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1. $u = (0,-2,2)$ $v = (1,3,-1)$

(a) $(2,2,2) = a(0,-2,2) + b(1,3,-1)$

$b = 2$

$-2a + 3b = 2$

$2a - b = 2$

Put $b = 2$ in $2a - b = 2$ to get

$2a - 2 = 2$

$2a = 4$

$a = 2$

A solution $(2,2)$ exists for the above system.
This implies that $(2,2,2)$ is a linear combination of u and v.

(c) $(0,0,0) = a(0,-2,2) + b(1,3,-1)$

This implies that

$b = 0$

$-2a + 3b = 0$

$2a - b = 0$

Put $b = 0$ in $2a - b = 0$ to get

$2a - 0 = 0$

$2a = 0$

$a = 0$

The solution $(0,0)$ exists.
Therefore $(0,0,0)$ is ~~is~~ a linear combination of u and v

3. $A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$

This implies that:

$$4a + b = 6$$
$$-b + 2c = -8$$
$$-2a + 2b + c = -1$$
$$-2a + 3b + 4c = -8$$

$$\begin{vmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} 6 \\ -8 \\ -1 \\ -8 \end{vmatrix}$$

Apply $R_3 \to R_3 + \frac{1}{2}R_1$ and $R_4 \to R_4 + \frac{1}{2}R_1$

$$\begin{vmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 5/2 & 1 \\ 0 & 7/2 & 4 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} 6 \\ -8 \\ 2 \\ -5 \end{vmatrix}$$

Apply $R_3 \to R_3 + \frac{5}{2} R_2$ and $R_4 \to R_4 + \frac{7}{2} R_2$

$$\begin{vmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 11 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} 6 \\ -8 \\ -18 \\ -33 \end{vmatrix}$$

The result of these eqs.

$c = -3$

$b = 2$

$a = 1$

This implies that $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of A, B and C

6. (a) $p = -9 - 7x - 15x^3$

$$-9 - 7x - 15x^2 = k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2)$$

$$= x^3(4k_1 + 3k_2 + 5k_2) + x(k_1 - k_2 + 2k_3) + (2k_1 + k_2 + 3k_3)$$

$-15 = 4k_1 + 3k_2 + 5k_3 \;\ldots\ldots (1)$

$-7 = k_1 - k_2 + 2k_3 \;\ldots\ldots (2)$

$-9 = 2k_1 + k_2 + 3k_3 \;\ldots\ldots (3)$

The augmented matrix

$$\begin{bmatrix} 4 & 3 & 5 & -15 \\ 1 & -1 & 2 & -7 \\ 2 & 1 & 3 & -9 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \\ 2 & 1 & 3 & -9 \end{bmatrix}$$

$R_2 \to -4R_1 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 7 & -3 & 13 \\ 2 & 1 & 3 & -9 \end{bmatrix}$$

$R_3 \to R_3 + (-2R_1)$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 7 & -3 & 13 \\ 0 & 3 & -1 & 5 \end{bmatrix}$$

$R_2 \to \frac{1}{7} R_2$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 1 & -3/7 & 13/7 \\ 0 & 3 & -1 & 5 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 11/7 & -36/7 \\ 0 & 1 & -3/7 & 13/7 \\ 0 & 3 & -1 & 5 \end{bmatrix}$$

$R_3 \to R_3 + (-3R_2)$

$$\begin{bmatrix} 1 & 0 & 11/7 & -\frac{36}{7} \\ 0 & 1 & -3/7 & 13/7 \\ 0 & 0 & 2/7 & -4/7 \end{bmatrix}$$

$R_3 \to \frac{7}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 11/7 & -36/7 \\ 0 & 1 & -3/7 & 13/7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 \to R_1 + (-11/7) R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -3/7 & 13/7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 3/7 R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$k_1 = -2,\ k_2 = 1;\ k_3 = -2$

Thus, $p = -2p_1 + p_2 - 2p_3$

(b) $p = 6 + 11x + 6x^2$

$p = k_1 p_1 + k_2 p_2 + k_3 p_3$

$6 + 11x + 6x^2 = x^2(4k_1 + 3k_2 + 5k_2) + x(k_1 - k_2 + 2k_3)$
$+ (2k_1 + k_2 + 3k_3)$

$6 = 4k_1 + 3k_2 + 5k_3 \quad \dots\dots (1)$

$11 = k_1 - k_2 + 2k_3 \quad \dots\dots (2)$

$6 = 2k_1 + k_2 + 3k_3 \quad \dots\dots (3)$

The augmented matrix is

$$\begin{bmatrix} 4 & 3 & 5 & 6 \\ 1 & -1 & 2 & 11 \\ 2 & 1 & 3 & 6 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \\ 2 & 1 & 3 & 6 \end{bmatrix}$$

$R_2 \rightarrow -4R_1 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 7 & -3 & -38 \\ 2 & 1 & 3 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 + (-2R_1)$

$$\begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 7 & -3 & -38 \\ 0 & 3 & -1 & -16 \end{bmatrix}$$

