ECE 361 Probability for Engineers (Fall, 2016)
Homework Solutions 3

Please answer the following questions:

1. (3 points) Let X, Y be a pair of independent and identically distributed random variables, each one uniformly distributed over $\{0, \ldots, N\}$, for $N = 10$. That is, $X \sim \text{Uni}(0, N)$ and $Y \sim \text{Uni}(0, N)$. Let $Z = X - Y$. Find the probability mass function for Z. *Hint: first find the support for Z, say $\mathcal{Z}$. Next, for any $z \in \mathcal{Z}$, find the set of all pairs (x, y) such that $x - y = z$. Finally, the probability that $(X, Y) = (x, y) = p_X(x)p_Y(y) = \frac{1}{N+1} \times \frac{1}{N+1}$, as (X, Y) are independent.*

   Solution. Observe $\mathcal{Z} = \{-N, \ldots, +N\}$. There are three ranges for z: $z < 0$, $z = 0$, and $z > 0$:

   - For $z \in \{-N, \ldots, -1\}$ the event $x - y = z$ holds for $(x, y) \in \{(0, -z), (1, -z+1), \ldots, (N+z, N)\}$, and there are $N + z + 1$ such points.
   - For $z = 0$ the event $x - y = 0$ holds for $(x, y) \in \{(0, 0), \ldots, (N, N)\}$, and there are $N + 1$ such points.
   - For $z \in \{+1, \ldots, +N\}$ the event $x - y = z$ holds for $(x, y) \in \{(z, 0), \ldots, (N, N - z)\}$, and there are $N - z + 1$ such points.

   As there are $(N+1) \times (N+1) = (N+1)^2$ possible values for the pair (X, Y), each of which is equally likely, it follows that Z has the following PMF:

$$p_Z(z) = \begin{cases} \frac{N+z+1}{(N+1)^2}, & z \in \{-N, \ldots, -1\} \\ \frac{1}{N+1}, & z = 0 \\ \frac{N-z+1}{(N+1)^2}, & z \in \{+1, \ldots, +N\} \end{cases} \tag{1}$$

   More compactly,

$$p_Z(z) = \frac{N - |z| + 1}{(N+1)^2}, \; z \in \{-N, \ldots, +N\}. \tag{2}$$

   This PMF is shown for $N = 10$ in Fig. 1.

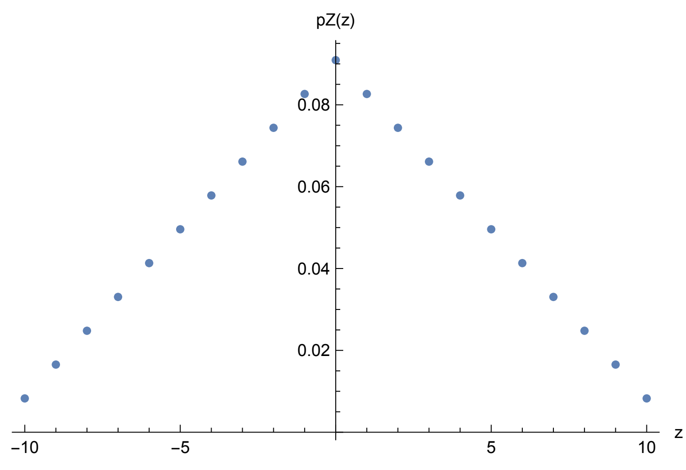

Figure 1: The PMF for problem 1 for $N = 10$.

2. (3 points) Let X, Y be as in the previous problem. Let $Z = X + Y$. Find the probability mass function for Z.

   Solution. Now $\mathcal{Z} = \{0, \ldots, 2N\}$. There are three ranges for z: $z < N$, $z = N$, and $z > N$.

- For $z \in \{0, \ldots, N-1\}$ the event $x+y=z$ holds for $(x,y) \in \{(0,z),(1,z-1),\ldots,(z-1,1),(z,0)\}$, and there are $z+1$ such points.
- For $z = N$ the event $x+y=N$ holds for $(x,y) \in \{(0,N),(1,N-1)\ldots,(N-1,1),(N,0)\}$, and there are $N+1$ such points.
- For $z \in \{N+1,\ldots,2N\}$ the event $x+y=z$ holds for $(x,y) \in \{(z-N,N),(z-N+1,N-1)\ldots,(N-1,z-N+1),(N,z-N)\}$, and there are $2N-z+1$ such points.

