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$A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$.

Characteristic eq. of B is $|B - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$ $(1-\lambda)(-2-\lambda) - 3(0) = 0$

$(1-\lambda)(-2-\lambda) = 0$

$\lambda = 1, -2$

Eigenvector for $\lambda = 1$

$BX = \lambda X \Rightarrow (B - 1 \cdot I)X = 0$

$\begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix} X = 0$ $\begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x_1 - 3x_2 = 0$. Put, $x_1 = k_1$ where k is the parameter.

Then, $x_2 = k_1$ Therefore, $X = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix}$

$X = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvector for $\lambda = 2$

$(B + 2 \cdot I) \cdot X = 0$

$\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix} X = 0$, $R_2 \rightarrow R_2 - R_3$

$\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $3x_1 = 0$

$3x_1 = 0$

Put, $x_2 = k_2$ Where k is the parameter

Therefore, $X = \begin{bmatrix} 0 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Calculation of $P^{-1} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$

$A \cdot P = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$

$P^{-1} \cdot A \cdot P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \neq B$

Therefore, A and B are not similar matrices.

2. $\det A = \begin{vmatrix} 4 & -1 \\ 2 & 4 \end{vmatrix} = 16 + 2 = 18$

$\det A \neq \det B$

$\det B = \begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} = 16 - 2 = 14$

Thus A, B are not similar matrices

4. $\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 0 & -1 \\ -2 & \lambda - 0 & -2 \\ -3 & 0 & \lambda - 3 \end{pmatrix}$

$= (\lambda - 1)(\lambda(\lambda - 3) - 0) - 0((-2)(\lambda - 3) - 6) - 1(0 + 3\lambda)$

$= \lambda(\lambda - 1)(\lambda - 3) - 3\lambda = \lambda(\lambda^2 - 4\lambda + 3) - 3\lambda$

$= \lambda^3 - 4\lambda^2$

$\Rightarrow \lambda^3 - 4\lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda - 4) = 0 \Rightarrow \lambda = 0, 0, 4$

So the eigenvalues of A are $0, 0, 4$.

$$\det(\lambda I - B) = \det\begin{pmatrix} \lambda-1 & -1 & 0 \\ -2 & \lambda-2 & 0 \\ 0 & -1 & \lambda-1 \end{pmatrix}$$

$= (\lambda-1)(\lambda-2)(\lambda-1) + 1(-2(\lambda-1) - 0) + 0(2-0) = (\lambda-1)(\lambda-2)(\lambda-1) - 2(\lambda-1)$

$= (\lambda-1)(\lambda^2 - 3\lambda + 2 - 2) = (\lambda-1)(\lambda^2 - 3\lambda)$

$\Rightarrow (\lambda-1)(\lambda^2 - 3\lambda) = 0$

$\Rightarrow \lambda(\lambda-1)(\lambda-3) = 0 \Rightarrow \lambda = 0, 1, 3$

So the Eigen values of A, B are different

Hence the matrices A, B are not similar

5. $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ $\det(\lambda I - A) = 0$

$\begin{vmatrix} \lambda-1 & 0 \\ -6 & \lambda+1 \end{vmatrix} = 0$ $\lambda^2 - 1 = 0$ $\lambda = 1, -1$

$(\lambda-1)(\lambda+1) = 0$

So, the eigenvalues of the matrix A are $\lambda_1 = 1$ and $\lambda_2 = -1$.

For $\lambda_1 = 1$

$(1 \cdot I - A)x_1 = 0$

$$\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Take,

$\Rightarrow -6x_1 + 2x_2 = 0$

$-3x_1 + x_2 = 0$

$\Rightarrow -3x_1 + x_2 = 0$

$x_2 = 3x_1$

Let $x_1 = t$ So, $x_2 = 3t$

Then the eigenvector for $\lambda_1 = 1$ is $= t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Therefore the basis for the Eigen space $\lambda_1 = 1$ is $\{(1,3)\}$

For $\lambda_2 = -1$

$(\lambda_2 I - A)x_2 = 0$ $(-1I - A)x_2 = 0$

$$\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2x_1 = 0$$

$x_1 = 0$

Let $x_2 = t$ Then the eigenvector corresponding to $\lambda_2 = -1$ is $t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Therefore the basis for the eigenspace for $\lambda_2 = -1$ is $\{(0,1)\}$

Hence the required matrix is $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

So, $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$

So, the diagonal elements are the eigenvalues

Therefore, a matrix P that diagonalizes A.

