Name :- Shreyas Srinivasa

ID :- 012551187

Chapter 5.1

1) (a) Sub. $k = 1$

$$a_1 = \frac{1}{10+1} = 1/11 \Rightarrow a_1 = 1/11$$

(b) Sub $k = 2$

$$a_2 = \frac{2}{10+2} = 2/12 = \frac{2}{6 \times 2} = 1/6$$

$\Rightarrow a_2 = 1/6$

(c) Sub. $k = 3$

$$a_3 = \frac{3}{10+3} = 3/13$$

$\Rightarrow a_3 = 3/13$

(d) Sub. $k = 4$

$$a_4 = \frac{4}{10+4} = 2/7 \Rightarrow a_4 = 2/7$$

The first four terms of the sequence :-
$1/11$, $1/6$, $3/13$ and $2/7$ equivalent to
$1/11$, $2/12$, $3/13$, $4/14$

3) (a) Sub $i = 0$

$$C_0 = \frac{(-1)^0}{3^0} = 1 \Rightarrow C_0 = 1$$

(b) Sub $i = 1$

$$C_1 = \frac{(-1)^1}{3^1} = -1/3 \Rightarrow C_1 = 1/3$$

(c) Sub $i = 2$

$$C_2 = \frac{(-1)^2}{3^2} = 1/9 \Rightarrow C_2 = 1/9$$

(d) Sub $i = 3$

$$C_3 = \frac{(-1)^3}{3^3} = -1/27 \Rightarrow C_3 = -1/27$$

$\therefore$ The first four terms in the seq. :- for $i \geq 0$ are $1, -1/3, 1/9$ and $-1/27$

11) $a_1 = 0 = (-1)^1(0)$

$a_2 = 1 = (-1)^{1+1}(1)$

$a_3 = -2 = (-1)^{2+1}(2)$

$a_4 = 3 = (-1)^{3+1}(3)$

$a_5 = -4 = (-1)^{4+1}(4)$

$a_6 = 5 = (-1)^{5+1}(5)$

Therefore, formula is!

$a_n = (-1)^{n+1}(n)$ for all integers $n \geq 0$

$a_n = n(-1)^{n+1}$ for all integers $n \geq 0$

Sub. $n = m-1$

$n+1 = m$ & $n \geq 0$

Also,

$m - 1 \geq 0$

$m \geq 1$

Therefore, the explicit formula!-

$a_m = (-1)^m (m-1)$ for all integers $m \geq 1$

2)

$a_1 = 1/4 = 1/2^2$

$a_2 = 2/9 = 2/3^2$

$a_3 = 3/16 = 3/4^2$

$a_4 = 4/25 = 4/5^2$

$a_5 = 5/36 = 5/6^2$

$a_6 = 6/49 = 6/7^2$

nth term of the sequence in the numerator is,
$a = 1$, $d = 2 - 1 = 1$; $t_n = 1 + (n-1)1 = n$

The sequence in the denominator:

$4, 9, 16, 25, 36, 49, \ldots$

This is in the form $2^2, 3^2, 4^2, 5^2, 6^2, 7^2, \ldots$

$= (1+1)^2, (2+1)^2, (3+1)^2, (4+1)^2, (5+1)^2, (6+1)^2, \ldots$

The nth term in the denominator is $(n+1)^2$

So, the formula for the sequence $1/4, 2/9, 3/16, 4/25, 5/36, 6/49$ is $a_n = \dfrac{n}{(n+1)^2}$ for all integers $n \geq 1$

So the formula for the sequence: $a_n = \dfrac{n}{(n+1)^2}$ for all integers $n \geq 1$

14) Initial terms :-

$$a_1 = 1/3 = 1/3^1$$

$$a_2 = 4/9 = 2^2/3^2$$

$$a_3 = 9/27 = 3^2/3^3$$

$$a_4 = 16/81 = 4^2/3^4$$

$$a_5 = \frac{25}{243} = 5^2/3^5$$

$$a_6 = 36/729 = \frac{6^2}{3^6}$$

Explicit formula :-

$$a_n = \frac{n^2}{3^n}$$, n is an integer and $n \geq 1$

19) $$\sum_{k=1}^{5} (k+1) = (1+1) + (2+1) + (3+1) + (4+1) + (5+1)$$

$$= 2 + 3 + 4 + 5 + 6 = 5 + 15 = 20$$

$$\therefore \sum_{k=1}^{5} (k+1) = 20$$

27) Rewrite the summation $\sum_{n=1}^{10} (1/n - 1/n+1)$ as:

$$= \left(1/1 - \frac{1}{1+1}\right) + \left(1/2 - \frac{1}{2+1}\right) + \left(1/3 - \frac{1}{3+1}\right)$$
$$+ \left(1/4 - \frac{1}{4+1}\right) + \left(1/5 - \frac{1}{5+1}\right) + \left(1/6 - \frac{1}{6+1}\right)$$
$$+ \left(1/7 - \frac{1}{7+1}\right) + \left(1/8 - \frac{1}{8+1}\right) + \left(1/9 - \frac{1}{9+1}\right)$$
$$+ \left(1/10 - \frac{1}{10+1}\right) = 1 - 1/11 = \frac{11-1}{11} = \frac{10}{11}$$

Therefore, the sum of $\sum_{n=1}^{10} \left(1/n - \frac{1}{n+1}\right)$ is $\frac{10}{11}$

33) Take the value $n = 1$

That is the first term of the summation

Hence, the sum is $1/1^2 = 1$

35) For $k = 3$, the given product has only three terms.

That is, the product of first three terms

Hence, the product of the first three terms is as follows. $\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right) = 1/4$

37) When $i = k+1$, $i^3 = (k+1)^3$

Therefore,

$$\sum_{i=1}^{k} i^3 + (k+1)^3 = \sum_{i=1}^{k+1} i^3$$

41) $$\sum_{k=1}^{m+1} k^2 = \sum_{k=1}^{m} k^2 + (m+1)^2$$

Chapter 5.2

3)(a) $$P(2) = \frac{2 \times (2-1) \times (2+1)}{3} = \frac{2 \times 1 \times 3}{3} = 2$$

$P(2) = 2$

From L.H.S

$$P(2) = \sum_{i=1}^{2-1} i(i+1) = \sum_{i=1}^{1} i(i+1) = 1 \times (1+1) = 1 \times 2 = 2$$

$P(2) = 2$

(b) $$P(k) = \sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

(c) $P(k+1)$ is $$\sum_{i=1}^{k} i(i+1) = \frac{(k+1)(k)(k+2)}{3}$$

d) If the property P(n) is true for n = k, then it is proven that P(n) is also true for n = k + 1 that is, for some integer $k \geq 2$

