ECE 361 Probability for Engineers (Fall, 2016)
Lecture 8b

§4.1 Derived distributions

Functions of two RVs

To find the PDF of $Z = g(X, Y)$ we use the same two step procedure: i) find the CDF F_Z, ii) differentiate to find f_Z.

Example. Two archers shoot at a target, the distance from the center of each shot is an independent uniform RV. What is the PDF of the distance of the losing shot? Let X, Y be the distances and define $Z = \max\{X, Y\}$. Then:

$$F_Z(z) = \mathbb{P}(\max\{X, Y\} \leq z) = \mathbb{P}(X \leq z, Y \leq z) = F_X(z)F_Y(z) = z^2, \; z \in [0, 1]. \tag{1}$$

and

$$f_Z(z) = 2z, \; z \in [0, 1]. \tag{2}$$

Example. Let X, Y be independent uniform RVs on $[0, 1]$. Find the PDF of $Z = Y/X$. Consider $z \in (0, 1]$ separately from $(1, \infty)$. See Fig. 4.5. Then:

$$F_Z(z) = \mathbb{P}(Y/X \leq z) = \begin{cases} \frac{z}{2}, & 0 \leq z \leq 1 \\ 1 - \frac{1}{2z}, & z > 1 \end{cases} \tag{3}$$

and

$$f_Z(z) = \begin{cases} \frac{1}{2}, & 0 \leq z \leq 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases} \tag{4}$$

Example. Let X, Y be independent exponential RVs with parameter λ Find the PDF of $Z = X - Y$. See Fig. 4.6. We consider separately the cases $z \geq 0$ and $z < 0$. For the first case:

$$F_Z(z) = \mathbb{P}(X - Y \leq z) = \int_0^\infty F_X(y + z) f_Y(y) \mathrm{d}y = 1 - \frac{1}{2}\mathrm{e}^{-\lambda z}. \tag{5}$$

For the second case:

$$F_Z(z) = 1 - \mathbb{P}(X - Y > z) = 1 - \int_0^\infty F_Y(x - z) f_X(x) \mathrm{d}x = \frac{1}{2}\mathrm{e}^{\lambda z}. \tag{6}$$

Differentiation gives the PDF:

$$f_Z(z) = \begin{cases} \frac{\lambda}{2}\mathrm{e}^{-\lambda z}, & z \geq 0 \\ \frac{\lambda}{2}\mathrm{e}^{\lambda z}, & z < 0 \end{cases} = \frac{\lambda}{2}\mathrm{e}^{-\lambda |z|}. \tag{7}$$

See Fig. 1.

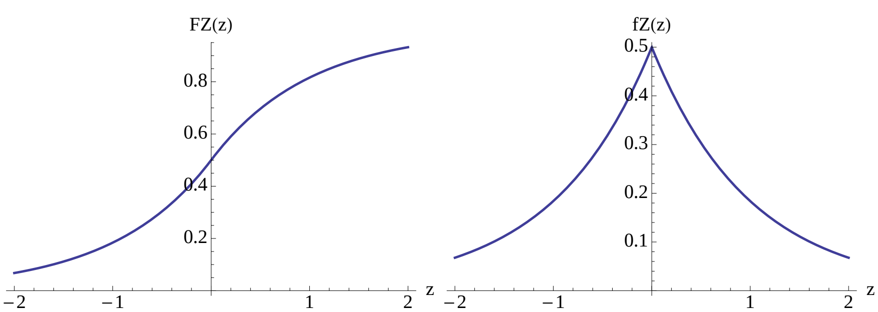

Figure 1: The CDF (left) and the PDF (right) for the example $Z = X - Y$ for X, Y independent exponential RVs.

Sums of independent RVs – convolution

Consider the special case where $Z = g(X,Y) = X+Y$ for X,Y independent RVs. First consider where X,Y are discrete and have PMFs $\mathbf{p}_X$ and $\mathbf{p}_Y$. Then for any integer z:

$$p_Z(z) = \mathbb{P}(X+Y=z) = \sum_{(x,y):x+y=z} \mathbb{P}(X=x,Y=y) = \sum_x \mathbb{P}(X=x,Y=z-x) = \sum_x p_X(x)p_Y(z-x) \quad (8)$$

This is the convolution of the PMFs $\mathbf{p}_X, \mathbf{p}_Y$. Now consider independent continuous RVs X,Y. Note:

$$F_{Z|X}(z|x) = \mathbb{P}(Z \leq z|X=x) = \mathbb{P}(X+Y \leq z|X=x) = \mathbb{P}(Y \leq z-x|X=x) = \mathbb{P}(Y \leq z-x). \quad (9)$$

Differentiation gives:

$$f_{Z|X}(z|x) = f_Y(z-x). \quad (10)$$

By the multiplication rule:

$$f_{X,Z}(x,z) = f_X(x)f_{Z|X}(z|x) = f_X(x)f_Y(z-x). \quad (11)$$

Marginalizing this joint disbn over X gives the marginal for Z:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(x,z)\mathrm{d}x = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\mathrm{d}x. \quad (12)$$

Note this is the analogue of the discrete case.

Example. Let X,Y be independent and uniform over $[0,1]$. The PDF $Z = X+Y$ is:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\mathrm{d}x = \int_{\max\{0,z-1\}}^{\min\{1,z\}} \mathrm{d}x = \min\{1,z\} - \max\{0,z-1\}, \ z \in [0,2]. \quad (13)$$

This has the shape given in Fig. 4.9.

Example. Let X,Y be independent normal RVs with parameters μ_x, μ_y and σ_x, σ_y, and define $Z = X+Y$. It follows that $Z \sim N(\mu_x + \mu_y, \sqrt{\sigma_x^2 + \sigma_y^2})$.

Example. Let X,Y be independent RVs, then $X-Y$ has PDF

$$f_{X-Y}(z) = \int_{-\infty}^{\infty} f_{-Y}(z-x)f_X(x)\mathrm{d}x = \int_{-\infty}^{\infty} f_Y(x-z)f_X(x)\mathrm{d}x. \quad (14)$$

Suppose $X,Y \sim \mathrm{Exp}(\lambda)$. Then:

$$f_{X-Y}(z) = \int_{-\infty}^{\infty} f_Y(x-z)f_X(x)\mathrm{d}x = \int_z^{\infty} \lambda \mathrm{e}^{-\lambda x}\lambda \mathrm{e}^{-\lambda(x-z)}\mathrm{d}x = \lambda^2 \mathrm{e}^{\lambda z}\int_z^{\infty} \mathrm{e}^{-2\lambda x}\mathrm{d}x = \frac{\lambda}{2}\mathrm{e}^{-\lambda z}, \ z \geq 0. \quad (15)$$

References

[1] *Introduction to Probability, 2nd Edition* by Dimitri P. Bertsekas and John N. Tsitsiklis, Athina Scientific Press, 2008.