6. $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$ $\quad \det(\lambda I - A) = 0$

$\lambda I - A = \begin{bmatrix} \lambda + 14 & -12 \\ 20 & \lambda - 17 \end{bmatrix}$

$\det(\lambda I - A) = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0$

$\Rightarrow \lambda = 1$ and $\lambda = 2$

For $\lambda = 1$

$\begin{bmatrix} 15 & -12 \\ 20 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = t$ and $x_1 = 4/5\, t$

Thus the eigenvectors of A corresponding to $\lambda = 1$ are the non zero vectors

$x = t \begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$

Since $\begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$ is linearly indipendent, these vectors form a basis for the eigenspace corr. to $\lambda = 1$

If $\lambda = 2$

$$\begin{bmatrix} 16 & -12 \\ 20 & -15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \tfrac{3}{4}t \text{ and } x_2 = t$$

Thus eigenvector of A $= t\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$. This vector also forms a basis for the eigenspace corresponding to $\lambda = 2$

Therefore, $P = \begin{bmatrix} 4/5 & 3/4 \\ 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 20 & -15 \\ -20 & 16 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 20 & -15 \\ -20 & 16 \end{bmatrix}\begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}\begin{bmatrix} 4/5 & 3/4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & -15 \\ -20 & 16 \end{bmatrix}\begin{bmatrix} 4/5 & 3/2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

So the diagonal elements are the Eigen values

7. $\lambda = 2$, $\lambda = 3$ ~~are~~ for A

The eigenvector of A ~~$(\lambda I - A)x = 0$~~ are the non zero solutions of $(\lambda I - A)x = 0$

$$\begin{bmatrix} \lambda - 2 & 0 & 2 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 3$, the Eigen vector is given as follows

$$\begin{bmatrix} 3-2\cdot 1 & 0 & 2 \\ 0 & 3-3=0 & 0 \\ 0 & 0 & 3-3=0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0 \Rightarrow x_2 = s, x_3 = t$. Then the solution to the system is, $x_1 = -2t$, $x_2 = s, x_3 = t$

That is,

$$x = \begin{bmatrix} -2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$ So, the Eigen vectors are,

$p_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Hence, $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the Eigen space of A corresponding to eigenvalue $\lambda = 3$. For $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{R_1}{2}, \frac{R_2}{-1}, R_1 \leftrightarrow R_2', R_3 + R_2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_2 = 0 \\ x_3 = 0 \end{array}$$

$x_1 = t, x_2 = 0, x_3 = 0; \quad x = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Therefore, $P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigenvector correspond-

..ing to the eigenvalue $\lambda = 2$

$\begin{vmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1(-1) \neq 0$. So the vectors are linearly independent

$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

8. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & -1 & \lambda - 1 \end{vmatrix} = 0$$

$(\lambda - 1)(\lambda^2 - 2\lambda) = 0$

$\lambda(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 0, 1, 2$

So, the eigenvalues of A are $\lambda = 0, 1, 2$

For $\lambda_1 = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 0, \\ -x_2 - x_3 = 0 \end{matrix}$$

Put $x_3 = t \Rightarrow x_2 = -t$

$x_1 = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Therefore, the basis for the eigenspaces for $\lambda_1 = 0$ is $\{(0,-1,1)\}$

For $\lambda_2 = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0, x_3 = 0$$

Put $x_1 = t \Rightarrow x_2 = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

The basis is $\{(1,0,0)\}$

For $\lambda_3 = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 0, x_2 - x_3 = 0, \\ -x_2 + x_3 = 0 \end{array}$$

Put $x_3 = t \Rightarrow x_2 = t$. So $x_2 = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Therefore, the basis for the eigenspaces for $\lambda_3 = 2$ is $\{(0,1,1)\}$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

So, $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Therefore, a matrix P that diagonalizes A.

