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FINAL EXAM

NOTE :- Answering easy questions first!

Problem 4

Ans :- X_1, X_2, X_3 are independent random variables.

$E(X_i) = 4$, $V(X_i) = 1$

$V = X_1 + X_3 + 5$, $W = X_1 - X_2 - 3X_3$

(a) $E(V) = E(X_1 + X_3 + 5)$

$= E(X_1) + E(X_3) + 5$

$= 4 + 4 + 5$

$= \underline{\underline{13}}$

$E(W) = E(X_1 - X_2 - 3X_3)$

$= E(X_1) - E(X_2) - 3E(X_3)$

$= 4 - 4 - (3 \times 4)$

$= \underline{\underline{-12}}$

(b) $V(V) = V(X_1 + X_3 + 5)$

$= V(X_i) + V(X_3) + 2\,Cov(X_1, X_3)$

$= V(X_i) + V(X_3) + 0$

[$\because$ X_i are indepen-dent]

$= 1 + 1$

$= 2$

$V(W) = V(X_1 - X_2 - 3X_3)$

$= V(X_1) + V(X_2) + 9V(X_3)$

$- 2\,Cov(X_1, X_2) - 6\,Cov(X_1, X_3)$

$+ 6\,Cov(X_2, X_3)$

$= V(X_1) + V(X_2) + 9V(X_3) + 0$

$= 1 + 1 + 9$

$= \underline{\underline{11}}$

(c) $Cov(V, W) = Cov(X_1 + X_3 + 5, X_1 - X_2 - 3X_3)$

$\Rightarrow Cov(X_1, X_1) - Cov(X_1, X_2)$

$- 3\,Cov(X_1, X_3)$

$- Cov(X_3, X_2)$

$- 3\,Cov(X_3, X_3)$

$+ 0$

$\Rightarrow V(X_1) - 3V(X_3) + 0$

$\Rightarrow 1 - 3$
$\Rightarrow -2$

(d) $$\rho(v,w) = \frac{\text{Cov}(v,w)}{\sqrt{v(v)} \cdot \sqrt{v(w)}}$$

$$= \frac{-2}{\sqrt{2}\sqrt{11}} = -0.4264$$

$$\therefore \rho(v,w) = -0.4264$$

Problem 2

Ans: Let X be a discrete random variable taking values $-4, 0, 12$ with probabilities :-

$$P(-4) = \frac{1}{2}, \quad P(0) = \frac{1}{6}, \quad P(12) = \frac{1}{3}$$

X	-4	0	12
$P(x)$	$1/2$	$1/6$	$1/3$

(i) $E(x) = \Sigma x \, P(x)$

$$= -\cancel{4}^{2} \times \frac{1}{\cancel{2}_1} + 0 \times \frac{1}{6} + \cancel{12}^{4} \times \frac{1}{\cancel{3}_1}$$

$$= -2 + 0 + 4$$

$$= 2$$

(ii) $\text{Var}(x) = E(x^2) - (E(x))^2$

$E(x^2) = \sum x^2 \cdot p(x)$

$E(x^2) = (-4)^2 \times \frac{1}{2} + 0 \times \frac{1}{6} + (12)^2 \times \frac{1}{3}$

$E(x^2) = \frac{16}{2} + 0 + \frac{144}{3}$

$= \frac{336}{6}$

$E(x^2) = \underline{\underline{56}}$

$Var(X) = 56 - (2)^2$

$Var(X) = \underline{\underline{52}}$

(iii) $\sigma_x = \sqrt{Var(X)}$

$\sigma_x = \sqrt{52}$

$\sigma_x = \underline{\underline{7.21}}$

<u>Problem 1</u>

<u>Ans:</u> $X_i \sim U(0,1)$, $i = 1, 2, 3$.

Let cdf of $V = \max\{X_1, X_2, X_3\}$ is G

& pdf be g.

Then, $G(V) = P(V \leq v)$

$= P(\max\{X_1, X_2, X_3\} \leq v)$

(5)

$$\Rightarrow P(X_1 \le v, X_2 \le v, X_3 \le v)$$

$$= (F_{X_1}(v))^3$$

pdf of V

$$g(v) = \frac{d}{dv} G(v)$$

$$= 3(F_X(v))^2 f_{X_1}(v)$$

$$= 3(v)^2 \cdot 1$$

$$= 3v^2$$

$$g(v) = \begin{cases} 3v^2 & ; 0 < v < 1 \\ 0 & ; \text{otherwise.} \end{cases}$$

$$P\left(v > \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} 3v^2 \, dv$$

$$= \left| v^3 \right|_{\frac{1}{2}}^{1}$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

and $W = \min\{X_1, X_2, X_3\}$

(6)

Let cdf of W is $H(w)$ and pdf $h(w)$.

