SPRING 2006 — MA 227 — FINAL

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1. Part I

There are 10 problems in Part I, each worth 4 points. Place your answer on the line below the question. In Part I, there is no need to show your work, since only your answer on the answer line will be graded.

(1) Find the cross product of the vectors $\langle 1, 2, -3\rangle$ and $\langle -1, 0, 1\rangle$.

$\langle 1, 2, -3\rangle \cdot \langle -1, 0, 1\rangle = -1 + 0 - 3$

-4

(2) Find the derivative of the vector function $\langle 1, 2\cos t, t\sin t\rangle$.

$\langle 0, -2\sin t, t\cos t + \sin t\rangle$

(3) A particle starts at the origin at time $t = 0$. Its velocity is given by $v(t) = \langle t, t^2, e^{-2t}\rangle$. What is the position vector of the particle at time T?

$\langle \frac{1}{2}t^2 + c, \frac{1}{3}t^3 + c, -\frac{1}{2}e^{-2t} + c\rangle$

$\langle \frac{1}{2}t^2, \frac{1}{3}t^3, -\frac{1}{2}e^{-2t} + \frac{1}{2}\rangle$

$c = 0$ $\quad c = 0$ $\quad c = \frac{1}{2}$

(4) Find the partial derivatives of the function $f(x, y) = xy\log(x + y)$.

$f_x = y\log(x+y) + \frac{xy}{(x+y)\log(x+y)}$

$f_y = x\log(x+y) + \frac{xy}{(x+y)\log(x+y)}$

(5) Find the gradient of $f(x, y, z) = \frac{x-y}{1+z^2}$.

$\nabla f = \langle \frac{-y}{1+z^2}, \frac{x}{1+z^2}, -\frac{2z(x-y)}{(1+z^2)^2}$

$\frac{\partial}{\partial z}(x-y)(1+z^2)^{-1} = -2z(x-y)(1+z^2)^{-2}$

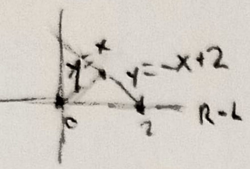
(6) Find the linearization $L(x, y)$ of $f(x, y) = x/y$ at the point $\langle 1, 1\rangle$.

$L(x,y) = x - y + 1$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f_x = \frac{1}{y}$ $\quad f(1,1) = 1$ $\quad f_x(1,1) = 1$ $\quad f_y(1,1) = -1$

$f_y = -\frac{x}{y^2}$ $\quad L(x,y) = 1 + (x-1) - (y-1)$



(7) Write down the iterated integral for $\iint_D 3y\,dA$ where D is the triangular region with vertices $(0,0)$, $(1,1)$ and $(2,0)$. You do not have to compute the integral.

$\int_0^2 \int_x^{-x+2} 3y\,dy\,dx$

(8) Evaluate $\iint_D dA$ where $D = \{(x,y) : 0 \le x^2 + y^2 \le 4, 0 \le y, 0 \le x\}$.

π

(9) Find the Jacobian of the transformation $x = 6u^2 - 2v^2$, $y = 2u^2 + 3v^2$.

$J = \begin{vmatrix} 12u & -4v \\ 4u & 6v \end{vmatrix} = 72uv + 16uv$

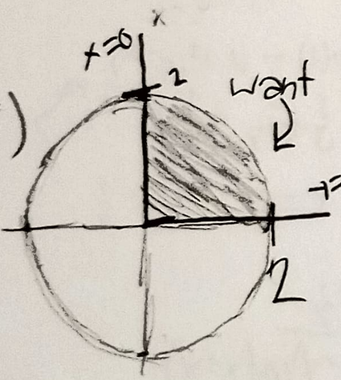
$88uv$

(10) Compute div **F** when $\mathbf{F}(x,y,z) = \langle \log(xy), 2\cos(xyz), -2e^{xy} \rangle$.

$\frac{y}{xy\log(xy)} - 2xz\sin(xyz)$

$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{y}{xy\log(xy)} - 2xz\sin(xyz) + 0$

(8)



want this area

$\iint_D dA = \int_0^2 \int_0^{\frac{\pi}{2}} r\,d\theta\,dr = \frac{\pi}{2}\int_0^2 r\,dr = \frac{\pi}{4}\left[r^2\right]_0^2 =$

$\frac{\pi}{4}(4) = \pi$

(2) Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^3 + xy^2 + 3x^2 + y^2.$$

