12/19/2021 MATH-245 ①

NAME:- SHREYAS SRINIVASA
PALOMAR ID:- 012551187.

1 Ans: Note:- Answers not in order. Answering what I know well first.

2. Ans: Counter examples below:-

(a) $x = 2, 4, 6$ are even numbers, but they are also less than 7.

(b) -3 is odd but it is not divisible by 5.

(c) 2 is prime, but it is not an odd number.

3. Ans Using the following ~~formula~~ formula:-

$P(n, r) = n.(n-1).(n-2)\ldots\ldots (n-r-1)$ or

$$P(n,r) = \frac{n!}{(n-r)!}$$

(a) $P(7,3) = \frac{7!}{(7-3)!}$

$= \frac{7!}{4!}$

$= 7 \times 6 \times 5$

$= 210$

$\therefore$ There are 210 different three-letter strings that are possible.

(b) $P(2,1) = \frac{2!}{(2-1)!} = \frac{2!}{1!}$

$= 2$

$\therefore$ There are 2 ways to select the middle letter.

For the remaining two letters, there are $7-1=6$ distinct magnets available after selecting the middle letter. Thus, the two remaining letters can be selected in $P(6,2)$ different ways.

$$P(6,2) = \frac{6!}{(6-2)!}$$

$$= \frac{6!}{4!}$$

$$= 6 \times 5$$

$$= 30$$

$\therefore$ There are 30 ways to select the remaining two letters.

Thus, the three-letters of the string can be formed in $2 \times 30 = 60$ ways.

$\therefore$ There are 60 different three lette strings that can be formed if the middle letter must be a vowel.

1 Ans. (a) x is a rational number.

Let $x + \sqrt{2}$ is a rational number

So, we can write :-

$$x + \sqrt{2} = \frac{a}{b} \quad \text{--- (1)}$$

Here 'a' and 'b' are two co-prime numbers and $b \neq 0$.

From equation 1 :-

$$\sqrt{2} = \frac{a}{b} - x$$

$$\sqrt{2} = \frac{a - bx}{b}$$

Here a and b are integers, so $\left[\frac{a-bx}{b}\right]$

is a rational number.

So, $\sqrt{2}$ should be a rational number too.

However $\sqrt{2}$ is an irrational number.

$\therefore$ It is contradictory.

$\therefore$ $x + \sqrt{2}$ is an irrational number.

(b) y is a rational number.

Lets assume that $y\sqrt{2}$ is a rational number.

We can write :- $y\sqrt{2} = \frac{a}{b}$

$\downarrow$
(2)

where 'a' and 'b' are two co-prime numbers and $b \neq 0$.

From equation (2) :-

$$y\sqrt{2} = \frac{a}{b}$$

$$\sqrt{2} = \frac{a}{by} \longrightarrow (3)$$

Here, a and b are two integers, so $\frac{a}{by}$ is a rational number.

So, $\sqrt{2}$ should be a rational number too.

However, $\sqrt{2}$ is an irrational num-ber.

Hence, it is contradictory.

$\therefore$ $y\sqrt{2}$ is an irrational number

4. Ans :- Using induction, we have to show that $8^n - 1$ is divisible by 7 $\forall$ $n \in I^+$.

i.e., $\frac{8^n - 1}{7} = 0$ (remainder)

(i) Now for $n = 1$,

$$8^1 - 1 = \frac{7}{7} = 0 \text{ (remainder)}$$

So, $8^1 - 1 = 7$ is divisible by 7.

(ii) Assume $n = k$ for $8^n - 1$ is divi- -sible by 7.

$\Rightarrow 8^k - 1 = 7m$

or, $8^k = 7m + 1 \rightarrow (1)$

where m is some integer.

Now, setting $n = k + 1$

$\Rightarrow 8^{k+1} - 1 = 8 \times (8^k - 1)$

From the above equation (1):

$8^{k+1} - 1 = 8.(8^k - 1) = 8(7m+1) -$

$= 7(8m) + 8 - 1$

$= 7(8m+1)$

Since $8m+1$ is divisible by 7.

So, $8^{k+1} - 1$ is divisible by 7.

Hence, $8^n - 1$ is divisible by 7.

for all integers $n \geq 1$.