11. $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$, $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$(-1-\lambda)[(4-\lambda)(3-\lambda)-1] - 4[-3(3-\lambda)+0] - 2[-3+3(4-\lambda)] = 0$

$(-1-\lambda)[12-7\lambda+\lambda^2] - 4[-9+3\lambda] + 6 - 6(4-\lambda) = 0$

$-\lambda^2+7\lambda-12 - 12\lambda+7\lambda^2-\lambda^2+36-12\lambda+6-24+6\lambda = 0 \Rightarrow \lambda^3-6\lambda^2+11\lambda-6 = 0$

$(\lambda-1)(\lambda^2-5\lambda+6) = 0$

$(\lambda-1)(\lambda-2)(\lambda-3)=0 \Rightarrow \lambda=1,2,3$

Hence, algebraic multiplicity of each eigenvalue is. 1.

$$\begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots\ldots (1)$$

Sub $\lambda=1$ in (1)

$$\begin{bmatrix} -1-1 & 4 & -2 \\ -3 & 4-1 & 0 \\ -3 & 1 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1+4x_2-2x_3=0$

$-3x_1+3x_2=0$

$-3x_1+x_2+2x_3=0$

Solve to get $x_1=1$, $x_2=1$ and $x_3=1$

There eigen vector for $\lambda=1$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Hence geometric multiplicity of $\lambda=1$ is also 1.

Substitute $\lambda = 2$ in matrix system (1)

$$\begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left.\begin{aligned} -3x_1 + 4x_2 - 2x_3 &= 0 \\ -3x_1 + 2x_2 &= 0 \\ -3x_1 + x_2 + x_3 &= 0 \end{aligned}\right\}$$ Solve to get $x_1 = 2/3$, $x_2 = 1$ and $x_3 = 1$

Therefore eigen vector corresponding to $\lambda = 1$ is

$$= \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

Hence, geometric multiplicity of $\lambda = 2$ is also 1.

Substitute $\lambda = 3$ in matrix system (1)

$$\begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -4x_1 + 4x_2 - 2x_3 &= 0 \\ -3x_1 + x_2 &= 0 \\ -3x_1 + x_2 &= 0 \end{aligned}$$

Solve to get

$x_1 = 1/4$, $x_2 = 3/4$ and $x_3 = 1$

Therefore, the eigen vector corresponding to $\lambda=3$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Hence geometric multiplicity of $\lambda=3$ is also 1.

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 & -2 \\ 3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

14. $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$ $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 1 & 5-\lambda & 0 \\ 0 & 1 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)\begin{vmatrix} 5-\lambda & 0 \\ 1 & 5-\lambda \end{vmatrix}$$

$$- 0 \begin{vmatrix} 1 & 0 \\ 0 & 5-\lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & 5-\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$$(5-\lambda)[(5-\lambda)(5-\lambda) - 0] - 0 + 0 = 0$$

$(5-\lambda)^3 - 0 = 0$

$\lambda = 5, 5, 5$

Let $v = (x_1, x_2, x_3)^T$ be the eigenvector.

$(A - 5 \cdot I)v = 0$

$$\begin{pmatrix} 5-5 & 0 & 0 \\ 1 & 5-5 & 0 \\ 0 & 1 & 5-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This leads to system, $x_1 = 0$, $x_2 = 0$

~~that~~ $x_3 = r$, $r \in R$

Basis for the eigenspace corresponding to eigenvalue $\lambda = 5$ is,

$$V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} : r \in R \right\} = \left\{ r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : r \in R \right\}$$

Therefore; geometric multiplicity of the eigenvalue $\lambda = 5$ is 1.

Algebraic multiplicity $\neq$ geometric multiplicity

Hence, the given matrix is not diagonalizable.

15. (a) The characteristic equation is $(\lambda-1)(\lambda+3)(\lambda-5)=0$, so this is an cubic equation, therefore the size of the matrix A is 3X3.

The Eigen values of matrix A is $\lambda_1=1$, $\lambda_2=-3$, and $\lambda_3=5$. Clearly these Eigen value have algebriac multiplicity 1.

Now A is an 3X3 matrix with 3 distinct Eigenvalues therefore, by the above result matrix A is diagonalizable, therefore by the above theorem the geometric multiplicity of each eigen value of A is 1.

This shows that for each eigin value λ_1, its corresponce Eigenspace A has dimension 1.

(b) Polynomial in the left hand side has degree 1 therefore, matrix A is of order 6X6.

The eigenvalues of A are $\lambda_1=0$, $\lambda_2=1$ and $\lambda_3=2$. Algebriac multiplicity is 2, 1 and 3 respectively.

