Name :- Shreyas Srinivasa

Blazer ID :- SSRINIVA

1) (a) $u_2 = (5, -10, -20)$

$u_2 = -5(-1, 2, 4)$

$u_2 = -5u_1$

Here u_2 is a vector which is a multiple of vector u_1 with scalar -5.

Therefore, the set $\{u_1, u_2\}$ is linearly dependent.

(b) We know that in n dimensional vector space more than n vectors form a linearly dependent set.

Here we have two dimensional vector space R^2 and 3 vectors.

Therefore set of 3 vectors in $R2$ must be a linearly dependent set.

(c) $p_2 = 6 - 4x + 2x^2$

$p_2 = 2(3 - 2x + x^2)$

$p_2 = 2(p_1)$

Here p_2 is a vector which is a multiple of vector p_1 with scalar 2.

Therefore, the set $\{p_1, p_2\}$ is linearly depende

(d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} = -1 \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$

$A = -1B = -B$ Here A is a vector which is a multiple of a vector B with scalar -1

Therefore, the set $\{A, B\}$ is linearly dependent.

2. (a) $a(-3,0,4) + b(5,-1,2) + c(1,1,3) = 0$

Then, $-3a + 5b + c = 0$

$-b + c = 0$

$4a + 2b + 3c = 0$

The augmented matrix

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right]$$

Gauss Jordan method

$R_1 \to -\frac{1}{3} R_1$, $R_3 \to -4R_1 + R_3$, $R_2 \to -1 \times R_1$

$$\left[\begin{array}{ccc|c} 0 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 26/3 & 13/3 & 0 \end{array}\right]$$

$R_3 \to R_3 - \frac{26}{3} R_2$

$$\left[\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right]$$

$R_3 \to 1/13 R_3$, $R_2 \to R_3 + R_2$, $R_1 \to 1/3 R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & -5/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_1 \to 5/3 R_2 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This can be written as a system of eq. $a=0, b=0, c=0$

Therefore, the vectors are linearly independent.

(b) $a(-2,0,1) + b(3,2,5) + c(6,-1,1) + d(7,0,-2) = 0$

Then,

$-2a + 3b + 6c + 7d = 0$

$2b - c = 0$

$a + 5b + c - 2d = 0$

The augmented matrix

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right]$$

$R_1 \to -1/2 R_1$

$R_3 \to -R_1 + R_3$

$R_2 \to 1/2 R_2$

$$\left[\begin{array}{cccc|c} 1 & -3/2 & -3 & -7/2 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 13/2 & 4 & 3/2 & 0 \end{array}\right] \quad R_3 \to -13/2 R_2 + R_3$$

$$\left[\begin{array}{cccc|c} 1 & -3/2 & -3 & -7/2 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 29/4 & 3/2 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to 4/29 R_3 \\ R_2 \to 1/2 R_3 + R_2 \\ R_1 \to 3R_3 + R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -3/2 & 0 & -167/58 & 0 \\ 0 & 1 & 0 & 3/29 & 0 \\ 0 & 0 & 1 & 6/29 & 0 \end{array}\right] \quad R_1 \to 3/2 R_2 + R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -79/29 & 0 \\ 0 & 1 & 0 & 3/29 & 0 \\ 0 & 0 & 1 & 6/29 & 0 \end{array}\right]$$

This can be written as a system of equations

$a - 79/29 = 0 \Rightarrow a = 79/29$

$b = -3/29$

$c = -6/29$

Therefore, the vectors are linearly dependent.

4a) $k_1(2-x+4x^2)+k_2(3+6x+2x^2)+k_3(2+10x-4x^2)=0$

$(2k_1+3k_2+2k_3)+(-k_1+6k_2+10k_3)x+(4k_1+2k_2-4k_3)x^2=0$

$2k_1+3k_2+2k_3=0$ $-k_1+6k_2+10k_3=0$

$4k_1+2k_2-4k_3=0$

$2k_1+k_2-2k_3=0$

Augmented matrix $\begin{bmatrix} 2 & 3 & 2 & 0 \\ -1 & 6 & 10 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}$

$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 2 & 1 & -2 \end{vmatrix} = 2(-10-10)-3(2-20)+2(-1-12)$
$=16$

Here, the determinent value of the matrix is non-zero and hence, the given set of vectors in P_2 are linearly independent.

