ECE 361 Probability for Engineers (Fall, 2016)
Homework Solutions 7

Please answer the following questions:

1. (3 points) Let (U, V) be independent continuous random variables, both uniformly distributed on $[0, 1]$, and define the pair of RVs (X, Y) where $X = \min(U, V)$ and $Y = \max(U, V)$. Compute the PDF of $Z = X/Y$. *Hint: first find the joint CDF $F_{X,Y}(x, y)$ for $0 \leq x \leq y \leq 1$, then find the joint PDF $f_{X,Y}(x, y)$, then find $F_Z(z) = \mathbb{P}(X/Y \leq z)$, for $z \in [0, 1]$, by conditioning on (X, Y), via $F_Z(z) = \int_0^1 \int_0^y \mathbb{P}(X/Y \leq z | X = x, Y = y) f_{X,Y}(x, y) \mathrm{d}x\mathrm{d}y$.*

Solution. The support of Z is $[0, 1]$. We first find the joint CDF for (X, Y): for $0 \leq x \leq y \leq 1$:

$$\begin{aligned}
F_{X,Y}(x, y) &= \mathbb{P}(X \leq x, Y \leq y) \\
&= \mathbb{P}(\min(U, V) \leq x, \max(U, V) \leq y) \\
&= \int_0^1 \mathbb{P}(\min(U, V) \leq x, \max(U, V) \leq y | U = u) f_U(u) \mathrm{d}u \\
&= \int_0^1 \mathbb{P}(\min(u, V) \leq x, \max(u, V) \leq y) f_U(u) \mathrm{d}u \\
&= \int_0^x \mathbb{P}(\min(u, V) \leq x, \max(u, V) \leq y) \mathrm{d}u + \int_x^y \mathbb{P}(\min(u, V) \leq x, \max(u, V) \leq y) \mathrm{d}u \\
&\quad + \int_y^1 \mathbb{P}(\min(u, V) \leq x, \max(u, V) \leq y) \mathrm{d}u \\
&= \int_0^x \mathbb{P}(V \leq y) \mathrm{d}u + \int_x^y \mathbb{P}(V \leq x, V \leq y) \mathrm{d}u + \int_y^1 0 \mathrm{d}u \\
&= \int_0^x y \mathrm{d}u + \int_x^y x \mathrm{d}u \\
&= xy + x(y - x) = x(2y - x) \qquad (1)
\end{aligned}$$

As an aside, observe this joint distribution has the following marginal distributions:

$$\begin{aligned}
F_X(x) &= F_{X,Y}(x, 1) = x(2 - x) \\
F_Y(y) &= F_{X,Y}(y, y) = y^2 \qquad (2)
\end{aligned}$$

The joint PDF is

$$f_{X,Y}(x, y) = 2, \ 0 \leq x \leq y \leq 1. \qquad (3)$$

The CDF for Z is

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(X/Y \leq z) \\
&= \int_0^1 \int_0^y \mathbb{P}(X/Y \leq z | X = x, Y = y) f_{X,Y}(x, y) \mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \int_0^{yz} 2 \mathrm{d}x\mathrm{d}y \\
&= 2 \int_0^1 yz \mathrm{d}y \\
&= 2z \frac{1}{2} y^2 |_0^1 \mathrm{d}y = z \qquad (4)
\end{aligned}$$

It follows that the PDF for Z is $f_Z(z) = 1$, for $0 \leq z \leq 1$. In summary, the ratio of the min over the max of two independent uniformly distributed RVs is itself uniformly distributed.

2. (2 points) Let X_1, X_2 be independent and identically distributed RVs, both exponentially distributed with parameter $\lambda > 0$. Define $X = X_1 + X_2$. Please do the following:

   - Find the PDF for X.

     Solution. Using the expression for the PDF of the sum of two independent RVs:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(x - x_1) \mathrm{d}x_1 \\ &= \int_0^x \lambda \mathrm{e}^{-\lambda x_1} \lambda \mathrm{e}^{-\lambda(x - x_1)} \mathrm{d}x_1 \\ &= \lambda^2 \mathrm{e}^{-\lambda x} \int_0^x \mathrm{d}x_1 \\ &= \lambda^2 x \mathrm{e}^{-\lambda x} \end{aligned} \tag{5}$$

   - Read about the gamma(k, θ) probability distribution, where k is the "shape" parameter and θ is the "scale" parameter. Show that the distribution of X is gamma, and find the appropriate values for (k, θ).

