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Exercise 3.3

1.(a) $u = (6, 1, 4)$
$v = (2, 0, -3)$

$u \cdot v = 6 \cdot 2 + 1 \cdot 0 + 4 \cdot -3 = 12 + 0 - 12 = 0$

The vectors are orthogonal.

(b) $u = (0, 0, -1)$
$v = (1, 1, 1)$

$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots u_n v_n$
$= 0 \cdot 1 + 0 \cdot 1 + (-1) \cdot 1 = 0 + 0 - 1 = -1$

Since $u \cdot v \neq 0$

The vectors are not orthogonal

(c) $u = (3, -2, 1, 3)$ $v = (-4, 1, -3, 7)$

$u \cdot v = 3 \cdot (-4) + (-2) \cdot 1 + 1 \cdot (-3) + 3 \cdot 7$
$= -12 + (-2) - 3 + 21 = 4$

The vectors are not orthogonal vectors

(d) $u = (5, -4, 0, 3)$ $v = (-4, 1, -3, 7)$

$u \cdot v = 5 \cdot (-4) + (-4) \cdot 1 + 0 \cdot (-3) + 3 \cdot 7$
$= -20 - 4 + 0 + 21 = -3$

The vectors are not orthogonal

3. $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$P(x_0, y_0, z_0) = (-1, 3, -2)$

$n(a, b, c) = (-2, 1, -1)$

Therefore, the equation is $\Rightarrow -2(x+1)+1(y-3)-1(z+2)=0$

4. $(x_1, y_1, z_1) = (1, 1, 4)$;

$(a, b, c) = (1, 9, 8)$

$1(x-1)+9(y-1)+8(z-4)=0$

$x-1+9y-9+8z-32=0$

$x+9y+8z-42=0$

7. $a_1 = 4, b_1 = -1, c_1 = 2$

$a_2 = 7, b_2 = -3, c_2 = 4$

So $a_1/a_2 = 4/7$, $b_1/b_2 = -1/-3 = 1/3$ and

$c_1/c_2 = 2/4 = 1/2$

Therefore $a_1/a_2 \neq b_1/b_2 \neq c_1/c_2$

Hence given planes are not parallel

8. $u = (1,-4,3)$

$v = (3,-12,-9)$

$v = 3(1,-4,-3) = 3u$

Since we can multiply the first vector by 3 and get the second, so the given planes are parallel. Therefore, the planes $x-4y-3z-2=0$ and $3x-12y-9z-7$ are parallel.

11. $a_1 = 3,\ b_1 = -1,\ c_1 = 1,\ a_2 = 1,\ b_2 = 0,\ c_2 = 2$

$a_1a_2 + b_1b_2 + c_1c_2 = 3(1) + (-1)(0) + 1(2) = 3 - 0 + 2$
$= 5 \neq 0$

$3x - y + z - 4 = 0$ and $x + 2z - 1$ are not perpendicular.

13. (a) $u = (1,-2)$ $a = (-4,-3)$

$$\|\text{proj}_a u\| = \frac{|u \cdot a|}{\|a\|} = \frac{|(-1,-2)\cdot(-4,-3)|}{\sqrt{(-4)^2 + (-3)^2}}$$

$$= \frac{1(-4) + (-2)(-3)}{\sqrt{16+9}} = \frac{-4+6}{\sqrt{25}} = 2/5$$

$\|\text{proj}_a u\| = 2/5$

15. $u \cdot a = 6 \cdot 3 + 2 \cdot (-9) = 0$

$\|a\| = \sqrt{3^2 + (-9)^2} = \sqrt{90}$

$\|a\|^2 = 90$

$$\text{proj}_a u = \frac{u \cdot a}{\|a\|^2} a = 0/90\,(3,-9) = (0,0)$$

$w_2 = u - w_1 = u - \text{proj}_a u$

$w_2 = (6,2) - (0,0) = (6-0, 2-0) = (6,2)$

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Exercise 4.1

(a) $u = (-1, 2)$ $v = (3, 4)$ $k = 3$

$u + v = (2, 6)$
$ku = (0, 6)$

(b) On $V$, the addition and scalar multiplication are defined as components of real numbers and $u_1, u_2, v_1, v_2$ are real numbers, so $u_1 + v_1$, $u_2 + v_2$ and $ku_2$ are also real numbers

Thus, $u + v = (u_1 + v_1, u_2 + v_2) \in R$
$ku = (0, ku_2) \in R$

By general real number properties, it is clear that $V$ is closed under addition and scalar multiplication.

(c) $u = (u_1, u_2)$, $v = (v_1, v_2)$

Axiom 1: Since $u = (u_1, u_2)$, $v = (v_1, v_2) \in V$
So
$u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \in V$
Therefore, $V$ is closed under addition

Axiom 2: Let $u(u_1, u_2)$, $v = (v_1, v_2) \in V$

Then

$$u + v = (v, u_2) + (v_1, v_2)$$

$$= (u_1 + v_1; u_2 + v_2)$$

$$= (v_1 + u_1, v_2 + u_2)$$

$$= (v_1, v_2) + (u_1, u_2) = v + u$$

Therefore, the commutative property holds on V

Axiom 3: Let $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ $\in V$

$$(u+v) + w = (u_1 + v_1, u_2 + v_2) + (w_1, w_2)$$

$$= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2)$$

$$= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2))$$

$$= (u_1; u_2) + (v_1 + w_1, v_2 + w_2)$$

$$= (u_1, u_2) + ((v_1, v_2) + (w_1, w_2))$$

$$= u + (v + w)$$

Therefore, associative law is satisfied in V.

Axiom 4: There is an object 0 in V, called zero vector for V, such that

$0 + u = (0,0) + (u_1, u_2)$

$= (0 + u_1, 0 + u_2)$

$= (u_1, u_2)$

$= u$

Therefore, 0 is the additive identity of V.

Axiom 5: For each $u \in V$, there is an object $-u \in V$, called negative of u such that

$u + (-u) = (-u) + u = 0$

And since $u_1, u_2$ are real numbers
So $-u_1, -u_2 \in \mathbb{R}$

Now

$u + (-u) = (u_1, u_2) + (-u_1, -u_2)$

$= (u_1 - u_1, u_2 - u_2) = (0,0) = 0$

Thus, for every $u \in V$, there is an additive inverse $-u \in V$

(d) Axiom 7: For every $u, v \in V$ and k is any scalar, $k(u+v) = ku + kv$.
Let $u = (u_1, u_2), v = (v_1, v_2) \in V$ and k be any scalar

Then

$k(u+v) = (0, k(u_2+v_2))$

$= (0, ku_2+kv_2)$

$= (0, ku_2) + (0, kv_2)$

$= ku+kv$

Hence $k(u+v) = ku+kv$

Axiom 8: For every $u \in V$ and $k, m$ are any scalars, $(k+m)u = ku+mu$

Let $u = (u_1, u_2) \in V$ and $k, m$ are any scalars:

$(k+m)u = (0, (k+m)u_2)$

$= (0, ku_2+mu_2)$

$= (0, ku_2) + (0, mu_2)$

$= ku+mu$

Hence $(k+m)u = ku+mu$

Axiom 9: For every $u \in V$ and $k, m$ are any scalars, $k(mu) = (km)u$

Let $u = (u_1, u_2) \in V$ and $k, m$ are any scalars

By the definition in (2), $mu = (0, mu_2)$

Now

$k(mu) = (0, k(mu_2))$

$= (0, kmu_2)$

$= (0, (km)u_2) = (km)(u)$

Hence $k(mu) = (km)u$

(e) Axiom 10: For every $u \in V$, $1u = u$

Let $u = (u_1, u_2) \in V$

By definition in (2),

$1u = 1(u_1, u_2)$

$= (0, 1u_2) = (0, u_2) \neq (u_1, u_2) = u$

Hence the Axiom 10 does not hold on V.

Therefore, V is not a Vector Space under the given operations.