(b) $k_1(1+3x+3x^2)+k_2(x+4x^2)+k_3(5+6x+3x^2)+k_4(7+2x-x^2)=0$

$(k_1+5k_3+7k_4)+(3k_1+k_2+6k_3+2k_4)x+(3k_1+4k_2+3k_3-k_4)x^2=0$

$k_1+5k_3+7k_4=0$, $3k_1+k_2+6k_3+2k_4=0$,

$3k_1+4k_2+3k_3-k_4=0$

Augmented matrix

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 3 & 1 & 6 & 2 & 0 \\ 3 & 4 & 3 & -1 & 0 \end{bmatrix}$$

Using Gauss elimination method

$R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & 9 & -19 & 0 \\ 3 & 4 & 3 & -1 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 4 & -12 & -22 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 0 & 24 & 54 & 0 \end{bmatrix}$$

that is eq. to

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 0 & 4 & 9 & 0 \end{bmatrix}$$

$\rightarrow k_1 + 5k_3 + 7k_4$

$\rightarrow k_2 - 9k_3 - 19k_4$

$\rightarrow 4k_3 + 9k_4 = 0$

$k_3 = -9/4\,k_4$, $k_2 = -5/4\,k_4$, $k_1 = 17/4\,k_4$, $k_4 = k_4$

Since one variable k_4 is expressed as a linear combination of other variables and not all scalar values are zero, the given set of vectors in P_2 are linearly dependent.

5. a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}.

Consider the linear combination equal to zero

$$a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$a + b = 0$

$2b + c = 0$

$a + 2b + 2c = 0$

$2a + b + c = 0$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

$R_3 \to R_3 - R_1, \; R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$a + b = 0$

$2b + c = 0$

$b + 2c = 0$

$3c = 0$

Hence, clearly $a = b = c = 0$.

Therefore the matrices are linearly independent

(b) $a\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

That is

$a = 0$

$b = 0$

$c = 0$

Hence, clearly $a = b = c = 0$

Therefore, the matrices are linearly independent.

9. (a) $v_1 = (0,3,1,-1)$, $v_2 = (6,0,5,1)$, $v_3 = (4,-7,1,3)$

$k_1(0,3,1,-1) + k_2(6,0,5,1) + k_3(4,-7,1,3) = 0$

$6k_2 + 4k_3 = 0$

$3k_1 - 7k_3 = 0$

$k_1 + 5k_2 + k_3 = 0$

$-k_1 + k_2 + 3k_3 = 0$

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + k_2 + 3k_3 &= 0 \\ \hline 6k_2 + 4k_3 &= 0 \end{aligned}$$

$6k_2 + 4k_3 = 0 \Rightarrow k_2 = -2/3\, k_3$

$3k_1 - 7k_3 = 0 \Rightarrow k_1 = 7/3\, k_3$

k_3 is a free variable

Therefore v_1, v_2 and v_3 are linearly dependent

(b) $v_1 = (0,3,1,-1)$ $v_2 = (6,0,5,1)$, $v_3 = (4,-7,1,3)$

Take v_1

Then $(0,3,1,-1) = k_1(6,0,5,1) + k_2(4,-7,1,3)$

That is,

$6k_1 + 4k_2 = 0$, $-7k_2 = 3$, $5k_1 + k_2 = 1$, $k_1 + 3k_2 = -1$

Put $k_2 = -3/7$ in $6k_1 + 4k_2$ to get.

$6k_1 + 4(-3/7) = 0 \quad k_1 = 2/7$

Therefore, the solution $(k_1, k_2) = (2/7, -3/7)$ satisfy the above system. Hence $v_1 = 2/7 v_2 - 3/7 v_3$

Take $v_2 = (6, 0, 5, 1)$

$(6, 0, 5, 1) = k_1(0, 3, 1, -1) + k_2(4, -7, 1, 3)$

That is;

$4k_2 = 6$, $3k_1 - 7k_2 = 0$, $k_1 + k_2 = 5$,
$-k_1 + 3k_2 = 1$

Put $k_2 = 3/2$ in $3k_1 - 7k_2 = 0$ to get

$3k_1 - 7 \cdot 3/2 = 0, \quad k_1 = 7/2$

Therefore, the solution $(k_1, k_2) = (7/2, 3/2)$ satisfies the above system. Hence $v_2 = 7/2 v_1 + 3/2 v_3$

Take $v_3 = (4, -7, 1, 3)$

Then $(4, -7, 1, 3) = k_1(0, 3, 1, -1) + k_2(6, 0, 5, 1)$

That is,

$6k_2 = 4$

$3k_1 = -7$

$k_1 + 5k_2 = 1$

$-k_1 + k_2 = 3$

$(k_1, k_2) = (-7/3, 2/3)$

This solution satisfies the system.

Hence $v_3 = -7/3\, v_1 + 2/3\, v_2$

13. (a)

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$T_A(x) = Ax$ $\quad u_1 = (1,2)$ and $u_2 = (-1,1)$

$$T_A(u_1) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T_A(u_2) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$a\begin{bmatrix} -1 \\ 4 \end{bmatrix} + b\begin{bmatrix} -2 \\ 2 \end{bmatrix} = 0$$

$-a - 2b = 0$

$4a + 2b = 0$

$$\begin{array}{r} -a - 2b = 0 \\ 4a + 2b = 0 \\ \hline 3a = 0 \end{array}$$

This implies that $a = b = 0$

Hence the set $\left\{\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$ is linearly independent

(b)

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

$u_1 = (1,2)$ and $u_2 = (-1,1)$

$$T_A(u_1) = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$T_A(u_2) = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$a\begin{bmatrix} -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -2 \\ 4 \end{bmatrix} = 0$$

$-a-2b=0$

$2a+4b=0$

$a=-2b$

As solution clearly depends on b a free variable which can take any value other than zero also.

