NAME :- SHREYAS SRINIVASA
PALOMAR ID :- 012551187

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MATH-141
CALCULUS-2

EXAM - 3
SPRING 2021

1) Ans. Given integral :- $\int_2^\infty y e^{-3y}\, dy$

Using integration by parts, we get :-

$$y \int e^{-3y}\, dy - \int \left[\frac{d}{dy}(y) \int e^{-3y}\, dy \right] dy$$

$$\Rightarrow \frac{y e^{-3y}}{-3} - \int 1 \times \frac{e^{-3y}}{-3}\, dy$$

$$\Rightarrow \frac{-y e^{-3y}}{3} + \frac{1}{3} \int e^{-3y}\, dy$$

$$\Rightarrow -\frac{1}{3} y e^{-3y} - \frac{1}{9} e^{-3y} + C$$

$$\therefore \int_2^\infty y e^{-3y}\, dy = \left[-\frac{1}{3} y e^{-3y} - \frac{1}{9} e^{-3y} \right]_2^\infty$$

$$= \left[\frac{-e^{-3y}}{9} (3y+1) \right]_2^\infty$$

$$= 0 + \frac{e^{-6}}{9} \cdot (7)$$

$$= \frac{7}{9e^6}$$, which is finite.

$\therefore$ The given integral is convergent!

2) Ans: $\int_0^{\infty} \sin\theta\, e^{\cos\theta}\, d\theta$ is the given integral.

Let $\cos\theta = t$

$-\sin\theta\, d\theta = dt$

Now, $\int \sin\theta\, e^{\cos\theta}\, d\theta = -\int e^t\, dt$

$= -e^t$

$= -e^{\cos\theta}$

$\therefore \int_0^{\infty} \sin\theta\, e^{\cos\theta}\, d\theta = \left[-e^{\cos\theta}\right]_0^{\infty}$

$= -\left(e^{\cos\infty} - e^0\right)$

$= -(\alpha - 1)$ $\left[\because \cos\infty = \alpha, \text{ where } -1 \le \alpha \le 1 \text{ as } |\cos\theta| \le 1\right]$

$=$ finite

$\therefore \int_0^{\infty} \sin\theta\, e^{\cos\theta}\, d\theta$ is convergent.

3) Ans: Given series:- $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

Since it is a positive series, we will use ratio test.

So, $\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+1)^2} \times \frac{n^2}{e^n}$

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \cdot \frac{n^2}{(n+1)^2}$

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$$\Rightarrow \lim_{n\to\infty} e^{n+1-n} \cdot \frac{n^2}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$\Rightarrow \lim_{n\to\infty} e \cdot \frac{1}{\left(1+\frac{1}{n}\right)^2}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = e > 1$$

$\therefore \sum_{n=1}^{\infty} \frac{e^n}{n^2}$ is divergent.

4) Ans. $\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \sum_{n=1}^{\infty} u_n$

where $u_n = \frac{n}{n^2+1} = f(n)$

$$\therefore f(x) = \frac{x}{x^2+1}$$

$\therefore$ It is continuous & positive at every point, since every value of x gives a real & positive number.

Now, $f'(x) = \frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$

$$\Rightarrow \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$\therefore$ For every value of $x > 1$, $f'(x) < 0$, so $f(x)$ is decreasing.

Now, $u_n = \frac{n}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{1}{n\left(1+\frac{1}{n^2}\right)}$

$\therefore$ Let $\frac{1}{n} = v_n$, then

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n^2}}\right) = \frac{1}{1+0} = 1$ which is finite.

So, Σu_n & Σv_n can be convergent or divergent.

Now, $\Sigma v_n = \Sigma \frac{1}{n}$ is in the form of $\Sigma \frac{1}{n^p}$, $p = 1$.

$\therefore$ Σv_n is divergent.

Hence, Σu_n is also divergent.

