ECE 361 Probability for Engineers (Fall, 2016)
Midterm Exam Solution

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1. **4 points.** Presents. Fix integers k, n with $0 < k < n$. A mischevious parent places k toys in n (closed) boxes and wraps up each box as a present. Thus k of the boxes hold toys and $n-k$ of the boxes are empty. A child is then allowed to open these presents one by one, stopping as soon as *any one* of the toys is found. Let the RV X denote the number of presents opened. *Hint: the event that x presents are opened means the first $(x-1)$ presents opened did not contain a toy and the xth present did contain a toy. There are $\binom{n}{x}$ ways of selecting x of the n presents and $x!$ ways of ordering them, for a total of $\binom{n}{x}x!$ ways of selecting x ordered presents. Compute how many of these ways result in first finding a toy on the xth present.*

   (a) **2 points.** Give the support $\mathcal{X}$ of the RV X in terms of k, n.
   Solution. $\mathcal{X} = \{1, \ldots, n-k+1\}$.

   (b) **2 points.** Give the probability mass function $p_X = (p_X(x), x \in \mathcal{X})$.
   Solution. The answer is:
$$p_X(x) = \frac{\binom{n-k}{x-1}\binom{k}{1}(x-1)!1!}{\binom{n}{x}x!} = \frac{\binom{n-k}{x-1}k}{\binom{n}{x}x}. \tag{1}$$
   To see this, suppose first the presents are labeled $1, \ldots, n$, and observe the event that x presents are opened means the first $x-1$ presents opened did not contain a toy, while the xth present did contain a toy. There are $\binom{n}{x}x!$ distinct ways of selecting the ordered list of x presents to open: $\binom{n}{x}$ ways to select the x presents and $x!$ ways to order them. Of these, there are $\binom{n-k}{x-1}\binom{k}{1}(x-1)!1!$ ways to select an ordered list of $x-1$ presents that don't contain a toy and ending with the xth present holding a toy. That is, $\binom{n-k}{x-1}$ ways to select the $x-1$ presents not containing a toy, each with $(x-1)!$ orderings, and $\binom{k}{1}$ ways to select the xth present containing a toy, with $1!$ ordering.

2. **4 points.** Rock, paper, scissors. Alejandro and Brunhilda play the game "rock paper scissors". That is, the two players each select one of these three items simultaneously and the winner of the game is determined by the following rule: rock beats scissors, scissors beats paper, paper beats rock, and if the two players select the same item then the outcome is a tie. Alejandro selects his item $X \in \{r, p, s\}$ (independently of Brunhilda) with probabilities $p_X = (p_X(r), p_X(p), p_X(s))$, where p_X is nonnegative and sums to one. Brunhilda selects her item $Y \in \{r, p, s\}$ (independently of Alejandro) with probabilities $p_Y = (p_Y(r), p_Y(p), p_Y(s))$, where p_Y is nonnegative and sums to one.

   (a) **2 points.** What is the probability that Alejandro wins the game?
   Solution. Let the ordered pair $(a, b) \in \{r, p, s\}^2$ denote the selections of Alejandro and Brunhilda, respectively. The event that Alejandro wins corresponds to outcomes $A = \{(r, s), (p, r), (s, p)\}$, which has probability
$$\mathbb{P}(A) = p_X(r)p_Y(s) + p_X(p)p_Y(r) + p_X(s)p_Y(p). \tag{2}$$

   (b) **2 points.** What is the probability of a tie?
   Solution. Using the same notation as above, the event of a tie corresponds to outcomes $T = \{(r, r), (p, p), (s, s)\}$, which has probability
$$\mathbb{P}(T) = p_X(r)p_Y(r) + p_X(p)p_Y(p) + p_X(s)p_Y(s). \tag{3}$$

3. **4 points.** Mini World Series. Two teams, the Aardwolves (a South African fox-like animal) and the Babirusas (a large wild pig of Indonesia), play a series of games, where in each game the Aardwolves win with probability p or the Babirusas win with probability $1-p$, for some $p \in [0,1]$. The outcomes of the games are independent of one another. The two teams play a "best of three" series, meaning the first team to win two games is the winner of the series.

   (a) **2 points.** What is the probability that the Aardwolves win the series?
   Solution. The event A that the Aardwolves win corresponds to series $A = \{(aa),(aba),(baa)\}$ and the probability of this event is thus
   $$\mathbb{P}(A) = \mathbb{P}(aa) + \mathbb{P}(aba) + \mathbb{P}(baa) = p^2 + 2p^2(1-p) = p^2(3-2p). \tag{4}$$
   As an aside, the event B that the Babirusas win corrresponds to the series $B = \{(bb),(bab),(abb)\}$ and has probability
   $$\mathbb{P}(B) = \mathbb{P}(bb) + \mathbb{P}(bab) + \mathbb{P}(abb) = \bar{p}^2 + 2p\bar{p}^2 = \bar{p}^2(3-2\bar{p}). \tag{5}$$
   Straightforward algebra establishes $\mathbb{P}(A) + \mathbb{P}(B) = 1$, as expected, but this was not required for credit.

