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1) $x_1 + x_2 - x_3 = 0$
$-2x_1 - x_2 + 2x_3 = 0$
$-x_1 + x_3 = 0$

Augmented matrix
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_1 \to -R_2 + R_1$
$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The obtained eqs.

$x_1 - x_3 = 0$, $x_2 = 0$. Here x_3 is a free variable

So $x_3 = k$

$(x_1, x_2, x_3) = (k, 0, k) = k(1, 0, 1)$

The basis for the solution space of the system is $(1, 0, 1)$. The dimension of the space is 1.

2)

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[A|0] = \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right] \quad R_2 \to 3R_2 - 5R_1$$

$$= \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & -8 & -2 & -8 & 0 \end{array}\right] \quad R_2 \to -1/8\, R_2$$

$$= \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1/4 & 1 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$= \left[\begin{array}{cccc|c} 3 & 0 & 3/4 & 0 & 0 \\ 0 & 1 & 1/4 & 1 & 0 \end{array}\right] \quad R_1 \to 1/3\, R_1$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & 1/4 & 1 & 0 \end{array}\right]$$

$x_1 + 1/4\, x_3 = 0 \qquad x_2 + 1/4\, x_3 + x_4 = 0$

2 eqs so 2 free variables

Suppose that x_3, x_4 are free variables

That is, $x_3 = r$, $x_4 = s$. Here $s, t \in \mathbb{R}$

$x_1 = -1/4\, r \qquad x_2 = -1/4\, r - s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left\{ r\begin{bmatrix} -1/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} : r, s \in R \right\}$$

Hence the basis for the solution space of the homogeneous linear system is $\{(-1/4, -1/4, 1, 0), (0, -1, 0, 1)\}$

Hence the dimension of the solution space is, 2

3. $$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

$R_3 \to -2R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -7 & 0 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix}$$

$R_3 \to R_3 / -8$

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 + x_3 = 0 \Rightarrow x_2 = 0$ $\quad v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 5x_3 = 0 \Rightarrow x_1 = 0$

Thus, the vector $(0,0,0)$ does not form a basis and the dimension is zero.

7) a) $x = 2/3\, y - 5/3\, z$

Suppose $y = s$ and $z = t$

$(x,y,z) = \left(2/3\, s - 5/3\, t, s, t\right)$, for all s and t.

$(x,y,z) = \left(2/3, 1, 0\right)s + \left(-5/3, 0, 1\right)t$

Therefore the vectors $(2/3, 1, 0)$ and $(-5/3, 0, 1)$ space is the solution space

$k_1 v_1 + k_2 v_2 = 0$

$k_1 \left(2/3, 1, 0\right) + k_2 \left(-5/3, 0, 1\right) = (0,0,0)$

$2/3\, k_1 - 5/3\, k_2 = 0$

$k_1 + 0 \cdot k_2 = 0$

$0 \cdot k_1 + k_2 = 0$

This gives the trivial solutions $k_1 = 0, k_2 = 0$, therefore these vectors are linearly independent.

Hence a basis is $(2/3, 1, 0), (-5/3, 0, 1)$ and its dimension is 2.

(b) $x = y$

Suppose $y = s$ and $z = t$, vector eq. can be written as

$(x, y, z) = (s, s, t)$; for all s and t

$(x, y, z) = (1, 1, 0)s + (0, 0, 1)t$

$k_1 v_1 + k_2 v_2 = 0$

$k_1(1,1,0) + k(0,0,1) = (0,0,0)$

$1 \cdot k_1 + 0 \cdot k_2 = 0$

$1 \cdot k_1 + 0 \cdot k_2 = 0$

$0 \cdot k_1 + 1 \cdot k_2 = 0$

This gives the trivial solutions $k_1 = 0, k_2 = 0$. The vectors are linearly independent.

Hence a basis $(1,1,0), (0,0,1)$ and its dimension is 2.

(c) $x = 2t, y = -t, z = 4t$

Points on this line $\Rightarrow t(2,-1,4)$ where $t \in R$

Thus the vector $(2,-1,4)$ is the sol. space for this line. There is only one vector and it is independent.

Hence a basis is $(2,-1,4)$ and its dimension is 1.

d) The vector eq. $\Rightarrow (a,b,c) = (a, a+c, c)$

$(a,b,c) = (1,1,0)a + (0,1,1)c$

The vectors $(1,1,0)$ and $(0,1,1)$ is the solution space.

