MA-485

HW8

NAME :- SHREYAS SRINIVASA
BLAZER ID :- SSRINIVA

Exercise 7.1

Ans :- X = exponential (1).

$$f(x) = \begin{cases} e^{-x} & , x \in [0, \infty] \\ 0 & , \text{otherwise}. \end{cases}$$

The range of x is $[0, \infty]$.

(a) $y = 2x - 1$

if $x = 0$, $y = 2(0) - 1$
$= 0 - 1$
$= -1$

If $x = \infty$, $y = 2(\infty) - 1$
$= \infty$

$\therefore$ Range of Y is $[-1, \infty]$

(b) $y = \frac{1}{x+1}$

if $x = 0 \Rightarrow y = \frac{1}{0+1} = \frac{1}{1} = 1$

if $x = \infty$, $\Rightarrow y = \frac{1}{\infty + 1} = 0$

Range of Y is $0 \leq y \leq 1$

$y = \frac{1}{X+1}$ with range $0 \le y \le 1$

Exercise 7.2

Ans: $Y = \frac{1}{X-1}$

$X = U(0,2)$

Range of Y y :-

If $X = 0$, then $Y = \frac{1}{0-1} = -1$

If $X = 2$, then $Y = \frac{1}{2-1} = 1$

$\therefore$ Range of y :- -1 to 1 or
$-1 \le y \le 1$

Exercise 7.3

Ans: Suppose X is a uniform random variable on the interval $(0,1)$.

(3)

$F(x) = \frac{1}{1-0} = 1 \quad , \quad 0 < x < 1$

Here $Y = X^3$

$\therefore X = Y^{1/3}$

$= g^{-1}(y)$

Finding $\left|\frac{dx}{dy}\right| = \left|\frac{1}{3} y^{-2/3}\right|$

$= \frac{1}{3 y^{2/3}}$

Range of Y if $X=0$, $Y=0$ & $X=1$ & $Y=1$.

$\therefore \quad 0 < x < 1 \Rightarrow 0 < y < 1$

$h(y) = f(x) \left|\frac{dx}{dy}\right|$

$\therefore h(y) = F[g^{-1}(y)] \left|\frac{dx}{dy}\right|$

$\therefore h(y) = 1 \times \frac{1}{3 y^{2/3}} \quad , \quad 0 < y < 1$

$\therefore h(y) = \dfrac{1}{3y^{2/3}}, \quad 0 < y < 1$

Exercise 7.4

Ans: $0 < x < 1 \Rightarrow 0 < 4x < 4 \Rightarrow 0-7 < 4x-7 < 4-7$

$\Rightarrow -7 < 4x-7 < -3$

Range $(Y) = (-7, -3)$

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \begin{cases} \int_{-\infty}^{x} 0\,dx & ; \text{ if } x < 0 \\ \int_{0}^{x} 1\,dx & ; \text{ if } 0 \le x < 1 \\ \int_{0}^{1} 1\,dx & ; \text{ if } x \ge 1 \end{cases}$$

$$F_x(x) = \begin{cases} 1 ; & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & ; \text{ if } x < 0 \\ x & ; \text{ if } 0 \le x < 1 \\ 1 & ; \text{ if } x \ge 1 \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(4x - 7 \le y)$$

$$= P\left(x \le \frac{y+7}{4}\right)$$

$$F_X\left(\frac{y+7}{4}\right) = \begin{cases} 0 & ; \text{ if } \frac{y+7}{4} < 0 \\ \frac{y+7}{4} & , \text{ if } 0 \le \frac{y+7}{4} < 1 \\ 1 & , \text{ if } \frac{y+7}{4} \ge 1 \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } y \le -7 \\ \frac{y+7}{4} & , \text{ if } -7 \le y < -3 \\ 1 & , \text{ if } y \ge -3 \end{cases}$$

$$F_y(y) = F'_y(y) = \begin{cases} 0 & ; \text{ if } y < -7 \\ 1/4 & ; \text{ if } -7 < y < -3 \\ 0 & ; \text{ if } y > -3 \end{cases}$$

$$= \begin{cases} 1/4 & ; \text{ if } -7 < y < -3 \\ 0 & ; \text{ otherwise} \end{cases}$$

