(\frac{y^3}{3} + \tan z\right)\hat{j} + (x^2 z + y^2)\hat{k} \right\} \cdot (0\hat{i} + 0\hat{j} - 1\hat{k}) \, dA$$

$$= -\iint_{S_1} x^2 z + y^2 \, dA$$

since z is 0, $-\iint y^2 \, dA$

Using polar coordinates :-

$x = r\cos\theta \ , \ y = r\sin\theta \ , \ dA = r\,dr\,d\theta$

$x^2+y^2 \le 1$

$r^2 \le 1$

$r \le \pm 1$

$\therefore 0 \le r \le 1 \ ; \quad 0 \le \theta \le 2\pi$ (one revolution)

$\therefore -\int_0^{2\pi}\int_0^1 r^2\sin^2\theta \,.\, r\,dr\,d\theta$

$= -\int_0^{2\pi}\left(\frac{r^4}{4}\right)_0^1 \sin^2\theta\,d\theta$

$= -\frac{1}{4}\int_0^{2\pi}\sin^2\theta\,d\theta$

since, $\cos 2x = \cos^2 x - \sin^2 x$

$= (1-\sin^2 x) - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\therefore -\frac{1}{4}\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = -\frac{1}{4}\left[\frac{1}{2}.2\pi - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)_0^{2\pi}\right]$

$$= -\frac{1}{4}(\pi - 0) = -\frac{\pi}{4}$$ $\because$ $\sin 4\pi = 0$ & $\sin 0$ is also 0.

$$\therefore \iint_S \vec{F}.d\vec{S} = \iint_{S_2} \vec{F}.d\vec{S} - \iint_{S_1} \vec{F}.d\vec{S}$$

$$= \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{2\pi}{5}^{\times 4} + \frac{\pi}{4}^{\times 5}$$

$$= \frac{8\pi + 5\pi}{20} = \frac{13\pi}{20}$$