Name :- Shreyas Srinivasa

Blazer ID :- SSRINIVA

Exercise 5.1

3) If X is an eigenvector corresponding to eigenvalue λ of the matrix A then, $AX = \lambda X$.

So,

$$\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$ By using matrix multiplication

$$\begin{bmatrix} 4+0+1 \\ 2+6+2 \\ 1+0+4 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ \lambda \end{bmatrix}$$

Equating both sides, $\lambda = 5$.

Therefore the eigenvalue corresponding to the eigenvector X is, $\lambda = 5$

4) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Ax = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(2)+1(-2)+1(-1) \\ 1(-1)+1(2)+1(-1) \\ 1(-1)+1(-1)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0x$$

Therefore the corresponding eigenvalue of the matrix is $\lambda = 0$

5. a) $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Consider $|\lambda I - A| = 0$

$$\left[\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\right] = 0$$

$$\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\right] = 0$$

$$\left[\begin{bmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{bmatrix}\right] = 0$$

$$(\lambda-1)(\lambda-3)-8=0$$

$$(\lambda-1)(\lambda-3)-8=0$$

$$\lambda^2-3\lambda-\lambda+3-8=0$$

$$\lambda^2-4\lambda-5=0$$

$$\lambda(\lambda-5)+(\lambda-5)=0$$

$$(\lambda+1)(\lambda-5)=0$$

$$\lambda=-1,\ \lambda=5$$

Therefore, the values of $\lambda = -1, 5$.

Sub. $\lambda = -1$ in $(\lambda I - A)X = 0$

$$\left(\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} (-1)-1 & -4 \\ -2 & (-1)-3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ This can be written as,

$-2x_1 - 4x_2 = 0$; $x_1 = -2x_2$

Substitute the value $x_2 = 1$ in the equation
$x_1 = -2x_2 \Rightarrow -2$

Therefore $X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an Eigen vector of the matrix A corresponding to $\lambda = -1$. $\{(-2,1)\}$ is a basis for the eigenspace corresponding to $\lambda = -1$

Sub $\lambda = 5$ in $(\lambda I - A)X = 0$

$$\left(\begin{bmatrix} (5)-1 & -4 \\ -2 & (5)-3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 4x_1 - 4x_2 = 0$$

$x_1 = x_2 \Rightarrow x_2 = 1,$
$x_1 = 1$

Therefore $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix A corresponding to $\lambda = 5$ are the non-zero vectors of the form $\begin{bmatrix} t \\ t \end{bmatrix}$. Therefore $\{(1,1)\}$ is a basis for the eigenspace.

b) $A = \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$. Consider $|\lambda I - A| = 0$

$$\left|\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} \lambda+2 & 7 \\ -1 & \lambda-2 \end{bmatrix}\right| = 0 \qquad \left|\begin{bmatrix} \lambda+2 & 7 \\ -1 & \lambda-2 \end{bmatrix}\right| = 0$$

$(\lambda+2)(\lambda-2) + 7 = 0$

$\lambda^2 - 2\lambda + 2\lambda - 4 + 7 = 0$

$\lambda^2 + 3 = 0$

The values of λ are not real, so the Eigen value does not exist.

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad |\lambda I - A| = 0$

$$\left|\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad \left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix}\right| = 0 \qquad (\lambda-1)^2 = 0$$

$(\lambda-1)(\lambda-1) = 0$. $\lambda = 1$

Hence Eigen values of the matrix A are $\lambda = 1, 1$

Sub $\lambda = 1$ in $(\lambda - A)X = 0$

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0x_1 + 0x_2 = 0$

Therefore vectors x_1, x_2 are non-zero

Therefore, take values $x_1 = 1, x_2 = 0$

Therefore $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an Eigen vector of the matrix A corresponding to $\lambda = 1$ ∴

$\{(1,0)\}$ is a basis for the eigenspace corresponding to $\lambda = 1$

Similarly $X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is also an Eigenvector of matrix A in the form $\begin{bmatrix} 0 \\ t \end{bmatrix}$