2. (a) $u = (0, 4)$ and $v = (1, -3)$

Now find the vector $u+v$

$u + v = (0,4) + (1,-3) = (0+1+1, 4-3+1)$

$= (2, 2)$

Therefore $u + v = (2, 2)$

Now find the vector $ku$, here $k = 2$

$2u = 2(0,4) = (0,8)$

(b) Suppose $(0,0)$ is identity element in $V$, then it should satisfy $(u_1, u_2) + (0,0) = (u_1, u_2)$

Now consider $(u_1, u_2) + (0,0)$

$$(u_1, u_2) + (0,0) = (u_1 + 0 + 1, u_2 + 0 + 1)$$
$$= (u_1 + 1, u_2 + 1) \neq (u_1, u_2)$$

Therefore, $(0,0)$ is not an identity element in $V$. Hence $(0,0) \neq 0$

(c) ~~Show that (...)~~

Suppose $(-1,-1)$ is identity element in $V$, then it should satisfy $(u_1, u_2) + (-1,-1) = (u_1, u_2)$

Now consider $(u_1, u_2) + (-1,-1)$

$$(u_1, u_2) + (-1,-1) = (u_1 - 1 + 1, u_2 - 1 + 1)$$

$$= (u_1, u_2)$$

Therefore, $(-1,-1)$ is the identity element in $V$.

Hence $(-1,-1) = 0$

(d) $-u = (-u_1 - 2, -u_2 - 2)$

$u + (-u) = (u_1, u_2) + (-u_1 - 2, -u_2 - 2)$

$= (u_1 - u_1 - 2 + 1, u_2 - u_2 - 2 + 1)$

$= (-1, -1) = 0$

Now consider $(-u) + u$

$u + (-u) = (-u_1 - 2, -u_2 - 2) + (u_1, u_2)$

$= (-u_1 - 2 + u_1 + 1, -u_2 - 2 + u_2 + 1)$

$= (-1, -1) = 0$

Therefore $u + (-u) = (-u) + u = 0$.

(e) Let $k = 1, m = 2$ and $u = (3, 4)$.

Consider $(k + m)u$

$(k+m)u = (1+2)(3,4)$

$= 3(3,4) = (9,12)$

Consider $ku + mu$

$ku + mu = 1(3,4) + 2(3,4)$

$= (3,4) + (6,8)$

$= (3+6+1, 4+8+1) = (10, 13)$

Therefore $(k+m)u \neq ku + mu$

Let $k=2$, $u=(3,4)$ and $v=(-1,5)$

Consider $k(u+v)$

$$k(u+v) = 2((3,4)+(-1,5))$$
$$= 2(3-1+1, 4+5+1) = 2(3,10) = (6,20)$$

Consider $ku+kv$

$$ku+kv = 2(3,4) + 2(-1,5)$$
$$= (6,8) + (-2,10)$$
$$= (6-2+1, 8+10+1) = (5,19)$$

Therefore, $k(u+v) \neq ku+kv$

Therefore, the set $V$ does not satisfy the vector space axioms $(k+m)u = (ku+mu)$ and $k(u+u) = ku+kv$

3. Let $u = x_1$, $v = x_2 \in V$

Consider,

$u+v = x_1 + x_2$

Since the addition of two number is real

Hence, Axiom 1 is satisfied.

Commutative:

$u+v = x_1+x_2 = x_2+x_1 = v+u$

Hence, Axiom 2 is satisfied

Associative:

Let $u = x_1$, $v = x_2$, and $w = x_3 \in V$

Consider,

$$u + (v+w) = x_1 + (x_2 + x_2) = x_1 + x_2 + x_2$$
$$= (x_1 + x_2) + x_2 = (u+v) + w$$

That is,

$$u + (v+w) = (u+v)+w$$

Hence, axiom 3 is satisfied

Identity:

Let 0 be the zero vector in V

Then,

$$0 + u = 0 + x = x + 0 = u + 0 = u$$

That is,

$$0 + u = u + 0 = u$$

Hence, Axiom 4 is satisfied

Inverse,

Let the negative of u be $-u$

Then,

$$-u = -x$$

So,

$$u + (-u) = x + (-x) = 0 = (-u) + (u)$$

Hence, Axiom 5 is satisfied

Let $k$ be a scalar and $u = x_1 \in V$

Consider, $ku = kx = kx \in V$

So, $ku$ is also in $V$

Hence, Axiom 6 is satisfied

Let $k$ be a scalar and $u = x_1, v = x_2 \in V$

$k(u+v) = k(x_1 + x_2) = kx_1 + kx_2 = ku + kv$

Hence, axiom 7 is satisfied.

Let $k, m$ be a scalar and $u = x_1 \in V$

$(k+m)u = (k+m)x = kx + mx = ku + mu$

Hence, Axiom 8 is satisfied

$k(mu) = k(mx) = (km)x = (km)u$

Hence Axiom 9 is satisfied

Find $1u$

$1u = 1(x) = x = u$

Hence, axiom 10 is satisfied

So, the set of all real numbers of the form $x$ is a vector space.

4. For this consider, $(x,0),(y,0)$ and $(z,0)$ be any vector in V then clearly,

$$(x,0)+(y,0)=((x+y),0)\in V$$

So, closure under addition holds.

Also,

$$[(x,0)+(y,0)]+(z,0)=(x+y,0)+(z,0)$$
$$=(x+y+z,0)=(x,0)+[(y,0)+(z,0)]$$

So, the associative law holds.

Also, $(0,0)$ is the additive identity as,

$$(x,0)+(0,0)=(x,0)=(0,0)+(x,0).$$

And, for any $(x,0)\in V$, there exists $(-x,0)\in V$ such that

$$(x,0)+(-x,0)=(0,0)=(-x,0)+(x,0)$$

Moreover,

$$(x,0)+(y,0)=((x+y),0)=((y+x,0))$$
$$=(y,0)+(x,0)$$

This shows that V is commutative with respect to addition.

Also V satisfies associative property under multiplication.

$1 \cdot (x, 0) = (x, 0) \in V$

Now consider that $\alpha$ and $\beta$ be any scalar

then, $\alpha [(x, 0) + (y, 0)] = \alpha (x + y, 0)$

$= \alpha x + \alpha y, 0 = (\alpha x, 0) + (\alpha y, 0)$

$= \alpha (x, 0) + \alpha (y, 0)$

And,

$(\alpha + \beta)(x, 0) = \alpha (x, 0) + \beta (x, 0)$

This shows that V forms a vector space

5. Let $u = (x, y)$, $v = (x', y')$ and $w = (x'', y'')$

where $x, x', x'' \geq 0$

$u + v = (x, y) + (x', y') = (x + x', y + y')$

Therefore, $u + v$ is in V.

Hence, the axiom 1 is satisfied.

$v + u = (x' + x, y' + y) = (x + x', y + y')$

Hence, the axiom 2 is satisfied.

$u + (v + w) = (u + v) + w$

$u + (v + w) = (x, y) + ((x', y') + (x'', y''))$

$= (x, y) + (x' + x'', y' + y'')$

$= (x + x' + x'', y + y' + y'')$

$= (x + x', y + y') + (x'', y'')$

$= (u+v)+w$

Therefore, $u+(v+w) = (u+v)+w$

Hence, the axiom 3 is satisfied

Let the $(0,0)$ be the zero vector in $V$

$u+0 = (x,y)+(0,0) = (x+0, y+0) = (x,y) = u$

$u+0 = 0+u = u$

Hence; the axiom 4 is satisfied

$-u = (-x, -y)$

As $x \geq 0$, $-x \leq 0$.

But for every $x \geq 0$, the ordered pair $(x,y)$ lies in the set

So, there does not exists any object $-u$ in $V$

Therefore, $-u$ is not in $V$.

Hence the axiom 5 fails.

Let $k = -2$

$ku = (-2)(x,y) = (-2x, -2y)$

As $x \geq 0$; $-2x \leq 0$

But for every $x \geq 0$, the ordered pair $(x,y)$ is in $V$ otherwise not in $V$.

So, there does not exist any object $ku$ in $V$ for some $k$.