$k = n - 1$

If $\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$

$$P(k+1) = \sum_{i=1}^{k} i(i+1) = \frac{(k+1)(k)(k+2)}{3}$$

Substitute k = n−1 in the above formula

$$\sum_{i=1}^{n-1} i(i+1) = \frac{(n-1+1)(n-1)(n-1+2)}{3}$$

$$\sum_{i=1}^{n-1} i(i+1) = \frac{(n)(n-1)(n+1)}{3}$$

Hence it is true for n = k + 1

6) LHS

When we substitute 1 for n,

The left-hand side is the sum of all the even integers from 2 to 2.1. The sum is 2

RHS

$1^2 + 1 = 1 + 1 = 2$

∴ LHS = RHS ∴ The property is true for n = 1

Show that for all integers $k \geq 1$, if the property is true for $n = k$, then it is also true for $n = k+1$

Let k be any integer where $k \geq 1$ and suppose that the property is true for $n = k$

$$2 + 4 + 6 + \cdots\cdots + 2k = k^2 + k$$

We must show that the property is true for $n = k+1$

$$2 + 4 + 6 + \cdots\cdots + 2k + 2(k+1) = (k+1)^2 + (k+1)$$

$$= (k^2 + 2k + 1) + (k+1) = k^2 + 3k + 2 \Rightarrow$$

$$2 + 4 + 6 + \cdots\cdots + 2k + \cancel{\&}\, 2(k+1) = k^2 + 3k + 2 \rightarrow (1)$$

LHS of (1)

$$2 + 4 + 6 + \cdots\cdots + 2k + 2(k+1) \cancel{\&} = 2 + 4 + 6 + \cdots\cdots + (2k+2)$$

$$= 2 + 4 + 6 + \cdots\cdots + 2k + (2k+2)$$

$$= [2 + 4 + 6 + \cdots\cdots + 2k] + (2k+2)$$

$$= (k^2 + k) + (2k+2) = k^2 + k + 2k + 2$$

$$= k^2 + 3k + 2 = \text{RHS of } (1),$$

Thus the property is true for $n = k+1$, $n \geq 1$

8) For the given statement, the property is the equation

$1 + 2 + 2^2 + \ldots\ldots 2^n = 2^{n+1} - 1$. For all positive integers

Assume that the property is true for $n = 0$

LHS

Add the terms from 1 upto $2^0 (=1)$

RHS

$2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1$

LHS = RHS

Hence, the property is true for $n = 0$

Assume that for all integers $k \geq 0$, if the property is true for $n = k$, then it is also true for $n = k+1$

Let k be any integer where $k \geq 0$

Suppose that the property is true for $n = k$

$1 + 2 + 2^2 + \ldots\ldots + 2^k = 2^{k+1} - 1$

Prove that

$1 + 2 + 2^2 + \ldots\ldots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1$

$LHS = 1 + 2 + 2^2 + \ldots\ldots + 2^k + 2^{k+1}$

$= [1 + 2 + 2^2 + \ldots\ldots + 2^k] + 2^{k+1}$

$= [2^{k+1} - 1] + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

$= 2^{k+1} + 2^{k+1} - 1 = 2(2^{k+1}) - 1 = 2^{1+k+1} - 1$

$= 2^{(k+1)+1} - 1 = RHS$

Thus the property is true for $n = k+1$, $n \geq 0$

10) When $k = 1$,

$P(1) = 1^2 = 1 = LHS$

$$RHS = \frac{(1)(1+1)(2+1)}{6} = 6/6 = 1$$

$LHS = RHS$

Hence, the property is true for $P(1)$ .... (1)

Assume that $P(m)$ is true

i.e. $1^2 + 2^2 + 3^2 + \ldots + m^2 = \dfrac{m(m+1)(2m+1)}{6}$

..... (2)

When $k = m+1$; $P(m+1)$

LHS = Sum of the squares of $(m+1)$ terms in the given summation. Add the $m+1$th term on both sides of equation (2)

$$\{1^2 + 2^2 + 3^2 + \ldots + m^2\} + (m+1)^2 = \left\{\frac{m(m+1)(2m+1)}{6}\right\} + (m+1)^2$$

$$= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6}$$

$$= \frac{(m+1)\{m(2m+1) + 6(m+1)\}}{6}$$

$$= \frac{(m+1)\{2m^2 + 7m + 6\}}{6}$$

$$= \frac{(m+1)\{2m^2 + 4m + 3m + 6\}}{6}$$

$$= \frac{(m+1)[2m(m+2) + 3(m+2)]}{6}$$

$$= \frac{(m+1)(m+1+1)\{2(m+1)+1\}}{6}$$

This ~~ea~~ expression is in the required form.
So, $P(m+1)$ is true ..... (3)

So, by (1), (2) and (3), the given statement is true for all integers $n \geq 1$.

11) When $n = 1$, $P(1) = 1^3 = 1 = LHS$

$$\left\{\frac{1(1+1)}{2}\right\}^2 = 1^2 = 1 = RHS$$

$LHS = RHS$

So $P(1)$ is true ...... (1)

Assume that $P(m)$ is true for m.

i.e., $1^3 + 2^3 + 3^3 + \ldots\ldots + m^3 = \left\{\frac{m(m+1)}{2}\right\}^2$

When $k = m+1$, the sum of the $\rightarrow (2)$
$m+1$ terms is nothing but adding the $m+1$th term on both sides

$$\{1^3 + 2^3 + 3^3 + \ldots + m^3\} + (m+1)^3 = \frac{m^2(m+1)^2}{4} + (m+1)^3$$

$$= \frac{m^2(m+1)^2 + 4(m+1)^3}{4}$$

$$= \frac{m^2(m+1)^2 + 4(m+1)^2(m+1)}{4}$$

$$= \frac{(m+1)^2(m^2 + 4m + 4)}{4} = \frac{(m+1)^2(m+2)^2}{2^2}$$

$$= \left\{\frac{(m+1)(m+2)}{2}\right\}^2$$

This is the required form

So, $P(m+1)$ is true .... (3)

(1), (2) and (3) satisfy the hypothesis of mathematical induction

So by the result of the mathematical induction, the given statement is true for every natural value of n

i.e., $1^3+2^3+3^3+ \dots\dots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

12) When $n=1$,

$P(1) = \frac{1}{1\cdot 2} = 1/2 = LHS$

$RHS = \frac{n}{n+1} = \frac{1}{1+1} = 1/2$

LHS = RHS. Hence, the statement is true for $n=1$. ...... (1)

Suppose that the statement is true for $n=m$

That is,

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots + \frac{1}{m(m+1)} =$

$\frac{m}{m+1}$ ...... (2)

To prove the result by mathematical induction it is enough to show that the result is true for $n = m+1$

When $n = m+1$, the sum of the $m+1$ terms is calculated by adding the $m+1$th term on both sides,

$$\left\{ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots\ldots + \frac{1}{m(m+1)} \right\} + \left(\frac{1}{(m+1)(m+2)} \right) = \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} =$$

$$\frac{m(m+2)+1}{(m+1)(m+2)} = \frac{m^2+2m+1}{(m+1)(m+2)} = \frac{(m+1)^2}{(m+1)(m+2)}$$

$$= \frac{m+1}{m+2} = \frac{m+1}{(m+1)+1}$$

This expression is in the required form.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(m+1)((m+1)+1)}$$

$$= \frac{m+1}{(m+1)+1}$$

Hence, $P(m+1)$ is true ..... (3)

From equations (1), (2) and (3) satisfy the hypothesis of mathematical induction.

