MATH - 245
TEST #3

1. Ans) The given universal set is U.

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

And we have, $A = \{1, 3, 5\}$, $B = \{1, 2\}$

& $C = \{2, 5\}$

(a) $A = \{1, 3, 5\}$ and $B = \{1, 2\}$

$\therefore A \cap B = \{1\}$

Now, $A^c = U - A$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 3, 5\}$
$= \{2, 4, 6, 7, 8, 9, 10\}$

And $B^c = U - B$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 2\}$
$= \{3, 4, 5, 6, 7, 8, 9, 10\}$

$\therefore A^c \cup B^c = \{2, 4, 6, 7, 8, 9, 10\} \cup \{3, 4, 5, 6, 7, 8, 9, 10\}$

$= \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \longrightarrow (1)$

Also, $(A \cap B)^c = U - (A \cap B)$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1\}$
$= \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \rightarrow (2)$

$\therefore (A \cap B)^c = A^c \cup B^c$ [By using (1) and (2)]

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(b) $A = \{1, 3, 5\}$, $B = \{1, 2\}$ and $C = \{2, 5\}$

Now, $A \cap B = \{1, 3, 5\} \cap \{1, 2\} = \{1\}$

$A \cap C = \{1, 3, 5\} \cap \{2, 5\} = \{5\}$

$\therefore (A \cap B) \cup (A \cap C) = \{1\} \cup \{5\} = \{1, 5\}$ $\rightarrow$ (3)

$B \cup C = \{1, 2\} \cup \{2, 5\} = \{1, 2, 5\}$

$A \cap (B \cup C) = \{1, 3, 5\} \cap \{1, 2, 5\}$
$= \{1, 5\}$ $\rightarrow$ (4)

From (3) and (4), it follows that :-
$(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

2. Ans. $A = \{3, 4\}$
$B = \{4, 5\}$

(a) $P(A) = \{\{\}, \{3\}, \{4\}, \{3, 4\}\}$

(b) $A \cap B = \{3, 4\} \cap \{4, 5\} = \{4\}$

$P(A \cap B) = \{\{\}, \{4\}\}$

(c) $A \cup B = \{3, 4, 5\}$
$P(A \cup B) = \{\{\}, \{3\}, \{4\}, \{5\}, \{3, 4\},$
$\{4, 5\}, \{3, 5\}, \{3, 4, 5\}\}$

5. Ans. Note :- Answers are not in order. Answering what I know well first. This is the 5^{th} Answer.

P.T.O

Define a relation on $\mathbb{Z}$ by xRy if xy is even.

(a) R is not Reflexive.

Counterexample :- For $x = 1 \in \mathbb{Z}$

$x.x = 1.1 = 1$ and 1 is not even.

$\therefore$ $x.x$ is not even.

$\therefore$ $x \not R x$, where R is not reflexive.

(b) R is not transitive.

Counterexample :- For $x = 1$, $y = 2$ and $z = 3$

We have $x.y = 1.2 = 2$ is even $\Rightarrow xRy$
$y.z = 2.3 = 6$ is even $\Rightarrow yRz$

But, $x.z = 1.3 = 3$ is not even $\Rightarrow x \not R z$

Hence, xRy and yRz but not xRz.

$\therefore$ R is not transitive.

3. Ans: Note :- This is the 3rd answer.

The given relation is :-
$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b - a \text{ divisible by } 5\}$

(i) Let $a \in \mathbb{Z}$. Then, $a - a = 0$, is divisible by 5.

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As $5 \mid 0$
aRa hold $\forall$ $a \in \mathbb{Z}$.

$\therefore$ R is reflexive.

(ii) Let $a, b \in \mathbb{Z}$.
Let aRb hold.

$\therefore$ $a-b$ divisible by 5.
i.e. $5 \mid a-b$

$\Rightarrow 5 \mid -(b-a)$

$b-a$ is divisible by 5.
$\therefore$ bRa hold.