Proved.

5. Ans:- We have $A = \{m, n\}$,

$B = \{3, 4\}$ & $C = \{4, 5\}$.

$\therefore A \times B = \{m, n\} \times \{4, 5\}$

$= \{(m, 4), (m, 5), (n, 4), (n, 5)\}$

$\therefore (A \times B) \cap (A \times C) = \{(m, 3), (m, 4),$
$(n, 3), (n, 4)\}$
$\cap \{(m, 4), (m, 5),$
$(n, 4), (n, 5)\}$

$\Rightarrow (A \times B) \cap (A \times C) = \underbrace{\{(m, 4), (n, 4)\}}_{(1)}$

Also, $B \cap C = \{3, 4\} \cap \{4, 5\}$

$\Rightarrow B \cap C = \{4\}$

Now, $A \times (B \cap C) = \{m, n\} \times \{4\}$

$= \{(m, 4), (n, 4)\}$

$\Rightarrow A \times (B \cap C) = \underbrace{\{(m, 4), (n, 4)\}}_{(2)}$

6.

From (1) and (2), we get :-

$(A \times B) \cap (A \times C) = A \times (B \cap C)$

Hence verified

6 Ans Given that :- $A = \{2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ & $U = \{1, 2, 3, \ldots 9\}$

(a) $(A \cup B)^c = U - (A \cup B)$

$A \cup B = \{2, 3, 4\} \cup \{3, 4, 5, 6\}$

$= \{2, 3, 4, 5, 6\}$

$\Rightarrow (A \cup B)^c = U - (A \cup B)$

$= \{1, 2, 3, 4, \ldots 9\} - \{2, 3, 4, 5, 6\}$

$= \{1, 7, 8, 9\}$

(b) $A^c \cap B^c$:-

$A^c = U - A = \{1, 2, 3, \ldots 9\} - \{2, 3, 4\}$

$= \{1, 5, 6, 7, 8, 9\}$

Now, $B^c = U - B$

$= \{1, 2, 3, \ldots 9\} - \{3, 4, 5, 6\}$

$= \{1, 2, 7, 8, 9\}$

$A^c \cap B^c = \{1, 5, 6, 7, 8, 9\} \cap \{1, 2, 7, 8, 9\}$

$= \{1, 7, 8, 9\}$

(c) $\phi^c = U - \phi = \{1, 2, \ldots 9\} - \{\phi\}$

$= \{1, 2, \ldots 9\} = U$

(d) $U^c = \phi$

7. Ans We can write 2 as $2 = 2 \times 1$

$\therefore$ 2 is even.

Suppose as induction hypothesis that some $x \in X$ is even.

Induction step :- As $x \in X$ is even,

then $x = 2k$ for some k, so

$x + 10 = 2k + 10 = 2(k+5)$, which is even.

$\therefore$ By induction, all elements of X are even.

8. Ans :- Solving question 8 later.

9. Ans. Let $P(n)$ be the statement $a_n = 2(3^n) - 4^n$, $n \geq 0$

Now, $P(0)$: $a_0 = 2 - 1 = 1$, which is true.

(13)

$P(1) = a_1 = 2(3) - 4 = 2$, which is true.

$P(2) = a_2 = 7a_1 - 12a_0 = (7 \cdot 2) - 12$

$= 2$

$= 2(3^2) - 4^2$

So, $P(2)$ is true.

$P(3) : a_3 = 7a_2 - 12a_1 = (7 \times 2) - (12 \times 2)$

$= -10$

$= 2 \cdot (3^3) - 4^3$

So $P(3)$ is true.

Let $P(0), P(1), P(2), \ldots, P(k)$ be true.

We will show that $P(k+1)$ is also true.

(14)

Since P(k-1) and P(k) are true,

so

$$a_{k-1} = 2(3^{k-1}) - 4^{k-1}$$

and $a_k = 2(3^k) - 4^k$.