So, by the result of mathematical induction, the given statement is true for all integers $n \geq 1$

Chapter 5.3

(a) $5 = 1 \times 5$
$8 = 1 \times 8$
$10 = 2 \times 5$
$13 = 1 \times 5 + 1 \times 8$
$15 = 3 \times 5$
$16 = 2 \times 8$
$20 = 4 \times 5$
$21 = 2 \times 8 + 1 \times 5$
$24 = 3 \times 8$
$25 = 5 \times 5$

Hence Proved

(b) $P(28)$ is true, since 28 stamps can be represented using four 5-stamp packages and one 8-stamp package

$k (\geq 28) \in \mathbb{Z}$ and suppose k stamps can be obtained by collection of 5-stamp and 8-stamp packages

Case 1:- In this case replace three 5-stamp packages by two 8-stamp packages to make a collection of $k+1$ stamps. (The collection of k stamps includes three 5-stamp packages.

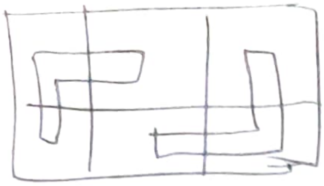
Case 2:- (The collection of k stamps does not include three 5-stamp packages):

In this case the collection includes at most two 5-stamp packages. Since there are at least 28 stamps. and at most 10 come from 5-stamp packages, at least 18 stamps come from 8-stamp packages. And because the smallest multiple of 8 that is at least 18 is 24; at least three 8-stamp packages must be in the collection. Replace three 8-stamp packages by ~~the~~ five 5-stamp packages to make a collection of $k+1$ stamps

Thus in either of the two possible cases a collection of $k+1$ stamps can be obtained by collection of 5-stamp and 8-stamp packages.

Hence $P(n)$ is true for all $n \geq 28$.

6) (a)



By observation P(1) is true.

(b) Any checkerboard whose dimension is $2 \times 3k$ can be fully covered with L-shaped trominoes.

(c) Any checkerboard whose dimension is $2 \times 3(k+1)$ can be fully covered with L-shaped trominoes.

(d) If any checkboard whose dimension is $2 \times 3k$ can be fully covered with L-shaped trominoes then any ~~che~~checkerboard whose dimension is $2 \times 3(k+1)$ can also be covered with L-shaped trominoes.

8) $P(n) = 5^n - 1$ is divisible by 4.
$P(0) = 5^0 - 1 = 1 - 1 = 0$ is divisible by 4.
So P(0) is true.

Consider that P(k) is true for any integer $k \geq 0$
Assume k be any integer with $k \geq 0$ for

$5^k - 1$ is divisible by 4.

Using definition of divisibility, we have

$5^k - 1 = 4r$ for integer r.

$P(k+1) = 5^{k+1} - 1$ is divisible by 4.

Since

$$5^{k+1} - 1 = 5^k \cdot 5 - 1$$
$$= 5^k(4+1) - 1$$
$$= 5^k \cdot 4 + (5^k - 1)$$

$$= 5^k \cdot 4 + 4r = 4(5^k + r)$$

As $5^{k+1} - 1$ is a factor of 4 and $(5^k + r)$

so, by definition of divisibility, $5^{k+1} - 1$ is divisible by 4.

10) When $n = 0$, the property is the sentence

$0^3 - 7 \times 0 + 3$

This is divisible by 3.

~~However, $0^3 - 7 \times 0 + 3 = 0 - 0 + 3 = 3$~~

~~And 3 is divisible by 3 because~~

Thus, the property is true for $n = 0$

Let k be any integer where $k \geq 0$

Assume that the property is true for $n = k$

We must show that the property is true for $n = k+1$

Now,

$(k+1)^3 - 7(k+1) + 3$

$= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3$

$= (k^3 - 7k + 3) + (3k^2 + 3k + 1 - 7)$

$= (k^3 - 7k + 3) + (3k^2 + 3k - 6)$

$= (k^3 - 7k + 3) + 3(k^2 + k - 6) \rightarrow (1)$

$k^3 - 7k + 3$ is divisible by 3

And $k^3 - 7k + 3 = 3r$ for some integer r.

Substituting into (1),

$(k+1)^3 - 7(k+1) + 3 = 3 \cdot r + 3(k^2 + k - 2)$

$= 3(r + k^2 + k - 2)$

Since k and r are integers, $r + k^2 + k - 2$ is clearly an integer

By the definition of divisibility,

$(k+1)^3 - 7(k+1) + 3$ is divisibe by 3

$\therefore$ The property is true for $n = k+1$

Thus, the property is true for all integers $n \geq 0$

11) For $n=0$

$P(0): 3^{2(0)}-1=1-1=0$

Thus, 0 is divisible by 8 since zero is divisible by all integers.

Hence, $P(0)$ is true.

Let $P(k)$ be true for some k for an arbitrary integer $k \geq 0$

That is, $P(n): 3^{2n}-1$ is divisible by 8

Thus, by the definition of divisibility, there exists an integer p such that $3^{2k}-1=8p \ldots (1)$

Now, need to prove that $P(k+1)$ is true

Consider $3^{2(k+1)}-1$.

$3^{2(k+1)}-1 = 3^{2k+2}-1 = 3^{2k} \cdot 3^{2}-1 = 9(8p+1)-1$

$= 72p+9-1 = 72p+8 = 8(9p+1)$

The value of $(9p+1)$ is an integer since product or sum of integers is also an integer. Hence, by the definition of divisibility, $3^{2(k+1)}$ is divisible by 8 since there is an integer $q = 9p+1$

~~such that~~ Thus $P(k+1)$ is true

Therefore, the state is true for all $n \geq 0$

12) For $n = 0$,

$P(0): 7^0 - 2^0 = 1 - 1 = 0$

Thus, 0 is divisible by 5 since zero is divisible by all integers.
Hence, $P(0)$ is true.

