SUMMER 2021 — MA 227 — FINAL

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1. Part I

There are 6 problems in Part I, each worth 3 points. Place your answer on the line below the question. In Part I, there is no need to show your work, since only your answer on the answer line will be graded.

(1) Find the cross product (or vector product) of the vectors $\langle 1, -3, 0\rangle$ and $\langle -2, 0, 3\rangle$.

$$\begin{vmatrix} i & j & k \\ 1 & -3 & 0 \\ -2 & 0 & 3 \end{vmatrix} = i(-9) - j(3) + k(7)$$

$-9i - 3j + 7k$

(2) Find the gradient of the function $f(x, y, z) = \sqrt{xy^2 + yz^2}$.

$\nabla f = \langle$

$(xy^2+yz^2)^{\frac{1}{2}}$ $\quad \frac{1}{2}(xy^2+yz^2)^{-\frac{1}{2}} \cdot (y^2+yz^2)$

$\nabla f = \left\langle \frac{y^2+yz^2}{2\sqrt{xy^2+yz^2}}, \frac{2xy+z^2}{2\sqrt{xy^2+yz^2}}, \frac{yz}{\sqrt{xy^2+yz^2}} \right\rangle$

(3) Compute the Jacobian of the transformation $x = uv$, $y = u/v$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = \left(-\frac{u}{v} - \frac{u}{v}\right) = -\frac{2u}{v}$$

(4) Compute div F when $F = \langle y + z^2, y, y + x^2\rangle$.

div: $\nabla \cdot \vec{F}$

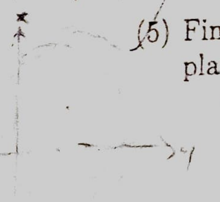
$\nabla F = \langle 0, 1, 0\rangle$

$\nabla \cdot \vec{F} = (0 + (1)(y) + 0) = y$

div $\vec{F} = y$

(5) Find a parametrization of a circle of radius 3 centered at the point $(3, 2)$ in the x-y-plane.

$r = 3$ $\quad$ cos $\quad$ sin



$\vec{r} = \langle 3\cos(\theta), 3\sin(\theta)\rangle$ cir centered @ 0,0

$\langle 3\cos(\theta) + 3, 3\sin(\theta) + 2\rangle$

$\vec{r} = \langle 3\cos(\theta) + 3, 3\sin(\theta) + 2\rangle$

(6) Evaluate $\iint_D xy \, dA$ where $D = \{(x, y) : 0 \le x \le 2, 0 \le y \le 3\}$.

$$\int_0^2 \int_0^3 xy \, dy \, dx = \frac{1}{2}\int_0^2 \left[xy^2\right]_{y=0}^{y=3} dx = \frac{9}{2}\int_0^2 x \, dx =$$

$$\frac{9}{4}\left[x^2\right]_0^2 = 9$$

9

2. Part II

There are 7 problems in Part II with the weight indicated. On Part II problems show all your work! Your work, as well as the answer, will be graded. Your solution must include enough detail to justify any conclusions you reach in answering the question.

(1) [12 points] A surface S is given parametrically by $\boldsymbol{r}(u, v) = \langle v+u, v-u, v+3\rangle$ where $0 \le u \le 1$ and $0 \le v \le u$. Evaluate the surface integral

$$\iint_S \underbrace{\langle x, y, z\rangle}_{\vec{F}} \cdot \underbrace{d\boldsymbol{S}}_{(r_u \times r_v)\, dA}.$$

First:

$\vec{r}_u = \langle 1, -1, 0\rangle$

$\vec{r}_v = \langle 1, 1, 1\rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \langle -1, 1, 2\rangle$

Parametrize F given u, v

$\vec{F}(\vec{r}(u,v)) = \langle v+u, v-u, v+3\rangle$

$\vec{F} \cdot d\vec{S} = \langle v+u, v-u, v+3\rangle \cdot \langle -1, 1, 2\rangle$

$= \cancel{-v}-u+\cancel{v}-u+2v+6 = 2v-2u+6$

Integrate:

$$\int_0^1\int_0^1 2v-2u+6\,dv\,du = \int_0^1 \left[v^2-2vu+6v\right]_0^1 du = \int_0^1 -2u+7\,du = \left[-(u^2)+7u\right]_0^1 = -1+7=6$$

$$\underbrace{\langle \overset{f_x}{2y + 3x}, \overset{f_y}{2x + y} \rangle}_{\vec{F}}$$

(2) [11 points] Find a potential function for the vector field $\langle 2y + 3x, 2x + y\rangle$.

Is $\vec{F}$ conservative?

$f_{xy} = 2 \quad f_{yx} = 2 \qquad \vec{F}$ is conservative ✓

Now we find f: $\int f_x \, dx = \int 2y + 3x \, dx = 2yx + \frac{3}{2}x^2 + g(y)$

$f_y = 2x + g'(y) = 2x + y$, so $y = g'(y)$

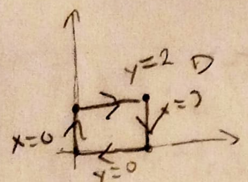
Now integrate $g'(y)$ to get $g(y)$: $\int y \, dy = \frac{1}{2}y^2 + k$

So f (our potential function) $= 2yx + \frac{3}{2}x^2 + \frac{1}{2}y^2 + k$

(3) [11 points] Evaluate the line integral

$$\int_C \left[\underbrace{(y^2 + \ln(1+\sqrt{x}))}_{P}\, dx + \overbrace{(\arctan(y^2 + ye^y) - 4xy)}^{Q}\, dy\right]$$

where C is the square with vertices $(0,0)$, $(0,2)$, $(3,0)$, and $(3,2)$. Assume the clockwise orientation for C.



negative integral

Green's Thm: $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dxdy = \int_C P dx + Q dy$

$Q_x - P_y$

$Q_x = \frac{\partial}{\partial x}\left(\arctan(y^2 + ye^y) - 4xy\right) = -4y$

$P_y = 2y$

$\iint_D -4y - 2y\, dA = \iint_D -6y\, dA = \int_0^2 \int_0^3 -6y\, dx dy = \int_0^2 -18y\, dy = -36$

-36

(4) [12 points] Suppose that you are climbing a hill whose shape is given by the graph of the function

$$z = 1000 - ax^2 - by^2, \quad a = 0.015, \quad b = 0.010$$

where the positive x-axis points east, the positive y-axis points north, and the positive z-axis points up. Now you are standing at the point with coordinates $(-20, 30, 985)$.