Thus, $aRb \Rightarrow bRa$ $\forall a, b \in \mathbb{Z}$.

$\therefore$ R is symmetric.

(iii) Let $a, b, c \in \mathbb{Z}$.
Let aRb & bRc holds.

i.e., $5 \mid a-b$ & $5 \mid b-c$

$\therefore a-b = 5k_1$
$b-c = 5k_2$, $k_1, k_2 \in \mathbb{Z}$

$\therefore a-c = (a-b)+(b-c)$
$= 5(k_1+k_2) = 5k_3$

$k_3 = k_1+k_2 \in \mathbb{Z}$

$\therefore 5 \mid a-c$
$\therefore aRc$ holds $\forall a, b, c \in \mathbb{Z}$.

$\therefore$ R is transitive.

Since, R is reflexive, symmetric & transitive

$\Rightarrow$ R is equivalence relation on Z.
Proved!

2nd part :- There are five distinct equivalence classes, they are:-

$Cl(0) = \{x \in Z : xR0 \text{ holds}\}$
$= \{x \in Z : 5 \mid x-0\}$
$= \{x \in Z : x = 5k, \text{ for } k \in Z\}$
$= \{5k : k \in Z\}$
$= \{0, \pm 5, \pm 10, \ldots\}$

$Cl(1) = \{x \in Z : xR1 \text{ holds}\}$
$= \{x \in Z : 5 \mid x-1\}$
$= \{x \in Z : x = 5k+1 \text{ for } k \in Z\}$
$= \{5k+1 : k \in Z\}$
$= \{1, 1 \pm 5, 1 \pm 10, \ldots\}$

$Cl(2) = \{x \in Z : xR2 \text{ holds}\}$
$= \{x \in Z : x = 5k+2\} \text{ for } k \in Z.$

$= \{5k+2 : k \in Z\}$
$= \{2, 2 \pm 5, 2 \pm 10, \ldots\}$

$Cl(3) = \{x \in Z : xR3 \text{ holds}\}$
$= \{x \in Z : x = 5k+3, \text{ for } k \in Z\}$
$= \{5k+3 ; k \in Z\}$
$= \{3, 3 \pm 5, 3 \pm 10, \ldots\}$

[redacted]

$Cl(4) = \{x \in Z : xR4 \text{ holds}\}$
$= \{x \in Z : x = 5k+4, k \in Z\}$
$\Rightarrow \{5k+4 ; k \in Z\}$
$\Rightarrow \{4, 4 \pm 5, 4 \pm 10, \ldots\}$

$\therefore Z = Cl(0) \cup Cl(1) \cup Cl(2) \cup Cl(3) \cup Cl(4)$

$\therefore Cl(0), Cl(1), Cl(2), Cl(3), Cl(4)$ are the distinct equivalence classes of R.

6. Ans: Note :- This is the 6^{th} answer.

Given :- $a_n = 7a_{n-1} - 10a_{n-2}$, $a_0 = 0$, $a_1 = 3$

$\Rightarrow a_n - 7a_{n-1} + 10a_{n-2} = 0$

The characteristic equation for the above given linear recurrence relation is :-

$$r^n - 7r^{n-1} + 10r^{n-2} = 0$$

Dividing both the sides by r^{n-2}, we get

$\Rightarrow r^2 - 7r + 10 = 0$
$\Rightarrow (r-2)(r-5) = 0$
$\therefore r = 2, 5$