As there are $(N+1)\times(N+1) = (N+1)^2$ possible values for the pair (X,Y), each of which is equally likely, it follows that Z has the following PMF:

$$p_Z(z) = \begin{cases} \frac{z+1}{(N+1)^2}, & z \in \{0,\ldots,N-1\} \\ \frac{1}{N+1}, & z = 0 \\ \frac{2N-z+1}{(N+1)^2}, & z \in \{N+1,\ldots,2N\} \end{cases} \tag{3}$$

More compactly,

$$p_Z(z) = \frac{1}{N+1}\left(1 - \frac{|z-N|}{N+1}\right), \; z \in \{0,\ldots,2N\}. \tag{4}$$

This PMF is shown for $N = 10$ in Fig. 2.

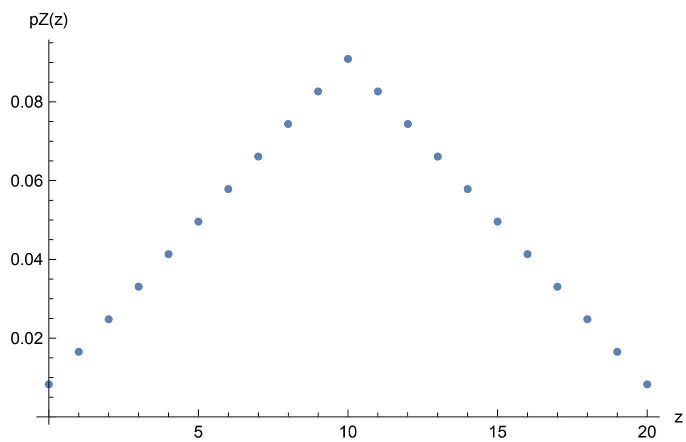

Figure 2: The PMF for problem 2 for $N = 10$.

3. (2 points) Let X, Y be as in the previous problem. Find $\mathbb{E}[X-Y]$ and $\mathbb{E}[X+Y]$.

   Solution. It is evident from Fig. 1 and Fig. 2 that $\mathbb{E}[X-Y] = 0$ and $\mathbb{E}[X+Y] = N$. The same conclusions also follow immediately by linearity of expectation.

4. (4 points) Fix $N = 10$. Let X be a random variable with support $[N]$ where

$$p_X(x) = \frac{2}{N}\left(1 - \frac{x}{N+1}\right), \; x \in [N].$$

   Please do the following:

   - Plot the PMF $p_X = (p_X(x), x \in \mathcal{X})$.
   - Establish whether or not p_X is a valid PMF.
   - Find $\mathbb{E}[X]$.
   - Find $\mathrm{Var}(X)$.

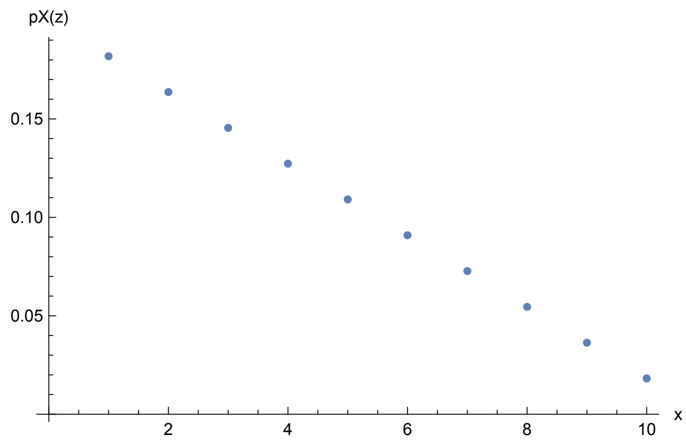

Figure 3: The PMF for problem 4 for $N = 10$.

Solution.