Let $P(k)$ be true for some k for an arbitrary integer $k \geq 0$
That is, $P(k): 7^k - 2^k$ is divisible by 5.

Thus, by the definition of divisibility, there exists an integer p, such that $7^k - 2^k = 5p$

Consider $7^{k+1} - 2^{k+1} =$

$7^k \cdot 7 - 2^k \cdot 2$

$= 7^k \cdot 7 + 7 \cdot 2^k - 7 \cdot 2^k - 2^k \cdot 2$

$= 7(7^k - 2^k) + 2^k(7 - 2)$

$= 7(7^k - 2^k) + 2^k \cdot 5$

$= 7(5p) + 2^k \cdot 5 = 5(7p + 2^k)$

By the definition of divisibility, $7^{k+1} - 2^{k+1}$ is divisible by 5 since there is an integer $q = (7p + 2^k)$. Thus $P(k+1)$ is true.

Thus the statement is true for $n \geq 0$

(14) $P(n)$: $n^3 - n$ is divisible by 6.

For $n = 0$

$n^3 - n = 0^3 - 0 = 0$

Since 0 is divisible by 6.

Therefore, $n^3 - n$ is divisible by 6 and

$P(n)$ is true for $n = 0$

&. Assume that $P(n)$ is true for any positive integer $n = k$. Since $n^3 - n$ is divisible by 6 for $n = k$. Therefore, $k^3 - k = 6m$ for any integer m.

For $n = k + 1$

$n^3 - n = (k+1)^3 - k - 1 = k^3 + 1^3 + 3k^2 + 3k - k - 1$

$= (k^3 - k) + (3k^2 + 3k) = 6m + 3k(k+1)$

The product $k(k+1)$ must be even.

Thus $k(k+1) = 2r$ for $r \in \mathbb{Z}$

Thus $n^3 - n = 6m + 3k(k+1) = 6m + 3(2r)$

$= 6m + 6r = 6(m + r)$

$(m - r)$ must be an integer

This implies that $n^3 - n$ is divisible by 6 for $n = k+1$ and $P(n)$ is true for $n = k+1$

Therefore by mathematical induction, the statement is true for each integer $n \geq 0$

Chapter 5.4

1) Now, $a_1 = 1$ and $a_2 = 3$ are both odd
Thus $P(1)$ and $P(2)$ are true.

Now, show that for any integer $k \geq 2$, if $P(i)$ is true for all integers i with $1 \leq i \leq k$, then $P(k+1)$ is true.

Let $k \geq 2$ be any integer.

Then by inductive hypothesis, a_i is odd for all integer i with $1 \leq i \leq k$.

Now, we must show that a_{k+1} is odd.

We know that

$a_{k+1} = a_{k-1} + 2a_k$ by the definition of $a_1, a_2, a_3, \ldots$

Since $k-1$ is less than $k+1$ and is greater than or equal to 1. Because $k \geq 2$

Thus, by the inductive hypothesis, a_{k-1} is odd.

Also, every term of the sequence is an integer. Because sum of products of integers is also an integer.

And, by the definition of even integer, $2a_k$ is even.

Thus, a_{k+1} is the sum of an odd integer and an even integer.

Hence, a_{k+1} is odd. Because sum of even integer and odd integer is odd.

5) $P(n): e_n = 5 \cdot 3^n + 7 \cdot 2^n$

Since e_0 and e_1 are given, take $a=0$ and $b=1$

$P(0) = e_0 = 5 \cdot 3^0 + 7 \cdot 2^0 = 5+7 = 12$

$P(0)$ is true

$P(1) = e_1 = 5 \cdot 3^1 + 7 \cdot 2^1 = 15+14 = 29$

$P(1)$ is true

~~P(k+1)~~

Suppose that $P(0), P(1), \ldots P(k)$ are all true.

$P(k+1) = e_{k+1} = 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}$

Consider e_{k+1}

Now,

$$e_{k+1} = 5e_{k+1-1} - 6e_{k+1-2}$$

$$= 5e_k - 6e_{k-1}$$

$$= 5(5 \cdot 3^k + 7 \cdot 2^k) - 6(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1})$$

$= 25 \cdot 3^k + 35 \cdot 2^k - 30 \cdot 3^{k-1} - 42 \cdot 2^{k-1}$

$= 25 \cdot 3^k + 35 \cdot 2^k - \frac{30}{3} \cdot 3^k - \frac{42}{2} \cdot 2^k$

$= 25 \cdot 3^k + 35 \cdot 2^k - 10 \cdot 3^k - 21 \cdot 2^k$

$= (25-10) \cdot 3^k + (\cancel{8}35-21) \cdot 2^k$

$= 15 \cdot 3^k + 14 \cdot 2^k = 5 \cdot 3 \cdot 3^k + 7 \cdot 2 \cdot 2^k$

$= 5 \cdot 3 \cdot 3^k + 7 \cdot 2 \cdot 2^k$

$= 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}$

Since $e_{k+1} = 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}$, $P(K+1)$ is true

$P(n)$ is true for all integers $n \geq 0$

6) Let $n = 0$

Substitute $n = 0$ is $P(n)$

$f_0 = 3 \cdot 2^0 + 2 \cdot 5^0 = 3 \cdot 1 + 2 \cdot 1 = 3 + 2 = 5$

Therefore, the result $P(n)$ is true for $n = 0$

Let $n = 1$

Substitute $n = 1$ in $P(n)$

$f_1 = 3 \cdot 2^1 + 2 \cdot 5^1 = 3 \cdot 2 + 2 \cdot 5 = 6 + 10 = 16$

Therefore, the result $P(n)$ is true for $n = 1$

The inductive hypothesis is to assume that the result $P(n)$ is true for $n = k$

That is $f_k = 3 \cdot 2^k + 2 \cdot 5^k$, for $k \geq 0$

Now, prove that $P(n)$ is true for $n = k+1$

That is to prove that $f_{k+1} = 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}$

By the definition of the sequence f_n, the obtained result is $f_{k+1} = 7f_k - 10f_{k-1}$

$$f_{k+1} = 7f_k - 10f_{k-1}$$

$$= 7(3 \cdot 2^k + 2 \cdot 5^k) - 10(3 \cdot 2^{k-1} + 2 \cdot 5^{k-1})$$

$$= 21 \cdot 2^k + 14 \cdot 5^k - 30 \cdot 2^{k-1} - 20 \cdot 5^{k-1}$$

$$= 21 \cdot 2^k + 14 \cdot 5^k - 15 \cdot 2 \cdot 2^{k-1} - 4 \cdot 5 \cdot 5^{k-1}$$

$$= 21 \cdot 2^k + 14 \cdot 5^k - 15 \cdot 2^k - 4 \cdot 5^k$$

$$= 6 \cdot 2^k + 10 \cdot 5^k = 3 \cdot 2 \cdot 2^k + 2 \cdot 5 \cdot 5^k$$

$$= 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}$$

Therefore, $P(n)$ is true for $n = k+1$

Hence, the sequence satisfies the relation $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$

7) $P(n)$: "$g_n = 2^n + 1$ for all integers $n \geq 1$."