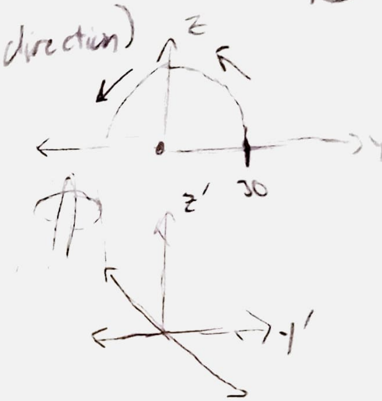
(a) If you were to walk due south would you ascend or descend? Find the rate of ascent or descent.

(b) There is a spring at this point. In which direction does its water flow?

greatest rate of negative change

$z = 1000 - 0.015x^2 - 0.010y^2$ at point $(-20, 30, 985)$ height

a) Due south: (aka in negative y direction)



$\nabla f = \langle 2ax, -2by \rangle$

$-2b(30) = -60 \quad y = 0$

function decreasing in positive y direction, increasing in negative y direction. So we would ascend.

b) Find direction of water flow: $1000 - ax^2 - by^2$ at $(-20, 30, 985)$

$f_x(-20, 30) = 2ax = a(-40)$

$f_y = -2by = b(-60)$

$$-\frac{\nabla F(x,y,z)}{|\nabla F(x,y,z)|} = \text{direction of water flow}$$

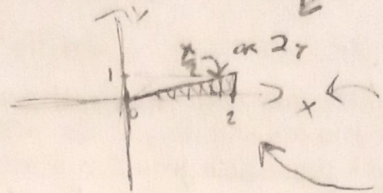
$$-\frac{\langle -40a, -60b \rangle}{\sqrt{40^2a^2 + 60^2b^2}} = \text{direction of water flow}$$

(5) [12 points] Evaluate the integral $\int_0^2 \int_{x/2}^1 e^{x/y}\, dy dx$.

Want to switch limits

$\frac{x}{2} = y$

$$\int_0^2 \int_{\frac{x}{2}}^1 e^{\frac{x}{y}}\, dy\, dx$$



$x = 2y$

$0 \leq y \leq 1$

$0 \leq x \leq 2y$

now integrate with new limits:

$$\int_0^1 \int_0^{2y} e^{\frac{x}{y}}\, dx\, dy = \int_0^1 y e^{\frac{x}{y}} \Big]_0^{2y} dy = \int_0^1 \left(y e^{\frac{2y}{y}} - y e^0\right) dy =$$

$$\int_0^1 y e^2 - y\, dy = \frac{1}{2}\left[y^2 e^2 - y^2\right]_0^1 = \frac{1}{2}\left(e^2 - 1\right)$$

(6) [13 points] Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 3y - 8z$ subject to the constraint $\underbrace{16x^2 + y^2 + 4z^2 = 25}_{g(x,y,z)}$.

$\nabla f = \lambda \nabla g$

$\nabla f = \langle 0, 3, -8 \rangle = \lambda \langle 32x, 2y, 8z \rangle$

$\langle 0, 3, -8 \rangle = \langle 32x\lambda, \lambda 2y, \lambda 8z \rangle$

$0 = 32x\lambda$ $\quad$ $3 = \lambda 2y$ $\quad$ $-8 = \lambda 8z$

$x = 0$ so λ not 0 here $\quad$ $\lambda \neq 0$ $\quad$ $-\frac{1}{\lambda} = z$

$\frac{3}{2\lambda} = y$

$16(0)^2 + \left(\frac{3}{2\lambda}\right)^2 + 4\left(\frac{-1}{\lambda}\right)^2 = 25$

$0 + \frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 25 \rightarrow \frac{9}{4} + 4 = 25\lambda^2$

$\frac{25}{4} = 25\lambda^2 \rightarrow \frac{1}{4} = \lambda^2 \rightarrow \pm\frac{1}{2} = \lambda$

plug lambda in to eq's

$\frac{3}{2(\frac{1}{2})} = y = 3$ $\quad$ $-\frac{1}{(\frac{1}{2})} = z = -2$ $\quad$ first point: $(0, 3, -2)$

Sec point: $(0, -3, 2)$

$f(0, 3, -2) = 9 - (-16) = 25$ max

$f(0, -3, 2) = -9 - 16 = -25$ min

Fg+g 5

(7) [11 points] Find the linear approximation to the surface given by $z = 3xe^{2xy}$ at the point $(1, 0, 3)$. (x_0, y_0, z_0)

$f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) =$

$L(x,y) = 3 + 3(x-1) + 6(y-0) = \cancel{3} + 3x - \cancel{3} + 6y = 3x + 6y$

$3(1) + 6(0) = 3 = z_0$

point is on the plane

$f_x = 3e^{2xy} + 6xye^{2xy}$ $\quad$ $f_y = 6x^2e^{2xy}$

$f_x(1,0) = 3e^0 + 0 = 3$ $\quad$ $f_y(1,0) = 6$