The PMF is shown in Fig. 3. Observe

$$\begin{aligned}
\sum_{x=1}^{N} p_X(x) &= \sum_{x=1}^{N} \frac{2}{N}\left(1 - \frac{x}{N+1}\right) \\
&= \frac{2}{N}\sum_{x=1}^{N}\left(1 - \frac{x}{N+1}\right) \\
&= \frac{2}{N}\left(N - \sum_{x=1}^{N}\frac{x}{N+1}\right) \\
&= \frac{2}{N}\left(N - \frac{1}{N+1}\sum_{x=1}^{N} x\right) \\
&= \frac{2}{N}\left(N - \frac{1}{N+1}\frac{N(N+1)}{2}\right) \\
&= \frac{2}{N}\left(N - \frac{N}{2}\right) \\
&= \frac{2}{N}\frac{N}{2} \\
&= 1
\end{aligned} \tag{5}$$

The expected value is found as:

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{N} x p_X(x) \\
&= \sum_{x=1}^{N} x \frac{2}{N}\left(1 - \frac{x}{N+1}\right) \\
&= \frac{2}{N}\sum_{x=1}^{N} x \left(1 - \frac{x}{N+1}\right) \\
&= \frac{2}{N}\left(\sum_{x=1}^{N} x - \frac{1}{N+1}\sum_{x=1}^{N} x^2\right) \\
&= \frac{2}{N}\left(\frac{N(N+1)}{2} - \frac{1}{N+1}\frac{N(N+1)(2N+1)}{6}\right) \\
&= N+1-\frac{2N+1}{3} \\
&= \frac{3N+3-2N-1}{3} \\
&= \frac{N+2}{3}
\end{aligned}
\tag{6}
$$

The second moment is

$$
\begin{aligned}
\mathbb{E}[X^2] &= \sum_{x=1}^{N} x^2 p_X(x) \\
&= \sum_{x=1}^{N} x^2 \frac{2}{N}\left(1 - \frac{x}{N+1}\right) \\
&= \frac{2}{N}\left(\sum_{x=1}^{N} x^2 - \frac{1}{N+1}\sum_{x=1}^{N} x^3\right) \\
&= \frac{2}{N}\left(\sum_{x=1}^{N} x^2 - \frac{1}{N+1}\left(\sum_{x=1}^{N} x\right)^2\right) \\
&= \frac{2}{N}\left(\frac{N(N+1)(2N+1)}{6} - \frac{1}{N+1}\left(\frac{N(N+1)}{2}\right)^2\right) \\
&= \frac{(N+1)(2N+1)}{3} - \frac{N(N+1)}{2} \\
&= (N+1)\left(\frac{2N+1}{3} - \frac{N}{2}\right) \\
&= (N+1)\frac{4N+2-3N}{6} \\
&= \frac{1}{6}(N+1)(N+2)
\end{aligned}
\tag{7}
$$

The only difficulty is the sum of the first N cubes; the fact that this equals the square of the sum of the first

N numbers is called Nicomachus's theorem. Finally:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{1}{6}(N+1)(N+2) - \left(\frac{N+2}{3}\right)^2 \\
&= (N+2)\left(\frac{N+1}{6} - \frac{N+2}{9}\right) \\
&= (N+2)\frac{9N+9-6N-12}{54} \\
&= \frac{1}{18}(N+2)(N-1) \qquad (8)
\end{aligned}$$

5. (3 points) A wireless channel flips each bit sent over the channel independently with probability 1/1000. A packet of length 1000 bits is sent over the channel. Error correction allows errors in a packet to be corrected provided there are five or fewer bit errors in a packet, otherwise the packet is in error. Find the probability that a packet is in error. *Hint: use the binomial distribution.*

   Solution. More generally, suppose the bit error probability is $p \in [0,1]$, a packet has N bits, and a packet error occurs whenever there are more than k errors. Let $X \sim \mathrm{Bin}(N,p)$ be the number of bit errors in the packet. The packet error event corresponds to $\{X > k\}$. Thus

$$\mathbb{P}(X > k) = 1 - \mathbb{P}(X \leq k) = 1 - \sum_{i=0}^{k} \mathbb{P}(X = i) = 1 - \sum_{i=0}^{k} \binom{N}{i} p^i (1-p)^{N-i}. \qquad (9)$$

   For the particular case of $p = 1/1000$, $N = 1000$, and $k = 5$ we compute this probability to be approximately 0.00058807.

6. (3 points) Recall the setup of the previous problem. To reduce the packet error rate the transmitter sends each packet twice. The information in the packet is lost precisely when *both* packets holding that information have a packet error. Find the probability that the information in a pair of packets is lost.

   Solution. Let E_1 and E_2 be the events that the first and second packets have errors, and let E be the event that the information in the pair of packets is lost. Then $E = E_1 \cap E_2$, where E_1, E_2 are independent events. Thus

$$\mathbb{P}(E) = \mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2) \approx 0.00058807^2 \approx 3.45826 \times 10^{-7}. \qquad (10)$$

7. (3 points) A fair coin is tossed $N = 10$ times. Let X be the number of heads and $N - X$ be the number of tails. Let $Z = |X - (N - X)| = |2X - N|$ be the discrepancy between the number of heads and tails.