Now, $a_{k+1} = 7a_k - 12a_{k-1}$

$$= 7\{2(3^k) - 4^k\} - 12\{2(3^{k-1}) - 4^{k-1}\}$$

$$= 14(3^k) - 7(4^k) - 24 \cdot \frac{3^k}{3} + 12 \cdot \frac{4^k}{4}$$

$$= 14(3^k) - 7(4^k) - 8(3^k) + 3(4^k)$$

$$= 6(3^k) - 4(4^k)$$

i.e. $a_{k+1} = 2(3^{k+1}) - 4^{k+1}$

=> P(k+1) is true.

Hence by strong induction, we can say that $P(n)$ is true for all $n \geq 0$

2. Ans $x_1 + x_2 + x_3 + x_4 = 30$

Given $x_1 \geq 1$, $x_2 > 2$, $x_3 > 3$

and $x_4 \geq 2$

If $x_1 \geq 1$ and if $x_2 > 2$,

$x_1 - 1 = k_1$ then $x_2 \geq 3$

$x_2 - 3 = k_2$

Similarly $x_3 > 3$ and $x_4 \geq 2$

$x_3 \geq 4$ $x_4 - 2 = k_4$

$x_3 - 4 = k_3$

So now, $k_1 + k_2 + k_3 + k_4$

$= (x_1 - 1) + (x_2 - 3) + (x_3 - 4) + (x_4 - 2)$

$k_1 + k_2 + k_3 + k_4 = (x_1 + x_2 + x_3 + x_4) - 10$

$k_1 + k_2 + k_3 + k_4 = 30 - 10$

$k_1 + k_2 + k_3 + k_4 = 20$

So, no of integer solutions are:-

$n = 20$
$r = 4$

$^{n+r-1}C_{r-1} = {}^{20+4-1}C_{4-1}$

$= {}^{23}C_3$

$= \dfrac{23!}{3!\,(23-3)!}$

$= \dfrac{23!}{3! \times 20!} = \dfrac{23 \times 22 \times 21 \times 20!}{(3 \times 2) \times 20!}$

$= 1771$

10. Ans $x_1 + x_2 + x_3 + x_4 = 12$

Here, $0 \le x_1 \le 4 \Rightarrow 4 - x_1 \ge 0$

let $y_1 = 4 - x_1 \Rightarrow y_1 \ge$

$0 \le x_2 \le 5 \Rightarrow 5 - x_2 \ge 0$ let y_2

$= 5 - x_2$

$\Rightarrow y_2 \ge 0$

$0 \le x_3 \le 8 \Rightarrow 8 - x_3 \ge 0$ let $y_3 =$

$8 - x_3$

$\Rightarrow y_3 \ge 0$

$0 \le x_4 \le 9 \Rightarrow 9 - x_4 \ge 0$ let $y_4 = 9 - x_4$

$\Rightarrow y_4 \ge 0$

$\therefore x_1 = 4 - y_1$

$x_2 = 5 - y_2$

$x_3 = 8 - y_3$

$x_4 = 9 - y_4$

$\therefore x_1 + x_2 + x_3 + x_4 = 12$ becomes

$4 - y_1 + 5 - y_2 + 8 - y_3 + 9 - y_4$

$= 12$

$\Rightarrow 26 - 12 = y_1 + y_2 + y_3 + y_4$

$\Rightarrow y_1 + y_2 + y_3 + y_4 = 14$ where

$y_1, y_2, y_3, y_4 \geq 0$

We know that the number of non-negative integral solutions to

$x_1 + x_2 + \dots + x_n = r = \dfrac{(n+r-1)!}{r!\,(n-1)!}$

$\therefore$ for $y_1 + y_2 + y_3 + y_4 = 14$,

$n = 4,\ r = 14$

$\therefore$ Total non-negative integral solutions $= \dfrac{(4+14+1)!}{(14)!\ (4-1)!}$

$= \dfrac{19!}{14!\ 3!}$

$= \dfrac{19 \times \cancel{18}^{3} \times 17 \times 16 \times 15 \times \cancel{14!}}{\cancel{14!}\ (3 \times 2 \times 1)}$

$= 19 \times 3 \times 17 \times 16 \times 15$

$= \underline{\underline{232560}}$

$\therefore$ Total number of non-negative integral solutions to $x_1 + x_2 + x_3 + x_4 = 12$ satisfying given condi-