Therefore, the axiom 6 fails.

Axiom 7, 8, 9 fail because the scalar multiplication property fails.

$1u = 1(x, y) = (x, y) = u$

Hence the axiom 10 is satisfied

As the axioms 5, 6, 7, 8, 9 fails, the set of all pairs of real numbers $(x, y)$ where $x \geq 0$ with the standard operations is not a vector space.

7. $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

$u+v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Therefore, $u+v$ is in $V$

Thus, $V$ is closed under addition

Hence, axiom 1 is satisfied

$v + u = (x_2, y_2, z_2) + (x_1, y_1, z_1)$
$= (x_2 + x_1, y_2 + y_1, z_2 + z_1)$

Observe that $u+v = v+u$

Thus, commutative law holds in V

Hence, axiom 2 is satisfied

$u+(v+w) = (x_1, y_1, z_1) + (x_2+x_3 + y_2+y_3 + z_2+z_3)$

$= (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$

$(u+v)+w = ((x_1, y_1, z_1) + (x_2, y_2, z_2)) + (x_3, y_3, z_3)$

$= (x_1+x_2, y_1+y_2, z_1+z_2) + (x_3, y_3, z_3) = (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$

Observe that $u+(v+w) = (u+v)+w$

Therefore, the associative law holds in V.

Hence axiom 3 is satisfied

$0+u = (0+x_1, 0+y_1, 0+z_1) = (x_1, y_1, z_1)$
$= u$

$u+0 = (x_1, y_1, z_1) + (0, 0, 0)$
$= (x_1+0, y_1+0, z_1+0) = (x_1, y_1, z_1)$
$= u$

Thus, $(0,0,0)$ is the zero vector of $V$.
Hence, axiom 4 is satisfied.

$u+(-u) = (x_1-x_1, y_1-y_1, z_1-z_1) = (0,0,0)$
$= 0$

$(-u)+u = (-x_1, -y_1, -z_1) + (x_1, y_1, z_1)$
$= (0,0,0) = 0$

Therefore, $-u$ is the additive inverse of $u$ in $V$.
Hence, axiom 5 is satisfied.

$ku = (k^2 x_1, k^2 y_1, k^2 z_1)$.
Therefore, $ku \in V$
Hence, axiom 6 is satisfied

$k(u+v) = k((x_1, y_1, z_1) + (x_2, y_2, z_2))$
$= k(x_1+x_2, y_1+y_2, z_1+z_2)$
$= (k^2(x_1+x_2), k^2(y_1+y_2), k^2(z_1+z_2))$

$ku+kv = k(x_1, y_1, z_1) + k(x_2, y_2, z_2)$

$= (k^2x_1+k^2x_2, k^2y_1+k^2y_2, k^2z_1+k^2z_2)$
$= (k^2(x_1+x_2), k^2(y_1+y_2), k^2(z_1+z_2))$

$k(u+v) = ku+kv$, for all $u, v \in V$ and for any scalar $k$. Hence, axiom 7 is satisfied

$(k+m)u = (k+m)(x_1, y_1, z_1)$

$= ((k+m)^2 x_1, (k+m)^2 y_1, (k+m)^2 z_1)$

$ku + mu = k(x_1, y_1, z_1) + m(x_1, y_1, z_1)$

$= (k^2 x_1, k^2 y_1, k^2 z_1) + (m^2 x_1, m^2 y_1, m^2 z_1)$

$= ((k^2+m^2)x_1, (k^2+m^2)y_1, (k^2 + m^2)z_1)$

Observe that, for $k, m \neq 0$

$((k+m)^2 x_1, (k+m^2)y_1, (k+m)^2 z_1) \neq$

$((k^2+m^2)x_1, (k^2+m^2)y_1, (k^2+m^2)z_1)$

Hence $(k+m)u \neq ku + mu$ for all $u \in V$ and $k, m$ be any scalars

Hence, axiom 8 was not satisfied

$k(mu) = k(m(x_1, y_1, z_1))$

$= k(m^2 x_1, m^2 y_1, m^2 z_1)$

$= (k^2(m^2 x_1), k^2(m^2 y_1), k^2(m^2 z_1))$

$= (k^2 m^2 x_1, k^2 m^2 y_1, k^2 m^2 z_1)$

$= ((km)^2 x_1, (km)^2 y_1, (km)^2 z_1)$

$= km(x_1, y_1, z_1)$

Thus $k(mu) = (km)u$, for all $u \in V$ and for any scalars $k, m$

Hence axiom 9 is satisfied.

~~Let $u = (x_1, y_1, z_1)$~~

$1u = 1(x_1, y_1, z_1) = (1^2 x_1, 1^2 y_1, 1^2 z_1)$

$= (x_1, y_1, z_1) = u$

So, axiom 10 is satisfied.

Observe that; axiom 8 $(k+m)u = ku + mu$ does not hold

Hence the set V of all triplets of real numbers is not a vector space under the given operations.

8. Let V be the set of all $2\times2$ invertible matrices with the standard matrix addition and scalar multiplication.

Let us take the two invertible matrices.

$u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Both u and v are invertible matrices belong to the set of vectors V.

Now consider the sum,

$$u+v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1-1 & 0+0 \\ 0+0 & 1-1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

The resultant matrix is a zero matrix and it is not invertible. So $0 \notin V$

Thus by the definition of V, $u+v$ is not a vector in V

So, the given set fails to hold the axiom, if u and v are objects in V, then $u+v \in V$

Hence, the set V is not a vector space.

10. Property 1

$(f+g)(1) = f(1) + g(1)$

Since f and g are elements in A so, $f(1) = 0$ and $g(1) = 0$. Therefore, $(f+g)(1) = 0$. Thus, $f+g$ is in A.

Property 2

$(f+g)(x) = f(x) + g(x)$. Also $(g+f)(x) = g(x) + f(x)$

Since, $f(x)$ and $g(x)$ are real numbers then by commutative property of real numbers, $f(x) + g(x) = g(x) + f(x)$

Therefore, $(f+g)(x) = (g+f)(x)$, that is $f+g = g+f$

If $f, g$ and $h$ are elements in $A$, then $f+(g+h) = (f+g)+h$

Property 3

Consider,

$[f+(g+h)](x) = f(x) + (g(x) + h(x))$

Since, $f(x)$, $g(x)$ and $h(x)$ are real numbers, then by associative property of addition

$(f(x)+g(x))+h(x) = [(f+g)+h](x)$

Therefore, $f+(g+h) = (f+g)+h$

Property 4.

The zero vector $z$ is zero everywhere. For all $x \in R$ $z(x)=0$. Then it is obvious that $z(1)=0$

Hence, the vector $z$ belongs to $A$

$(f+z)(x) = f(x) + z(x)$. Since $z(x)=0$,

$(f+z)(x) = f(x)$: That is $f+z=f$

Similarly, by commutative property

$z+f=f$

Property 5

$f(1)=0$, thus,

$(-f)(1) = -(f(1))$. This implies that $(-f)(1)$

$=0$

Hence $-f$ belongs to $A$.

$[f+(-f)](x) = f(x) + (-f(x))$. This can be further simplified as,

$f(x) - f(x) = 0$, that is $[f+(-f)](x) = 0$

Similarly, by the commutative property it can be shown that

$[(-f)+f](x) = 0$

## Property 6

Let $f$ belong to $A$, then $f$ is a real valued function defined everywhere, then $kf$ is also a real valued function defined every where

Also, $(kf)(1) = k(f(1))$. Since $f(1) = 0$, $(kf)(1) = k(0)$

Therefore $(kf)(1) = 0$

Hence, $kf$ belongs to $A$.