$R_2 \rightarrow 1/7 R_2$

$$\begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 1 & -3/7 & -38/7 \\ 0 & 3 & -1 & -16 \end{bmatrix}$$

$R_2 \rightarrow R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 11/7 & 39/7 \\ 0 & 1 & -3/7 & -38/7 \\ 0 & 3 & -1 & -16 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 11/7 & 39/7 \\ 0 & 1 & -3/7 & -38/7 \\ 0 & 0 & 2/7 & 2/7 \end{bmatrix}$$

$R_3 \rightarrow 2/7 R_3$

$$\begin{bmatrix} 1 & 0 & 11/7 & 39/7 \\ 0 & 1 & -3/7 & -38/7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 11/7 R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & -3/7 & -38/7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow 3/7 R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$k_1 = 4, k_2 = -5, k_3 = 1$

$p = 4p_1 - 5p_2 + p_3$

7. (a) $(b_1, b_2, b_3) = k_1(2,2,2) + k_2(0,0,3) + k_3(0,1,1)$

$= (2k_1, 2k_1, 2k_1) + (0,0,3k_2) + (0, k_3, k_3) = (2k_1, 2k_1 + k_3, 2k_1 + 3k_2 + k_3)$

$2k_1 = b_1$

$2k_1 + k_3 = b_2$

$2k_1 + 3k_2 + k_3 = b_3$

$Ak = b$

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2(0-3) - 0(2-2) + 0(6-0) = -6 \neq 0$$

Therefore the system is consistent.

Hence, v_1, v_2 and v_3 span R^3

b) $(b_1, b_2, b_3) = k_1(2,-1,3) + k_2(4,1,2) + k_3(8,-1,8)$

$= (2k_1, -k_1, 3k_1) + (4k_2, k_2, 2k_2) + (8k_3, -k_3, 8k_3)$

$= (2k_1 + 4k_2 + 8k_3, -k_1 + k_2 - k_3, 3k_1 + 2k_2 + 8k_3)$

$$Ak = b$$

$2k_1 + 4k_2 + 8k_3 = b_1$
$-k_1 + k_2 - k_3 = b_2$
$3k_1 + 2k_2 + 8k_3 = b_3$

$$\begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 2(8+2) - 4(-8+3) + 8(-2-3) = 20 + 20 - 40 = 0$$

Therefore, the system is inconsistent.
Hence, v_1, v_2 and v_3 do not span the vector space R^3

8 (a) $(2, 3, -7, 3) = k_1(2, 1, 0, 3) + k_2(3, -1, 5, 2) + k_3(-1, 0, 2, 1)$
$= (2k_1 + 3k_2 - k_3, k_1 - k_2, 5k_2 + 2k_3, 3k_1 + 2k_2 + k_3)$

$2 = 2k_1 + 3k_2 - k_3$
$3 = k_1 - k_2$
$-7 = 5k_2 + 2k_3$
$3 = 3k_1 + 2k_2 + k_3$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -7 \\ 3 \end{bmatrix}$$

Reduce the system into echelon form

$2R_2 - R_1,\ 2R_4 - 3R_1$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 1 \\ 0 & 5 & 2 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -7 \\ 0 \end{bmatrix}$$

$R_3 + R_2,\ R_4 - R_2$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \\ -4 \end{bmatrix}$$

$3R_4 - 4R_3$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 2k_1 + 3k_2 - k_3 &= 2 \\ -5k_2 + k_3 &= 4 \\ 3k_3 &= -3 \end{aligned}\right\} \begin{aligned} & k_3 = -1 \\ &\Rightarrow -5k_2 - 1 = 4 \Rightarrow k_2 = -1 \\ &\Rightarrow 2k_1 + 3(-1) - (-1) = 2 \Rightarrow k_1 = 2 \end{aligned}$$

Therefore, $k_1 = 2$, $k_2 = -1$ and $k_3 = -1$

So, write $(2,3,-7,3) = 2(2,1,0,3) - 1(3,-1,5,2) - 1(-1,0,2,1)$. It lies in the span

(b) $(0,0,0,0) = k_1(2,1,0,3) + k_2(3,-1,5,2) + k_3(-1,0,2,1)$

$= (2k_1 + 3k_2 - k_3, k_1 - k_2, 5k_2 + 2k_3, 3k_1 + 2k_2 + k_3)$

$0 = 2k_1 + 3k_2 - k_3$

$0 = k_1 - k_2$

$0 = 5k_2 + 2k_3$

$0 = 3k_1 + 2k_2 + k_3$

$k_1 = 0$, $k_2 = 0$ and $k_3 = 0$

Therefore $(0,0,0,0)$ lies in the span $\{(2,1,0,3), (3,-1,5,2), (-1,0,2,1)\}$

(c) $(1,1,1,1) = k_1(2,1,0,3) + k_2(3,-1,5,2) + k_3(-1,0,2,1)$

$= (2k_1 + 3k_2 - k_3, k_1 - k_2, 5k_2 + 2k_3, 3k_1 + 2k_2 + k_3)$

$1 = 2k_1 + 3k_2 - k_3$

$1 = k_1 - k_2$

$1 = 5k_2 + 2k_3$

$1 = 3k_1 + 2k_2 + k_3$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$2R_2 - R_1,\ 2R_4 - 3R_1$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 1 \\ 0 & 5 & 2 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$R_3 + R_2,\ R_4 - R_2$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$$

