ECE 361 Probability for Engineers (Fall, 2016)
Lecture 3b

Some hints for homework 3

Hints on problems 1 and 8...

§2.4 Expectation, mean, and variance

Mean and variance of some common RVs

Example. The mean and variance of a Poisson RV. Let $X \sim \mathrm{Po}(\lambda)$. Then:

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k p(k) = \sum_{k=0}^{\infty} k \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{m=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^m}{m!} = \lambda. \tag{1}$$

Next,

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 p(k) = \sum_{k=1}^{\infty} k^2 \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} k \mathrm{e}^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{m=0}^{\infty} (m+1) \mathrm{e}^{-\lambda} \frac{\lambda^m}{m!} = \lambda(\lambda+1). \tag{2}$$

Finally:

$$\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda. \tag{3}$$

Decision making using expected values

Given a set of decisions from which to choose, each with an associated random reward, it is often advantageous to choose the one with the highest expected value. We illustrate this using an example.

Example. The quiz problem. A quiz game requires the contestant to answer two questions and the person must decide which question to answer first. Question 1 will be answered correctly with probability 0.8 and yields prize 100, while question 2 will be answered correctly with probability 0.5 and yields prize 200. If the first question is answered incorrectly the quiz terminates, while if answered correctly the contestant is given a chance to answer the second question. Which question should be answered first to maximize the expected reward? Note the tradeoffs make the answer non-obvious. See Fig. 2.9 in the text. Let X be the total reward received. If we answer question 1 first the PMF is

$$p(0) = 0.2,\ p(100) = 0.8 \times 0.5,\ p(300) = 0.8 \times 0.5. \tag{4}$$

If we answer question 2 first the PMF is

$$p(0) = 0.5,\ p(200) = 0.5 \times 0.2,\ p(300) = 0.5 \times 0.8. \tag{5}$$

The expected values are $\mathbb{E}[X] = 160$ and $\mathbb{E}[X] = 140$ respectively.

Now generalize the example to success probabilities p_1, p_2 and rewards v_1, v_2. Selecting question 1 first:

$$\mathbb{E}[X] = p_1(1-p_2)v_1 + p_1 p_2 (v_1 + v_2) = p_1 v_1 + p_1 p_2 v_2. \tag{6}$$

Selecting question 2 first:

$$\mathbb{E}[X] = p_2(1-p_1)v_2 + p_2 p_1 (v_1 + v_2) = p_2 v_2 + p_2 p_1 v_1. \tag{7}$$

It is optimal to answer question 1 first iff

$$p_1 v_1 + p_1 p_2 v_2 > p_2 v_2 + p_2 p_1 v_1 \Leftrightarrow \frac{p_1 v_1}{1-p_1} \geq \frac{p_2 v_2}{1-p_2}. \tag{8}$$

Thus answer the questions in the order of their value index, where the value index for question i is $p_i v_i / (1-p_i)$.

§2.5 Joint PMFs of multiple RVs

There may be multiple RVs associated with a given random experiment, each RV is simply a function from the sample space Ω to the reals $\mathbb{R}$. Recall that for a single RV X with PMF $\mathbf{p} = (p_X(x),\ x \in \mathcal{X})$, the PMF elements $p_X(x)$ are shorthand for probabilities of events: $p_X(x) = \mathbb{P}(X = x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$. We now extend this concept to two RVs, X, Y, through the joint PMF

$$\mathbf{p}_{X,Y} = (p_{X,Y}(x,y),\ x \in \mathcal{X}, y \in \mathcal{Y}). \tag{9}$$

Here the elements of the joint PMF $p_{X,Y}(x,y)$ are the probabilities of the joint event $\mathbb{P}(\{\omega : X(\omega) = x\} \cap \{\omega : Y(\omega) = y\})$. If $A \subset \mathcal{X} \times \mathcal{Y}$ then we find its probability by simply summing the joint PMF over the corresponding pairs:

$$\mathbb{P}((X,Y) \in A) = \sum_{(x,y) \in A} p_{X,Y}(x,y) \tag{10}$$

Given a joint PMF $\mathbf{p}_{X,Y}$ we call the the PMFs $\mathbf{p}_X$ and $\mathbf{p}_Y$ marginal PMFs. We can obtain the marginals from the joint, but not vice-versa[1]:

$$p_X(x) = \sum_{y \in \mathcal{Y}} p_{X,Y}(x,y),\ \forall x \in \mathcal{X}, \quad p_Y(y) = \sum_{x \in \mathcal{X}} p_{X,Y}(x,y),\ \forall y \in \mathcal{Y}. \tag{11}$$

A useful device for representing joint PMFs is the tabular method shown in Fig. 2.10 in the text. Note how the marginal PMFs for X, Y are computed by summing the rows and columns, respectively.

References

[1] *Introduction to Probability, 2nd Edition* by Dimitri P. Bertsekas and John N. Tsitsiklis, Athina Scientific Press, 2008.

[1] Unless X, Y are independent, to be discussed later.