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MA-485

HW 9

Exercise 10.1

Ans:- We have $X_i \sim \text{Exp}(\lambda)$, $i = 1, 2, 3$

$V = \max\{X_1, X_2, X_3\}$ and $W = \min\{X_1, X_2, X_3\}$

We know that $f_{X_i}(x) = \lambda e^{-\lambda x}$, $x > 0$, $\lambda > 0$

$$P\left(V > \frac{3}{\lambda}\right) = 1 - P\left(V \leq \frac{3}{\lambda}\right)$$

$$P\left(V > \frac{3}{\lambda}\right) = 1 - \underbrace{P\left(X_1 \leq \frac{3}{\lambda}\right) P\left(X_2 \leq \frac{3}{\lambda}\right) \times P\left(X_3 < \frac{3}{\lambda}\right)}_{\text{are all independent}}$$

$$= 1 - \left(\underbrace{P\left(X_1 < \frac{3}{\lambda}\right)}_{\text{are all identical}}\right)^3$$

We also know that $P(X \le x) = 1 - e^{-\lambda x}$ for exponential random variable with parameter λ.

$$= 1 - \left(1 - e^{-\lambda x 3/\lambda}\right)^3$$

$$= 1 - \left(1 - e^{-3}\right)^3$$

Now, $P(W > 3/\lambda) = P\left(X_1 > \frac{3}{\lambda}, X_2 > \frac{3}{\lambda}, X_3 > \frac{3}{\lambda}\right)$

are all independent.

$$= P\left(X_1 > \frac{3}{\lambda}\right) P\left(X_2 > \frac{3}{\lambda}\right) P\left(X_3 > \frac{3}{\lambda}\right)$$

$\Rightarrow P(X_1 > x) = e^{-\lambda x}$ for exponential random variable with parameter λ.

$$P\left(W > \frac{3}{\lambda}\right) = \underbrace{\left(P\left(X_1 > \frac{3}{\lambda}\right)\right)^3}_{\text{are all identical}}$$

$$= \left(\left(e^{-\lambda \times \frac{3}{\lambda}}\right)\right)^3$$

$$= \left(e^{-3}\right)^3 = \underline{\underline{e^{-9}}}$$

Exercise 11.1

Ans, His expected winnings, $E(x) = \text{Exp}(x)$

$$= -1\left(\frac{20}{38}\right) + 1\left(\frac{18}{38}\right)$$

$$= \frac{-2}{38}$$

$$= -\underline{\underline{\$0.053}}$$

Exercise 11.2

Ans A player bets \$1 and if he wins he gets ~~\$~~ \$10

Let p be the probability of winning for this bet.

$$E(x) = \sum x P(x)$$
$$= (-1)(1-p) + 10p$$
$$= -1 + p + 10p$$
$$= 11p - 1$$

$\therefore$ $E(x) = 0$

$\Rightarrow$ $11p - 1 = 0$
$11p = 1$
$p = \frac{1}{11}$

Exercise 11.3

Ans: (a) Balls are numbered from 1 to 11.
Total number of balls = 11

X be the number on ball $P(X = N) = \frac{1}{11}$,

$n = 1, 2, \ldots 11$

Thus, $E(x) = \sum_{1}^{11} x\, P(x) = \frac{1}{11}\{1 + 2 + 3 + \ldots + 11\}$

$= 6$

(b) Balls are numbered from 0 to 10.
Total balls = 11.
Probability of every ball of selecting is $\frac{1}{11}$.

Thus, $E(x) = \sum_{0}^{10} x\, p(x) = \frac{1}{11}\{0+1+2+3+\dots+10\}$

$= \underline{\underline{5}}$

(c) Balls are numbered from 1 to 10.
Total balls = 10
$P(\text{Ball selected}) = \frac{1}{10}$

$E(x) = \sum x\, p(x) = \frac{1}{10}\{1+2+3+\dots+10\}$

$= \underline{\underline{5.5}}$

Exercise 11.4

Ans: X denotes winning amount of player.

