NAME :- SHREYAS SRINIVASA

1

BLAZER ID :- SSRINIVA

04/23/2022

MA 485

HW 11

Exercise 15.1

Ans:- (a) The probability that at least 5 accidents will occur next month using Markov's inequality is :-

$$P(X \geq a) \leq \frac{E(X)}{a}$$

$$P(X \geq 5) \leq \frac{2}{5}$$

(b) Using the Poisson distribution, the same probability that at least 5 accidents will occur next month is :-

$$P(X \geq 5) = 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3) - P(X=4)$$

$$= 0.0527$$

(c) Variance = 2 per month.

Thus, using Chebyshev's inequality, we have :-

$$Pr(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

This can be written as:-

$$P(X \geq 5) = P(X-2 \geq 5-2)$$
$$= P(X-2 \geq 3)$$

We can equate 3 and k x sigma and we get the value of k as:-

$$3 = k \times 2$$
$$\frac{3}{2} = k$$

From the value of k, we get the desired probability as:-

$$P(X \geq 5) \leq \frac{1}{k^2}$$

$$P(X \geq 5) \leq \frac{1}{(1.5)^2}$$

$$P(X \geq 5) \leq 0.4444$$

Exercise 15.2

Ans:- Let X be the IQ score for a student.

That is, $P(x > 140)$

$$P(x > 140) = \left(\frac{x-\mu}{\sigma} > \frac{140-110}{15}\right)$$

By using LLN :-

$1 - P(z < 2)$

$= 1 - 0.9973$
$= 0.0227$

Exercise 15.3

Ans:- Average weight of sample $= \frac{304}{150}$

$= 2.0267$

We are trying to find the probability that average weight ≥ 2.0267

(a) Mean of total weight $= 2 \times 150 = 300$ grams

Standard deviation of sample total $= 150 \times$ standard deviation

$= 150 \times 0.1$
$= 15$ grams

P (sample total ≥ 304) :-

$\mu = 300$, $\sigma = 15$

We need to compute $P(X \geq 304)$. The corresponding z-value needed to be computed is :-

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{304 - 300}{15}$$

$$= \underline{0.2667}$$

$\therefore$ We get that

$$P(x \geq 304) = P\left(z \geq \frac{304-300}{15}\right)$$

$$= P(z \geq 0.2667)$$

$$= 1 - 0.6051$$

$$= \underline{0.3949}$$

(b) Standard deviation of sample $= \dfrac{\text{Standard deviation}}{n^{0.5}}$

$$= \frac{0.1}{150^{0.5}}$$

$$= \underline{0.0082}$$

$\mu = 2$, $\sigma = 0.0082$

We need to compute $P(x \geq 2.0267)$. The correspon- -ding z-value needed to be computed is :-

$$Z = \frac{X - \mu}{\sigma} = \frac{2.0267 - 2}{0.0082}$$

$$= \underline{3.2561}$$

$\therefore$ We get that :-

$$P(X \geq 2.0267) = P\left(Z \geq \frac{2.0267 - 2}{0.0082}\right)$$

$$= P(Z \geq 3.2561)$$
$$= 1 - 0.9994$$
$$= \underline{0.0006}$$

<u>Exercise 15.4</u>

<u>Ans:</u> Given $Var(X) = \sigma_x^2 = 19$

$$\bar{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$\therefore E(\bar{X}_n) = \frac{E(X_1 + X_2 + \dots + X_n)}{n}$$

$$= \frac{E(x_1) + E(x_2) + \dots + E(x_n)}{n}$$

$\left[\text{Assuming } E(x_1) = E(x_2) \dots = E(x_n) = \mu\right]$

$$= \frac{\mu + \mu + \mu + \dots + \mu}{n}$$

$$= \frac{n\mu}{n} = E(X)$$

(6)

and $Var(\bar{X}_n) = Var\left(\frac{X_1+X_2+\dots+X_n}{n}\right)$

$$= \frac{Var(X_1+X_2+\dots+X_n)}{n^2}$$

$$= \frac{Var(X_1)+Var(X_2)+\dots+Var(X_n)}{n^2}$$

$$= \frac{1+1+\dots+1}{n^2}$$

$$= \frac{n}{n^2} \quad \left[\because Var(X_1) = Var(X_2) = \dots = Var(X_n) = 1\right]$$

$$= \frac{1}{n}$$

(a) By using Weak Law of Large number :-

$$P\left[\,|\bar{X}_n - E(X)| < \varepsilon\,\right] \geq 1 - \frac{Var(X_1+X_2+\dots+X_n)}{n^2\varepsilon^2}$$

$$= 1 - \frac{1+1+\dots+1}{n^2\varepsilon^2}$$

$$= 1 - \frac{1}{n\varepsilon^2}$$

(7)