     Solution. The PDF for a gamma(k, θ) distribution is

$$f_X(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} \mathrm{e}^{-x/\theta}. \tag{6}$$

     It is clear that $X_1 + X_2$ above has a gamma distribution, with parameters $\lambda = 1/\theta$ and $k = 2$.

3. (2 points) Let X_1, X_2 be independent and identically distributed RVs, both distributed as standard normals. Define $X = X_1^2 + X_2^2$. Please do the following:

   - Find the PDF for X. *Hint: first find the CDF for X_1^2 and the PDF for X_1^2, then use the convolution formula to find the PDF for X. You may find it useful to use the fact that $\int_0^x \frac{1}{\sqrt{x_1(x-x_1)}} \mathrm{d}x_1 = \pi$.*

     Solution. We first find the distribution of X_1^2 for $X_1 \sim N(0,1)$. For $x \geq 0$:

$$\begin{aligned} F_{X_1^2}(x) &= \mathbb{P}(X_1^2 \leq x) \\ &= \mathbb{P}(-\sqrt{x} \leq X_1 \leq +\sqrt{x}) \\ &= F_{X_1}(\sqrt{x}) - F_{X_1}(-\sqrt{x}) \\ &= F_{X_1}(\sqrt{x}) - (1 - F_{X_1}(\sqrt{x})) \\ &= 2F_{X_1}(\sqrt{x}) - 1 \\ &= 2\Phi(\sqrt{x}) - 1, \end{aligned} \tag{7}$$

     where $\Phi(z)$ is the standard normal CDF. Thus the PDF for X_1^2 is, with $\phi(z)$ the standard normal PDF,

$$f_{X_1^2}(x) = \frac{1}{\sqrt{x}} \phi(\sqrt{x}) = \frac{1}{\sqrt{2\pi x}} \mathrm{e}^{-\frac{x}{2}} \tag{8}$$

Using the expression for the PDF of the sum of two independent RVs:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X_1^2}(x_1) f_{X_2^2}(x-x_1)\mathrm{d}x_1 \\ &= \int_0^x \frac{1}{\sqrt{2\pi x_1}}\mathrm{e}^{-\frac{x_1}{2}} \frac{1}{\sqrt{2\pi(x-x_1)}}\mathrm{e}^{-\frac{(x-x_1)}{2}}\mathrm{d}x_1 \\ &= \frac{1}{2\pi}\mathrm{e}^{-\frac{x}{2}} \int_0^x \frac{1}{\sqrt{x(x-x_1)}}\mathrm{d}x_1 \\ &= \frac{1}{2}\mathrm{e}^{-\frac{x}{2}}, \end{aligned} \tag{9}$$

where in the last step we used $\int_0^x \frac{1}{\sqrt{x_1(x-x_1)}}\mathrm{d}x_1 = \pi$.

- Read about the chi-squared $\chi^2(k)$ probability distribution, where $k \in \mathbb{N}$ is the "degrees of freedom" parameter. Show that the distribution of X is gamma, and find the appropriate values for k.

  Solution. The PDF for a $\chi^2(k)$ probability distribution is $f_X(k) = \frac{1}{2^{\frac{k}{2}}\Gamma(k/2)} x^{\frac{k}{2}-1}\mathrm{e}^{-\frac{x}{2}}$, for $k \in \mathbb{N}$. It is clear that X above is $\chi^2(2)$, i.e., with $k=2$. Also, observe $\chi^2(2)$ is equal to the exponential distribution with rate $1/2$.