Yes, the variable Y is familiar and $Y \sim U(-7,-3)$

Exercise 7.5

Ans: Given $X \sim U(-2,2)$

so, $f_X(x) = \begin{cases} \frac{1}{4} & , -2 < x < 2 \\ 0 & , \text{ otherwise} \end{cases}$

Then ~~the~~ pdf of Y:

$$F_Y(y) = P(Y \leq -1)$$

$$= P(|x| \leq y)$$

$$= P(-y < x < y)$$

$$= \int_{-y}^{y} \frac{1}{4}\, dx = \frac{2Y}{4} = \frac{Y}{2}, \quad 0 < Y < 2$$

Hence, $f_{\cancel{X}}(y) = \frac{\partial}{\partial y} F_Y(Y) = \frac{1}{2}, \quad 0 < y < 2$

Yes Y is a uniform R.V. in $(0, 2)$, i.e.,
$\cancel{X} \sim U(0, 2)$.

Exercise 7.6

Ans: $F_Y(y) = P(Y \leq y) = P(\tan(X) \leq y)$

This happens precisely when $P(Y \leq y) = P(X \leq \arctan(y))$

But as $X \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have:

$P(X \le \arctan(y)) = \frac{1}{2} + \frac{1}{\pi}\arctan(y)$, as the cumulative distribution of this variable is :-

$$F_x(x) = \frac{x + \frac{\pi}{2}}{\pi} = \frac{1}{2} + \frac{x}{\pi}$$

$\therefore$ $F_x(y) = \frac{1}{2} + \frac{1}{\pi}\arctan(y)$ is the required cumulative distribution of this variable.

PDF on differentiating :-

$$f_x(y) = \frac{\partial}{\partial y} F_x(y) = \frac{\partial}{\partial y}\left(\frac{1}{2} + \frac{1}{\pi}\arctan(y)\right)$$

i.e. $f_x(y) = \frac{1}{\pi(1+y^2)}$ is the distribution function.

Exercise 8.1

Ans: By the normal distribution table :-

(a) $P(-0.9 < Z < 2.65) = \phi(2.65) - \phi(-0.9)$

$= 0.811915$

(b) $P(Z > -0.29) = 1 - \phi(-0.29) = 0.614092$

(c) $P(-1.65 < Z < -0.18) = \phi(-0.18) - \phi(-1.65)$

$= 0.379105$

(d) $P(|Z| \leq 2.2) = \phi(2.2) - \phi(-2.2)$

$= 0.972193$

Exercise 8.2

Ans:- Given :- $X \sim N(3,9)$ i.e. $\mu = 3$, $\sigma = 3$

(a) $P(2 < X < 5) = P\left[\frac{2-3}{3} < \frac{X-\mu}{\sigma} < \frac{5-3}{3}\right]$

$= P[-0.3333 < Z < 0.6667]$

$= P[Z < 0.6667] - P[Z < -0.3333]$

$= 0.7475 - 0.3694$

$= 0.3781$

(b) $P[X > 0] = 1 - P[X < 0]$

$$= 1 - P\left[\frac{X-\mu}{6} < \frac{0-3}{3}\right]$$

$$= 1 - P[Z < -1]$$

$$= 1 - 0.1587$$

$$= 0.8413$$

(c) $P[|X-3| > 6] = P[X-3 < -6] + P[X-3 > 6]$

$$= P[X < -6+3] + P[X > 6+3]$$

$$= P[X < -3] + P[X > 9]$$

$$= P\left[\frac{X-\mu}{6} < \frac{-3-3}{3}\right]$$

$$+ P\left[\frac{X-\mu}{6} > \frac{9-3}{3}\right]$$

$$= P[Z < -2] + P[Z > 2]$$

$$= 0.0227 + 0.0227$$

$$= \underline{\underline{0.0454}}$$

Exercise 8.3

Ans: Given :- Mean, $\mu = 180$
Variance, $\sigma^2 = 900$

Standard deviation, $\sigma = 30$

$P(X > 240) = 1 - P(X < 240)$

$= 1 - P[(x-\mu)/\sigma < (240-180)/30]$

$= 1 - P(z < 2)$

$= 1 - 0.9772$

$= 0.0228 \times 100$

$= 2.28\%$

Exercise 8.4

Ans: $\mu = 270$

$\sigma^2 = 100$

$\sigma = \sqrt{100} = 10$

$P(245 < x < 295) = ?$

$z = \frac{x-\mu}{\sigma} = \frac{245-270}{10} = -2.5$

$z = \frac{x-\mu}{\sigma} = \frac{295-270}{10} = 2.5$

So, $P(245 < x < 295) = P(-2.5 < z < 2.5)$

$= P(z < 2.5) - [1 - P(z < 2.5)]$

$= 0.9938 - 0.0062$

$= 0.9876$

$\therefore$ Probability that he could nevertheless be the father of the child $= 1 - P(245 < x < 295)$