AVQ, $\epsilon = 0.5$ and $n = 20$

$$\therefore P\left[|\bar{X}_n - E(X)| < 0.5\right] \geq 1 - \frac{1}{20 \times 0.5^2}$$

$$= 1 - 0.2$$

$$\Rightarrow P\left[|\bar{X}_n - E(X)| < 0.5\right] \geq 0.8$$

(b) By using Central Limit Theorem :-

$$\bar{X}_n \sim N\left(E(\bar{X}_n), Var(\bar{X}_n)\right)$$

where $E(\bar{X}_n) = E(X)$

$$Var(\bar{X}_n) = \frac{1}{n} = \frac{1}{20} = 0.05$$

$$\therefore P\left[|\bar{X}_n - E(X)| < 0.5\right]$$

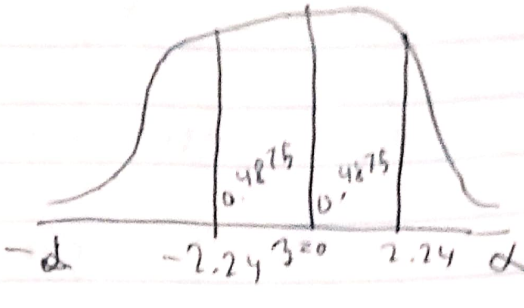
$$= P\left[-0.5 < \bar{X}_n - E(X) < 0.5\right]$$

$$= P\left[\frac{-0.5}{\sqrt{Var(\bar{X}_n)}} < z < \frac{0.5}{\sqrt{Var(\bar{X}_n)}}\right]$$

$$= P\left[\frac{-0.5}{\sqrt{0.05}} < z < \frac{0.5}{\sqrt{0.05}}\right]$$

$$= P[-2.24 < z < 2.24]$$

$$= 0.4875 + 0.4875$$
$$= 0.975$$



From the results (a) :-

$$P[|\bar{X}_n - E(X)| < 0.5] = 0.975 > 0.8$$

(c) By using Weak Law of Large Numbers:-

$$P[|\bar{X}_n - E(X)| < 0.5] \geq 1 - \frac{1}{n \times 0.5^2}$$

[∵ By using (a)]

A/Q, $1 - \dfrac{1}{n \times 0.5^2} = 0.90$

$\Rightarrow \dfrac{1}{n \times 0.25} = 0.10$

$\Rightarrow n = \dfrac{1}{0.25 \times 0.10} = 40$

∴ The required number of measurements is $\underline{\underline{40}}$.

(d) By using Central Limit Theorem :-

$$\bar{X}_n \sim N\left(E(\bar{X}_n), Var(\bar{X}_n)\right)$$

where $E(\bar{X}_n) = E(X)$

$Var(\bar{X}_n) = \frac{1}{n}$ [∵ n is not given]

$$\therefore P\left[|\bar{X}_n - E(x)| < 0.5\right]$$

$$= P\left[-0.5 < \bar{X}_n - E(x) < 0.5\right]$$

$$= P\left[\frac{-0.5}{\sqrt{Var(\bar{X}_n)}} < Z < \frac{0.5}{\sqrt{Var(\bar{X}_n)}}\right]$$

$$= P\left(\frac{-0.5}{\frac{1}{\sqrt{n}}} < Z < \frac{0.5}{\frac{1}{\sqrt{n}}}\right)$$

$$= P\left(-0.5\sqrt{n} < Z < 0.5\sqrt{n}\right)$$

$$= 2 \times P\left(0 < Z < 0.5\sqrt{n}\right)$$ (By symmetric property)

A/Q, $2 \times P\left(0 < Z < 0.5\sqrt{n}\right) = 0.90$

$$\Rightarrow P\left(0 < Z < 0.5\sqrt{n}\right) = 0.45$$

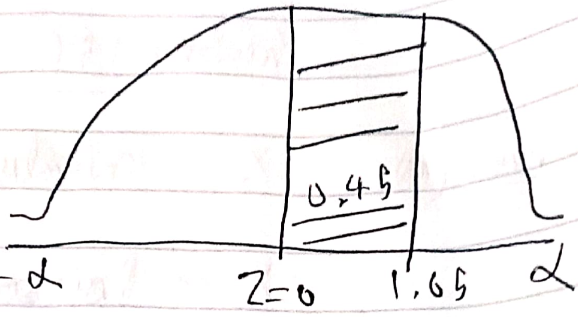
According to the Z-table :-

$$P\left(0 < Z < 1.65\right) = 0.45$$

$\therefore \; 0.5\sqrt{n} = 1.65$

$\Rightarrow \sqrt{n} = \dfrac{1.65}{0.5}$

$\sqrt{n} = 3.3$



$\therefore$

$n = 10.89$

$n \simeq 11$

$\therefore$ The required number of measurement is 11.