## Property 7:

For all $x \in R$, $u(x) = 0$ and $v(x) = 0$. Then it is obvious that $u(1) = 0$ and $v(1) = 0$

Consider the ~~take~~ the L.H.S

$$
\begin{aligned}
[k(u+v)](1) &= k[(u+v)(1)] \\
&= k[u(1) + v(1)] \\
&= k\cancel{f}u(1) + kv(1)
\end{aligned}
$$

From the above $[k(u+v)](1) = 0$

Now let us consider R.H.S

$(ku+kv)(1) = ku(1) + kv(1) = 0$

Therefore, $k(u+v) = ku+kv$

Property 8:

Consider,

$[(k+m)u](1) = (k+m)u(1) \rightarrow B$

$(ku+mu)(x) = ku(x) + mu(x)$

Then, $[(k+m)u](1) = ku(1) + mu(1) \rightarrow C$

Therefore from B and C

$(k+m)u = ku+mu$

Property 9:

To show that for any scalar $k, m$ and any element $u$ in $A$, then $k(mu) = (km)u$

Consider,

$[k(mu)](1) = k[(mu)(1)]$.

$k[mu(1)] = kmu(1) = (km)u(1)$.

Therefore, $k(mu) = (km)u$

Property 10:

Consider.

$[1(u)](1) = 1 \times [u(1)]$

Since, 1 is a unit element in multiplication

$[1(u)](1) = u(1)$

Therefore, $1(u) = u$

Hence, all 10 properties are satisfied so this set A is a vector space

13. Let $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$, $v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$,

$w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$

Now consider,

$$u + v = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{12} & u_{22} + v_{22} \end{pmatrix}$$

And

$$v + w = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} + w_{11} & v_{12} + w_{12} \\ v_{21} + w_{12} & v_{22} + w_{22} \end{pmatrix}$$

Then

$$u + (v + w) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \left(\begin{pmatrix} v_{11} + w_{11} & v_{12} + w_{12} \\ v_{21} + w_{21} & v_{22} + w_{22} \end{pmatrix}\right)$$

$$= \begin{pmatrix} u_{11} + (v_{11} + w_{11}) & u_{12} + (v_{12} + w_{12}) \\ u_{21} + (v_{21} + w_{21}) & u_{22} + (v_{22} + w_{22}) \end{pmatrix}$$

Rearranging the elements in the above matrices gives

$$u+(v+w) = \begin{pmatrix} (u_{11}+v_{11})+w_{11} & (u_{12}+v_{12})+w_{12} \\ (u_{21}+v_{21})+w_{21} & (u_{22}+v_{22})+w_{22} \end{pmatrix}$$

$$= \left(\begin{pmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{pmatrix}\right) + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

$$= (u+v)+w$$

So, axiom $u+(v+w) = (v+u)+w$ holds well in 2X2 vector matrices

Consider,

$$k(u+v) = k\begin{pmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} k(u_{11}+v_{11}) & k(u_{12}+v_{12}) \\ k(u_{21}+v_{21}) & k(u_{22}+v_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} ku_{11}+kv_{11} & ku_{12}+kv_{12} \\ ku_{21}+kv_{21} & ku_{22}+kv_{22} \end{pmatrix}$$

Then

$$k(u+v) = \begin{pmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{pmatrix} + \begin{pmatrix} kv_{11} & kv_{12} \\ kv_{21} & kv_{22} \end{pmatrix}$$

$$= k\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + k\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$= ku + kv$

Therefore $k(u+v) = ku + kv$

Therefore the axioms $k(u+v) = ku + kv$ holds well in 2X2 vector matrices.

Let $k, m$ be any scalars and $u$ be 2X2 vector matrices

$$(k+m)u = (k+m)\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (k+m)u_{11} & (k+m)u_{12} \\ (k+m)u_{21} & (k+m)u_{22} \end{pmatrix}$$

Then $(k+m)u = \begin{pmatrix} ku_{11} + mu_{11} & ku_{12} + mu_{12} \\ ku_{21} + mu_{21} & ku_{22} + mu_{22} \end{pmatrix}$

$$= \begin{pmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{pmatrix} + \begin{pmatrix} mu_{11} & mu_{12} \\ mu_{21} & mu_{22} \end{pmatrix}$$

$$= k\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + m\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$= ku + mu$

Therefore the Axioms $(k+m)u = ku + mu$ hold well in 2X2 vector matrices

Let $k, m$ be any scalars and $u$ be $2\times 2$ vector matrices.

$$k(mu) = k\left[m\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}\right]$$

$$= k\begin{pmatrix} mu_{11} & mu_{12} \\ mu_{21} & mu_{22} \end{pmatrix} = \begin{pmatrix} kmu_{11} & kmu_{12} \\ kmu_{21} & kmu_{22} \end{pmatrix}$$

Then

$$k(mu) = \begin{pmatrix} (km)u_{11} & (km)u_{12} \\ (km)u_{21} & (km)u_{22} \end{pmatrix}$$

$$= (km)\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = (km)u$$

Therefore the axiom $k(mu) = (km)u$ holds well in $2\times 2$ vector matrices.

Hence verified

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## Exercise 4.2

1. (a) $V = \{(a,0,0) : a \in R\}$

Let $u = (u_1, 0, 0)$

$v = (v_1, 0, 0)$ be two vector spaces in $R^3$

So,

$u + v = (u_1, 0, 0) + (v_1, 0, 0)$

$= (u_1 + v_1, 0, 0)$

For any scalar $k$

$ku = k(u_1, 0, 0) = (ku_1, 0, 0)$

The resultant vectors $u+v$; $ku$, are in the defined form and belong to $R^3$.
Hence, it is a vector subspace.

(b) $V = \{(a,1,1) : a \in R\}$

Let,

$u = (u_1, 1, 1)$
$v = (v_1, 1, 1)$

$u + v = (u_1, 1, 1) + (v_1, 1, 1)$

$= (u_1 + v_1, 2, 2) \notin V$

u+v is not in the defined form and does not belong to $R^3$

Hence, it is not a vector subspace

(c) $V = \{(a,b,c) : a,b,c \in R \text{ and } b = a+c\}$

Let,

$u = (a_1, b_1, c_1)$

$v = (a_2, b_2, c_2)$

$b_1 = a_1 + c_1 \qquad b_2 = a_2 + c_2$

So,

$u+v = (a_1, b_1, c_1) + (a_2, b_2, c_2)$

$= (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

Where,

$(a_1 + c_1) + (a_2 + c_2) = b_1 + b_2$

For any scalar k,

$ku = k(a_1, b_1, c_1)$

$= (ka_1, kb_1, kc_1)$

Where $kb_1 = ka_1 + kc_1$

The resultant vectors u+v, ku are defined and belong to $R^3$

Hence it forms ~~a~~ vector subspace.

(d) $V = \{(a,b,c) : a, b, c \in R \text{ and } b = a+c+1\}$

$b_1 = a_1 + c_1 + 1$

$b_2 = a_2 + c_2 + 1$

$u+v = (a_1+a_2, b_1+b_2, c_1+c_2) \notin V$

Since

$b_1 + b_2 = (a_1 + c_1 + 1) + (a_2 + c_2 + 1)$

$= (a_1 + a_2) + (c_1 + c_2) + 2 \notin V$

Hence, it is not a vector subspace of $R^3$.

(e) $V = \{(a,b,0) : a, b \in R\}$

Let,

$u = (a_1, b_1, 0)$

$v = (a_2, b_2, 0)$

So,

$u+v = (a_1, b_1, 0) + (a_2, b_2, 0)$

$= (a_1 + a_2, b_1 + b_2, 0) \in V$

For any scalar $k$,

$ku = k(a,b,0) = (ka, kb, 0) \in V$

$u+v$, $ku$ are defined and belong to $R^3$

Hence it forms vector subspace.

