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HW-10

Exercise 1.1

Ans: For independent events, $Cov(X, Y) = 0$

$Var(3X - 2Y + 7) = 3^2 Var(X) + 2^2 Var(Y)$

$= 9 \times 4 + 4 \times 3$
$= 48$

Exercise 1.2

Ans: $f(x) = (r-1) x^{-r}$, $x > 1$ elsewhere.
$= 0$

$E(x) = \int_1^{\infty} x f(x) dx = \int_1^{\infty} (r-1) x^{-r+1} dx$

$= \lim_{R \to \infty} \int_1^{R} (r-1) x^{-r+1} dx$

$= \lim_{R \to \infty} \frac{r-1}{-r+2} x^{-r+2} \Big|_1^R$

$\left[\because -r+2 < 0 \quad -r < -2 \quad r > 2 \right]$

$= \lim_{R \to \infty} \frac{r-1}{-r+2} (R^{-r+2} - 1)$

$$= \frac{r-1}{-r+2}(0-1)$$

$$= \frac{r-1}{r-2} \quad \text{if } r>2$$

$$E(X^2) = \int_1^\infty x^2 f(x)\,dx = \int_1^\infty (r-1)x^{-r+2}\,dx$$

$$= \lim_{R\to\infty} \int_1^R (r-1)x^{-r+2}\,dx$$

$$= \lim_{R\to\infty} \frac{r-1}{-r+3} x^{-r+3}\Big|_1^R$$

$$= \lim_{R\to\infty} \frac{r-1}{-r+3}\left(R^{-r+3}-1\right) \quad \left[\because -r+3<0 \atop r>3\right]$$

$$= \frac{r-1}{-r+3}(0-1) = \frac{r-1}{r-3} \quad \text{if } r>3.$$

$$Var(x) = E(X^2) - (E(x))^2$$

$$= \frac{r-1}{r-3} - \left(\frac{r-1}{r-2}\right)^2 ; \text{ if } r>3$$

$$= \frac{(r-1)(r-2)^2 - (r-1)^2(r-3)}{(r-3)(r-2)^3}$$

$$= \frac{r^3 - 5r^2 + 8r - 4 - (r^3 - 5r^2 + 7r - 3)}{(r-3)(r-2)^2}$$

$$= \frac{r-1}{(r-3)(r-2)^2}$$

<u>Exercise 1.3</u>

<u>Ans</u> Variance $= 8 - 3^2 = -1$

But, the variance can never be negative, since it is a squared value of the standard deviation.

$\therefore$ This is <u>impossible</u>. $E(X)=3$ & $E(X^2)=8$ cannot happen.

<u>Exercise 1.4</u>

<u>Ans</u> Let $X_1, \ldots, X_n$ are independent with $E(X_i) = \mu$ & $V(X_i) = \sigma^2 \ \forall \ i = 1, \ldots, n$.

Let $Y = X_1 + \ldots + X_n$

$$E(Y) = E(X_1 + \ldots + X_n)$$
$$= E(X_1) + \ldots + E(X_n)$$

$= \mu + \dots + \mu$

$E(Y) = n\mu$

$V(Y) = V(X_1 + \dots + X_n)$

$= V(X_1) + \dots + V(X_n)$ {∵ Independent random variables}

$= \sigma^2 + \dots + \sigma^2$

$V(Y) = n\sigma^2$

$E[(x_1 + \dots + x_n)^2] = E[y^2]$

$= [E(y)^2] + V(y)$

$= n^2\mu^2 + n\sigma^2$

$= n(n\mu^2 + \sigma^2)$

$E[(x_1 + \dots + x_n)^2] = n^2\mu^2 + n\sigma^2$

Exercise 2.1

Ans Let $f(x)$ be the pdf of the uniform distribution

$$f(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in (a, b) \\ 0 \end{cases} \longrightarrow (A)$$

Here, $a = -1$ & $b = 1$.

so, $f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in (-1, 1) \\ 0 \end{cases} \longrightarrow$ (B)

For uniform distribution in (Form A)

We know :-

$$\text{mean} = E(X) = \frac{b+a}{2}$$

$$\text{Variance} = V(X) = \frac{1}{12}(b-a)^2$$

r^{th} order raw moment.

$$\mu_r' = \int_a^b x^r \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^{r+1} - a^{r+1}}{r+1}\right)$$