First, show that $P(1)$ and $P(2)$ ~~is~~ are true

$P(1)$: $g_1 = 2^1 + 1 = 3$
$P(2)$: $g_2 = 2^2 + 1 = 5$

From (1), by the given definition also, the values of $g_1 = 3$ and $g_2 = 5$

Hence, the statements $P(1)$ and $P(2)$ are true.

Let k be any integer with $k \geq 1$ and suppose that $g_i = 2^i + 1$, for all integers i from 1 through k".

$k + 1 \geq 2$

$g_{k+1} = 3g_k - 2g_{k-1}$ from (1)

$= 3(2^k + 1) - 2(2^{k-1} + 1)$

$= 3 \cdot 2^k + 3 - 2 \cdot 2^{k-1} - 2 = 3 \cdot 2^k - 2^k + 1$

$= (2^k + 2^k + 2^k) - 2^k + 1$

$= 2^k + 2^k + 1 = 2 \cdot 2^k + 1 = 2^{k+1} + 1$

Hence $P(k+1)$ is also true.

The statement $g_n = 2^n + 1$ for $n \geq 1$ is true

Chapter 5.6

1) $a_2 = 2a_1 + 2 = 2\times 1 + 2 = 2 + 2 = 4$

$a_3 = 2a_2 + 3 = 2\times 4 + 3 = 8 + 3 = 11$

$a_4 = 2a_3 + 4 = 2\times 11 + 4 = 22 + 4 = 26$

3) $c_1 = 1(c_0)^2 = 1\cdot 1^2 = 1$

$c_2 = 2(c_1)^2 = 2\cdot 1^2 = 2$

$c_3 = 3(c_2)^2 = 3\cdot(2)^2 = 3\cdot 4 = 12$

5) $s_2 = s_1 + 2s_0 = 1 + 2 + 1 = 1 + 2 = 3$

$s_3 = s_2 + 2s_1 = 3 + 2\times 1 = 3 + 2 = 5$

$s_0 = 1, \quad s_1 = 1$

$s_2 = 3, \quad s_3 = 5$

7) $u_3 = 3u_2 - u_1 = 3\times 1 - 1 = 3 - 1 = 2$

$u_4 = 4u_3 + u_2 = 4\times 2 - 1 = 8 - 1 = 7$

$u_1 = 1 \quad u_2 = 1$

$u_3 = 2 \quad u_4 = 7$

9) For all integers $k \geq 1$, $a_k = 3k+1$

Substituting $n = k-1$ into equation (1), we get

$a_{k-1} = 3(k-1)+1 = 3k - 3+1$

$a_{k-1} = 3k-2$

Adding 3 to both sides, we get

$a_{k-1}+3 = 3k-2+3$

$a_{k-1}+3 = 3k+1$

$a_{k-1}+3 = a_k$

$\therefore a_k = a_{k-1}+3$, for all $k \geq 1$

11) $c_k = 2^k - 1$ ..... (1)

$c_{k-1} = 2^{k-1} - 1$ ..... (2)

Consider the RHS of $c_k = 2c_{k-1}+1$

$2c_{k-1}+1 = 2(2^{k-1}-1)+1$ [By (2)]

$= 2^k - 2+1 = 2^k - 1 = c_k = \text{LHS}$ [By (1)]

Thus the formula $c_n = 2^n - 1$ satisfies the recurrence relation $c_k = 2c_{k-1}+1$ for every integer $k \geq 1$.

13) Substitute $n = k-2$ into equation (1)

$t_{k-2} = 2 + (k-2) = k \quad \ldots (2)$

Substitute $n = k-1$ into equation (1)

$t_{k-1} = 2 + (k-1) = k+1 \quad \ldots (3)$

Substitute $n = k$ into equation (1)

$t_k = 2 + k = 2 + 2k - k = 2(k+1) - k$

$= 2t_{k-1} - t_{k-2}$ Use equations (2) and (3)

Hence the sequence $\{t_n\}$ satisfies the recurrence relation is $t_k = 2t_{k-1} - t_{k-2}$

Chapter 5.7

3) $a_1 = 1 \cdot a_0 = 1 \times 1 = 1$ $\quad {}^{2}\#$

$a_2 = 2 \cdot a_1 = 2 \times 2 = 1$

$a_3 = 3 \cdot a_2 = 3 \times 2 = 6 = 3 \times 2 \times 1$

$a_4 = 4 \cdot a_3 = 4 \times 6 \cancel{8} = 24 = 4 \times 3 \times 2 \times 1$

$a_5 = 5 \cdot a_4 = 5 \times 24 = 120 = 5 \times 4 \times 3 \times 2 \times 1$

$a_n = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!$

Hence the explicit formula is $a_n = n!$ for $n \geq 1$

5) $c_k = 3c_{k-1} + 1$ for all integers $k \geq 2$

Again, the given value is

$c_1 = 1$

Now substituting $k = 2, 3, 4, \ldots, n$

For $k = 2$

$c_2 = 3c_{2-1} + 1 = 3c_1 + 1 = 3(1) + 1 = 3 + 1$

For $k = 3$

$c_3 = 3c_{3-1} + 1 = 3c_2 + 1 = 3(3+1) + 1$

$= 9 + 3 + 1 = 3^2 + 3 + 1$

For $k = 4$

$c_4 = 3c_{4-1} + 1 = 3c_3 + 1 = 3(3^2 + 3 + 1) + 1$

$= 27 + 9 + 3 + 1 = 3^3 + 3^2 + 3 + 1$

Thus $k = n$

$c_n = 3^{n-1} + 3^{n-2} + \ldots + 3^3 + 3^2 + 3 + 1$

$= \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}$

10) Substitute $k = 1$ in $h_k = 2^k - h_{k-1}$

$h_1 = 2^1 - h_{1-1} = 2 - h_0 = 2 - 1$ since $h_0 = 1$

Substitute $k = 2$ in $h_k = 2^k - h_{k-1}$

$h_2 = 2^2 - h_{2-1} = 2^2 - (2-1) = 2^2 - 2 + 1$

Substitute $k = 3$ in $h_k = 2^k - h_{k-1}$

$$h_3 = 2^3 - h_{3-1} = 2^3 - (2^2 - 2 + 1) = 2^3 - 2^2 + 2 - 1$$

Substitute $k = 4$ in $h_k = 2^k - h_{k-1}$

$$h_4 = 2^k - h_{4-1} = 2^4 - (2^3 - 2^2 + 2 - 1)$$

$$= 2^4 - 2^3 + 2^2 - 2 + 1$$

$$h_n = 2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \dots + (-1)^{n-1} \cdot 2 + (-1)^n = (-1)^n \left[(-2)^n + (-2)^{n-1} + (-2)^{n-2} + \dots + (-2)^1 + (-2)^0 \right]$$