5) Ans:- Given series $\Rightarrow \sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$

Using limit comparison test, with :-

$u_n = \frac{1+2^n}{1+3^n}$, $v_n = \frac{2^n}{3^n}$ [$\because$ For n sufficiently large n, we can ignore the addition of 1]

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1+2^n}{1+3^n}\right) \cdot \left(\frac{3^n}{2^n}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2^n} + \frac{2^n}{2^n}}{\frac{1}{3^n} + \frac{3^n}{3^n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2^n} + 1}{\frac{1}{3^n} + 1}$$

$$= \frac{0+1}{1+0}$$

$$= 1$$

Since the limit exists & $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2^n}{3^n}$ is a convergent geometric series $\left(r = \frac{2}{3} < 1\right)$,

$\therefore \sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ also converges

c) Ans. Using Alternating series test :-

given series :- $\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{n^2}{n^3+1}}_{a_n}$

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Then, $\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\cancel{1/n}^{\,0}}{1+\cancel{1/n^3}^{\,0}}$

$$= \frac{0}{1+0} = \underline{\underline{0}}$$

$\therefore$ The given series is convergent.
[$\because a_{n+1} < a_n$ for all n.]

__Note__ :- Using the simplest tests available to solve each problem because I don't have a lot of time left.

7) __Ans__ Given series :- $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$

$$\Rightarrow a_n = \frac{n(-3)^n}{4^{n-1}}$$

Now, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(-3)^{n+1}}{4^{n+1-1}} \times \frac{4^{n-1}}{(n)(-3)^n}\right|$

$$\Rightarrow \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right) \frac{(-3)\cdot\cancel{(-3)^n} \times \cancel{4^{n-1}}}{\cancel{4^{n-1}} \times 4 \times \cancel{(-3)^n}}\right|$$

$$\Rightarrow \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)\left(-\frac{3}{4}\right)\right| = \underline{\underline{\frac{3}{4}}} < 1$$

$\therefore$ By the Ratio test, Σa_n converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\therefore \sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$ is a convergent series.

8). Ans: Note :- Will get back to this later.

9). Ans: Given series :- $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n 4^n}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n \cdot 4^n}}{\frac{1}{(n+1)4^{n+1}}}\right|$$

$$\Rightarrow \lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right) \cdot 4\right|$$

$$\Rightarrow \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)4\right|$$

$$\Rightarrow (1+0)4$$

$$\Rightarrow 4$$

$\therefore$ Radius of convergence $= 4$

Interval of convergence $= (-2-4, -2+4) \cup \{-6\}$

$= (-6, 2) \cup \{-6\}$

$= [-6, 2)$

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8). Ans: given series :- $\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$

Let $\sum_{n=1}^{\infty} u_n$ be the given series,

Then $u_n = \left(\frac{-2n}{n+1}\right)^{5n}$

Now, $|u_n| = \left|\left(\frac{-2n}{n+1}\right)^{5n}\right|$

$$= \left(\frac{2n}{n+1}\right)^{5n}$$

$\therefore |u_n|^{\frac{1}{n}} = \left\{\left(\frac{2n}{n+1}\right)^{5n}\right\}^{1/n}$

$$= \left(\frac{2n}{n+1}\right)^{5}$$

Now, $\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{2n}{n+1}\right)^{5}$

$$= \lim_{n\to\infty}\left\{\frac{2\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)}\right\}^{5}$$

$$= \lim_{n\to\infty}\left(\frac{2}{1+\frac{1}{n}}\right)^{5}$$

$$= \left(\frac{2}{1+0}\right)^{5} \left[\because \lim_{n\to\infty}\frac{1}{n} = 0\right]$$

$$= 2^5$$

$\therefore \lim_{n\to\infty} |u_n|^{\frac{1}{n}} = 2^5 > 1 \; [\because 2^5 = 32 > 1]$

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Hence, by root test, $\sum_{n=1}^{\infty} u_n$ is divergent, i.e.

the series $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1} \right)^{4n}$ is divergent.

10) Ans No time! It's 2:35 PM. Don't want to lose 10 points!