$k_1v_1 + k_2v_2 = 0 \Rightarrow k_1(1,1,0) + k_2(0,1,1) = (0,0,0)$

$1 \cdot k_1 + 0 \cdot k_2 = 0$, $1 \cdot k_1 + 1 \cdot k_2 = 0$, $0 \cdot k_1 + 1 \cdot k_2 = 0$

The trivial solutions are $k_1 = 0$, $k_2 = 0$

The vectors are linearly independent.

Hence a basis is $(1,1,0), (0,1,1)$ and its dimension is 2.

9) a) The vector space of $n \times n$ diagonal matrices consists at most n linearly independent, solutions since they have n different diagonals each is independent vector.

Therefore, the dimension of the system is n.

b) Suppose V be the vector space of all symmetric $n \times n$ matrices

Let $V = E_{ij}$, here $E_{ij} = (a_{ij})$, such that $a_{ij} = a_{ji}$

All the entries below the diag. of symmetric matrix equals to their reflections above the diagonals.

That is, $\{E_{ij} : 1 \le i, j \le n, i \ne j\}$ and the "n" diagonal elements.

Therefore, the elements in the set $\{E_{ij} : 1 \le i, j \le n, i \ne j\}$ are,

Hence, the required dimension is, $\frac{n(n-1)}{2}$

$$\frac{n(n-1)}{2} + n = \frac{n(n-1)+2n}{2} = \frac{n^2-n+2n}{2}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

Therefore, the dimension of the vector space of all symmetric $n \times n$ matrices is $\frac{n(n+1)}{2}$.

(c) Suppose $\{E_{ij}\}$ is set of upper triangular matrices, here E_{ij} is the matrix with 1 in $(i-j)^{th}$ position and 0's elsewhere for $i \le j$.

These matrices are linearly independent.

There is 1 for every diagonal entry position. So, there exists n matrices.

There is 1 for every super-diagonal entry position. So, there exist $n-1$ matrices and so on.

That is sum of matrices is, $n + (n-1) + (n-2) + \dots + 1 = \frac{n(n+1)}{2}$

Since the sum of first n natural numbers is, $\frac{n(n+1)}{2}$. Therefore, the dimension of the vector space of all upper triangle $n \times n$ matrices is,

$$\frac{n(n+1)}{2}$$

10. The subspace of P_3 consists of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$

Then $a_0 + a_1x + a_2x^2 + a_3x^3 = a_1x + a_2x^2 + a_3x^3$

So, three variables a_1, a_2, a_3 are free variables.

Thus, dimension being equal to number of free variables.

Therefore, the dimension of the subspace is 3.

12. (a) $v_1 = (-1, 2, 3)$, $v_2 = (1, -2, -2)$

A basis set of R^3 is with a minimum of 3 linearly independent vectors.

$A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & -2 \end{bmatrix}$ with $\{v_1, v_2\}$

A into echolon form

$R_2 \to R_2 + R_1$

$R_1 \to -R_1$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Since all rows in the echelon form of A are nonzero, So the set $\{v_1, v_2\}$ is linearly independent.

Check whether the set $\{v_1, v_2, e_3\}$ is linearly independent. Assume e_2 is desired vector that is not linear combination of v_1 and v_2.

$$B = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Since det $B = -1 \neq 0$, so the set $\{v_1, v_2, e_2\}$ is linearly independent.

Hence, the vector e_2 can be added to the set to produce a basis for R^3.

(b) $v_1 (1, -1, 0)$, $v_2 = (3, 1, -2)$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix} \text{ with } \{v_1, v_2\}$$

$R_2 \to R_2 - 3R_1$ $\quad$ $R_2 \to 1/4 R_2$ $\quad$ $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \end{bmatrix}$$

Since all rows in the echelon form of A are nonzero, so the set $\{v_1, v_2\}$ is linearly independent

$$C = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since det $C = 1 \neq 0$, the set $\{v_1, v_2, e_3\}$ is linearly independent

Hence, the vector e_3 can be added to the set $\{v_1, v_2\}$ to produce a basis for R^3.

13. $v_1 = (1, -4, 2, -3)$, $v_2 = (-3, 8, -4, 6)$

Form a matrix A with row entries as the vectors $\{v_1, v_2\}$

$$A = \begin{bmatrix} 1 & -4 & 2 & -3 \\ -3 & 8 & -4 & 6 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$ $\quad$ $R_1 \to R_1 - R_2$ $\quad$ $R_2 \to -\frac{1}{4} R_2$

$$\begin{bmatrix} 1 & -4 & 2 & -3 \\ 0 & -4 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 3/4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 3/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ from $\{v_1, v_2, e_2, e_3\}$

Since $\det B = 3/4 \neq 0$, so the set is linearly independent.