So, from the question :- X can take 4 values
= \$0, \$30, \$800, \$1,200,000

$P(X=30) = \frac{4000}{\cancel{2000000}\ 500} = \frac{1}{500}$

$$P(X=800) = \frac{\overset{1}{\cancel{5\times 9}}}{\underset{4000}{\cancel{2000000}}} = \frac{1}{4000}$$

$$P(X=1200000) = \frac{1}{2000000}$$

$$P(X=0) = \frac{1995499}{2000000}$$

Expected winning amount $= E(x)$

$$= \sum x\, P(X=x)$$

$$= 0\times P(X=0) + 30\times P(X=30) + 800\times P(X=800) + 1200000\, P(X=1200000)$$

$$= 0 + 30\times\frac{1}{500} + 800\times\frac{1}{4000} + 1200000\times\frac{1}{2000000}$$

$$= 0.06 + 0.2 + 0.6$$

$$= \$0.86$$

Exercise 11.5

Ans:- X denotes the number of correct questions.

p = probability that a question is correct = $\frac{1}{4}$

N = number of problems = 10

Thus, $X \sim Bin(10, \frac{1}{4})$

Expected number of correct answers = $E(X)$

for $X \sim Bin(n, p)$,
$E(X) = np$

$= 10 \times \frac{1}{4}$

$= \underline{\underline{2.5}}$

Exercise 11.6

Ans. In a poisson random variable, we know:-

$\lambda = np$, here $n = 20000$, $p = \frac{1}{5000}$

$\lambda = \cancel{20000}^{4} \times \frac{1}{\cancel{5,000}}$

$= 4$

X denotes number of calls in an hour.

$X \sim Poisson(4)$

number of calls ~~in~~ expected in an hour

$= E(X)$

$\Rightarrow X \sim P(\lambda)$ then $E(X)$

$= \lambda$

$= \underline{\underline{4}}$

Exercise 11.7

Ans $f_X(x) = \begin{cases} kx(2-x) & , 0 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$