∴ Hence the x_1, x_2, x_3 & x_4 are

2 3 2 5 6 0

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11. Ans Given linear recurrence relation is $a_n = 8a_{n-1} - 15a_{n-2}$

$a_0 = 3$, $a_1 = 11$

The characteristic will be $r^2 - 8r + 15 = 0$

$\Rightarrow (r-3)(r-5) = 0$

$\Rightarrow r = 3, 5$ are roots

Hence, the general solution will be, $a_n = A(3)^n + B(5)^n$

Now, $a_0 = 3$ i.e. $3 = A(3)^0 + B(5)^0$

$= A + B$ — (1)

$a_1 = 11$ i.e. $11 = A(3)^1 + B(5)^1$

$= 3A + 5B$ — (2)

From (1) and (2), $B = 1$, $A = 2$

Hence, $a_n = 2(3)^n + 1(5)^n$

12. Ans A partition of a set 'A' induces an equivalence relation on A.

Given $P = \{\{3,4\}, \{0,1\}, \{2\}\}$

The induced equivalence relation is defined as $\{3,4\} \times \{3,4\}$

$= \{(3,3), (4,3), (3,4), (4,4)\}$

$\{0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\}$

$\{2\} \times \{2\} = \{(2,2)\}$

All of the above are ordered

have in the relation R.

$\therefore R = \{(3,3), (4,3), (3,4), (4,4), (0,0), (0,1), (1,0), (1,1), (2,2)\}$

(2) Given set $A = \{1, 2, 3, 4, 0\}$ & equivalence relation $R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$

with observation, we find that:

$\{1,3\} \times \{1,3\} = \{(1,1), (1,3), (3,1), (3,3)\}$

& $\{0,4\} \times \{0,4\} = \{(0,4), (0,0), (4,0), (4,4)\}$

$\{2\} \times \{2\} = \{(2,2)\}$

Then $P = \{\{1,3\}, \{0,4\}, \{2\}\}$

is the induced partition by R.

13. Ans Suppose $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ and R is defined on A by $aRb \iff a^2 - b^2$ is divisible by 5.

It is enough to show that R is Reflexive, symmetric & transitive.

Reflexive :- $aRa \iff a^2 - a^2 = 0$ is divisible by 5

$\therefore$ R is reflexive.

Symmetric :- $aRb \iff a^2 - b^2$ is divisible by 5.

$\iff -(b^2 - a^2)$ is divisible by 5

$\iff b^2 - a^2$ is divisible by 5.

$\iff bRa.$

$\therefore$ R is symmetric.

Transitive:- Let aRb and bRc.

Then $a^2 - b^2$ and $b^2 - c^2$ are both divisible by 5.

$\Rightarrow$ $(a^2 - b^2) + (b^2 - c^2) = a^2 - c^2$ is also divisible by 5.

$\Rightarrow$ aRc.

$\therefore$ R is transitive.

We know that the set of all equivalence classes is a partition of set A.

Since, $|A| = 9$, therefore let the number of equivalence classes be :-

$B_9 = \sum_{k=0}^{8} \binom{n}{k} B_k$, where B_k

is the bell number.

$= \underline{\underline{4140}}$

Thus, the number of distinct equivalence classes of A with respect to R is 4140.

14. Ans $U = \{ x \in N : x \leq 200 \}$

No of elements that are divisible by 2 $= \frac{200}{2} = 100$

No of elements that are divisible by 3 = 66

No of elements that are divisible by 7 = 28.

No of elements that are divisible by 2 & 3 $= \frac{200}{\operatorname{lcm}(2,3)} = 33\frac{2}{3}$

$\therefore$ 33 numbers are divisible by 2 and 5.

No of elements that are divisible by 3 and 7 $= \frac{200}{lcm(3,7)} = 9\frac{11}{21}$

$\therefore$ 9 numbers are divisible by 7 and 3.

No of elements that are divisible by 2 and 7. $= \frac{200}{lcm(2,7)} = 14\frac{4}{14}$

$\therefore$ 14 numbers are divisible by 2 and 7.

Nos that are divisible by 2, 3 & 7 $= \frac{200}{lcm(2,3,7)} = 4\frac{32}{42}$

$\therefore$ 4 numbers are divisible by (2, 3, & 7.)