$\det \{v_1, v_2, e_2, e_4\} = 1/2 \neq 0$

$\det \{v_1, v_2, e_3, e_4\} = 1 \neq 0$

Hence the vectors e_2 and e_3 or e_2 and e_4 or e_3 and e_4 can be added to the set $\{v_1, v_2\}$ to produce a basis for R^4.

15) Let scalars k_1, k_2 and k_3 such that

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$k_1(1,-2,3) + k_2(0,5,-3) + k_3(a,b,c) = 0$

$(k_1 + ak_3, -2k_1 + 5k_2 + bk_3, 3k_1 - 3k_2 + ck_3) = 0$

This gives

$k_1 + 0k_2 + ak_3 = 0$
$-2k_1 + 5k_2 + bk_3 = 0$
$3k_1 - 3k_2 + ck_3 = 0$

$$\begin{bmatrix} 1 & 0 & a & 0 \\ -2 & 5 & b & 0 \\ 3 & -3 & c & 0 \end{bmatrix}$$

$R_2 \to 2R_1 + R_2$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 5 & b+2a & 0 \\ 0 & -3 & c-3a & 0 \end{bmatrix}$$

$R_2 \to R_2/5$

$$\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2/5a + 1/5b & 0 \\ 0 & -3 & c-3a & 0 \end{bmatrix}$$

$R_3 \to 3R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2/5a + 1/5b & 0 \\ 0 & 0 & -9/5a + 3/5b + c & 0 \end{bmatrix}$$

Trivial solution of homogeneous system if

$-9/5a + 3/5b + c \neq 0$

Adding any vector $v_3 = (a,b,c)$ with this condition to the set $\{v_1, v_2\}$ will create a basis of $\mathbb{R}^3$

16. $v_3 = (0,0,1,0)$, $v_4 = (0,0,0,1)$

$$[v_1, v_2, v_3, v_4] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The determinant of the above matrix

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$$

Therefore v_3, v_4 are the desired vectors and $\{v_1, v_2, v_3, v_4\}$ form a basis for $\mathbb{R}^4$.

17. $v_1 = (1,0,0)$, $v_2 = (1,0,1)$, $v_3 = (2,0,1)$,
$v_4 = (0,0,-1)$

$v_1 + v_2 = (1+0, 0+0, 0+1) = (2,0,1) = v_3$

$v_1 + (-1)v_2 = (1-1, 0+0, 0-1) = (0,0,-1) = v_4$

v_3 and v_4 can be discarded

$k_1 v_1 + k_2 v_2 = 0$

$k_1(1,0,0) + k_2(1,0,1) = (0,0,0)$

This gives the linear system

$1 \cdot k_1 + 1 \cdot k_2 = 0$

$0 \cdot k_1 + 0 \cdot k_2 = 0$

$0 \cdot k_1 + 1 \cdot k_2 = 0$

This gives the trivial solutions $k_1 = 0$, $k_2 = 0$ therefore these vectors are linearly independent.

Hence the basis for subspace R^3 is $v_1 = (1, 0, 0)$, $v_2 = (1, 0, 1)$

19. (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Consider that $Ax = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

System of equation

$x_1 + x_2 = 0$ Here, x_2 is a free variable.

$-x_2 + x_3 = 0$ Thus, $x_1 = -x_2$, $x_3 = x_2$

Therefore, the vector x can be written as,

$$x = (-x_2, x_2, x_2) = x_2(-1,1,1)$$

~~There~~ This shows that $x \in \text{span}\{(-1,1,1)\}$

And the vector $(-1,1,1)$ is linearly independent.

Therefore, the basis of the subspace of $\mathbb{R}^3$ is $\{(-1,1,1)\}$. Hence the dimension of the subspace of $\mathbb{R}^3$ is 1.

(b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ Consider $Ax = 0$

Augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives $x_1 + 2x_2 = 0$ and x_2, x_3 are free variable

Therefore $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Both are linearly independent set of vectors,
Hence, the dimension of the subspace of R^3
is 2.

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Consider that $Ax = 0$

This implies that,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Solve this to get,

$x_1 = 0$
$x_1 = x_2$
$x_1 + x_2 + x_3 = 0$

Simplify this to get $x_1 = x_2 = x_3 = 0$
Therefore, this system has only the trivial solution.

Hence, the dimension of the subspace of R^3
is 0.