Therefore, $\{(1,0)\}, \{(0,1)\}$ ~~are~~ is the basis for the eigenspace corresponding to $\lambda = 1$

d) $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ $\quad |\lambda I - A| = 0$

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \right| = 0 \Rightarrow \left| \begin{bmatrix} \lambda-1 & 2 \\ 0 & \lambda-1 \end{bmatrix} \right| = 0$$

$(\lambda - 1)(\lambda - 1) = 0 \Rightarrow \lambda = 1, 1$

Hence Eigen values of the matrix A are $\lambda = 1, 1$

Sub $\lambda = 1$ in $(\lambda I - A)X = 0$

$(\lambda I - A)X = 0$

$$\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\left(\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \qquad 0x_1 + 2x_2 = 0$$

$$x_2 = 0$$

Then put the value $x_1 = k = 1$

$X = \begin{bmatrix}1\\ 0\end{bmatrix}$ is an Eigen vector of the matrix A corresponding to $\lambda = 1$

8) $A = \begin{bmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 4\end{bmatrix}$ $\quad |\lambda I - A| = 0$

$$\left|\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} - \begin{bmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 4\end{bmatrix}\right| = 0$$

$$\begin{vmatrix}\lambda - 1 & 0 & 2\\ 0 & \lambda & 0\\ 2 & 0 & \lambda - 4\end{vmatrix} = 0 \qquad (\lambda - 1)(\lambda(\lambda - 4)) + 2(-2\lambda) = 0$$

$(\lambda - 1)(\lambda(\lambda - 4)) + 2(-2\lambda) = 0$

$(\lambda - 1)(\lambda^2 - 4\lambda) - 4\lambda = 0$

$\lambda^3 - 4\lambda^2 - \lambda^2 + 4\lambda = 0$

$\lambda^3 - 4\lambda^2 - \lambda^2 = 0$

$$\lambda^2(\lambda-5)=0$$

$$\lambda = 0,0,5$$

$$(\lambda I - A)x = 0 \quad \ldots\ldots\ldots (1)$$

Put the value $\lambda=0$ in the eq. (1)

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} -x_1+2x_3=0 \\ 2x_1-4x_3=0 \end{matrix}$$

These give $x_1 = 2x_3$ Then $x_1 = 2t$

Choose $x_2 = s$, $x_3 = t$

There fore $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ s \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Therefore, the basis for the eigenspace for $\lambda=0$ is $\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ Put $\lambda=5$ in eq. (1)

So, $(\lambda I - A)x = 0$

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_1 \to \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{bmatrix} \; R_3 \to R_3 - R_1 \; \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives: $2x_1 + x_3 = 0$; $x_2 = 0$

Choose $x_t = -t$ where t is a parameter.

Then $x_3 = 2t$

The eigenvector of the matrix A is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ 2t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

The basis for the eigenspace for $\lambda = 5$ is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

9. $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda - 6 & -3 & 8 \\ 0 & \lambda + 2 & 0 \\ -1 & 0 & \lambda + 3 \end{vmatrix} = 0$$

$\lambda^3 - \lambda^2 - 16\lambda - 20 = 0$

$(\lambda + 2)(\lambda^2 - 3\lambda - 10) = 0$

$(\lambda + 2)^2(\lambda - 5) = 0$

$(\lambda + 2)^2 = 0, \quad \lambda - 5 = 0$

$\lambda + 2 = 0; \ \lambda + 2 = 0, \ \lambda - 5 = 0$

$\lambda = -2, -2, 5$

The eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = -2$ and $\lambda_3 = 5$

For $\lambda_1 = -2$

$(\lambda_1 I - A)x_1 = 0$

$(-2I - A)x_1 = 0$

$$\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 8x_1 - 3x_2 + 8x_3 = \\ 0 - x_1 + x_3 = 0 \end{array}$$