2. (a) $W = \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} : a_{11}, a_{22} \cdots a_{nn} \in \mathbb{C} \right\}$

Let

$u = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, v = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

$\in W$

For $a_{11}, a_{22}, \cdots a_{nn}, b_{11}, b_{22}, b_{nn} \in \mathbb{C}$

$a_{11} + b_{11}, a_{22} + b_{22} \cdots, a_{nn} + b_{nn} \in \mathbb{C}$

So, by the definition of the set $W$

$u + w = \begin{bmatrix} a_{11}+b_{11} & 0 & 0 & 0 \\ 0 & a_{22}+b_{22} & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & a_{nn}+b_{nn} \end{bmatrix} \in W$

Let $k \in \mathbb{C}$,

For $a_{11}, a_{22} \cdots a_{nn} \in \mathbb{C}, k \in \mathbb{C}, ka_{11}, ka_{22} \cdots$
$ka_{nn} \in \mathbb{C}$

$$ku = \begin{bmatrix} ka_{11} & 0 & 0 & 0 \\ 0 & ka_{22} & 0 & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \dots & ka_{mn} \end{bmatrix} \in W$$

The subset is closed under scalar multiplication ~~also~~ and scalar addition

Therefore, W is a subspace of $M_{n \times n}$

(b) $W = \{A_{n \times n} : \det(A_{n \times n}) = 0\}$

Let $u, v \in W$

$\Rightarrow u = A_{n \times n}$, $v = B_{n \times n}$ and $\det(A_{n \times n}) = 0$, $\det(B_{n \times n}) = 0$

Then,

$u + v = A_{n \times n} + B_{n \times n} = C_{n \times n}$,

Also,

$$\det(C_{n \times n}) = \det(A_{n \times n} + B_{n \times n})$$
$$\neq \det(A_{n \times n}) + \det(B_{n \times n})$$

Thus, $\det(C_{n \times n}) \neq 0$

So by definition of the set W, $u + v \notin W$

Let, $u \in W$, $k \in C \Rightarrow u = A_{n\times n}$, $\det(A_{n\times n}) = 0$

$\Rightarrow ku = kA_{n\times n}$

Consider,

$\det(kA_{n\times n}) = k\det(A_{n\times n}) = k^2 \cdot 0 = 0$

~~ku~~ Therefore $ku \in W$

The subset is closed under scalar multiplication

~~Also~~

The set is not closed under addition.

Therefore, W is not a subspace of $M_{n\times n}$

(c) $W = \{A_{n\times n} : tr(A) = 0\}$

Let

$u, v \in W$

$\Rightarrow u = A_{n\times n}$, $v = B_{n\times n}$ and $tr(A) = 0$, $tr(B) = 0$

Then,

$u + v = A_{n\times n} + B_{n\times n} = C_{n\times n}$, say

Consider,

$tr(C_{nn}) = tr(A_{nn} + B_{nn}) = tr(A_{nn}) + tr(B_{nn})$

$= 0$

So $C_{n\times n}$ or $u + v \in W$

Let $u \in W, k \in C \Rightarrow u = A_{n \times n}, tr(A) = 0$

Then, $ku = kA_{n \times n}$

And,

$tr(kA_{n \times n}) = k\,tr(A_{n \times n}) = k \cdot 0 = 0$

$kA_{n \times n}$ or $ku \in W$

The set is closed under addition and scalar multiplication.

Therefore, $W$ is a subspace of $M_{n \times n}$

(d) Let $u, v \in W \Rightarrow u = A_{n \times n}, v = B_{n \times n}$

and $A = A^T$, $B = B^T$

Then,

$u + v = A_{n \times n} + B_{n \times n} = C_{n \times n}$.

Consider $(C)^T$

$(C)^T = (A+B)^T = A^T + B^T = A + B = C$

$C_{n \times n}$ or $u + v \in W$

Again let $u \in W, k \in C$

$\Rightarrow u = A_{n \times n}, A = A^T$

Then: $ku = kA_{n \times n}$

$(kA)^T = kA^T = kA$

Therefore $ku \in W$.

Thus, the set is closed under addition and Scalar multiplication.

Therefore, W is a subspace of $M_{n\times n}$

(e) Let $u, v \in W \Rightarrow u = A_{n\times n}$, $v = B_{n\times n}$ and $A^T = -A$, $B^T = -B$

Then,

$u+v = A_{n\times n} + B_{n\times n} = C_{n\times n}$, say

Consider $(C)^T$

$(C)^T = (A+B)^T = A^T + B^T$

$= -A - B = -(A+B) = -C$

So $u+v \in W$

$u \in W, k \in \mathbb{C}$

$\Rightarrow u = A_{n\times n}$, $A^T = -A$

Then $ku = kA_{n\times n}$

$(kA)^T = kA^T = -kA$

Therefore, $ku \in W$

Thus, the set is closed under addition and Scalar multiplication

Therefore, W is a subspace of $M_{n\times n}$

(f) Counter example:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$.

Note that, the matrix equations $Ax = 0$ and $Bx = 0$ have trivial solutions because determinant of A and B are not equal to zero.

But their sum is,

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 8 \end{bmatrix}$$

has infinitely many solutions because its determinant is zero. In this case W is not a subspace

(g) Let $u, v \in W$.

$\Rightarrow u = A_{n\times n}$, $v = C_{n\times n}$ and $AB = BA$, $CB = BC$

Then,

$u + v = A_{n\times n} + C_{n\times n} = D_{n\times n}$, say

Consider, DB for some fixed matrix B

$\therefore DB = (A + C)B = AB + CB = BA + BC = B(A + C)$

$= BD$

$u + v \in W$

Thus, the subset W is closed under addition

$u \in W$, $k \in C$

$\Rightarrow u = A_{n\times n}$, $AB = BA$, for some fixed matrix B

$ku = kA_{n\times n}$

Consider $(kA)B$

$(kA)B = k(AB)$

$= k(BA)$

$= (kB)A = B(kA)$.

$ku \in W$

Thus the set is closed under addition and scalar multiplication.

Therefore $W$ is a subspace of $M_{n\times n}$

3. (a) As $P = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_1, a_2, a_3$ are any scalars and $a_0 = 0\}$

Let $f = a_0 + a_1x + a_2x^2 + a_3x^3, g = b_0 + b_1x + b_2x^2 + b_3x^3 \in P$

Then, $a_0 = 0$ and $b_0 = 0$

Consider the sum as follows:

$f + g = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \in P$

As the first term $a_0 + b_0 = 0$

Let $k$ be any scalar and $f = a_0 + a_1x + a_2x^2 + a_3x^3 \in P$

Then,

$a_0 = 0$

Consider kf as follows:

$kf = k(a_0 + a_1x + a_2x^2 + a_3x^3)$

$= (ka_0) + (ka_1)x + (ka_2)x^2 + (ka_3)x^3 \in P$

Clearly, $kf \in P$ as first term equals zero.
Therefore, P is a sub space of vector space $P_3$

(b)

$P = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3$ are any scalars and $a_0 + a_1 + a_2 + a_3 = 0\}$

Let $f = a_0 + a_1x + a_2x^2 + a_3x^3$,

$g = b_0 + b_1x + b_2x^2 + b_3x^3 \in P$

Then,

$a_0 + a_1 + a_2 + a_3 = 0$ and $b_0 + b_1 + b_2 + b_3 = 0$

Consider the sum as follows:

$f + g = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \in P$

Since

$(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) = 0$

$f + g \in P$

Consider kf as follows

ace $P_3$

Let $k$ be any scalar and $f = a_0 + a_1x + a_2x^2 + a_3x^3 \in P$

Then,

$a_0 + a_1 + a_2 + a_3 = 0$

$kf = k(a_0 + a_1x + a_2x^2 + a_3x^3)$

$= (ka_0) + (ka_1)x + (ka_2)x^2 + (ka_3)x^3 \in P$

Since, $(ka_0) + (ka_1) + (ka_2) + (ka_3) = k(a_0 + a_1 + a_2 + a_3) = 0$

Therefore, $P$ is a sub space of vector space $P_3$

(c) $P = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3$ are Integers$\}$

Let $f = a_0 + a_1x + a_2x^2 + a_3x^3$, $g = b_0 + b_1x + b_2x^2 + b_3x^3 \in P$

Then, $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ are Integers

$f + g = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \in P$

Since $a_0, a_1, a_2, b_0, b_1, b_2$ are rational, the coefficients of $x$ are also rational

Let $k = \sqrt{2}$ be any scalar and $f = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Then, $a_0, a_1, a_2, a_3$ are rational numbers

$kf = \sqrt{2}(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$

$= (\sqrt{2}a_0) + (\sqrt{2}a_1)x + (\sqrt{2}a_2)x^2 + (\sqrt{2}a_3)x^3$

$\notin P$

Since, $(\sqrt{2}a_0), (\sqrt{2}a_1), (\sqrt{2}a_2), (\sqrt{2}a_3)$ are NOT rational numbers, therefore $P$ is NOT a subspace of vector space $P_3$

(d) $P = \{a_0 + a_1 x : a_0, a_1 \in R\}$

$f = a_0 + a_1 x$, $g = b_0 + b_1 x \in P$

Then, $a_0, a_1; b_0, b_1 \in R$

Consider the sum as follows.