$3R_4 - 4R_3$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -14 \end{bmatrix}$$

The last eq. is equivalent to $0 = -14$

Therefore $(1,1,1,1)$ does not belong to the span of v_1, v_2 and v_3

13. (a) $T_A(e_1) = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$T_A(e_2) = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$

Check if $u = (1,1,1)$ can be expressed as a linear combination of the vectors $T_A(e_1)$ and $T_A(e_2)$

Let
$$\begin{bmatrix}1\\1\\1\end{bmatrix} = a\begin{bmatrix}0\\1\\1\end{bmatrix} + b\begin{bmatrix}2\\-2\\0\end{bmatrix}$$

Now solve for a and b

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = a\begin{bmatrix}0\\1\\1\end{bmatrix} + b\begin{bmatrix}2\\-2\\0\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2b\\a-2b\\a\end{bmatrix}$$

There is no solution to this system of equations. Therefore the vector $u = (1,1,1)$ is not in the span of $\{Ta(e_1), Ta(e_2)\}$.

11 (a)
$$A = \begin{bmatrix}1 & 1 & 0 & 0\\0 & 1 & 1 & 0\\1 & 0 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ Apply $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix}1 & 1 & 0 & 0\\0 & 1 & 1 & 0\\0 & -1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ $R_3 \to R_3 + R_2$

$$A \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\Rightarrow \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Only (3) independent elements.
So, it does not span M_{22}

15) $u = (1, 0, -1, 0)$, $v = (0, 1, 0, -1)$

Span $W = \{Ax = 0;$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$= x_1 + x_2 + x_3 + x_4 = 0$

$x_1 + x_3 = 0$

$x_2 + x_4 = 0$

$$(1, 0, -1, 0) = \left\{ \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_3 = 0 \\ x_2 + x_4 = 0 \end{array} \right\} - (1)$$

Here $x_1 = 1$, $x_2 = 0$, $x_3 = -1$ and $x_4 = 0$

Satisfies the equation

Hence $u =$ Span w

$$(0,1,0,-1) = \left\{ \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_3 = 0 \\ x_2 + x_4 = 0, \end{array} \right\}$$

Here $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ and $x_4 = -1$
satisfies the equation.

hence set $\{u, v\} =$ span w

<u>True False Exercises</u>

(a) $k_1 v_1 + k_2 v_2 + \dots + k_r v_r$

Here, $v_1, v_2, v_3 \dots v_r \in V$ is the sum of the form $k_1 v_1 + k_2 v_2 + \dots + k_r v_r$ where $k_1, k_2, k_3 \dots k_r \in R$ are scalars
By definition it is clearly a linear combinatio
Hence, the given statement is true

(b) The span of a single vector in R^2 is a line only when this line passes through the origin, but is span is not passing through the origin, it is not necessarily a line
Hence, the given statement is false

(c) The span of two non-parallel vectors in R3 is a plane. But in case of parallel vectors, it may not necessarily be a plane Therefore, the given statement is incorrect. Hence, the given statement is false.

(d) The span of a non-empty set S of vectors in V is the smallest subspace of V that contains S. is true.

Consider a set k which is a vector subspace which contains every vector in S and is smaller than span S so that $k \subseteq span(S)$

Now, k is not just a set it's a vector space and it contains every member of S. $(k \geq S)$

So, it follows k contains every linear combination of vectors in S. But Span(S) is by definition the set of all linear combination of vector in S. So $k \geq Span(S)$, which is a contradiction to the assumption.

k is at least large as Span(S)

Hence, the given statement is true

(e) Any finite set of vectors in a vector space has a closed span under addition, which indicates that if any vectors are added together their total is also in the vector space.

Being closed under scalar multiplication indicates that all vectors in the vector space remain in the same vector space when multiplied by a scalar.

Addition and multiplication operations are always sensible they must be closed under Scalar multiplication.

Hence, the given statement is true.

(f) Two subsets of a vector space V that span the same subspace must not be equal.
Hence, the given statement is false.

(g) $P_1 = x - 1$

$P_2 = (x-1)^2 = x^2 - 2x + 1$

$P_3 = (x-1)^3 = x^3 - 3x^2 + 3x + 1$

Let $C_1P_1 + C_2P_2 + C_3P_3 = 0$

Sub.

$$C(x+1) + C_2(x-1)^2 + C_3(x-1)^3 = 0$$

$$[C_1 + C_2(x-1) + C_3(x-1)^2](x-1) = 0$$

Now, if $C_1 + C_2(x-1) + C_3(x-1)^2 = 0$

$$C_1 + C_2x - C_2 + C_3x^2 - 2C_3x + C_3 = 0$$

$$C_3x^2 + x(C_2 - 2C_3) + C_1 - C_2 + C_3 = 0$$

A polynomial is zero polynomial if and only if all its coefficients are 0.

Hence the given statement is false.