   (b) **2 points.** What is the probability mass function on the number of games played in the series?
   Solution. The number of games is a random variable, say N, with support $\mathcal{N} = \{2,3\}$. The event $N = 2$ corresponds to series $\{N=2\} = \{(aa),(bb)\}$, while the event $\{N=3\} = \{(abb),(aba),(baa),(bab)\}$. Observe:
   $$\begin{aligned} \mathbb{P}(N=2) &= \mathbb{P}(aa) + \mathbb{P}(bb) = p^2 + \bar{p}^2 \\ \mathbb{P}(N=3) &= \mathbb{P}(abb) + \mathbb{P}(aba) + \mathbb{P}(baa) + \mathbb{P}(bab) = 2p\bar{p}^2 + 2\bar{p}p^2 \end{aligned} \tag{6}$$
   Again, simple algebra establishes that $\mathbb{P}(N=2) + \mathbb{P}(N=3) = 1$, but this was not required for credit.

4. **12 points.** Colors. Each of three players (Adebayo, Baldemar, and Cuthbert), denoted A, B, C, simultaneously and independently picks one of two colors, coquelicot (a shade of orange) or smaragdine (a shade of green), denoted c, s. Let p be the probability that a player selects coquelicot, and $1-p$ be the probability a player selects smaragdine. Define the following RVs, where a majority means two or three players:

   - Define the RV X to equal: *i*) -1 if the majority select coquelicot, *ii*) $+1$ if the majority select smaragdine, or *iii*) 0 if there is no majority.
   - Define the RV Y to equal: *i*) -1 if all three players select coquelicot, *ii*) $+1$ if all three players select smaragdine, or *iii*) 0 if the players select a mixture of colors between them.
   - Define the RV $Z = XY$.

   (a) **2 points.** Give the joint PMF for (X,Y) in terms of p. *Hint: first fill in the remaining seven entries in the following table, and use it to construct the table $p_{X,Y}$ with three rows ($X \in \{-1,0,+1\}$) and three columns ($Y \in \{-1,0,+1\}$). Note, some entries in $p_{X,Y}$ may equal zero. Clearly label the rows and columns of the $p_{X,Y}$ table to indicate the corresponding values of X, Y.*

A	B	C	$p(A,B,C)$	X	Y
c	c	c	p^3	-1	-1

(7)

Solution. Although it is not required for credit, it may be helpful to construct the following table:

A	B	C	$p(A,B,C)$	X	Y
c	c	c	p^3	-1	-1
c	c	s	$p^2(1-p)$	-1	0
c	s	c	$p^2(1-p)$	-1	0
s	c	c	$p^2(1-p)$	-1	0
c	s	s	$p(1-p)^2$	$+1$	0
s	c	s	$p(1-p)^2$	$+1$	0
s	s	c	$p(1-p)^2$	$+1$	0
s	s	s	$(1-p)^3$	$+1$	$+1$

(8)

With this table one can construct the joint distribution on (X,Y) as:

$$p_{X,Y} = \begin{array}{c} \\ {\scriptstyle X=-1} \\ {\scriptstyle X=0} \\ {\scriptstyle X=+1} \end{array} \begin{array}{c} \begin{array}{ccc} {\scriptstyle Y=-1} & {\scriptstyle Y=0} & {\scriptstyle Y=+1} \end{array} \\ \left[\begin{array}{ccc} p^3 & 3p^2(1-p) & \\ & & \\ & 3p(1-p)^2 & (1-p)^3 \end{array} \right] \end{array} \quad (9)$$

(b) **2 points.** Give the PMF for X in terms of p

Solution. Marginalizing the above joint distribution for X gives:

X	-1	0	$+1$
$p_X(x)$	$p^3+3p^2(1-p)$		$(1-p)^3+3p(1-p)^2$

(10)

An alternative expression, equivalent to the above, is (with $\bar{p} = 1-p$):

x	-1	0	$+1$
$p_X(x)$	$p^2(3-2p)$	0	$\bar{p}^2(3-2\bar{p})$

(11)

(c) **2 points.** Give the PMF for Z in terms of p

Solution. The RV Z has support $\mathcal{Z} = \{0,+1\}$ with PMF

z	0	$+1$
$p_Z(z)$	$3p(1-p)$	$p^3+(1-p)^3$

(12)

(d) **2 points.** Give $\mathbb{E}[X]$ in terms of p

Solution. We compute

$$\begin{aligned} \mathbb{E}[X] &= (-1)\times p^2(3-2p) + 0\times 0 + (+1)\times \bar{p}^2(3-2\bar{p}) \\ &= 1-2p^2(3-2p) \end{aligned} \quad (13)$$

(e) **2 points.** Give $\mathbb{E}[Z]$ in terms of p

Solution. We compute

$$\begin{aligned} \mathbb{E}[Z] &= 0\times 3p(1-p) + (+1)\times(p^3+(1-p)^3) \\ &= p^3+(1-p)^3 \end{aligned} \quad (14)$$