Put $x_3 = t \Rightarrow x_1 = t$

Then $-8t - 3x_2 + 8t = 0 \Rightarrow x_2 = 0$

The eigenvector corr. to. $\lambda_1 = -2$, is

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Therefore, the bases for the eigenspaces for $\lambda_1 = -2$ is $\{(1, 0, 1)\}$

For $\lambda_3 = 5$

$(\lambda_3 I - A)x_3 = 0$ $\quad (5I - A)x_3 = 0$

$$\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x_1 - 3x_2 + 8x_3 = 0 \\ 7x_2 = 0, \\ -x_1 + 8x_3 = 0 \end{array}$$

$\Rightarrow x_2 = 0, -x_1 + 8x_3 = 0$

Put $x_3 = t \Rightarrow x_1 = 8t$

The eigenvector for $\lambda_3 = 5$ is

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8t \\ 0 \\ t \end{bmatrix}$$

Therefore the bases for the eigenspaces for $\lambda_3 = 5$ is $\{(8,0,1)\}$

12. $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ $\quad |\lambda I - A| = 0$

$$\left|\begin{bmatrix} \lambda-1 & 3 & -3 \\ -3 & \lambda+5 & -3 \\ -6 & 6 & \lambda-4 \end{bmatrix}\right| = 0$$

On further solving

$(\lambda-1)((\lambda+5)(\lambda-4)+18) - 3(-3\lambda+12-18) - 3(-18+6\lambda+30) = 0$

$(\lambda-1)(\lambda^2+\lambda-2) + 9\lambda + 18 - 18\lambda - 36 = 0$

Solve for λ

$(\lambda-1)(\lambda^2+\lambda-2) - 9\lambda - 18 = 0$

$(\lambda-1)(\lambda-1)(\lambda+2) - 9(\lambda+2) = 0$

$(\lambda+2)((\lambda-1)^2 - 9) = 0$ On further solving

$(\lambda+2) = 0, \ (\lambda-1-3) = 0, \ (\lambda-1+3) = 0$

$\lambda = -2, 4, -2$

Since Eigen values are repeating, there will be two Eigen spaces.

Put the value $\lambda = -2$ in the eq. $(\lambda I - A)x = 0$

$$\left(\begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 3x_2 - 3x_3 = 0$

$-3x_1 + 3x_2 - 3x_3 = 0$

$-6x_1 + 6x_2 - 6x_3 = 0$

Solve the above eq. for the values x_1, x_2 and x_3

$x_2 = (x_1 + x_3)$,

For $x_1 = 0$, $x_3 = 1$

Therefore Eigen vector of A is $X = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

For $x_1 = 1, x_3 = 0$

$X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Put the value $\lambda = 4$ in $(\lambda I - A)x = 0$

$$\left(\begin{bmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + 3x_2 - 3x_3 = 0$

$-3x_1 + 9x_2 - 3x_3 = 0$

$-6x_1 + 6x_2 + 0x_3 = 0$

$\Rightarrow$ $-6x_1 + 6x_2 + 0x_3 = 0$

$6x_1 = 6x_2$

$x_1 = x_2$

On further solving

$3x_1 + 3x_2 - 3x_3 = 0 \Rightarrow x_3 = x_1 + x_2$

Fix $x_1 = x_2 = k$

Therefore Eigen vector of the matrix A is

$$X = \begin{bmatrix} k \\ k \\ 2k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Hence the basic for the Eigen value $\lambda = 4$ is

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

13. $$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$$ Consider $\det(\lambda I - A)$

$$= \det\left(\begin{bmatrix} \lambda-3 & 0 & 0 \\ -2 & \lambda-7 & 0 \\ 4 & 8 & \lambda-1 \end{bmatrix}\right) = (\lambda-3)(\lambda-7)(\lambda-1)$$