Hence, the set $\left\{\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$ in not linearly independent.

16. (a) $c_1(3\sin^2 x)+c_2(2\cos^2 x)=6$

Let $c_1=2$ and $c_2=3$, then

$2(3\sin^2 x)+3(2\cos^2 x)=6\sin^2 x+6\cos^2 x$

$=6$

6 is a linear combination of vectors $3\sin^2 x$ and $2\cos^2 x$

Therefore the vectors are linearly dependent.

(b) $c_1 x+c_2\cos x=0$

This satisfies when $c_1=c_2=0$

Since, let $x=0$ in $(-\infty,\infty)$, then

$c_1(0)+c_2\cos(0)=0 \Rightarrow c_2=0$

Sub $c_2 = 0$. $c_1 x + (0)\cos x = 0 \Rightarrow c_1 x = 0$
$\Rightarrow c_1 = 0$

Since either of the vectors x, $\cos x$ cannot be written as a linear combination of each other, the vectors are not linearly dependent.

(d) $\cos 2x = c_1 \sin^2 x + c_2 \cos^2 x$

Let $c_1 = 1$ and $c_2 = -1$

$\cos 2x = (1)\cos^2 x + (-1)\sin^2 x$

$\cos 2x = \cos^2 x - \sin^2 x$

So $\cos 2x$ can be written as a linear combination of vectors $\sin^2 x$ and $\cos^2 x$

Therefore, the vectors $\cos 2x, \sin^2 x, \cos^2 x$ are linearly dependent.

(e)

$c_1(x^2 - 6x) + c_2(5) = (3-x)^2$ for all $x \in (-\infty, \infty)$, not all c_1, c_2 and c_3 is zero

Let $c_1 = 1$ and $c_2 = 9/5$, then

$(1)(x^2 - 6x) + (9/5)(5) = (3-x)^2$

$(x^2 - 6x) + 9 = (3-x)^2$

$x^2 - 6x + 9 = (3-x)^2$

$(3-x)^2 = (3-x)^2$

So $(3-x)^2$ is written as a linear combination of vectors x^2-6x and 5

Therefore, the vectors $(3-x)^2, x^2-6x, 5$ are linearly dependent.

True or False Exercises

(a) A set of single vectors is linearly independent if and only if that vector is not zero vector.
Then, the given statement does not hold for every case and is false.

(b.) $Y = x_1 A_1 + x_2 A_2 + \dots x_n A_n$
Here $x_1, x_2, \dots, x_n$ are scalars
By definition, for the vectors to be linearly independent, $Y=0$ if and only if $x_1, x_2, \dots x_n$ are zero
However, let x_1 not be equal to zero and the remaining $x_2, \dots x_n$ are zero then,
$Y = x_1 A_1 \neq 0$

Therefore, they can't be linearly independent and the given statement is true.

(c) Consider the vectors $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ which is

not a zero vector, but they are linearly dependent.
Hence, the given statement is false.

(d) Suppose a set of vectors $\{v_1, v_2, v_3\}$ are linearly independent.

$av_1 + bv_2 + cv_3 = 0$

This is only possible when, $a = b = c = 0$

Now suppose that,

$kav_1 + kbv_2 + kcv_3 = 0$

$k(av_1 + bv_2 + cv_3) = 0$

This implies that $a = b = c = 0$

Here, $\{kv_1, kv_2, kv_3\}$ are also linearly independent.
Hence, the given statement is true.

(e) Assume $c^1 v_1 + c^2 v_2 + c^n v_n = 0$

Take k to be the largest number for which c_k is not equal to zero. Therefore,

$c^1 v_1 + c^2 v_2 + \ldots\ldots + c^{k-1} v_{k-1} + c^k v_k = 0$

This implies that $k > 1$, which contradicts the assumption.

Therefore,

$$c^1 v_1 + c^2 v_2 + \ldots\ldots c^{k-1} v_{k-1} = -c^k v_k$$

$$-\frac{c^1}{c^k} v_1 - \frac{c^2}{c^k} v_2 - \ldots\ldots - \frac{c_{k-1}}{c_k} v_{k-1} = v_k$$

Thus v_k is expressed as a linear combination of the previous vectors

Hence, the given statement is true

(f) $M_{22} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ The matrix contains two 1's and two 0's, but is linearly dependent.

Hence, the given statement is false.

(g) Matrix w.r.t to the basis $\{1, x, x^2\}$

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_1 \to -\frac{1}{2}R_1$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Here, the rank of matrix A is 3.

Therefore, the given set of polynomial is linearly independent.

Hence, the given statement is true.

(h) f_1 and f_2 are said to be linearly independent, not dependent

Hence, the given statement is false.