Use $S = \dfrac{r^{n+1} - 1}{r - 1}$ when $n \geq 0$ and $r \neq 1$

$r = 2$

$$h_n = (-1)^n = \frac{(-2)^{n+1} - 1}{-2 - 1} = (-1)^n \frac{(-2)^{n+1} - 1}{-3}$$

$$= (-1)^{n+1} \cdot \frac{(-2)^{n+1} - 1}{3} = \frac{(-1)^{n+1}(-2)^{n+1} - (-1)^{n+1}}{3}$$

$$= \frac{2^{n+1} - (-1)^{n+1}}{3}$$

The formula is

$$h_k = \frac{2^{k+1} - (-1)^{k+1}}{3}$$

when $k \geq 0$

12) Let $S_k = S_{k-1} + 2k$ for $k \geq 1$, $S_0 = 3$

$S_1 = S_0 + (2 \cdot 1) = 3 + 2 \cdot 1$

$S_2 = S_1 + (2 \cdot 2) = 3 + 2 \cdot 1 + 2 \cdot 2$

$S_3 = S_2 + (2 \cdot 3) = 3 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3$

$S_4 = S_3 + (2 \cdot 4) = 3 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4$

$S_n = 2n + 2(n-1) + \ldots + 2 \cdot 4 + 2 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 + 3 = 3 + 2 \cdot (n + (n-1) + \ldots + 4 + 3 + 2 + 1)$

$1 + 2 + \ldots n = \dfrac{n(n+1)}{2}$

$S_n = 3 + 2 \cdot \dfrac{n(n+1)}{2} = 3 + n(n+1)$ for all integers $n \geq 1$

26) (a) $A_0 = 1000$

Interest at the end of 1st month $= A_0 \times \frac{3}{100} \times \frac{1}{12}$

Additional deposit \$200

So $A_1 = A_0 + 0.0025A_0 + 200$

$= 1.0025A_0 + 200 \quad \ldots\ldots (1)$

Interest at the end of 2^{nd} month $= A_1 \times \frac{3}{100} \times \frac{1}{12}$

Additional deposit $=$ \$200

So, $A_2 = A_1 + 0.0025A_1 + 200 = 1.0025A_1 + 200$ .... (2)

$A_n = 1.0025A_{n-1} + 200$

(b) $A_1 = 1.0025A_0 + 200$

$A_2 = 1.0025\{1.0025A_0 + 200\} + 200$

$= 1.0025^2A_0 + 200\{1 + 1.0025\}$

$A_3 = 1.0025A_2 + 200$

$= 1.0025\{1.0025^2A_0 + 200(1 + 1.0025)\} + 200$

$= 1.0025^3A_0 + 200\{1 + 1.0025 + 1.0025^2\}$

$A_n = 1.0025^nA_0 + 200\{1 + 1.0025 + 1.0025^2 + \ldots + 1.0025^{n-1}\}$ .... (3)

$= \frac{a(r^n - 1)}{r - 1} = \frac{1(1.0025^n - 1)}{1.0025 - 1} = 400(1.0025^n - 1)$

$A_n = 1.0025^n \times 81000 - 80000$

(c) Putting $n=0$, we get $A_0 = 1.0025^0 \times 1000 + 80\,000(1.0025^0 - 1)$

$= 1 \times 1000 + 80\,000(0) = 1000\$$

Putting $n=1$, $A_1 = (1.0025 \times 1000) + 80000(1.0025 - 1)$

$= 1202.5$

$A_0 + \frac{3}{1200}A_0 + 200 = 1000 + 0.0025(1000) + 200$

$= 1202.5$

The implicit formula is true for $n=1$

Suppose it is true for $n=k$

When $n=k+1$, we have $A_{k+1} = (1.0025^{k+1} \times 1000) + 80{,}000(1.0025^{k+1} - 1) = 1.0025^{k+1} \times 1000 + 80000(1.0025^{k+1} - 1.0025 + 0.0025)$

$= 1.0025^{k+1} \times 1000 + 80\,000(1.0025^{k+1} - 1.0025) + 80\,000 \times 0.0025 = 1.0025\{1.0025^k + 80\,000(1.0025^k - 1)\} + 200$

This can be written as $1.0025A_k + 200$

Thus $A_{k+1} = 1.0025A_k + 200$

When $n=k+1$ the statement is true

(d) 20 years = 240 months

$A_{240} = (1.0025^{240} \times 1000) + 80000(1.0025^{240} - 1)$

$= \$67481.15$

After 40 years, $A_{480} = (1.0025^{480} \times 1000) + 80000(1.0025^{480} - 1)$

$= \$188527.04$

(e) $10000 = (1.0025^n \times 1000) + 80000(1.0025^n - 1)$

$\Rightarrow 90000 = 1.0025^n \times 81000$

$\Rightarrow 1.0025^n = 10/9$

$\Rightarrow n \log 1.0025 = \log 10 - \log 9$

$\Rightarrow n = \dfrac{1 - \log 9}{\log 1.0025} = 42.19$

~~The~~ Time required is just above 42 months or three and half years

50) $a_1 = 0 \qquad a_k = 2a_{k-1} + k - 1$

$a_2 = 2 \cdot a_1 + 2 - 1 = 2 \times 0 + 1 = 1$

$a_3 = 2 \cdot a_2 + 3 - 1$

$= 2 \times 1 + 2 = 4$

$a_4 = 2 \cdot a_3 + 4 - 1 = 2 \times 4 + 3 = 11$

However, if $n=4$

According to the explicit formula $a_4 = (4-1)^2 = 3^2$
$= 9$

They are not equal.

$\therefore$ The formula does not satisfy the recurrence relation.