4. (2 points) Let X be a standard normal random variable and define $Y = 1/X$. Please do the following:

- Find the PDF and CDF for Y. *Hint: consider $y < 0$ and $y \geq 0$ separately. Consider $y \geq 0$, and observe the equivalence of the events $\{Y \leq y\}$ and $\{X \leq 0$ or $X > 1/y\}$.*

  Solution. For $y \geq 0$:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(1/X \leq y) \\ &= \mathbb{P}(X \leq 0 \text{ or } X > 1/y) \\ &= \mathbb{P}(X \leq 0) + \mathbb{P}(X > 1/y) \\ &= \mathbb{P}(X \leq 0) + 1 - \mathbb{P}(X \leq 1/y) \\ &= 3/2 - \Phi(1/y), \end{aligned} \tag{10}$$

  where $\Phi(z)$ is the CDF for the standard normal. For $y < 0$:

$$\begin{aligned} 1 - F_Y(y) &= \mathbb{P}(Y > y) \\ &= \mathbb{P}(1/X > y) \\ &= \mathbb{P}(X > 0 \text{ or } X < 1/y) \\ &= \mathbb{P}(X > 0) + \mathbb{P}(X < 1/y) \\ &= 1/2 + \Phi(1/y) \end{aligned} \tag{11}$$

  In summary:

$$F_Y(y) = \begin{cases} 1/2 - \Phi(1/y), & y < 0 \\ 3/2 - \Phi(1/y), & y \geq 0 \end{cases} \tag{12}$$

  The PDF for Y is

$$f_Y(y) = \frac{1}{y^2}\phi(1/y) = \frac{1}{\sqrt{2\pi}y^2}\mathrm{e}^{-\frac{1}{2y^2}},\ y \in \mathbb{R}. \tag{13}$$

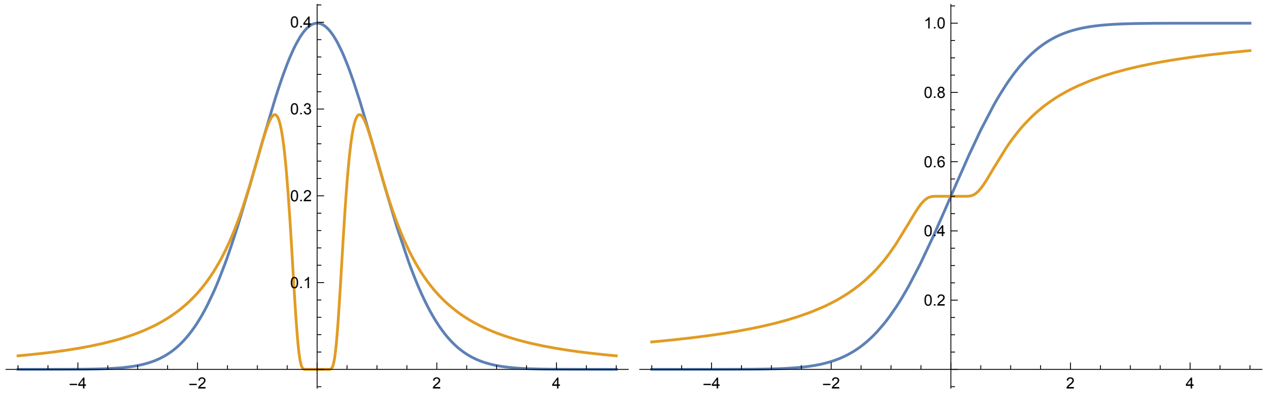

Figure 1: The PDF and CDF for a standard normal and a reciprocal of a standard normal.

- Create two plots. The first plot should show the PDF of Y and the PDF of the standard normal distribution over the interval $[-5, +5]$. The second plot should show the CDF of Y and the CDF of the standard normal distribution over the interval $[-5, +5]$.
  Solution.

5. (3 points) Let (X, Y) be a pair of Bernoulli RVs with joint PMF

$$\mathbb{P}(X = Y = 1) = p,\ \mathbb{P}(X = 1, Y = 0) = (1-p)/2,\ \mathbb{P}(X = 0, Y = 1) = (1-p)/2,\ \mathbb{P}(X = Y = 0) = 0 \quad (14)$$

for a parameter $p \in (0, 1)$. Find the correlation of X, Y in terms of p.