$\nabla f = \langle 3x^2 + y^2 + 6x,\ 2xy + 2y \rangle$

$3x^2 + 6x + y^2 = 0$ $\quad 3(1) - 6 + y^2 = 0 \rightarrow y^2 - 3 = 0 \rightarrow y^2 = 3 \rightarrow y = \pm\sqrt{3}$

plug into f_y

$2xy + 2y = 0$

$2xy = -2y$

$x = -1$ plug into f_x

$2x\sqrt{3} + 2\sqrt{3} = 0$

$x = -1$ ✓

$-2x\sqrt{3} + (-2\sqrt{3}) = 0$

$x = -1$ ✓

points: $(-1, \pm\sqrt{3})$

$D = f_{xx} f_{yy} - (f_{xy})^2$ $\quad f_{xx} = 6x + 6 \quad f_{yy} = 2x + 2 \quad f_{xy} = 2y$

$D(-1, \sqrt{3}) = \underbrace{(-6+6) f_{yy}}_{0} - (f_{xy})^2$ so $(-1, \sqrt{3})$ is a saddle pt

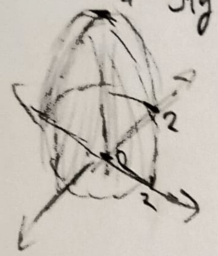
$D(-1, -\sqrt{3}) = \underbrace{(-6+6)}_{0} f_{yy} - (f_{xy})^2$ so $(-1, -\sqrt{3})$ is a saddle pt density function: function we will integrate over

(3) Find the mass m of the solid E lying below the paraboloid $x^2 + y^2 + z = 4$ but above the plane $z = 0$. The density function for E is $f(x, y, z) = \sqrt{x^2 + y^2}$. Argue why the center of mass is on the z-axis. Evaluate the z-component of the center of mass $\bar{z} = M_{xy}/m$.

The center of mass is on the z-axis b/c $z = 4 - x^2 - y^2$ is symmetrical across the zy and xz planes (see fig A)

below $x^2+y^2+z=4$ above $z=0$

fig A



$\rho^2 + z = 4$ so $0 \le \rho \le 2$

$z=0$

$\rho^2 + 0 = 4$

$\rho = 2$

$\rho^2 + z = 4$ so $0 \le z \le 4$

$0 + z = 4$

and we know $0 \le \theta \le 2\pi$

convert to cyl coords

$f(\rho, \theta, z) = \rho$

$$m = \int_0^2 \int_0^{2\pi} \int_0^4 \rho\,\rho \, dz\, d\theta\, d\rho = 4\int_0^2 \int_0^{2\pi} \rho^2 \, d\theta\, d\rho = 8\pi \int_0^2 \rho^2 \, d\rho = \frac{8}{3}\pi \left[\rho^3\right]_0^2 = \frac{64\pi}{3}$$

$\bar{z} = \frac{3}{64\pi} M_{xy}$ $\quad M_{xy} = \iiint_E z\, f(x,y,z)\, dV$

$$M_{xy} = \int_0^2 \int_0^4 \int_0^{2\pi} \rho^2 z \, d\theta\, dz\, d\rho = 2\pi \int_0^2 \int_0^4 \rho^2 z\, dz\, d\rho = \pi \int_0^2 \rho^2 \left[z^2\right]_{z=0}^{z=4} d\rho$$

$$16\pi \int_0^2 \rho^2 \, d\rho = \frac{16\pi}{3}\left[\rho^3\right]_0^2 = \frac{8(16)\pi}{3}$$

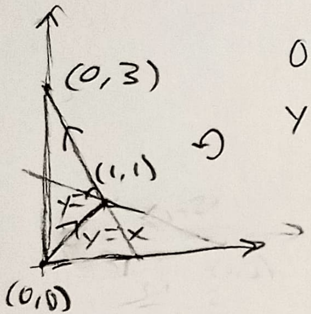
$$\bar{z} = \frac{3}{64\pi} \cdot \frac{8(16)\pi}{3} = 2\pi$$

Green's: $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$Q_x - P_y$

(5) Let C be the curve consisting of the sides of the triangle with vertices $(0,0)$, $(1,1)$, and $(0,3)$. Evaluate $\int_C (\underbrace{xy\,dx}_{P\,dx} + \underbrace{x^2\,dy}_{Q\,dy})$ by using Green's Theorem as well as by direct integration.

Greens:



$0 \le y \le 3$

$y \le x \le -\frac{y-3}{2}$

$$\int_0^3 \int_y^{-\frac{y-3}{2}} (2x - x)\,dx\,dy = \int_0^3 \int_y^{-\frac{y-3}{2}} x\,dx\,dy$$

$y = \frac{1-3}{1-0} = -2(x-0)$

$y = -2x + 3 \quad -\frac{y-3}{2}$