Hence, the characteristic equation is

$(\lambda-3)(\lambda-7)(\lambda-1) = 0$

14. $$A = \begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$ $R_1 \to R_1 - 8R_2 - 2R_3 - 3/7 R_4$

$$\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

So the eigenvalues of the matrix A are $\lambda = 9, -1, 3, 7$

Therefore the characteristic equation of the matrix A is,

$$(\lambda - 9)(\lambda + 1)(\lambda - 3)(\lambda - 7) = 0$$

16. $T(x, y, z) = (2x - y - z, x - z, -x + y + 2z)$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y - z \\ x - z \\ -x + y + 2z \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$T_A(x) = Ax$ can be written as $w = Ax$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

Characteristic eq. is as follows $|\lambda I - A| = 0$

$$\left|\begin{bmatrix} \lambda - 2 & 1 & 1 \\ -1 & \lambda & 1 \\ 1 & -1 & \lambda - 2 \end{bmatrix}\right| = 0$$

$$(\lambda - 2)(\lambda(\lambda - 2) + 1) - 1(-\lambda + 2 - 1) + 1(1 - \lambda) = 0$$

$$(\lambda - 2)(\lambda^2 - 2\lambda + 1) + \lambda - 1 + 1 - \lambda = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda - 2\lambda^2 + 4\lambda - 2 + \lambda - 1 + 1 - \lambda = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

Solve for λ

$$(\lambda-1)(\lambda^2-3\lambda+2)=0$$

$$(\lambda-1)(\lambda^2-2\lambda-\lambda+2)=0$$

$$(\lambda-1)(\lambda(\lambda-2)-(\lambda-2))=0$$

$$(\lambda-1)(\lambda-1)(\lambda-2)=0$$

$$\lambda=1,1,2$$

Sub $\lambda=1$ in $(\lambda I-A)x=0$

$$\left(\begin{bmatrix}-1 & 1 & 1\\ -1 & 1 & 1\\ 1 & 1 & -1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \to R_3+R_2$

$R_2 \to R_2-R_1$

$$\Rightarrow \left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

Solve the system of equation of above matrix

$$-x_1+x_2+x_3=0$$

$$x_2=s,\ x_3=t,\ x_1=s+t$$

$$x=\begin{bmatrix}s+t\\s\\t\end{bmatrix}=s\begin{bmatrix}1\\1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Therefore the eigenvectors of A for $\lambda=1$ are

$\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and they form a basis for the eigenspace

Put $\lambda = 2$ in $(\lambda I - A)x = 0$

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2 - R_1$
$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 + x_3 = 0$
$-x_1 + x_2 = 0$
$\Rightarrow$ $x_3 = s$
$x_2 = -s$
$x_1 = -s$

$$x = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, the eigenvector of A corresponding to $\lambda = 2$ are $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

and is a basis for the eigenspace.

19. (a) The image of $e_1(1,0)$ is the vector $u_1(\cos(t), \sin(t))$

The image of $e_2(0,1)$ is the vector $u_2(\sin(t), -\cos(t))$

The matrix of the orthogonal reflection is

$$A = \begin{bmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{bmatrix}$$

$\sin(t) = \dfrac{2m}{1+m^2}$, $\cos(t) = \dfrac{1-m^2}{1+m^2}$

Therefore

$$A = \begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} \\ \dfrac{2m}{1+m^2} & -\left(\dfrac{1-m^2}{1+m^2}\right) \end{bmatrix}$$

If $y = x$ then $t = \pi/2$

therefore $m = \tan\left(\dfrac{\pi/2}{2}\right)$

$= \tan(\pi/4) = 1$

Therefore the reflection matrix reduces to

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad |\lambda I - A| = 0$

$\left|\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\right| = 0 \Rightarrow (\lambda^2 - 1) = 0 \Rightarrow \lambda = 1, -1$

Hence, Eigen values of the matrix A are $\lambda = 1, -1$

Put the value $\lambda = -1$ in the equation $(\lambda I - A)x = 0$

$\left(\begin{bmatrix} (-1) & -1 \\ -1 & (-1) \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x_1 - x_2 = 0 \\ x_1 = -x_2 \end{matrix}$