   - Write the PMF p_Z
   - Plot the PMF p_Z.
   - Establish whether or not p_Z is a valid PMF.

   Solution. Observe that for N even $\mathcal{Z}_{\mathrm{ev}} = \{0, 2, 4, \ldots, N\}$, while for N odd $\mathcal{Z}_{\mathrm{od}} = \{1, 3, 5, \ldots, N\}$. It is more natural to write $z = 2|x - N/2|$. Observe $\{Z = z\}$ corresponds to $\{X = \frac{1}{2}(N \pm z)\}$ for any $z \neq 0$.

Thus, for $z \neq 0$:

$$\begin{aligned} p_Z(z) &= \mathbb{P}(X = \frac{1}{2}(N-z)) + \mathbb{P}(X = \frac{1}{2}(N+z)) \\ &= \binom{N}{\frac{1}{2}(N-z)}(1/2)^{\frac{1}{2}(N-z)}(1/2)^{N-\frac{1}{2}(N-z)} + \binom{N}{\frac{1}{2}(N+z)}(1/2)^{\frac{1}{2}(N+z)}(1/2)^{N-\frac{1}{2}(N+z)} \\ &= (1/2)^N \left(\binom{N}{\frac{1}{2}(N-z)} + \binom{N}{\frac{1}{2}(N+z)} \right) \\ &= (1/2)^{N-1} \binom{N}{\frac{1}{2}(N-z)} \end{aligned} \quad (11)$$

where in the last step we have used the identity $\binom{N}{k} = \binom{N}{N-k}$. For $z = 0$ we have $\{Z = 0\}$ corresponds to $\{X = N/2\}$, and so

$$p_Z(0) = \binom{N}{N/2}(1/2)^N. \quad (12)$$

The PMF for $N = 10$ is shown in Fig. 4.

We establish the PMF is valid for N odd:

$$\sum_{z \in \mathcal{Z}_{od}} p_Z(z) = (1/2)^{N-1} \sum_{z \in \mathcal{Z}_{od}} \binom{N}{\frac{1}{2}(N-z)} = 1. \quad (13)$$

The equality follows by observing two key facts. First, by the binomial theorem we have $2^N = (1+1)^N = \sum_{k=0}^{N} \binom{N}{k}$. Second, $\binom{N}{k} = \binom{N}{N-k}$. The argument for N even is similar.

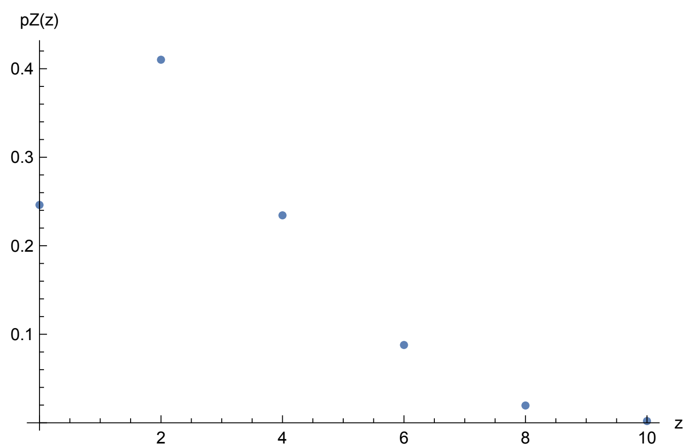

Figure 4: The PMF for problem 7 for $N = 10$.

8. (3 points) Harry and Mary take turns flipping a biased coin with bias $p \in (0,1)$, with Harry flipping the first coin. The first person to flip a heads wins the coin. Find the probability that Harry wins the coin. *Hint: Let* $X \sim \text{Geo}(p)$ *be the toss on which the first head occurs, and let* $E = \{X \text{ odd}\}$ *be the event that Harry wins. Then* $\mathbb{P}(E) = \mathbb{P}(X = 1) + \mathbb{P}(X = 3) + \cdots$. *Recall the geometric series formula from the first homework.*

Solution. Let $X \sim \text{Geo}(p)$ be the toss on which the first head occurs, and let $E = \{X \text{ odd}\}$ be the event that Harry wins the game. Then

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(X=1)+\mathbb{P}(X=3)+\cdots \\ &= p+(1-p)^2p+(1-p)^4p+\cdots \\ &= p\sum_{k=0}^{\infty}(1-p)^{2k} \\ &= p\sum_{k=0}^{\infty}\left[(1-p)^2\right]^k \\ &= p\frac{1}{1-(1-p)^2}=\frac{1}{2-p}. \end{aligned} \quad (14)$$

9. (3 points) Recall the setup of the previous problem. For which values of p is it beneficial to be the first person to flip the coin?