$f + g = a_0 + a_1 x + b_0 + b_1 x$

$= (a_0 + b_0), (a_1 + b_1) \in R$

Consider $kf$ as follows:

Let $k$ be any real scalar and $f = a_0 + a_1 x \in P$

Then, $a_0, a_1 \in R$

$kf = k(a_0 + a_1 x) = (ka_0) + (ka_1)x \in P$

Since, $(ka_0), (ka_1) \in R$

Therefore, $P$ is a sub space of vector space $P_3$

4. (a) $S = \{f \in F \mid f(0) = 0\}$

Let $f, g \in S$

Then $f(0) = 0,\ g(0) = 0$

Now

$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$

Therefore, $f+g \in S$

Let $k$ be any scalar and let $f \in S$.

Then $f(0) = 0$

Now $(kf)(0) = k(f(0)) = k(0) = 0$

Therefore, $kf \in S$

Hence, the given set is a subspace of $F(-\infty, \infty)$

(b) Let $S = \{f \in F \mid f(0) = 1\}$

Let $f, g \in S$

Then $f(0) = 1,\ g(0) = 1$

Now $(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$

Since $(f+g)(0) \neq 1$ Therefore $f+g \notin S$

Hence, the given set is not a subspace of $F(-\infty, \infty)$

(c) Let $S = \{f \in F \mid f(-x) = f(x)\}$

Let $f, g \in S$

Then $f(-x) = f(x)$. $g(-x) = g(x)$

Now $(f+g)(-x) = f(-x) + g(-x)$
$= f(x) + g(x) = (f+g)(x)$

Therefore, $f+g \in S$

Let $k$ be any scalar and let $f \in S$

Then $f(-x) = x$

Now $(kf)(-x) = k(f(-x)) = k(f(x)) =$
$(kf)(x)$

Therefore, $kf \in S$

Hence, the given set is a subspace of $F(-\infty, \infty)$

(d) Let $S$ be the set of all polynomials of degree 2. It is not a vector subspace.

Since $u = 1 + 2x + 3x^2$ and $v = 2 + 4x - 3x^2$

Then

$u + v = 1 + 2x + 3x^2 + 2 + 4x - 3x^2 = 3 + 6x$

And $3 + 6x$ is a polynomial of degree 1.

Therefore, $u + v \notin S$

Hence $S$ is not a subspace of $F(-\infty, \infty)$

6. $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, v_2, z_2)$.

$v_1 + v_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (at_1 + at_2, bt_1 + bt_2, ct_1 + ct_2)$

$= (a(t_1 + t_2), b(t_1 + t_2), c(t_1 + t_2))$

Here, $x = at$, $y = bt$ and $z = ct$ are parametric equation on the line L, which is in $R^3$

Therefore, $v_1 + v_2$ is also in $R^3$.

If k is any real number, then

$kv_1 = k(x_1, y_1, z_1) = (kat_1, kbt_1, kct_1)$

$= k(at_1, bt_1, ct_1)$ which is also in $R^3$

So, L is a subspace of $R^3$

7. $u = (0, -2, 2)$ and $v = (1, 3, -1)$

(a) If $(2, 2, 2) = a(0, -2, 2) + b(1, 3, -1)$ for some scalars a, b.

$(2, 2, 2) = a(0, -2, 2) + b(1, 3, -1)$

$b = 2$

$-2a + 3b = 2$

$2a - b = 2$

Put $b = 2$ in $2a - b = 2$ to get $2a - 2 = 2$

$2a = 4$, $a = 2$

Therefore, there is solution $(2, 2)$

This implies that $(2, 2, 2)$ is in a linear combination of $u = (0, -2, 2)$ and $v = (1, 3, -1)$

(b) $(0,4,5) = a(0,-2,2) + b(1,3,-1)$

$b = 0$ Put $b=0$ in $2a-b=5$ to get

$-2a+3b=4$ $2a-0=5$

$2a-b=5$ $2a=5$

$a = 5/2$

But when the value of $b=0$ is subbed in $-2a+3b=4$ it gives $a=-2$

Therefore, ~~the~~ no solution exists for the above system of equation

This implies that $(2,2,2)$ is not in a linear combination of $u=(0,-2,2)$ and $v=(1,3,-1)$

(c) $(0,0,0) = a(0,-2,2) + b(1,3,-1)$

This implies that

$b=0$ Put $b=0$ in $2a-b=0$ to get

$-2a+3b=0$ $2a-0=0$

$2a-b=0$ $2a=0$

$a=0$

Therefore, there is solution $(0,0)$

This implies that $(0,0,0)$ is in linear combination of $u=(0,-2,2)$ and $v=(1,3,-1)$

11. (a) $(b_1, b_2, b_3) = k_1(2,2,2) + k_2(0,0,3) + k_3(0,1,1)$

$= (2k_1, 2k_1, 2k_1) + (0, 0, 3k_2) + (0, k_3, k_3)$

$= (2k_1, 2k_1 + k_3, 2k_1 + 3k_2 + k_3)$

Equate the terms to get the 3 equations as:

$2k_1 = b_1$ ; $2k_1 + k_3 = b_2$ ; $2k_1 + 3k_2 + k_3 = b_3$

Write the system in matrix form as $Ak = b$

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2(0-3) - 0(2-2) + 0(6-0) = -6 \neq 0$$

Therefore, the system is consistent

So, every vector in $R3$ can be expressed as linear combination of vectors $v_1$, $v_2$ and $v_3$

Hence, the vectors $v_1 = (2,2,2)$, $v_2 = (0,0,3)$, $v_3 = (0,1,1)$ spans the vector space $R3$

(b) $(b_1, b_2, b_3) = k_1(2,-1,3) + k_2(4,1,2) + k_3(8,-1,8)$

$= (2k_1, -k_1, 3k_1) + (4k_2, k_2, 2k_2) + (8k_3, -k_3, 8k_3)$

$= (2k_1 + 4k_2 + 8k_3, -k_1 + k_2 - k_3, 3k_1 + 2k_2 + 8k_3)$

Equate the terms to get the thee equations as:

$2k_1 + 4k_2 + 8k_3 = b_1$

$-k_1 + k_2 - k_3 = b_2$

$3k_1 + 2k_2 + 8k_3 = b_3$

In matrix form $Ak = b$:

$$\begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 2(8+2) - 4(-8+3) + 8(-2-3) = 20 + 20 - 40 = 0$$

Therefore the system is inconsistent

So, every vector in $R^3$ can be expressed as linear combination of vectors $v_1, v_2, v_3$

Hence $v_1, v_2$ and $v_3$ do not span the vector space $R3$.

(c) $(b_1, b_2, b_3) = k_1(3,1,4) + k_2(2,-3,5) + k_3(5,-2,9)$

$= (3k_1, k_1, 4k_1) + (2k_2, -3k_2, 5k_2) + (5k_3, -2k_3, 9k_3)$

$= (3k_1 + 2k_2 + 5k_3, k_1 - 3k_2 - 2k_3, 4k_1 + 5k_2 + 9k_3)$

$3k_1 + 2k_2 + 5k_3 = b_1$
$k_1 - 3k_2 - 2k_3 = b_2$
$4k_1 + 5k_2 + 9k_3 = b_3$

$Ak = b$;

$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 4 & 5 & 9 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 4 & 5 & 9 \end{vmatrix}$$

$= 3(-27+10) - 2(9+8) + 5(5+12)$

$= -51 - 34 + 85 = 0$

Therefore, the system is inconsistent.
So, every vector in $R^3$ can be expressed as a linear combination of vectors $v_1, v_2$ and $v_3$

Hence these vectors do not span the vector space $\mathbb{R}^3$.