Put $x_2 = 1 \Rightarrow x_1 = -(1)$

Therefore $X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an Eigen vector of the matrix A

Put the value $\lambda = 1$ in equation

$\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - x_2 = 0 \\ x_1 = x_2 \end{matrix}$

$x_2 = 1 \Rightarrow x_1 = 1$ Therefore $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector of A.

c) $A = \begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ) \\ \sin(90^\circ) & \cos(90^\circ) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$|\lambda I - A| = 0$

$\left| \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \right| = 0 \quad (\lambda^2 + 1) = 0$

On solving $\lambda^2 = -1 \Rightarrow \lambda = \pm 1$

Hence Eigen values of the matrix A does not exist, since the values of λ are imaginary.

23. (a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \end{bmatrix}$

$(\lambda I - A)X = 0 \quad \left(\begin{bmatrix} \lambda - 4 & 1 \\ -2 & \lambda - 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{vmatrix} \lambda - 4 & 1 \\ -2 & \lambda - 1 \end{vmatrix} = 0 \Rightarrow (\lambda - 4)(\lambda - 1) - 1 \times (-2) = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow \lambda(\lambda - 2) - 3(\lambda - 2) = 0$

$(\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2, 3$

Sub. $\lambda = 2$

$\left(\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2x + y \\ -2x + y \end{bmatrix} = 0$

Solve, $y = 2x$

The eigenspace for the matrix A corresponding to $\lambda = 2$ is $y = +2x$

Now, sub. $\lambda = 3$

$$\left(\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -x+y \\ -2x+2y \end{bmatrix} = 0$$

$-x+y=0$ and $-2x+2y=0$

$y = x$ and $y = x$

The eigenspace for the matrix A corresponding to $\lambda = 3$ is, $y = x$

Hence, the equation for the lines in R^2 which are invariant under the matrix A is $y = 2x$ and $y = x$

(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \end{bmatrix}$

$$(\lambda I - A)X = 0$$

$$\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0$$

Solve this expression, $\lambda = \sqrt{-1} = \pm i$

Hence the equation for the lines in R^2 which are invariant under the matrix A is not possible or no lines because the eigenvalue of matrix A is imaginary.

24. (A) $\det(\lambda I - A) = \lambda^3 - 2\lambda^2 + \lambda + 5$

Put $\lambda = 0 \Rightarrow \det(0I - A) = 0^3 - 2(0)^2 + 0 + 5$

$\det(-A) = 5$

We know that $\det(KA) = K^n \det(A)$

Here $n = 3$. So, $(-1)^3 \det A = 5$

Therefore $\det(A) = -5$

(B) $\det(\lambda I - A) = \lambda^4 - \lambda^3 + 7$

Put $\lambda = 0 \Rightarrow \det(0I - A) = 0^4 - 0^3 + 7$

$\det(-A) = 7$. Here $n = 4$. So, $(-1)^4 \det A = 7$

Therefore, $\det(A) = 7$

25. Consider $p(\lambda) = 0$. Then.

$(\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3 = 0$

$\lambda - 1 = 0$, $(\lambda - 3)^2 = 0$, $(\lambda - 4)^3 = 0$

$\lambda = 1$, $(\lambda - 3)(\lambda - 3) = 0$, $(\lambda - 4)(\lambda - 4)(\lambda - 4) = 0$

$\lambda = 1$, $\lambda - 3 = 0$, $\lambda - 3 = 0$, $\lambda - 4 = 0$, $\lambda - 4 = 0$,

$\lambda - 4 = 0$ $\qquad \lambda = 1, 3, 3, 4, 4, 4$

The given matrix has 6 eigen values

Therefore, the size of the matrix A is 6×6.

Since the roots of $P(\lambda) = 1, 3, 3, 4, 4, 4$ are non-zeros, the square matrix A is invertible

Since the eigen values of A are 1, 3, 4

Hence the number of eigen spaces of A are 3.