Exercise 15.5

Ans:- The full binomial probability formula with the binomial coefficient is:-

$$P(X) = \frac{n!}{X!\,(n-X)!} \cdot p^{X} \cdot (1-p)^{n-X}$$

substituting in $n = 50$, $p = 0.80$ & $X = 42$.

$$P(42) = \frac{50!}{42!\,(50-42)!} \times 0.80^{42} \times (1-0.80)^{50-42}$$

$= 0.1169218$

Exercise 15.6

Ans:- (a) $X =$ number of defectives

$X \sim$ Poisson $(\lambda = np = 0.0015 \times 500 = 0.75)$

$\therefore P(X=x) = \dfrac{e^{-0.75}(0.75)^x}{x!}$; $x = 0, 1, 2, \ldots$

Now, $P(X \leq 1)$

$= P(X=0) + P(X=1)$

$= \dfrac{e^{-0.75} \times 0.75^0}{0!} + \dfrac{0.75 \times e^{-0.75}}{1!}$

$= 0.8266$

(b) By normal approximation :-

$X \sim N(\mu = np = 0.75, \sigma = \sqrt{np(1-p)}$

$= \sqrt{0.75(1-0.0015)}$

$= \sqrt{(0.865376)^2}$

$= 0.8654$

$\Rightarrow X \sim N(\mu = 0.75, \sigma = 0.8654)$

To find $P(X \leq 1) = P\left(\dfrac{X-\mu}{\sigma} \leq \dfrac{1-0.75}{0.8654}\right)$

$= P(z \leq -1.73)$, $Z \sim N(0,1)$

$= 0.6141$ ($\because$ from z table).

We see poisson approximation probability is closer to 0.829 than normal approximation probability. So, poisson approximation is better when n is large and p is small.

Exercise 15.7

Ans:- X_i = outcome of ith die. $i = 1(1)85$

$X_i = 1, 2, 3, 4, 5, 6$ w.p. $\frac{1}{6}$

$\therefore E(X_i) = \sum x_i P(x_i) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= 3.5$

$E(X_i)^2 = \sum x_i^2 P(x_i) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6}$

$= \frac{91}{6}$

(13)

$\therefore Var(X_i) = E(X_i^2) - E^2(X_i)$

$= \frac{91}{6} - (3.5)^2$

$= \underline{\underline{2.91667}}$

Let $X = \sum_{i=1}^{85} X_i$ = sum of outcomes in 85 rolls.

$\therefore E(X) = 85\, E(X_i) = 85 \times 3.5 = \underline{\underline{297.5}}$

$V(X) = 85\, V(X_i) = \underline{\underline{247.91695}}$

$SD(X) = \sqrt{V(X)} = \sqrt{247.91695}$

$= \underline{\underline{15.7454}}$

By normal approximation,

$X \sim N(\mu = 297.5,\ \sigma = 15.7454)$

To find $P(X \geq 300)$

$= P\left(\frac{X-\mu}{\sigma} \geq \frac{300-297.5}{15.7454}\right)$

$= P(Z \geq 0.16),\quad Z \sim N(0,1)$

$= 1 - P(Z \leq 0.16)$

$= 1 - 0.5636$

$= \underline{0.4364}$ (from z table)

Exercise 15.8

Ans: $X =$ number of free throws in 35 attempts.

$\therefore X \sim \text{Binomial}(n=35, p=0.7)$

$\therefore E(X) = np = 35 \times 0.7 = 29.5$

$Var(X) = np(1-p) = 24.5(1-0.7) = 7.35$

$\therefore SD(X) = \sqrt{V(X)}$

$= \sqrt{7.35}$

$= 2.71109$

By normal approximation :-

$X \sim N(\mu = 24.5, \sigma = 2.71109)$

To find $P(X \leq 25)$

$= P\left(\frac{X-\mu}{\sigma} \leq \frac{25-24.5}{2.71109}\right)$

$= P(z \leq 0.18) = 0.5714$ (from z table)