$= 1 - 0.9876$

$= 0.0124$

Exercise 8.5

Ans: $I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx dy$

$x = r\cos\theta$, $y = r\sin\theta$ with $0 < r < \infty$
$0 < \theta < 2\pi$

$\therefore I = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} . r \, dr d\theta$

$= [\theta]_0^{2\pi} \left[-e^{-r^2/2}\right]_0^{\infty}$

$= 2\pi(0+1)$

$= 2\pi$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \cdot \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = 2\pi$$

$$\Rightarrow \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right) = 1\times1$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1$$

Hence proved

Exercise 8.6

Ans (a) Let Z be a standard normal variable.

Probability density function is given by :-

$$f_Z(z) = \begin{cases} \frac{1}{y}\frac{1}{\sqrt{2\pi}} e^{-z^2/2}, & -\infty \le Z \le \infty \\ 0, & \text{otherwise} \end{cases}$$

Let $Y = e^x$

$P(Y \le y) = F_Y(y)$

$F_Y(y) = P[e^x \le y]$

$= P[x \le \ln y]$

$= \phi(\ln y)$

We get the density by differentiating :-

$f_Y(y) = \frac{1}{y} \frac{d}{dy} \phi(\ln y)$

$= \frac{1}{y} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(\ln y)^2}{2}\right\}$

$f_X(y) = \frac{1}{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} \; ; \; 0 < y < \infty$

which is the pdf of a log normal distribution with parameter $\mu = 0, \sigma = 1$

(b) Here, X is a standard normal variable, i.e. $X \sim N(0,1)$

$\therefore$ The pdf of X is :-

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \quad ; \quad -\infty < x < \infty$$

Let $\phi(x) = $ cdf of $X = P(X \leq x)$.

Let another continuous random variable Y be $Y = \sqrt{|X|}$

$\left[\because -\infty < x < \infty \Rightarrow 0 \leq |X| < \infty \Rightarrow 0 \leq \sqrt{|X|} < \infty \right.$

$\left[\Rightarrow 0 \leq Y < \infty \right.$ — Range (Y)

Now, the cdf of Y is :-

$$\Rightarrow F_Y(y) = P(Y \leq y)$$
$$= P(\sqrt{|X|} \leq y)$$
$$= P(|X| \leq y^2)$$
$$= P(-y^2 \leq X \leq y^2)$$
$$= \phi(y^2) - \phi(-y^2)$$

$\left[\because |a| \leq b \Rightarrow (-b \leq a \leq b) \right.$

$$= \phi(y^2) - [1 - \phi(y^2)]$$
$$= 2 \cdot \phi(y^2) - 1 \quad , \text{ from } 0 \leq y < \infty.$$

$\left[\because \phi(-k) = 1 - \phi(k) \quad (\forall k). \right.$

$\therefore$ We get :- $F_Y(y) = \begin{cases} 0 & ; y < 0 \\ [2 \cdot \phi(y^2) - 1] & ; y \geq 0 \end{cases}$

Hence the pdf of Y is

$$f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right]$$

$\left[\because \frac{d}{dx}[\Phi(x)] = f_X(x)\right]$

$$= \frac{d}{dy}\left[2.\Phi(y^2) - 1\right]$$

$$= 2.f_X(y^2).(2y)$$

$$= 4y\, f_X(y^2)$$

$\left[\text{Using the form of } f_X(x)\right]$

$$= 4y.\left[\frac{1}{\sqrt{2\pi}}.e^{-\frac{(y^2)^2}{2}}\right]$$

$$= 2.\frac{\sqrt{2}}{\sqrt{\pi}}.y.e^{-\frac{y^4}{2}}; \text{ from } y \geq 0.$$

$\therefore$ The pdf of Y is :-

$$\Rightarrow f_Y(y) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}}.y.e^{-\frac{y^4}{2}}; & \text{from } 0 \leq y < \infty \\ 0; & \text{otherwise.} \end{cases}$$