$\therefore$ The general solution of the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ is:

$$a_n = c_1(2)^n + c_2(5)^n \quad \text{—} (1)$$

Plugging $n=0$ in equation (1), we get :-

$$a_0 = c_1(2)^0 + c_2(5)^0$$

$$\Rightarrow a_0 = c_1 + c_2$$
$$\Rightarrow c_1 + c_2 = a_0$$
$$\Rightarrow c_1 + c_2 = 0 \quad [\because a_0 = 0]$$

$$\Rightarrow c_1 = -c_2 \quad \text{---} (2)$$

Plugging $n=1$ in equation (1), we get:-

$$a_1 = c_1(2)^1 + c_2(5)^1$$

$$\Rightarrow a_1 = 2c_1 + 5c_2$$

$$\Rightarrow 2c_1 + 5c_2 = 3 \quad (\because a_1 = 3)$$

$$\Rightarrow 2(-c_2) + 5c_2 = 3 \; [\text{from (2)}, c_1 = -c_2]$$

$$\Rightarrow 3c_2 = 3$$

$$\therefore c_2 = 1$$

Putting $c_2 = 1$ in equation (2), we get :-

$$c_1 = -1$$

Substituting $c_1 = -1$ and $c_2 = 1$ in equation (1), we get :-

$$a_n = -1(2)^n + 1(5)^n$$
$$\Rightarrow a_n = (5)^n - (2)^n$$
$$\therefore a_n = 5^n - 2^n$$

2. Ans $a_n = 8a_{n-1} - 16a_{n-2}$, $a_0 = 1$, $a_1 = 1$

This is a homogeneous recurrence relation.

Let $a_n = \alpha^n$

$\therefore$ We get :-

$$\alpha^n = 8\alpha^{n-1} - 16\alpha^{n-2}$$

dividing both sides by α^{n-2}, we get

$$\alpha^2 = 8\alpha - 16$$
$$\therefore \alpha^2 - 8\alpha + 16 = 0$$

is the required ~~[illegible]~~ initial equation.

$$\therefore \alpha = \frac{8 \pm \sqrt{64-64}}{2}$$

$$\alpha = \frac{8}{2} = 4 \text{ (twice)}$$

$$\therefore \alpha_1 = 4, \alpha_2 = 4.$$

$\therefore$ General solution to the recurrence relation is :-

$$a_n = A\alpha^n + nB\alpha^n$$

$$\therefore a_n = A4^n + nB4^n$$

$a_0 = 1$, $1 = A4^0 + (0)B4^0$

$$\therefore 1 = A$$

$a_1 = 1$, $1 = A4^1 + (1) B 4^1$

$\therefore 4A + 4B = 1$

But, $A = 1$

$\therefore 4(1) + 4B = 1$

$\therefore 4B = 1 - 4 = -3$

$\therefore B = -\frac{3}{4}$

$\therefore$ The solution of the given recurrence relation is

$$a_n = 4^n - \frac{3}{4} n 4^n = 4^n - 3n4^{n-1}$$

8. Ans: As $6 \in X$ and 6 is an even number. Again applying R_1, $\frac{6}{2} = 3$.

$\therefore$ $3 \in X$ and is an odd number.

According to R_2: $3+1=4$. So, $4 \in X$ and it is an even number.

Now, applying R_1, $\frac{4}{2} = 2$.

So, $2 \in X$ and 2 is an even number.

Again applying $R_1 = \frac{2}{2} = 1$. So, $1 \in X$.

Hence the set $X = \{12, 6, 3, 4, 2, 1\}$

10 Ans. Let set U of strings of a, b, c.

1. $b \in U$
2. If $x \in U$, then $axc \in U$

The elements of U = a, abc, b, aabcc, abc

To show that every element in U is of the form $a^n b c^n$ where n is a non-negative number, i.e. $(n \geq 0)$.

In the base case:-

The property of $a^n b c^n$ form is true.

Since, $b = a^0 b c^0$ & $n = 0$ is a non negative number.

Now suppose the condition is true that every element in U is of the form $a^n b c^n$.

Since U is set of strings of a, b, c

So, $a^n b c^n \in U$.

Let $a^n b c^n = x \in U$.

$\Rightarrow axc \in U \Rightarrow a \cdot a^n b c^n c \in U$

$\Rightarrow a^{n+1} b c^{n+1} \in U$ and $n+1 \geq 0$

Hence proved!