We know that,

$$\int_0^2 f_X(x)\,dx = 1$$

$$\int_0^2 kx(2-x)\,dx = 1$$

$$\int_0^2 (2kx - kx^2)\,dx = 1$$

$$\left(\frac{2kx^2}{2} - \frac{kx^3}{3}\right)_0^2 = 1$$

$$k(2)^2 - \frac{k}{3}(2^3) - 0 = 1$$

$$4k - \frac{8k}{3} = 1$$

$$\frac{12k - 8k}{3} = 1$$

$$\frac{4c}{3} = 1$$

$$c = \frac{3}{4}$$

$$E(x) = \int_0^2 x f_X(x)\,dx$$

$$= \int_0^2 x\,cx\,(2-x)\,dx$$

$$= \int_0^2 cx^2(2-x)\,dx$$

$$= \int_0^2 (2cx^2 - cx^3)\,dx$$

$$= \left(\frac{2cx^3}{3} - \frac{cx^4}{4}\right)_0^2$$

$$= \frac{2}{3}c(2)^3 - \frac{c}{4}(2)^4$$

$$= \frac{2}{3}\times 8c - \frac{c}{\cancel{4}_1}\times \cancel{16}^4$$

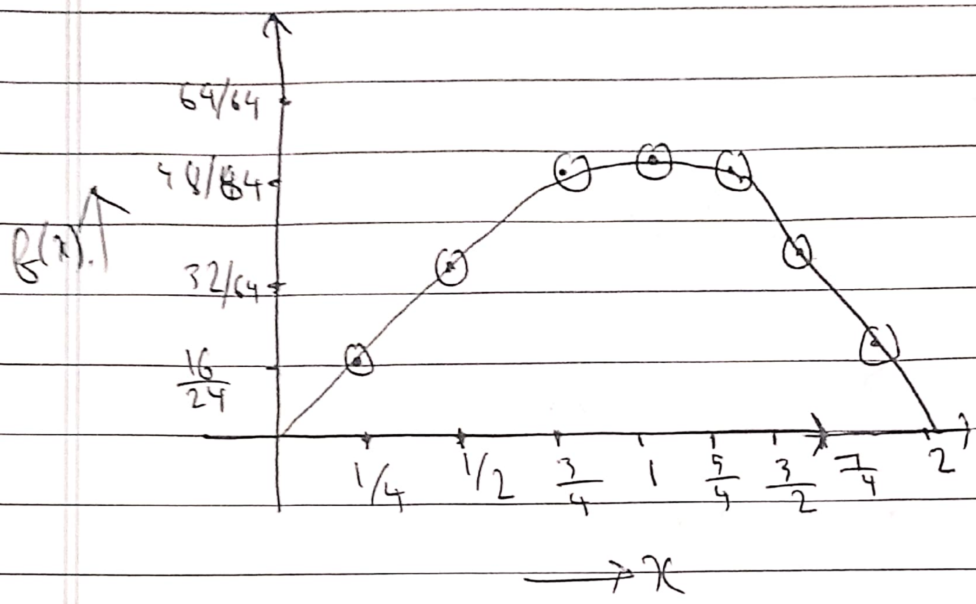
$$= \frac{16}{3}c - 4c$$

$$= \frac{16c-12c}{3} = \frac{4c}{3}$$

$$= \frac{4}{3}\times\frac{3}{4}$$

$$= 1$$

10



At $x = \frac{1}{4}$, $f\left(\frac{1}{4}\right) = \frac{3}{4}\left(2 - \frac{1}{4}\right)$

$= \frac{3}{4} \times \frac{1}{4}\left(\frac{7}{4}\right)$

$= \frac{21}{64}$

At $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = \frac{3}{2}\left(2 - \frac{1}{2}\right)$

$= \frac{3}{4} \times \frac{1}{2}\left(\frac{3}{2}\right)$

$= \frac{9}{16}$

$= \frac{36}{64}$

At $x = \frac{3}{4}$, $f\left(\frac{3}{4}\right) = \frac{3\cancel{x}}{4}\left(2 - \frac{3}{4}\right)$

$= \frac{3}{4} \times \frac{3}{4}\left(\frac{5}{4}\right) = \frac{45}{64}$

At $x = 1$, $f(1) = k(1)(2-1)$

$= \frac{3}{4}$

$= \frac{48}{64}$

Exercise 11.8

Ans: Given that :- Let $X_1, \dots X_n$ be independent and uniform random variables on $(0, 1)$.

Let us denote $V = \max\{X_1, \dots X_n\}$ and $W = \min\{X_1, \dots X_n\}$.

$\Rightarrow$ $f_X(x) = 1$; $0 < x < 1$
$F_X(x) = x$

Now, The cumulative distribution of $V = F_n(V)$

$= [F_X(V)]^n = V^n$

Thus, the probability density function of V is

$$f(V) = \frac{dF_n(V)}{dV} = nV^{n-1} ; 0 < v < 1$$

Hence, $E(V) = \int_{-\infty}^{\infty} v \cdot f(v)\, dv$

$$= \int_0^1 v\,(n v^{n-1})\, dv$$

$$= \left[\frac{n v^{n+1}}{n+1} \right]_0^1$$

$$\therefore E(V) = \frac{n}{(n+1)}$$

and, the cumulative distribution of $W = F_1(w)$

$$= [1 - F_X(w)]^n$$

$$= 1 - [1 - w]^n$$

Thus, the probability density function of W is :-

$$f(w) = \frac{dF_1(w)}{dw}$$

$$= n\,(1-w)^{n-1}, \quad 0 < w < 1$$

Hence, $E(W) = \int_{-\infty}^{\infty} w\, f(w)\, dw$

$$= \int_0^1 w\,[n(1-w)^{n-1}]\, dw$$

$= \dfrac{1}{(n+1)}$

$\therefore \quad E(W) = \dfrac{1}{(n+1)}$

Exercise 11.9

Ans: Let X be a random variable with standard normal distribution.

$$f_X(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2} & ; -\infty < x < \infty \\ 0 & ; \text{o.w} \end{cases}$$

Now, $E[|x|] = \int_{-\infty}^{\infty} |x| \, f_X(x) \, dx$

$$= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

Integration is positive in x, then we have:-

$$\therefore \quad E[|x|] = 2 \int_{0}^{\infty} x \frac{1}{2\pi} e^{-x^2/2} \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x^2/2} \, dx$$

Put $y = \frac{x^2}{2}$

$\Rightarrow x = \sqrt{2y}$

$dy = \frac{2x}{2} dx$

$\Rightarrow dx = \frac{1}{x} dx$ as $0 < x < \infty$ then $0 < y < \infty$

$$E[|x|] = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-y} dy$$

$$= \frac{2}{\sqrt{2\pi}} \left[-e^{-y}\right]_0^\infty$$

$$= \frac{2}{\sqrt{2\pi}} [0+1]$$

$$= \frac{2}{\sqrt{2\pi}}$$

$$E[|x|] = \sqrt{\frac{2}{\pi}}$$

$$\cong \sqrt{\frac{2}{3.14}}$$

$$E[|x|] = 0.7980$$