Chapter 5.8

1) Form of second-order linear homogeneous relation $a_k = Aa_{k-1} + Ba_{k-2}$ where $A = 2$ and $B = -5$

(a) Yes
(b) No
(c) No
(d) Yes
(e) No
(f) Yes

3) (a) $a_0 = 1$

Letting $n=0$, we get $a_0 = C \cdot 2^0 + D \cdot 1^0 = C + D = 1$ ...... (1)

Letting $n=1$, we get $a_1 = C \cdot 2^1 + D \cdot 1^1$
$= 2C + D = 3$ ... (2)

Solving, we get
$C = 2$. From this, $D = -1$

So the given solution is $a_n = 2\cdot 2^n + (-1)\cdot 1^n$

i.e, $a_n = 2^{n+1} - 1$

From this, $a_2 = 2^{2+1} - 1 = 7$

(b) $a_0 = 0$

Letting $n=0$, we get $a_0 = C\cdot 2^0 + D\cdot 1^0 = C + D = 0$ ......(1)

Letting $n=1$, we get $a_1 = C\cdot 2^1 + D\cdot 1^1 = 2C + D = 2$ ......(2)

Solving (1) and (2), we get

$C = 2$ and $D = -2$

$\therefore$ The solution is $a_n = 2\cdot 2^n + (-2)\cdot 1^n$

So $a_n = 2^{n+1} - 2$

From this $a_2 = 2^{2+1} - 2 = 6$

5) $a_n = C\cdot 2^n + D$, for $n \geq 0$

$n = k$

$a_k = C\cdot 2^k + D$

$n = k-1$

$a_{k-1} = C\cdot 2^{k-1} + D$

$a_{k-2} = C\cdot 2^{k-2} + D$

$3a_{k-1} - 2a_{k-2} = 3(C\cdot 2^{k-1} + D) - 2(C\cdot 2^{k-2} + D)$

$= 3C\cdot 2^{k-1} + 3D - 2C\cdot 2^{k-2} - 2D$

$= 3C\cdot 2^{k-2+1} - 2C\cdot 2^{k-2} + (3D - 2D)$

$= 3C\cdot 2^{k-2}\cdot 2 - 2C\cdot 2^{k-2} + D$

$= 6C\cdot 2^{k-2} - 2C\cdot 2^{k-2} + D$

$= (6C - 2C)\cdot 2^{k-2} + D$

$= 4C\cdot 2^{k} 2^{-2} + D = 4C\cdot 2^{k}(1/2^{2}) + D$

$= 4C\cdot 2^{k}(1/4) + D = C\cdot 2^{k} + D = a_k$

Therefore, $a_k = 3a_{k-1} - 2a_{k-2}$

b) $b_k = C\cdot 3^{k} + D\cdot(-2)^{k} \dots (1)$

$b_{k-1} = C\cdot 3^{k+1} + D\cdot(-2)^{k-1}$

$b_{k-1} = C\cdot 3^{k-2} + D(-2)^{k-2}$

Taking $n = k-1$ and $n = k-2$ in b_n

Thus the expression $b_{k-1} + 6b_{k-2}$ is calculated as follows:

$b_{k-1} + 6b_{k-2} = C\cdot 3^{k-1} + D\cdot(-2)^{k-1} + 6(C\cdot 3^{k-2} + D\cdot(-2)^{k-2}) = C\cdot 3^{k-1} + D\cdot(-2)^{k-1} +$

$2\cdot3\cdot C\cdot3^{k-2} + 2\cdot3\cdot D\cdot(-2)^{k-2}$

$= C\cdot3^{k-1} + D\cdot(-2)^{k-1} + 2\cdot C\cdot3^{k-1} - (-2)\cdot3\cdot D\cdot(-2)^{k-2}$

$= C\cdot3^{k-1} + D\cdot(-2)^{k-1} + 2\cdot C\cdot3^{k-1} - 3\cdot D\cdot(-2)^{k-1}$

$= 3\cdot C\cdot3^{k-1} - 2D\cdot(-2)^{k-1} = 3\cdot C\cdot3^{k-1} + (-2)\cdot D\cdot(-2)^{k-1}$

$= C\cdot3^{k} + D\cdot(-2)^{k} = b_k$

Hence $b_k = b_{k-1} + 6b_{k-2}$ for all $k \geq 2$

8) a) The characteristic equation of the given recurrence relation is $t^2-2t-3=0$

Solve it

$t^2-2t-3=0$

$(t-3)(t+1)=0$

$t-3=0$ or $t+1=0$

$t=3$ or $t=-1$

Hence, the possible values of t are -1 and 3

b) $a_n = C\cdot3^n + D(-1)^n \cdots (1)$

Put $n=0$

$a_0 = C\cdot3^0 + D(-1)^0$

$1 = C + D$

$C + D = 1 \cdots (1)$

Put $n=1$

$a_1 = C\cdot3^1 + D(-1)^1$

$2 = 3C - D$

$3C - D = 2 \cdots (2)$

Solve the equations (1) and (2)

$C = 3/4, D = 1/4$

Substitute in (1)

$a_n = 3/4 \cdot 3^n + 1/4(-1)^n = \frac{3^{n+1}}{4} + \frac{(-1)^n}{4}$

The solution is $a_n = \frac{3^{n+1}}{4} + \frac{(-1)^n}{4}$ for $n \geq 0$

a) The characteristic equation of the given recurrence relation is $t^2 - 7t + 10 = 0$

$t^2 - 7t + 10 = 0$

$(t-2)(t-5) = 0$

$t - 2 = 0$ or $t - 5 = 0$

$t = 2$ or $t = 5$

Hence, the possible values of t are 2 and 5

b) $b_n = C \cdot 2^n + D \cdot 5^n \ldots\ldots (1)$

Put $n = 0$

$b_0 = C \cdot 2^0 + D \cdot 5^0$

$2 = C + D$

$C + D = 2 \ldots\ldots (1)$

Put $n = 1$

$b_1 = C \cdot 2^1 + D \cdot 5^1$

$2 = 2C + 5D$

$2C + 5D = 2 \ldots\ldots (2)$

Solve the equations (1) and (2)

$C = 8/3, \; D = -2/3$

Sub. in eq (1),

$$b_n = C \cdot 2^n + D \cdot 5^n = \left(8/3\right) 2^n + \left(-2/3\right) \cdot 5^n$$

Hence, the required sol. is $b_n = \left(8/3\right) \cdot 2^n + \left(-2/3\right) 5^n$
for $n \geq 0$

