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Exercise 2.2

1. $A^T = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$

$\det(A) = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = (-2)(4) - (3)(1) = -11$

$\det(A^T) = \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = (-2)(4) - (3)(1)$
$= -11$

$\det(A) = \det(A^T) = -11$

2. $\det(A) = \begin{bmatrix} -6 & 1 \\ 2 & -2 \end{bmatrix} = (-6)(-2) - 1(2)$
$= 10$

$\det(A^T) = \begin{vmatrix} -6 & 2 \\ 1 & -2 \end{vmatrix} = (-6)(-2) - (1)(2)$
$= 10$

$\det(A) = 10 = \det(A^T)$

3. We know that $\det I = 1$; where I is $n \times n$ elementary matrix

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -5.$$

6. Let E be an $n \times n$ elementary matrix. If E results from adding a multiple of one row o I_n to another, then $\det(E) = 1$

Thus, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{vmatrix} = 1$

7. The matrix A is obtained by interchanging second and third row of the identity matrix I_4.

Use the theorem:

Let E be $n \times n$ elementary matrix

If E results from interchanging two rows of I_n, then $\det(E) = -1$

By the theorem, the determinant of matrix A is -1.

8.

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -\frac{1}{3}$$, since the

Second row of I_4 was multiplied by $-\frac{1}{3}$

9.

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} \Rightarrow 3\begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix}$$

Add 2 times the first row to the second row

$$3\begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} \Rightarrow 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix}$$

Interchanging the second and third rows

$$\Rightarrow -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{vmatrix}$$

Add -3 times second row to the third row

$$\Rightarrow -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{vmatrix}$$

Take the common factor -11 from the third row

$$-3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{vmatrix} \Rightarrow (-3)(-11)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

Since the resultant matrix is upper triangular, its determinant is the product of the entries on the main diagonal of the matrix.

Therefore,

$$(-3)(-11)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} = (-3)(-11) = 33$$

11. $$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$ Interchange the 1st and 2nd rows

Add -2 times the first row to the second row

$$\Rightarrow \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

Add -2 times the second row to the third row

$$\Rightarrow - \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

Add -1 times the second row to the fourth row

$$\Rightarrow \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$$

Add the third row to the fourth row

$$\Rightarrow \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

Since the resultant matrix is upper triangular, its determinant is the product of entries on the main diagonal.

$$-\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix} = (-)\times 1\times 1\times -1\times 6 = \underline{6}$$

15. Use the result that if B is the matrix obtained by interchanging two rows or two columns of the $n\times n$ matrix A, then $\det(B) = -\det(A)$

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} =$$

$$(-)(-)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-)(-)(-6)$$

$$= \underline{-6}$$

17. $|A| = 3\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

$= 3 \cdot -1 \begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix}$

$= 3 \cdot -1 \cdot 4 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= -12(-6)$, since $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$

$|A| = 72$

18. $\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix} = (-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= (-1)(-6) = 6$

(Here 2 cases arise. The second row was added to the first row. For that change, the determinant does not affect. Secondly the second row was multiplied by -1 times, So the

Sign of the determinant changes.

a) If B is the matrix that results when a multiple of one row of A is added to another row or column multiplied of one column is added to another column, then $\det(B) = \det(A)$

b) If B is the matrix that results when a single row of single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$

Thus $\begin{vmatrix} a+d & b+c & c+d \\ -d & -e & -f \\ g & h & i \end{vmatrix} = 6$

21. $|A| = \begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

$= (-3) \begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

$= (-3) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-3)(-6) = \underline{\underline{18}}$

29. $|A| = \begin{vmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 2(4) & 1 & 4 \\ 3 & 2(1) & 5 & 1 \\ 1 & 2(5) & 6 & 5 \\ 4 & 2(-3) & 4 & -3 \end{vmatrix}$

$= \begin{vmatrix} -2 & 4 & 1 & 4 \\ 3 & 1 & 5 & 1 \\ 1 & 5 & 6 & 5 \\ 4 & -3 & 4 & -3 \end{vmatrix}$ ← A common factor of 2 was taken from the Second column.

A is a square matrix with two proportional rows or two proportional columns.

Thus $|A| = 0$

30. $|A| = \begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix}$

Adding second, third, fourth and fifth rows to the first row gives

$$|A| = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix} = 0$$

Since the first row contains all zeroes,
$\therefore$ $\det(A) = 0$

32. $M = \left[\begin{array}{c|c} A & 0 \\ \hline C & B \end{array}\right]$ Therefore, $\det(M) = \det(A) \times \det(B)$

$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\det(A) = 1(1-0) - 2(0-0) + 0(0-0) = 1$

$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\det(B) = 1 - 0 = 1$

$\det(M) = \det(A)\det(B) = 1$

$\det(M) = 1$