1.(a)

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$$a_n = 2a_{n-1} + 3a_{n-2} + 25 \times 4^{n-2}$$

$$\Rightarrow a_n - 2a_{n-1} - 3a_{n-2} = 25 \times 4^{n-2}$$

particular : assume $25c \cdot 4^{n-2}$

characteristic : $x^2 - 2x - 3 = 0 \Rightarrow x = 3$ or -1

solve particular :

$$25c \cdot 4^{n-2} - 50c \cdot 4^{n-3} - 75c \cdot 4^{n-4} = 25 \times 4^{n-2}$$

$$\Rightarrow c \cdot 4^{n-2} - 2c \cdot 4^{n-3} - 3c \cdot 4^{n-4} = 4^{n-2}$$

$$\Rightarrow c \cdot 4^{n-2} - \frac{1}{2}c \cdot 4^{n-2} - \frac{3}{16}c \cdot 4^{n-2} = 4^{n-2}$$

$\Rightarrow c - \frac{1}{2}c - \frac{3}{16}c = 1 \Rightarrow c = \frac{16}{5}$ $\therefore$ particular sol: $\overset{5}{25} \times \frac{16}{5} \times 4^{n-2} = 80 \cdot 4^{n-2}$

total solution : $80 \cdot 4^{r-2} + A \cdot 3^r + B \cdot (-1)^r = a_r$

$$\begin{cases} a_0 = 80 \cdot 4^{-2} + A + B = 1 \\ a_1 = 80 \cdot 4^{-1} + 3A - B = 32 \end{cases} \Rightarrow \begin{cases} A + B = -4 \\ 3A - B = 12 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = -6 \end{cases}$$

total solution : $\therefore a_n = 80 \cdot 4^{n-2} + 2 \cdot 3^n - 6 \cdot (-1)^n$ #

1.(b)

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

particular sol : 0

charateristic equ : $x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$ (repeated root)

homogeneous : $(Ar + B) \cdot 3^r$

$$\begin{cases} a_0 = B = 1 \\ a_1 = (A+B) \cdot 3 = 15 \end{cases} \Rightarrow \begin{cases} A = 4 \\ B = 1 \end{cases}$$

$\therefore$ total solution $\Rightarrow a_n = (4n+1) \cdot 3^n$ #

2.

$$a_n a_{n-2} = (a_{n-1})^2 + 2a_{n-1}a_{n-2} \quad , \text{ for } n \geq 2$$

$\Rightarrow$ $a_0 = 2$

$) \times 2$

$a_1 = 4$

$) \times 4$

$a_2 = 16$

$) \times 6$

$a_3 = 96$

$) \times 8$

$a_4 = 768$

$) \times 10$

$a_5 = 7680$

$) \times 12$

$\vdots \qquad \vdots$

$$a_n = a_{n-1} \times (2n)$$

$$\Rightarrow a_n - 2n a_{n-1} = 0 \quad , \text{ for } n \geq 1$$

3、

show this by induction:

base case: $k=1$

$2020^2-1 = (2020 \times 2020)-1 = 4080399 = 2021 \times 2019$, true

Inductive case: $k \geq 1$

assume $k=n$ holds for $2020^{2n}-1$ is a multiple of 2021

$\Rightarrow 2020^{2n}-1 = 2021 \cdot m \ (m \in N)$

then for $k=n+1$

※ $\Rightarrow 2020^{2n} = 2021m+1$

$\Rightarrow 2020^{2(n+1)}-1 = (2020)^{2n+2}-1 = \left[(2020)^{2n} \cdot (2020)^2\right]-1$

$= \left\{\left[(2021 \cdot m)+1\right] \cdot (2020)^2\right\}-1 = \left[2021 \cdot 2020^2 m\right] + \left[(2020)^2-1\right]$

since we know that $\left[(2020)^2-1\right] = 2021 \times 2019$. then

$\Rightarrow \left[2021 \cdot 2020^2 m\right] + \left[2021 \times 2019\right] = 2021\left[2020^2 \cdot m + 2019\right]$, true

∴ we complete the proof #

4. prove this by pigeonhole:

since we need to choose 4 distinct numbers from 16 numbers, so we have $C^{16}_{4} = 1820$ holes.

5.

let a_1, a_2, a_3, a_4 be chosen, and they are all $\in [1, 9]$

$a_1^2 + a_2^2 = a_{12}^2$

$a_1^2 + a_3^2 = a_{13}^2$

$a_1^2 + a_4^2 = a_{14}^2$, and $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \in \mathbb{N}$

$a_2^2 + a_3^2 = a_{23}^2$

$a_2^2 + a_4^2 = a_{24}^2$

$a_3^2 + a_4^2 = a_{34}^2$

then $\begin{cases} a_1^2 = a_{12}^2 - a_2^2 = a_{13}^2 - a_3^2 = a_{14}^2 - a_4^2 \\ a_2^2 = a_{23}^2 - a_3^2 = a_{24}^2 - a_4^2 \\ a_3^2 = a_{34}^2 - a_4^2 \end{cases}$