Date: 11/17/2021

MATH - 245

TEST #3

1. Ans. The given universal set U is :-

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

And we have, $A = \{1, 3, 5\}$, $B = \{1, 2\}$

& $C = \{2, 5\}$

(a) $A = \{1, 3, 5\}$ and $B = \{1, 2\}$

$\therefore A \cap B = \{1\}$

Now, $A^c = U - A$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 3, 5\}$
$= \{2, 4, 6, 7, 8, 9, 10\}$

And $B^c = U - B$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 2\}$
$= \{3, 4, 5, 6, 7, 8, 9, 10\}$

$\therefore A^c \cup B^c = \{2, 4, 6, 7, 8, 9, 10\} \cup \{3, 4, 5, 6, 7, 8, 9, 10\}$

$= \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \longrightarrow (1)$

Also, $(A \cap B)^c = U - (A \cap B)$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1\}$
$= \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \rightarrow (2)$

$\therefore (A \cap B)^c = A^c \cup B^c$ [By using (1) and (2)]

P.T.O

(b) $A = \{1,3,5\}$, $B = \{1,2\}$ and $C = \{2,5\}$

Now, $A \cap B = \{1,3,5\} \cap \{1,2\} = \{1\}$

$A \cap C = \{1,3,5\} \cap \{2,5\} = \{5\}$

$\therefore (A \cap B) \cup (A \cap C) = \{1\} \cup \{5\} = \{1,5\} \rightarrow (3)$

$B \cup C = \{1,2\} \cup \{2,5\} = \{1,2,5\}$

$A \cap (B \cup C) = \{1,3,5\} \cap \{1,2,5\}$
$= \{1,5\} \rightarrow (4)$

From (3) and (4), it follows that:-
$(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

2. Ans: $A = \{3,4\}$
$B = \{4,5\}$

(a) $P(A) = \{\{\}, \{3\}, \{4\}, \{3,4\}\}$

(b) $A \cap B = \{3,4\} \cap \{4,5\} = \{4\}$

$P(A \cap B) = \{\{\}, \{4\}\}$

(c) $A \cup B = \{3,4,5\}$
$P(A \cup B) = \{\{\}, \{3\}, \{4\}, \{5\}, \{3,4\},$
$\{4,5\}, \{3,5\}, \{3,4,5\}\}$

5. Ans: Note:- Answers are not in order. Answering what I know well first. This is the 5th Answer.

P.T.O

Define a relation on $\mathbb{Z}$ by $x R y$ if xy is even.

(a) R is not Reflexive.

Counterexample :- For $x = 1 \in \mathbb{Z}$

$x . x = 1 . 1 = 1$ and 1 is not even.

$\therefore$ $x . x$ is not even.

$\therefore$ $x \not R x$, where R is not reflexive.

(b) R is not transitive.

Counterexample :- For $x = 1$, $y = 2$ and $z = 3$

We have $x . y = 1 . 2 = 2$ is even $\Rightarrow x R y$
$y . z = 2 . 3 = 6$ is even $\Rightarrow y R z$

But, $x . z = 1 . 3 = 3$ is not even $\Rightarrow x \not R z$

Hence, $x R y$ and $y R z$ but not $x R z$.

$\therefore$ R is not transitive.

3 Ans : Note :- This is the 3rd answer.

The given relation is :-
$R = \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} : b - a \text{ divisible by } 5 \}$

(i) Let $a \in \mathbb{Z}$. Then, $a - a = 0$, is divisible by 5.

[P.T.O]

As $5 \mid 0$

$\therefore aRa$ hold $\forall a \in Z$.

$\therefore$ R is reflexive.

(ii) Let $a, b \in Z$.
Let aRb hold.

$\therefore a-b$ divisible by 5.
i.e. $5 \mid a-b$

$\Rightarrow 5 \mid -(b-a)$

$b-a$ is divisible by 5.
$\therefore bRa$ hold.