13. $ax^2 + bx + c = k_1p_1 + k_2p_2 + k_3p_3 + k_4p_4$ or not

$ax^2 + bx + c = k_1(1 - x + 2x^2) + k_2(3 + x) +$
$k_3(5 - x + 4x^2) + k_4(-2 - 2x + 2x^2)$ ......(1)

$ax^2 + bx + c = (2k_1 + 4k_3 + 2k_4)x^2 +$
$(-k_1 + k_2 - k_3 - 2k_4)x + (k_1 + 3k_2 + 5k_3 - 2k_4)$

$2k_1 + 4k_3 + 2k_4 = a$
$-k_1 + k_2 - k_3 - 2k_4 = b$
$k_1 + 3k_2 + 5k_3 - 2k_4 = c$

$$\begin{bmatrix} 2 & 0 & 4 & 2 \\ -1 & 1 & -1 & -2 \\ 1 & 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The augmented matrix of the above system is,

$$\begin{bmatrix} 2 & 0 & 4 & 2 & a \\ -1 & 1 & -1 & -2 & b \\ 1 & 3 & 5 & -2 & c \end{bmatrix}$$

Multiply the first row by $1/2$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a/2 \\ -1 & 1 & -1 & -2 & b \\ 1 & 3 & 5 & -2 & c \end{bmatrix}$$

Add first row to the second row,

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a/2 \\ 0 & 1 & 1 & -1 & b+(a/2) \\ 1 & 3 & 5 & -2 & c \end{bmatrix}$$

Add -1 time the first row to the third row

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a/2 \\ 0 & 1 & 1 & -1 & b+(a/2) \\ 0 & 3 & 3 & -3 & c-(a/2) \end{bmatrix}$$

Add -3 times the second row to the third row

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a/2 \\ 0 & 1 & 1 & -1 & b+(a/2) \\ 0 & 0 & 0 & 0 & c-3b-2a \end{bmatrix}$$

$k_1 + 2k_3 + k_4 = a/2$

$k_2 + k_3 - k_4 = b + (a/2)$

$0 = c - 3b - 2a$

$0 = c - 3b - 2a$ is not true

Hence, the polynomials $p_1, p_2, p_3$ and $p_4$ do not span $P_2$.

15.

(a) $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$Ax = 0$

$$\begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the augmented matrix

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 3R_1$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & -4 & -5 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_1$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -3 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_2 + R_3$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow 2R_1 - R_2$

$$\begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

That is

$x = -\frac{1}{2}z \quad y = -\frac{3}{2}z$

The above solution represents the equation of a line passing through the origin

So,

$x = -\frac{1}{2}t \quad y = -\frac{3}{2}t$

$x = -\frac{1}{2}t, \; y = -\frac{3}{2}t$ and $z = t$

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 8 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & 8 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1 \qquad R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

That is:

$x + 3y = 0$

$y - 3z = 0$

$-z = 0$

Therefore, the above solution represents origin

(c)

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - 3y + z = 0$

The above sol. represents the plane passing through the origin. Hence it is the plane.

(d)

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -1 & 4 & 0 \\ 3 & 1 & 11 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 1 & 11 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 8 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = -3z$
$y = -2z$

$x = -3t \quad y = -2t \quad z = t$

16 (a) Let $S = \{f \in F \mid f$ is continuous$\}$
Let $f, g \in S$
Then $f$ and $g$ are continuous functions
So $f+g$ is also continuous
Therefore, $f+g \in S$
Let $k$ be any scalar and let $f \in S$
$f$ is continuous and $kf$ is continuous
$kf \in S$. Hence the given set is a subspace of $F(-\infty, \infty)$

(b) $S = \{ f \in F \mid f$ is differentiable$\}$

Let $f, g \in S$

Then $f$ and $g$ are differentiable functions

So $f+g$ is difference

Therefore $f+g \in S$.

Let $k$ be any scaler and let $f \in S$

$f$ is differentiable, $kf$ is also diff. able

$kf \in S$ and the given set is a subspace of $F(-\infty, \infty)$

(c) Let $S = \{ f \in F \mid f' + 2f = 0 \}$

Let $f, g \in S$

Then $f' + 2f = 0$ and $g' + 2g = 0$

Now

$(f+g)' + 2(f+g) = f' + g' + 2f + 2g$

$= (f' + 2f) + (g' + 2g) = 0 + 0 = 0$

Therefore, $f+g \in S$

Let $k$ be any scaller and let $f \in S$

Then $f' + 2f = 0$

Now,

$(kf)' + 2(kf) = kf' + 2kf$

$= kf' + k(2f) = k(f' + 2f) = k(0) = 0$

Therefore, $kf \in S$

Hence, the given set is a subspace of $F(-\infty, \infty)$

20. (a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$T_A(u_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T_A(u_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$T_A(u_3) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Consider $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

To show $a = b = 0$

Then,

$a + b = 0$

$-2b = 0$

$a = b = 0$

Hence the vectors are linearly independent.

Therefore the set $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ spans $R^2$

(1b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}$

$$T_A(u_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$T_A(u_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

And $T_A(u_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

Consider $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$$a\begin{bmatrix} 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$a - b = 0$

$-2a - 2b = 0$

$a = b = 0$

The vectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ ~~and the set~~ are linearly independent and the set spans $R^2$

21. Kernal $T_A = \{x \mid Ax = 0\}$

Now observe that the above system is a homogeneous system therefore, it has a trivial solution or non-trivial solution.

Consider the case when it has a trivial solution then in this case kernel will represent origin only.

Next if the system has non-trivial solution then either it has one free variable or two free variables.

In case of one free variable kernel will represent a line and in case of two free variable kernel will represents a plane.

Further, if A is a zero matrix then this case the kernel will contains all the points of $\mathbb{R}^3$ and hence kernel will be equal to $\mathbb{R}^3$.

5. (a) $S = \{v \in R^\infty \mid v = (v, 0, v, 0, v, 0 \dots)\}$

As $0 = (0, 0, 0, 0, 0 \dots)$ so $0 \in S$

Let $u, v \in S$

Then $u = (u, 0, u, 0, u, 0 \dots)$ and $v = (v, 0, v, 0, v, 0 \dots)$

Now

$$u + v = (u, 0, u, 0, u, 0) + (v, 0, v, 0, v, 0)$$
$$= (u + v, u + v, 0, u + v, 0, \dots)$$

Therefore, $u + v \in S$

Let $k$ be any scalar and $u \in S$

Then $u = (u, 0, u, 0, u, 0, \dots)$

Now

$$ku = k(u, 0, u, 0, u, 0, \dots)$$
$$= (ku, 0, ku, 0, ku, 0, \dots)$$

Therefore, $ku \in S$, for any scalar

Hence, the given set is a subspace of $R^\infty$

(b) $S = \{v \in R^\infty / v = (v, 1, v, 1, v, 1, \dots)\}$

Let $u, v \in S$

Then $u = (u, 1, u, 1, u, 1, \dots)$

$v = (v, 1, v, 1, v, 1, \dots)$

Now

$$u + v = (u, 1, u, 1, u, 1 \dots) + (v, 1, v, 1, v, 1, \dots)$$
$$= (u + v, 1 + 1, u + v, 1 + 1, u + v, 1 + 1 \dots)$$