11) The characteristic equation is

$t^k = 4t^{k-2}$

Dividing by t^{k-2}, we get

$$\frac{t^k}{t^{k-2}} = \frac{4t^{k-2}}{t^{k-2}}$$

$t^2 = 4$

$\Rightarrow t^2 - 4 = 0$

$\Rightarrow (t+2)(t-2) = 0$

$\Rightarrow t + 2 = 0$ or $t - 2 = 0$

$\Rightarrow t = -2$ or $t = 2$

$d_n = C \cdot 2^n + D(-2)^n \; \ldots\ldots \; (1)$

$d_0 = 1, \; n = 0$

$d_0 = C \cdot 2^0 + D \cdot (-2)^0 = 1$

$\Rightarrow C + D = 1$

$\Rightarrow D = 1 - C \; \ldots\ldots \; (2)$

$d_1 = -1, \; n = 1$

$d_1 = C \cdot 2^1 + D(-2)^1 = -1$

$\Rightarrow 2C - 2D = -1 \; \ldots\ldots \; (3)$

Subbing D from eq. (2) into eq. (3), we get

$2C - 2(1 - C) = -1$

$2C - 2 + 2C = -1$

$4C = -1 + 2$

$4C = 1$

$C = 1/4$

Subbing C into eq. (2), we get

$D = 1 - 1/4 = 3/4$

$C = 1/4$ and $D = 3/4$

$d_n = 1/4 \, 2^n + 3/4 (-2)^n$, for all $n \geq 0$

13) The characteristic equation is $t^k = 2t^{k-1} - t^{k-2}$

Dividing by t^{k-2},

$$\frac{t^k}{t^{k-2}} = \frac{2t^{k-1}}{t^{k-2}} - \frac{t^{k-2}}{t^{k-2}}$$

$\Rightarrow t^2 = 2t^1 - 1$

$\Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow (t-1)(t-1) = 0$

$\Rightarrow t - 1 = 0$ or $t - 1 = 0$

$\Rightarrow t = 1$ or $t = 1$

By the single root theorem,

$r_n = (C + Dn) \ldots (1)$

$r_0 = 1$

$\Rightarrow r_0 = (C + D \cdot 1 = 1$

$\Rightarrow C = 1 \ldots (2)$

$r_1 = 4$

$r_1 = (C + D \cdot 1 = 4$

$\Rightarrow C + D = 4 \ldots (3)$

Let $C = 1$ in eq (3)

$1 + D = 4$

$D = 4 - 1 = 3$

The explicit formula for the given sequence is
$r_n = 1 + 3n$ for all $n \geq 0$

15) Consider the recurrence relation as,

$t_k = 6t_{k-1} - 9t_{k-2}$

$A = 6$ and $B = -9$

The characteristic equation is,

$t^2 - 6t + 9 = 0$

$t^2 - 3t - 3t + 9 = 0$

$t(t-3) - 3(t-3) = 0$

$(t-3)(t-3) = 0$

$(t-3)^2 = 0$

$t = 3, 3$

By single root theorem,

$t_n = C \cdot (3^n) + Dn(3^n)$

$t_0 = 1, t_1 = 3$

$t_0 = (C \cdot (3^0) + D(0)(3^0)$

$1 = C \cdot (1) + 0$

$C = 1$

$t_1 = C \cdot (3^1) + D(1)(3^1)$

$3 = 3C + 3D = 3(1) + 3D$

$\Rightarrow D = 0$

Thus the explicit formula is,

$t_n = (1)\cdot(3^n) + (0)n(3^n) \Rightarrow t_n = 3^n$ for all $n \geq 0$

Chapter 5.9

9) Let the statement "The string has an even number of a's be the property of s in S. Need to show that the property is true for each string $s \in S$.

Step 1 - Basis

The only object in the base is ε, and it has an even number of a's. as it ~~contains~~ consists of no a's. So, the property is true for all BASE objects.

Step 2 - Induction!

The recursion of S has two components, II(a) and II(b)

Suppose a string ~~s~~ s in S satisfies the property. It means that there exists an s such that s has an even no. of a's.

When either the rule II(a) or II(b) is applied to s, the outcome is bs or sb

When either the rule II(c) or II(d) is applied to s, the outcome is saa or aas, which also has an even number of a's, since the outcome has 2 more a's to the number of a's in S.

So, when each rule is applied to a string in S that has an even no. of a's, the result is also a string of even no. of a's.

Thus, ~~the~~ property is true for each rule in the recursion for S.

Since the set S has only objects, which are defined above in I or II, every string in S contains an even number of a's.

Hence proved.

11) Let the statement "The string represents an odd integer be the property of s in S.

Need to show that the property is true for $s \in S$

Step 1 - Basis

The objects in the base of S are 1, 3, 5, 7 and 9. Since they are all odd, the property is true for all BASE objects.

Step-2 - Induction

The recursion for S has five components, from 11(a) to 11(e).

Suppose there are two strings s, and t is S such that s, and t represent integers. Then, the two strings s and t are ending in an odd integer.

When the rule II(a) is applied to s and t, the outcome is st, which also ends in an odd integer, since t is ending in an odd integer.

When any of the remaining ~~four~~ four rules is applied to s, the outcome is 2s, 4s, 6s or 8s., which also ends in an odd integer, since s is ending in an odd integer.

Thus, the property is true for each rule in the recursion for S.

Since the set S has only objects, which are defined above in I and II, every string in S represents an~~d~~ odd integer.
Hence, the theorem is proved.

5) ~~(a)~~ (1) By I, 2, ~~3~~4, 6, 6.1, 7 and 9 are real numbers and hence, they are arithmetic

(2) By (1) and II(c) (6.1+2) is arithmetic

By (1) and II~~(d)~~(e) 9.(6.1+2) is arithmetic

By (I) and II(d) $(4-7)$ is arithmetic

By (I) and II(c) $(4-7)\cdot 6$ is arithmetic

(3) By (2) and II(f) $\left(\frac{(9\cdot(6\cdot 1+2))}{((4-7)\cdot 6)}\right)$ is arithmetic

Hence, by using the given recursive definitions, it is proved as arithmetic.

i) I) Base: 5 is in S

II) Recursion: Given any integer n in S, n+4 is in S.

To prove $\forall n \in S$, $n \bmod 2 = 1$

Proof:- $5 = 2\times 2+1 = 1 \bmod 2$

So $5 \bmod 2 = 1 \bmod 2$

So $5 \bmod 2 = 1$

Now let $k \bmod 2 = 1$ for some $k \in S$

So $(k+4) \bmod 2 = k \bmod 2 + 4 \bmod 2$

$= k \bmod 2 + (2\times 2) \bmod 2$

$= k \bmod 2 + 0 = 1$

So $(K+4) \bmod 2 = 1$

So if $K \bmod 2 = 1$ then $(K+4) \bmod 2 = 1$

(this is structural induction step) and

$5 \bmod 2 = 1$ (Base step)

So by structured induction, we get that for every n in S $n \bmod 2 = 1$