Solution. The correlation is

$$\rho(X, Y) = \frac{\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]}{\mathrm{Std}(X)\mathrm{Std}(Y)}. \quad (15)$$

We first find the marginal PMFs (p_X, p_Y), the means $(\mathbb{E}[X], \mathbb{E}[Y])$, and the variances $(\mathrm{Var}(X), \mathrm{Var}(Y))$ of (X, Y). First, the marginal PMFs:

$$\begin{array}{ll} p_X(1) = (1+p)/2 & p_X(0) = (1-p)/2 \\ p_Y(1) = (1+p)/2 & p_Y(0) = (1-p)/2 \end{array} \quad (16)$$

Second, the expected values are:

$$\begin{aligned} \mathbb{E}[X] &= (1+p)/2 \\ \mathbb{E}[Y] &= (1+p)/2 \end{aligned} \quad (17)$$

Third, the variances are:

$$\begin{aligned} \mathrm{Var}(X) &= (1+p)/2(1-(1+p)/2) = \frac{1}{4}(1+p)(1-p) \\ \mathrm{Var}(Y) &= (1+p)/2(1-(1+p)/2) = \frac{1}{4}(1+p)(1-p) \end{aligned} \quad (18)$$

Fourth, we compute covariance, the numerator of the correlation:

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\
&= \sum_{(x,y)\in\{0,1\}^2} p_{X,Y}(x,y)(x-\mathbb{E}[X])(y-\mathbb{E}[Y]) \\
&= (1-(1+p)/2)(1-(1+p)/2)p + (0-(1+p)/2)(1-(1+p)/2)(1-p)/2 \\
&\quad +(1-(1+p)/2)(0-(1+p)/2)(1-p)/2 + (0-(1+p)/2)(0-(1+p)/2)0 \\
&= -\frac{1}{4}(1-p)^2 \qquad (19)
\end{aligned}$$

Finally, we compute the correlation:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Std}(X)\mathrm{Std}(Y)} = -\frac{(1-p)^2/4}{(1+p)(1-p)/4} = -\frac{1-p}{1+p}. \qquad (20)$$

6. (2 points) Let (X,Y) be a pair of discrete RVs, each with support $\{-1,0,+1\}$, with joint PMF:

$$\mathbb{P}(X=-1,Y=0) = \mathbb{P}(X=0,Y=-1) = \mathbb{P}(X=0,Y=+1) = \mathbb{P}(X=+1,Y=0) = \frac{1}{4}. \qquad (21)$$

Please do the following:

- Find the correlation of (X,Y).
  Solution. The marginal PMFs (p_X, p_Y) are:

	-1	0	$+1$
$p_X(x)$	$1/4$	$1/2$	$1/4$
$p_Y(y)$	$1/4$	$1/2$	$1/4$

(22)

  The means are:

$$\begin{aligned}
\mathbb{E}[X] &= (-1)(1/4) + (0)(1/2) + (+1)(1/4) = 0 \\
\mathbb{E}[Y] &= (-1)(1/4) + (0)(1/2) + (+1)(1/4) = 0 \qquad (23)
\end{aligned}$$

  The covariance is

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\
&= \mathbb{E}[XY] \\
&= \frac{1}{4}((-1)(0)+(0)(-1)+(0)(+1)+(+1)(0)) = 0 \qquad (24)
\end{aligned}$$

  The correlation $\rho(X,Y)$ is thus also equal to zero.
- Determine whether (X,Y) are dependent or independent.
  Solution. If (X,Y) are independent then $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for each pair (x,y). Observe:

x	y	$p_{X,Y}(x,y)$	$p_X(x)$	$p_Y(y)$	$p_X(x)p_Y(y)$
-1	0	$1/4$	$1/4$	$1/2$	$1/8$
0	-1	$1/4$	$1/2$	$1/4$	$1/8$
0	$+1$	$1/4$	$1/2$	$1/4$	$1/8$
$+1$	0	$1/4$	$1/4$	$1/2$	$1/8$

(25)

  Observe $p_{X,Y}(x,y) \neq p_X(x)p_Y(y)$ for *any* pair (x,y). Thus (X,Y) are *not* dependent but *are* uncorrelated.