Thus, $aRb \Rightarrow bRa \ \forall a, b \in Z$.

$\therefore$ R is symmetric.

(iii) Let $a, b, c \in Z$.
Let aRb & bRc holds.

i.e., $5 \mid a-b$ & $5 \mid b-c$

$\therefore a-b = 5k_1$
$b-c = 5k_2$, $k_1, k_2 \in Z$

$\therefore a-c = (a-b)+(b-c)$
$= 5(k_1+k_2) = 5k_3$

$k_3 = k_1+k_2 \in Z$

$\therefore 5 \mid a-c$
$\therefore aRc$ holds $\forall a, b, c \in Z$.

$\therefore$ R is transitive.

Since, R is reflexive, symmetric & transitive

$\Rightarrow$ R is equivalence relation on Z.
Proved!

2nd Part :- There are five distinct equivalence classes, they are:-

$$Cl(0) = \{x \in Z : xR0 \text{ holds}\}$$
$$= \{x \in Z : 5 \mid x-0\}$$
$$= \{x \in Z : x = 5k, \text{ for } k \in Z\}$$
$$= \{5k : k \in Z\}$$
$$= \{0, \pm 5, \pm 10, \dots\}$$

$$Cl(1) = \{x \in Z : xR1 \text{ holds}\}$$
$$= \{x \in Z : 5 \mid x-1\}$$
$$= \{x \in Z : x = 5k+1 \text{ for } k \in Z\}$$
$$= \{5k+1 : k \in Z\}$$
$$= \{1, 1 \pm 5, 1 \pm 10, \dots\}$$

$$Cl(2) = \{x \in Z : xR2 \text{ holds}\}$$
$$= \{x \in Z : x = 5k+2\} \text{ for } k \in Z.$$

$$= \{5k+2 : k \in Z\}$$
$$= \{2, 2 \pm 5, 2 \pm 10, \dots\}$$

$$Cl(3) = \{x \in Z : xR3 \text{ holds}\}$$
$$= \{x \in Z : x = 5k+3, \text{ for } k \in Z\}$$
$$= \{5k+3 ; k \in Z\}$$
$$= \{3, 3 \pm 5, 3 \pm 10, \dots\}$$

$Cl(4) = \{x \in Z : x R 4 \text{ holds}\}$
$= \{x \in Z : x = 5K+4, K \in Z\}$
$\Rightarrow \{5K+4 ; K \in Z\}$
$\Rightarrow \{4, 4 \pm 5, 4 \pm 10, \dots\}$

$\therefore Z = Cl(0) \cup Cl(1) \cup Cl(2) \cup Cl(3) \cup Cl(4)$

$\therefore Cl(0), Cl(1), Cl(2), Cl(3), Cl(4)$ are the distinct equivalence classes of R.

6. Ans: Note :- This is the 6^{th} answer.

Given :- $a_n = 7a_{n-1} - 10a_{n-2}$, $a_0 = 0$, $a_1 = 3$

$\Rightarrow a_n - 7a_{n-1} + 10a_{n-2} = 0$

The characteristic equation for the above given linear recurrence relation is :-

$$r^n - 7r^{n-1} + 10r^{n-2} = 0$$

Dividing both the sides by r^{n-2}, we get

$\Rightarrow r^2 - 7r + 10 = 0$
$\Rightarrow (r-2)(r-5) = 0$
$\therefore r = 2, 5$