Then, $H(w) = P(W < w)$

$$= 1 - P(\min(X_1, X_2, X_3) > w)$$

$$= 1 - (1 - F_{X_1}(w))^3$$

$$h(w) = \frac{d}{dw} H(w) = 3(1 - F_{X_1}(w))^2 f_{X_1}(x)$$

$$= 3(1-w)^2 \cdot 1$$

$$h(w) = \begin{cases} 3(1-w)^2 & ; \ 0 \leq w < 1 \\ 0 & ; \ \text{otherwise.} \end{cases}$$

$$P\left(W > \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} 3(1-w)^2 \, dw$$

$$= 3\left[\int_{\frac{1}{2}}^{1} (1 + w^2 - 2w) \, dw\right]$$

$$= 3\left[\frac{1}{2} + \frac{\left(1 - \frac{1}{8}\right)}{3} - \left(1 - \frac{1}{4}\right)\right]$$

$$= 3\left[\frac{1}{2} + \frac{7}{8 \times 3} - \frac{3}{4}\right]$$

$$= \frac{3}{2} + \frac{7}{8} - \frac{3}{4}$$

$$= \frac{12+7-18}{8} = \frac{1}{8}$$

Problem 3

Ans: $f_X(x) = \begin{cases} 1-\frac{x}{2} & ; 0 \le x \le 2 \\ 0 & ; \text{otherwise} \end{cases}$

$$E(X) = \int_0^2 \left(1-\frac{x}{2}\right) x\,dx$$

$$= \int_0^2 \left(x - \frac{x^2}{2}\right) dx$$

$$= \left[\frac{x^2}{2}\right]_0^2 - \left[\frac{x^3}{6}\right]_0^2$$

$$= 2 - \frac{4}{3}$$

$$E(X) = \frac{2}{3}$$

now, $V(X) = E(X^2) - (E(X))^2$

$$\therefore E(X^2) = \int_0^2 \left(1-\frac{x}{2}\right) x^2\,dx$$

$$= \int_0^2 \left(x^2 - \frac{x^3}{2}\right) dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 - \left[\frac{x^4}{8}\right]_0^2$$

$$\Rightarrow \frac{8}{3} - 2$$

$$\Rightarrow \frac{8-6}{3}$$

$$E(X^2) = \frac{2}{3}$$

Hence, $V(x) = E(x^2) - (E(x))^2$

$$= \frac{2}{3} - \left(\frac{2}{3}\right)^2$$

$$= \frac{2}{3} - \frac{4}{9}$$

$$= \frac{6-4}{9}$$

$$V(x) = \frac{2}{9}$$

Problem 5

Ans: $E(X) = 3$, $\sigma_X = 2$

(a) $P(X \geq a) \leq \frac{E(x)}{a}$

$$\Rightarrow P(X \geq 15) \leq \frac{3}{15}$$

$$\Rightarrow P(X \geq 15) \leq 0.2$$

(b) $P(|X-\mu| \geq k) \leq \dfrac{V(X)}{k^2}$

$\Rightarrow P(X \geq 15) = P((X-\mu) \geq 15-\mu) \leq \dfrac{V(X)}{k^2}$

$= P(X-\mu \geq 15-3) \leq \dfrac{4}{k^2}$

$= P(X-\mu \geq 12) \leq \dfrac{4}{(12)^2}$

$\Rightarrow P(X \geq 15) \leq 0.0277$

Problem 6 ~~[illegible]~~

~~[illegible]~~

Ans: For $n = 30 \geq 30$,

$\therefore$ we can apply the central limit theorem to approximate the distribution here as normal for the sum of 30 values, for which:-

$$E(S) = 30E(X)$$
$$= 30 \times 3.5$$
$$= 105$$

$Var(S) = 30\ Var(X)$
$= 30 \times 2.92$
$= 87.6$

$SD(S) = \sqrt{87.6}$
$= 9.3595$

Now applying the continuity correction, we get:

$P(89.5 < S < 110.5)$

~~From~~ Converting to standard normal variable $= P\left(\frac{(89.5 - 105)}{9.3595} < Z < \frac{(110.5 - 105)}{9.3595}\right)$

$= P(-1.66 < Z < 0.59)$

$= P(Z < 0.59) - P(Z < -1.66)$

From standard normal tables,
$= 0.7224 - 0.0485$
$= 0.6739$

$\therefore$ 0.6739 is the required probability.

Problem 7

Note:- Cannot figure it out.