7. (3 points) Let (U,V) be independent continuous random variables, both uniformly distributed on $[0,1]$, and define the pair $(X,Y))$ where $X = \min(U,V)$ and $Y = \max(U,V)$. Compute the correlation of (X,Y). *Hints: recall the joint* $(f_{X,Y}(x,y))$ *and marginal* $(f_X(x), f_Y(y))$ *distributions from an earlier problem. Compute the means* $(\mathbb{E}[X], \mathbb{E}[Y])$*, compute the expected squared values* $(\mathbb{E}[X^2], \mathbb{E}[Y^2])$*, compute the variances* $(\mathrm{Var}(X), \mathrm{Var}(Y))$*, compute the covariance* $\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$ *using the joint distribution* $f_{X,Y}(x,y)$*, and finally, compute the correlation.*

Solution. We already know from an earlier problem that

$$\begin{aligned} F_{X,Y}(x,y) = x(2y-x) \quad F_X(x) = x(2-x) \quad F_Y(y) = y^2 \\ f_{X,Y}(x,y) = 2 \quad f_X(x) = 2(1-x) \quad f_Y(y) = 2y \end{aligned} \tag{26}$$

First, find the expected values:

$$\begin{aligned} \mathbb{E}[X] &= \int_0^1 x f_X(x) \mathrm{d}x = \int_0^1 x2(1-x)\mathrm{d}x = \frac{1}{3} \\ \mathbb{E}[Y] &= \int_0^1 y f_Y(y) \mathrm{d}y = \int_0^1 y2y\mathrm{d}y = \frac{2}{3} \end{aligned} \tag{27}$$

Second, find the expected squared values:

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^1 x^2 f_X(x) \mathrm{d}x = \int_0^1 x^2 2(1-x)\mathrm{d}x = \frac{1}{6} \\ \mathbb{E}[Y^2] &= \int_0^1 y^2 f_Y(y) \mathrm{d}y = \int_0^1 y^2 2y\mathrm{d}y = \frac{1}{2} \end{aligned} \tag{28}$$

Third, find the variances:

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18} \\ \mathrm{Var}(Y) &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}. \end{aligned} \tag{29}$$

Fourth, find the covariance:

$$\begin{aligned} \mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\ &= \int_0^1 \int_0^y \left(x - \frac{1}{3}\right)\left(y - \frac{2}{3}\right) f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\ &= 2\int_0^1 \int_0^y \left(x - \frac{1}{3}\right)\left(y - \frac{2}{3}\right) \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{36} \end{aligned} \tag{30}$$

Finally, find the correlation

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Std}(X)\mathrm{Std}(Y)} = \frac{1/36}{1/18} = \frac{1}{2}. \tag{31}$$

8. (2 points) Let (X,Y) be the coordinates of a point chosen uniformly at random in the unit circle $D = \{(x,y) : \sqrt{x^2+y^2} \le 1\}$. Find the correlation of (X,Y).

Solution. It is clear that

$$f_{X,Y}(x,y) = \begin{cases} 1, & x^2+y^2 \leq 1 \\ 0, & \text{else} \end{cases} \tag{32}$$

From here we can find the marginal distributions

$$\begin{aligned} f_X(x) &= \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} f_{X,Y}(x,y)\mathrm{d}y = 2\sqrt{1-x^2} \\ f_Y(y) &= \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} f_{X,Y}(x,y)\mathrm{d}x = 2\sqrt{1-y^2} \end{aligned} \tag{33}$$

From here we can find the expected values:

$$\begin{aligned} \mathbb{E}[X] &= \int_{-1}^{+1} x f_X(x)\mathrm{d}x = 2\int_{-1}^{+1} x\sqrt{1-x^2}\mathrm{d}x = 0 \\ \mathbb{E}[Y] &= \int_{-1}^{+1} y f_Y(y)\mathrm{d}y = 2\int_{-1}^{+1} y\sqrt{1-y^2}\mathrm{d}y = 0 \end{aligned} \tag{34}$$

Next, we find the covariance

$$\begin{aligned} \mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\ &= \mathbb{E}[XY] \\ &= \int_{-1}^{+1}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} xy f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_{-1}^{+1} y \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} x\mathrm{d}x\mathrm{d}y \\ &= 0 \end{aligned} \tag{35}$$

It follows that the correlation is zero. Although (X,Y) are dependent (observe $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$), they are uncorrelated.