Section 4.5B. Coordinates and Basis Overview and To-Do List

The goals of Section 4.5 are to begin to establish what we mean by **dimension of a vector space** (seen in the next Section 4.6) and to investigate **coordinate systems** in a vector space by making use of what's called a **basis** for a vector space.

Definition 0. A vector space V is said to be **finite dimensional** if a finite set of vectors spans V.
Examples include R^n, P_n, and M_{mn} . A **non-example** is the vector space P_∞ which consists of all possible polynomials defined on the real line. It couldn't have a finite subset S of polynomials that span the vector space since there would be a polynomial in the set S of maximum degree, say M. A polynomial of degree M+1 would not then be a linear combination of the vectors in S.

Definition 1. A **basis** for a finite dimensional vector space V is a set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_r\} \subseteq V$ which is both independent and spans V.

Standard Bases for Common Vector Spaces. These vector spaces have other bases than the ones below. However, the following are used most often and are referred to as the standard bases for these vector spaces.

1. $V = R^n$. Standard basis is $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$.
2. $V = P_n$. Standard basis is $\{1, x, x^2, ..., x^n\}$.
3. $V = M_{23}$. Standard basis is $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, ..., \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Some other Bases

1. $V = \{\vec{0}\}$. The zero vector space has only one vector. This one vector will span the space. Unfortunately, it isn't independent, so we are pressed to find a basis for this vector space. We could say it doesn't have one. However, in this special case, we are going to make an assumption that the empty set $\varnothing$ is the basis for the zero vector space. This may seem a little weird, but our "agreement" to do this will help us make some general statements later and not have to rule out the zero vector space as a special case.

2. Let $H = \{ p(x) \in P_2 : p(0) = 0 \}$. If $p(x) = a_0 + a_1 x + a_2 x^2$ and $p(0) = 0$, then $a_0 = 0$. Then a basis for H is $\{x, x^2\}$. This set will span H. The set is independent since one vector is not a multiple of the other.

3. $2x - y + 3z = 0$ is the equation of a plane (and vector space) through the origin. Since $y = 2x + 3z$, we found earlier in the overview for Section 4.2, Part 2, that
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 0y \\ 2x + 3y \\ 0x + 1z \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$ Hence, $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a spanning set for the plane. Notice that S is also an independent set because, due to the placement of the 1's and 0's, no vector in S is a multiple of the other. Thus S is a basis for the vector space which is the plane.

4. What happened in #3 above can be generalized to find a basis for a larger system of homogeneous Equations. Suppose we solve a system of equations that has 4 equations in 5 unknowns. and we find that that the solutions in parametric form turns out to be $x_1 = -s - t,\ x_2 = s,\ x_3 = -t,\ x_4 = 0,\ x_5 = t$.

Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ 1s+0t \\ 0s-t \\ 0s+0t \\ 0s+t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Thus, $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, and \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ form a spanning set for the solution space.

The two vectors are also independent due to the placement of the l's and 0's as in #3. The set containing the two vectors is then a basis for the solution space of the homogeneous system.

Thm 4.5.1 tells us that a vector in a vector space V with a basis B can be represented in one and only one way as a linear combination of the vectors in B. The proof uses Definition 1 of linear independence to show this.

Definition 2 in this section defines what is mean by the coordinate vector $(\vec{v})_S$ of a vector $\vec{v}$ in a vector space V with a basis S. This is a vector associated with $\vec{v}$ whose components are the coefficients in the unique linear combination of basic vectors that yields $\vec{v}$.
For this to make sense, we must consider our basis as **ordered,** i.e. the basis vectors listed in a particular order. See the **Remark** about this which follows Definition 2 in the text.

Examples 7-9 are good examples about coordinate vectors.

Read Section 4.5
Work Text Problems 1,2,4-6, 7(a,b), 8-9, 11(a,b), 14(a,b), 17, 19 (c,d)
Take Check Quiz 19