E_λ denotes the Eigenspace for Eigen value λ, then the possible dimensions are as follows $\dim(E_0)=1,2$ $\dim(E_1)=1$ $\dim(E_2)=1,2,3$

16. (a) $\lambda^3(\lambda^2-5\lambda-6)=0$

$\lambda^3(\lambda^2-5\lambda-6)=0$

$\lambda^3(\lambda^2-6\lambda+\lambda-6)=0$

$\lambda^3(\lambda(\lambda-6)+(\lambda-6))=0$

$\lambda^3((\lambda+1)(\lambda-6))=0$

$\lambda=0,0,0,-1,6$

$$A=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

For $\lambda=0$.

$(A-\lambda I)=A$. Rank of A is 2.
Therefore dimension of Eigen space corresponding to Eigen value $\lambda=0$ is 3.

For $\lambda=-1$

$$(A+\lambda I)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&7\end{bmatrix}$$

Rank of the matrix is 4

Therefore dimension of the Eigen space corresponding to Eigen value $\lambda=-1$ is 1.

For $\lambda=6$

$(A - \lambda I) = \begin{bmatrix} -6 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Rank of the matrix is 4

Therefore dimension of Eigen space corresponding to Eigen value $\lambda = 6$ is 1.

(b) $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda - 1)^3 = 0 \Rightarrow \lambda = 1, 1, 1$ $A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(A - \lambda I)$ For $\lambda = 1$

$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Rank is 0.

Therefore dimension of Eigen space corresponding to Eigen value $\lambda = 0$ is 3.

18. $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

Characteristic eq. of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda = 1, \lambda = 2$

Eigenvector corresponding to $\lambda = 1$

$AX = \lambda X$

$\Rightarrow (A - \lambda I)X = 0 \Rightarrow (A - I)X = 0$

$$\Rightarrow \left[\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] X = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} X = 0, \; R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -x_1 + x_2 = 0$ ...... (1)

Let $x_2 = k$ where k is the parameter

Plug in x_2 in (1) then $x_1 = k$

Therefore, $X = \begin{bmatrix} k \\ k \end{bmatrix} = k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

For $\lambda = 2$

$AX = \lambda X \Rightarrow (A - \lambda I)X = 0 \Rightarrow (A - 2I)X = 0$

$$\Rightarrow \left[\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] X = 0$$

$$\Rightarrow \left[\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right] X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} X = 0, \quad R_2 \rightarrow R_2 - R_1$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow x_1 = 0 \quad \ldots\ldots (1)$$

Let $x_2 = k$ where k is the parameter

Therefore, $X = \begin{bmatrix} 0 \\ k \end{bmatrix} \Rightarrow X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the Eigenvector corresponding to $\lambda = 2$

$$P = [x_1 \; x_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad |P| = 1$$

$$\text{adj } P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1} A \cdot P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= D$$

$D = P^{-1} \cdot A \cdot P$

$A^{10} = P \cdot D^{10} \cdot P^{-1}$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1024 & 1024 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 1 & 0 \\ -1023 & 1024 \end{bmatrix}$$

19. $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 5 \end{bmatrix}$

$|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda+1 & -7 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & 15 & \lambda+2 \end{vmatrix} = 0$$

$(\lambda+1)((\lambda-1)(\lambda+2) - 15(0)) - (-7)(0) + 1(0) = 0$

$(\lambda+1)(\lambda-1)(\lambda+2) = 0$

$\lambda = 1, 1, 2$

Hence the eigenvalues of A are $-1, 1$ and 2

$$P^{-1} = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix P diagonalizes the matrix A.

Now find the matrix A^{11}.

As $P^{-1}AP = D$, the matrix A can be written as $A = PDP^{-1}$. $\Rightarrow A^{11} = PD^{11}P^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} -2048 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$$ is the required matrix.

20.

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 1 & 7 & -8 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix} = 0$$

$$(\lambda-1)[(\lambda+1)(\lambda+1) - 0(0)] - (-7)(0) - 8(0) = 0$$

$$(\lambda-1)(\lambda+1)(\lambda+1) = 0 \quad \lambda = -1, 1$$

Hence, the eigenvalues of the matrix A are -1 and 1.

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As $P^{-1}AP = D$ so the matrix P diagonalizes the matrix A.

a) $A^{1000} = PD^{1000}P^{-1}$

$$= \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, which is the required matrix

$A^{-1000} = PD^{-1000}P^{-1}$

$$= \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ which is the required matrix