$\therefore$ The general solution of the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ is :

$$a_n = c_1(2)^n + c_2(5)^n \quad \text{--- (1)}$$

Plugging $n=0$ in equation (1), we get :-

$$a_0 = C_1(2)^0 + C_2(5)^0$$

$$\Rightarrow a_0 = C_1 + C_2$$
$$\Rightarrow C_1 + C_2 = a_0$$
$$\Rightarrow C_1 + C_2 = 0 \quad (\because a_0 = 0)$$

$$\Rightarrow C_1 = -C_2 \quad \text{—— (2)}$$

Plugging $n=1$ in equation (1), we get :-

$$a_1 = C_1(2)^1 + C_2(5)^1$$

$$\Rightarrow a_1 = 2C_1 + 5C_2$$

$$\Rightarrow 2C_1 + 5C_2 = 3 \quad (\because a_1 = 3)$$

$$\Rightarrow 2(-C_2) + 5C_2 = 3 \quad [\text{from (2)}, C_1 = -C_2]$$

$$\Rightarrow 3C_2 = 3$$

$$\therefore \underline{\underline{C_2 = 1}}$$

Putting $C_2 = 1$ in equation (2), we get :-

$$\underline{\underline{C_1 = -1}}$$

Substituting $C_1 = -1$ and $C_2 = 1$ in equation (1), we get :-

$$a_n = -1(2)^n + 1(5)^n$$
$$\Rightarrow a_n = (5)^n - (2)^n$$
$$\therefore \underline{\underline{a_n = 5^n - 2^n}}$$

(7)

2. Ans $a_n = 8a_{n-1} - 16a_{n-2}$, $a_0 = 1$, $a_1 = 1$

This is a homogenous recurrence relation.

Let $a_n = \alpha^n$

$\therefore$ We get :-

$\alpha^n = 8\alpha^{n-1} - 16\alpha^{n-2}$

dividing both sides by α^{n-2}, we get :-

$\alpha^2 = 8\alpha - 16$

$\therefore \alpha^2 - 8\alpha + 16 = 0$

is the required initial equation.

$\therefore \alpha = \dfrac{8 \pm \sqrt{64-64}}{2}$

$\alpha = \dfrac{8}{2} = 4$ (twice)

$\therefore \alpha_1 = 4$, $\alpha_2 = 4$.

$\therefore$ General solution to the recurrence relation is :-

$a_n = A\alpha^n + nB\alpha^n$

$\therefore a_n = A4^n + nB4^n$

$a_0 = 1$, $1 = A4^0 + (0)B4^0$

$\therefore 1 = A$

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$a_1 = 1$, $1 = A4^1 + (1)B4^1$

$\therefore\ 4A + 4B = 1$

But, $A = 1$

$\therefore\ 4(1) + 4B = 1$

$\therefore\ 4B = 1 - 4 = -3$

$\therefore\ B = -\frac{3}{4}$

$\therefore$ The solution of the given recurrence relation is :-

$$a_n = 4^n - \frac{3}{4}n4^n = 4^n - 3n4^{n-1}$$

8. Ans As $6 \in X$ and 6 is an even number.

Again applying R_1, $\frac{6}{2} = 3$.

$\therefore\ 3 \in X$ and is an odd number.

According to R_2 : $3 + 1 = 4$. So, $4 \in X$ and it is an even number.

Now, applying R_1, $\frac{4}{2} = 2$.

So, $2 \in X$ and 2 is an even number.

Again applying $R_1 = \frac{2}{2} = 1$. So, $1 \in X$.

Hence the set $X = = \{12, 6, 3, 4, 2, 1\}$

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10 Ans. Let set U of strings of a, b, c.

B. $b \in U$

R. If $x \in U$, then $axc \in U$

The elements of U = a, aoc, b, abc, c, acc

To show that every element in V a of the form at $a^n b c^n$ where n is a non-negative number, i.e. $(n \geq 0)$.

In the base case:-

The property of $a^n b c^n$ form is true.

Since, $b^0 a c^0$ & $n = 0$ is a non-negative number.

Now suppose the condition is true that every element in U is of the form $a^n b c^n$.

Since U is set of strings of a, b & c.

So, $a^n b c^n \in U$.

Let $a^n b c^n = x \in U$.

$\Rightarrow axc \in U \Rightarrow b.\ a^n b c^n \in V.$

$\Rightarrow b^{n+1} a c^{n+1} \in C$ and $n+1 \geq 0$

Hence proved!