$= (u+v, 2, u+v, 2, u+v, 2, \dots)$

Since $u+v$ is not in the form of $(a,1,a,1,a,1)$, therefore, $u+v \notin S$

Hence, the given set is not a subspace of $R^\infty$

(c) $0 = (0,0,0,0,\dots)$

$= (2, 2(0), 4(0), 8(0), \dots)$

Therefore, $0 \in S$

Let $u, v \in S$

Then $u = (u, 2u, 4u, 8u, 16u, \dots)$

$v = (v, 2v, 4v, 8v, 16v, \dots)$

Now

$u+v = (u+v, 2(u+v), 4(u+v), 8(u+v), 16(u+v), \dots)$

Therefore $u+v \in S$

Let $k$ be any scalar and $u \in S$

Then $u = (u, 2u, 4u, 8u, 16u, \dots)$

Now

$ku = k(u, 2u, 4u, 8u, 16u, \dots)$

$= (ku, k(2u), k(4u), k(16u) \dots)$

$= (ku, 2(ku), 4(ku), 8(ku), 16(ku), \dots)$

Therefore, $ku \in S$ for any scalar

Hence, the given set is a subspace of $R^\infty$

(d) $S = \{v \in R^\infty \mid$ The components of $v$ are $0$ from some point on.$\}$

Let $u, v \in S$

Then $u = (u_1, u_2, 0, 0, 0, \ldots)$
$v = (v_1, v_2, v_3, 0, 0, 0, \ldots)$

Now

$u+v = (u_1, u_2, 0, 0, 0 \ldots) + (v_1, v_2, v_3, 0, 0, 0 \ldots)$
$= (u_1+v_1, u_2+v_2, 0+v_3, 0, 0, \ldots)$
$= (u_1+v_1, u_2+v_2, v_3, 0, 0, \ldots)$

Therefore $u+v \in S$

Let $k$ be any scalar and $u \in S$

Then $u = (u_1, u_2, 0, 0, 0)$

Now

$ku = k(u_1, u_2, 0, 0, 0, \ldots)$
$= (ku_1, ku_2, k(0), k(0), \ldots)$
$= (ku_1, ku_2, 0, 0, 0, \ldots)$

Therefore, for any scalar $k$, $ku \in S$

Hence, the given set is a subspace of $R^\infty$

8) (a) $u = (2,1,4)$, $v = (1,-1,3)$ $w = (3,2,5)$

Let $c_1, c_2, c_3 \in \mathbb{R}$

$$(-9,-7,-15) = c_1(2,1,4) + c_2(1,-1,3) + c_3(3,2,5)$$

$$(-9,-7,-15) = (2c_1, c_1, 4c_1) + (c_2, -c_2, 3c_2) + (3c_3, 2c_3, 5c_3)$$

$$(-9,-7,-15) = (2c_1 + c_2 + 3c_3,\ c_1 - c_2 + 2c_3,\ 4c_1 + 3c_2 + 5c_3)$$

$$2c_1 + c_2 + 3c_3 = -9$$
$$c_1 - c_2 + 3c_3 = -7$$
$$4c_1 + 3c_2 + 5c_3 = -15$$

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 3 & -15 \end{array}\right] R_1 \to \tfrac{1}{2}R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & -9/2 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & -9/2 \\ 0 & -3/2 & 1/2 & -5/2 \\ 0 & 1 & -1 & 3 \end{array}\right] R_2 \to (-2/3)R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & -9/2 \\ 0 & 1 & -1/3 & 5/2 \\ 0 & 1 & -1 & 3 \end{array}\right] R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & -9/2 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & -2/3 & 4/3 \end{array}\right] R_3 \to (-3/2) R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & -9/2 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$c_1 + 1/2\, c_2 + 3/2\, c_3 = -9/2$ ....... (1)

$c_2 - 1/3\, c_3 = 5/3$ ....... (2)

$c_3 = -2$ ....... (3)

Substitute $c_3 = -2$ in equation (2), then

$c_2 - 1/3(-2) = 5/3$

$c_2 = 5/3 - 2/3$

$c_2 = 1$

Now substitute $c_3 = -2$ and $c_2 = 1$ values in (1)

$c_1 + \frac{1}{2}(1) + \frac{3}{2}(-2) = -\frac{9}{2}$

$c_1 + \frac{1}{2} - 3 = -\frac{9}{2}$

$c_1 = -\frac{9}{2} + 3 - \frac{1}{2}$

$c_1 = -2$

Therefore the vector $(-9, \cancel{8} -7, -15)$ can be expressed as a linear combination of given vectors as $(-9, -7, -15) = -2(2, 1, 4) + 1(1, -1, 3) - 2(3, 2, 5) = -2u + v - 2w$

(b) $u(2,1,4)$, $v = (1, -1, 3)$, $w = (3, 2, 5)$

$(6, 11, 6) = c_1(2,1,4) + c_2(1,-1,3) + c_3(3,2,5)$

$(6, 11, 6) = (2c_1, c_1, 4c_1) + (c_2, -c_2, 3c_2) + (3c_3, 2c_3, 5c_3)$

$(6, 11, 6) = (2c_1 + c_2 + 3c_3, c_1 - c_2 + 2c_3; 4c_1 + 3c_2 + 5c_3)$

$2c_1 + c_2 + 3c_3 = 6$

$c_1 - c_2 + 2c_3 = 11$

$4c_1 + 3c_2 + 5c_3 = 6$

$$[A\,|\,b] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \quad R_1 \to \tfrac{1}{2}R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 3 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 3 \\ 0 & -3/2 & 1/2 & 8 \\ 0 & 1 & -1 & -6 \end{array}\right] \quad R_2 \to (-2/3)R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 3 \\ 0 & 1 & -1/3 & -16/3 \\ 0 & 1 & -1 & -6 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 3 \\ 0 & 1 & -1/3 & -16/3 \\ 0 & 0 & -2/3 & -2/3 \end{array}\right] \quad R_3 \to (-3/2)R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 3 \\ 0 & 1 & -1/3 & -16/3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$c_1 + \frac{1}{2}c_2 + \frac{3}{2}c_3 = 3$ ...... (1)

$c_2 - \frac{1}{3}c_3 = -\frac{16}{3}$ ..... (2)

$c_3 = 1$ ........ (3).

Sub. $c_3 = 1$ in eq (2), then

$c_2 - \frac{1}{3}(1) = -\frac{16}{3}$

$c_2 = -\frac{16}{3} + \frac{1}{3} = -5$

Sub. $c_3 = 1$ and $c_2 = -5$ in (1), then

$c_1 + \frac{1}{2}(-5) + \frac{3}{2}(1) = 3$

$c_1 = \frac{5}{2} - \frac{3}{2} + 3 = 4$

$(6, 11, 6) = 4(2, 1, 4) - 5(1, -1, 3) + 1(3, 2, 5)$

$= 4u - 5v + w$

(c) $u = (2, 1, 4)$; $v = (1, -1, 3)$ and $w = (3, 2, 5)$

$(0, 0, 0) = (2c_1 + c_2 + 3c_3;\ c_1 - c_2 + 2c_3,\ 4c_1 + 3c_2 + 5c_3)$

$2c_1 + c_2 + 3c_3 = 0$

$c_1 - c_2 + 2c_3 = 0$

$4c_1 + 3c_2 + 5c_3 = 0$

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] R_1 \to \tfrac{1}{2} R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 0 \\ 0 & -3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] R_2 \to (-2/3) R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & -2/3 & 0/3 \end{array}\right] R_3 \to (-3/2) R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$c_1 + \frac{1}{2}c_2 + \frac{3}{2}c_3 = 0 \quad \ldots\ldots (1)$$

$$c_2 - \frac{1}{2}c_3 = 0 \quad \ldots\ldots (2)$$

$$c_3 = 0 \quad \ldots\ldots (3)$$

Sub. $c_3 = 0$ in eq. (2)

$$c_2 - \frac{1}{3}(0) = 0$$

$$c_2 = 0$$

Now substitute $c_3 = 0$ and $c_2 = 0$ values in (1), then

$$c_1 + \frac{1}{2}(0) + \frac{3}{2}(0) = 0$$

$$c_1 = 0$$

$$(0,0,0) = 0(2,1,4) + 0(1,-1,3) + 0(3,2,5)$$

$$= 0u + 0v + 0w$$