(f) **2 points.** Give $\mathrm{Var}(Z)$ in terms of p
Solution. We compute

$$\begin{aligned} \mathbb{E}[Z^2] &= 0^2 \times 3p(1-p) + (+1)^2 \times (p^3 + (1-p)^3) \\ &= p^3 + (1-p)^3 \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} \mathrm{Var}(Z) &= \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 \\ &= (p^3 + (1-p)^3) - (p^3 + (1-p)^3)^2 \\ &= 3p - 12p^2 + 18p^3 - 9p^4 \\ &= 3p(1 - p(4 - 3p(2-p))) \end{aligned} \tag{16}$$

Any of the above three forms being acceptable.

5. **4 points.** Coins. Agathangelos and Benedykta hold separate biased coins and perform the following experiment. Agathangelos flips his coin n_A times (for $n_A \in \mathbb{N}$); his coin comes up heads with probability $p_A \in [0,1]$ or tails with probability $1 - p_A$. Benedykta flips her coin n_B times (for $n_B \in \mathbb{N}$); her coin comes up heads with probability $p_B \in [0,1]$ or tails with probability $1 - p_B$. Let the RV X be the number of times that Agathangelos's coin shows a head and let the RV Y be the number of times that Benedykta's coin shows a head. Let the RV $Z = aX + bY$, for (a,b) real numbers.

   (a) **2 points.** Give an expression for $\mathbb{E}[Z]$ in terms of $(a, b, p_A, p_B, n_A, n_B)$.
   Solution. Clearly $X \sim \mathrm{bin}(n_A, p_A)$ and $Y \sim \mathrm{bin}(n_B, p_B)$, with $\mathbb{E}[X] = n_A p_A$ and $\mathbb{E}[Y] = n_B p_B$, and thus by linearity of expectation

   $$\mathbb{E}[Z] = \mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y] = a n_A p_A + b n_B p_B. \tag{17}$$

   (b) **2 points.** Give an expression for $\mathrm{Var}(Z)$ in terms of $(a, b, p_A, p_B, n_A, n_B)$.
   Solution. As (X, Y) are clearly independent we can use the rule for the variance of a linear combination of independent RVs:

   $$\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) = a^2 n_A p_A (1 - p_A) + b^2 n_B p_B (1 - p_B). \tag{18}$$

6. **Extra Credit: 4 points.** Voting. Two political candidates, Akulina and Beulah, are competing in an election with 3 voters named Chinatsu, Damayanti, and Edgard. Each of the voters votes randomly and independently, casting a vote for Akulina with probability p or for Beulah with probability $1-p$, for $p \in [0,1]$. The winner is the candidate with the majority vote.

   (a) **2 points.** What is the probability that Chinatsu voted for Akulina given that Akulina wins the election?
   Solution. Let C be the event that Chinatsu voted for Akulina and let A be the event that Akulina wins the election. Then $\mathbb{P}(A|C) = 1 - (1-p)^2$ and $\mathbb{P}(A|C^c) = p^2$. To see this, observe $\mathbb{P}(A|C)$ is the probability that either or both of Damayanti and Edgard voted for Akulina, or equivalently, one minus the probability that neither of them voted for Akulina. By Bayes' rule:

   $$\mathbb{P}(C|A) = \frac{\mathbb{P}(A|C)\mathbb{P}(C)}{\mathbb{P}(A|C)\mathbb{P}(C) + \mathbb{P}(A|C^c)\mathbb{P}(C^c)} = \frac{(1-(1-p)^2)p}{(1-(1-p)^2)p + p^2(1-p)} = \frac{2-p}{3-2p}. \tag{19}$$

(b) **2 points.** What is the probability that all three voters voted for Akulina given that Akulina wins the election?

Solution. Let F be the event that all three vote for Akulina. Then $\mathbb{P}(A|F) = 1$ and $\mathbb{P}(A|F^c) = 3p^2(1-p)/(1-p^3)$. To see this, observe there are seven possible outcomes for the votes conditioned on not all three votes being for Akulina. First, the three outcomes with two votes for a are (aab, aba, baa), with total probability $3p^2(1-p)/(1-p^3)$. Second, the three outcomes with one vote for a are (bba, bab, abb), with total probability $3(1-p)^2p/(1-p^3)$. Third, the one outcome with zero votes for a, i.e., bbb, has probability $(1-p)^3/(1-p^3)$. Now, by Bayes' rule:

$$\mathbb{P}(F|A) = \frac{\mathbb{P}(A|F)\mathbb{P}(F)}{\mathbb{P}(A|F)\mathbb{P}(F) + \mathbb{P}(A|F^c)\mathbb{P}(F^c)} = \frac{1 \times p^3}{1 \times p^3 + 3p^2(1-p)/(1-p^3) \times (1-p^3)} = \frac{p}{3-2p}. \quad (20)$$