   Solution. It is *always* beneficial. Let $F=\{X \text{ even}\}$ be the event that Mary wins the game. Then $\mathbb{P}(F)=1-\mathbb{P}(E)=1-\frac{1}{2-p}=\frac{1-p}{2-p}$. These two curves are shown in Fig. 5.

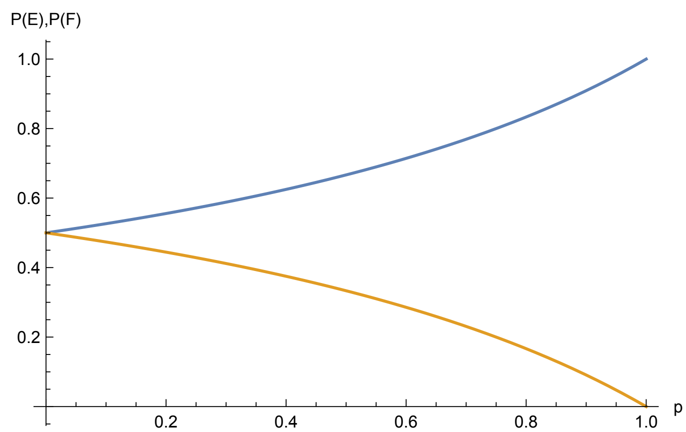

Figure 5: The probabilities of Harry (blue) and Mary (yellow) winning in problem 8.

10. (3 points) Recall the setup of the previous problem. Now, Harry flips the coin first, then Mary flips twice, then Harry flips once, then Mary flips twice, etc. Find the probability that Harry wins the coin.

    Solution. Again, let $X \sim \text{Geo}(p)$ be the toss on which the first head occurs. Now let $E=\{X \in \{1,4,7,10,\ldots\}\}$. Then

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(X=1)+\mathbb{P}(X=4)+\mathbb{P}(X=7)+\mathbb{P}(X=10)+\cdots \\ &= p+(1-p)^3p+(1-p)^6p+(1-p)^9p+\cdots \\ &= p\sum_{k=0}^{\infty}(1-p)^{3k} \\ &= p\sum_{k=0}^{\infty}\left[(1-p)^3\right]^k \\ &= \frac{p}{1-(1-p)^3}=\frac{1}{3-p(3-p)} \end{aligned} \quad (15)$$

11. (3 points) Recall the setup of the previous problem. For which values of p is it beneficial to be the first person to flip the coin?

Solution. It follows that the probability that Mary wins is $1 - \frac{1}{3-p(3-p)}$. These two curves are shown in Fig. 6. These two curves intersect at $p^* = \frac{3-\sqrt{5}}{2} \approx 0.381966$. Thus, it is beneficial to be the first person to flip the coin for $p > p^*$, while it is beneficial to be second for $p < p^*$.

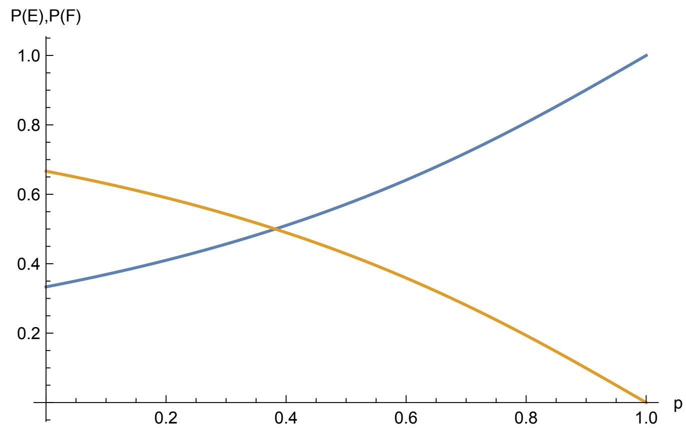

Figure 6: The probabilities of Harry (blue) and Mary (yellow) winning in problem 10.