For (B) :-

$$E(X) = \frac{1 + (-1)}{2} = 0$$

$$V(X) = \frac{1}{12}[1 - (-1)]^2 = \frac{1}{3}$$

$$\mu_3' = \frac{1}{2}\left[\frac{(1)^{3+1} - (-1)^{3+1}}{3+1}\right] = E(X^3)$$

$$= \frac{1}{8}[1 - 1] = 0$$

$$\text{Cov}(X, X^2) = \text{Cov}(X, Y).$$

$$= E(X \cdot X^2) - E(X)\,E(X^2)$$

$= E(X^3) - E(X)\,E(X^2)$

$= 0 - 0$

$= 0.$ [So, X & Y are independent].

Exercise 2.2

Ans: (i) $\mathrm{cov}(aX + bY, cX + dY)$

$= \mathrm{cov}(X, X)\,ac + ad\,\mathrm{cov}(X, Y) + bc\,\mathrm{cov}(Y, X) + bd\,\mathrm{cov}(Y, Y)$

(ii) $\mathrm{cov}(X, X) = \mathrm{var}(X)$

(iii) $\mathrm{cov}(X, Y) = \mathrm{cov}(Y, X)$

Using result (i) :-

$\mathrm{cov}(X+Y, X-Y) = \mathrm{cov}(X, X) - \mathrm{cov}(X, Y) + \mathrm{cov}(Y, X) - \mathrm{cov}(Y, Y)$

Using result (ii) & (iii) :-

$\mathrm{Var}(X) - \mathrm{cov}(X, Y) + \mathrm{cov}(X, Y) - \mathrm{Var}(Y)$

$= 5 - 7$

$= -2$

$\therefore \mathrm{cov}(X+Y, X-Y) = -2$

Exercise 2.3

Ans: Suppose X_1, X_2 & X_3 are independent.

Also, $V(X_i) = i \quad , i = 1, 2, 3.$

Now, $Cov(X_1 - X_2, X_2 + X_3) = Cov(X_1, X_2) + Cov(X_1, X_3) - Cov(X_2, X_2) - Cov(X_2, X_3)$

$= 0 + 0 - V(X_2) - 0$

$= -V(X_2)$

$= \underline{-2}$ [Since X_i's are independent. $Cov(X_i, X_j) = 0$, if $i \neq j$, $i = 1, 2, 3$, $j = 1, 2, 3$]

Exercise 2.4

Ans: $\rho(X_1 - X_2, X_2 + X_3) = \dfrac{Cov(X_1 - X_2, X_2 + X_3)}{\sqrt{V(X_1 - X_2)} \cdot \sqrt{V(X_2 + X_3)}}$

$= \dfrac{-2}{\sqrt{3} \cdot \sqrt{5}}$

$= -\dfrac{2}{\sqrt{15}}$

$= \underline{-0.516}$

[$\because V(X_1 - X_2) = V(X_1) + V(X_2) + Cov(X_1, X_2)$

$= V(X_1) + V(X_2)$

$= 1 + 2$

$= 3$

$V(X_2 + X_3) = V(X_2) + V(X_3) + 2\,Cov(X_2, X_3)$

$= 2 + 3 + 0 = \underline{5}$]

(8)

Exercise 2.5

Ans:- Let $mu = 0$.

Then, $Es(K) = E(S_n) = 0$

Let $n >= k$

Let $S = $ Sigma a

$Var(SK) = KS^2$

$Var(S_n) = nS^2$

Then, $E(Sx, S_n) = E\ x_1 + x_2 \dots x + k_1, x_1 + k_2 \dots x_k + x_{k+1} + x_n]$

$\Rightarrow E[x_1 + x_1 \dots x_k]^2 + E[x_1 + x_2 + \dots x_k]$

$(x_{k+1} \dots x_n) = k s^2$

Then $\rho = $ cov as (S_k, S_n) sqrt(S_k) Var$(S_n))$

$= \frac{KS^2}{Var(SK)\,Var(S_n)}$

$= \frac{KS^2}{\sqrt{S^2 K \times nS^2}}$

$= \frac{KS^2}{\sqrt{(Kn)}\,S^2}$

$= \sqrt{\frac{K}{n}}$

If $K \leq n$, then the solution is $\sqrt{n/K}$.