ECE 361 Probability for Engineers (Fall, 2016)
Homework 2

Assigned:	Thursday September 29, 2016	
Due:	Thursday October 6, 2016	**(at the beginning of class 11am)**
Returned:	Monday October 17, 2016	**(in recitation)**

Your full name ______________________________

Your Drexel student ID ______________________________

ECE 361 homework policies:

1. There will be eight homework assignments, roughly one per week, but check the Course Calendar for exact dates.
2. Each homework will count equally towards your overall homework grade, although the lowest homework will be dropped.
3. Homework must be done INDIVIDUALLY. Although you are free to discuss course content in general with your classmates and peers, you are expected to NOT discuss particulars about homework problems with them or anyone else.
4. Similarly, although you are encouraged to read external sources (online or offline), it is expected that your homework handed in reflects YOUR work, and not work that you found elsewhere.
5. Homework will always be due at the BEGINNING of lecture. One of the Teaching Assistants will be in the lecture hall at the beginning of class to collect the assignments. At 11:10am (ten minutes into the start of lecture) the TA will leave the lecture with the assignments.
6. No submissions will be accepted after that time. No exceptions.
7. Homework will be returned to you during recitations.
8. Homework assignments should have your full NAME, your STUDENT ID, the ASSIGNMENT NUMBER, should be STAPLED, and should be written CLEARLY, with your final ANSWER clearly indicated, and all supporting WORK provided. Sloppy, unclear, or illegible work will not be graded.

Please answer the following questions:

1. (3 points) A group of n businesswomen are out for lunch and they decide on the following scheme to pay for the meal. Each puts her business card in a hat, and each one draws one card from the hat. Whoever draws her own card must pay (if multiple women draw their own cards then that subset of women split the bill). The outcome of the experiment is the number of women drawing their own cards, i.e., $\Omega_n = \{0, 1, 2, \ldots, n\}$, where 0 means no one draws her own card. Find the $n+1$ probabilities $(p_{k,n}, k \in \Omega_n)$, with $p_{k,n}$ denoting the probability that exactly k of the n women split the bill for each $k \in \Omega_n$ for the special case of $n = 5$. *Hint: there are $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ (read as "n factorial") ways of ordering the elements in the set $[n]$. E.g., the (unordered) set $\{1, 2, 3\}$ may be ordered in $3! = 6$ distinct ways: (123), (132), (213), (231), (312), and (321). An ordering of a set is called a permutation. Thus, an n-element set has $n!$ permutations. Think of the set $[n] = \{1, 2, \ldots, n\}$ as the n women, and think of a randomly selected permutation as an assignment of cards to women, e.g., (132) means the first woman gets her own card, and the other two exchange cards. It should be clear that the $n!$ permutations are equally likely. Hint: you may write a computer program to answer this question.*
2. (3 points) The restaurant at which the n women are dining decides to offer to pick up the lunch tab if the random experiment results in no women holding their own card. Thus $p_{0,n}$ is the probability the lunch is free for n women. Find $p_{0,n}$ for $n \in \{1, \ldots, 8\}$. *Hint: either write a compute program to answer this question (easy, recommended) or seek to find an expression for $p_{0,n}$ (this is tricky).*
3. (3 points) There are N ($N \in \mathbb{N}$) families in a city, labeled $\{1, 2, \ldots, N\}$, and each family has a number of children between 1 and K ($K \in \mathbb{N}$). In fact, the number of families with k children is given by the vector $\mathbf{n} = (n_k, k \in [K])$, where n_k is the number of families with exactly k children. Observe $n_1 + \cdots + n_K = N$. This information is used by pollsters who wish to ask questions of particular types of children, e.g., the oldest child, the youngest child, the second youngest, etc. We simplify matters by assuming that there are no twins, triplets, etc. We also assume that each child is equally likely to answer the phone when it rings. A pollster picks one of the N families uniformly at random from the list and calls the home. What is the probability that the child answering the phone is i) the oldest child in the house? ii) has a younger sibling? *Hint: condition on the event that there are B_k children in the house that is called, and use the Total Probability Theorem.*
4. (3 points) The pollster wishes to speak with a child that is the mth oldest in the house, for some $m \in [K]$. She repeatedly selects families uniformly at random (sampling with replacement) until she reaches such a child. What is the probability she must make n calls (for $n \in \mathbb{N}$)? *Hint, recall the biased coin problem from the previous homework.*
5. (3 points) A boy and a girl play hide and seek. There are N possible locations for the boy to hide. The girl picks one such location uniformly at random. The boy is not thorough, however, and may not see the girl when he looks in a location. Label the locations 1 through N and let $p_n \in [0, 1]$ be the probability that the boy spots the girl when she is hiding in location $n \in [N]$. Suppose the boy looks in location n and does not find the girl – what is the probability the girl is in that location? Suppose the girl is found – what is the probability the girl is in location n? *Hint: define events A_n that the girl is in location n, and B_n that the boy looks in location n and does not find the girl there. Use Bayes' rule.*
6. (3 points) Recall the setup and notation from problem 5 above. The boy uses the two strategies below. For each strategy, give the probability that the girl is found.
   - The boy chooses a single location uniformly at random and looks once.
   - The boy chooses a single location uniformly at random and looks at it twice.

   Hint: let F be the event that the girl is found, and let C_n be the event that the boy chooses to look in location n. Condition on all possible values of C_n (the boy's location) and A_m (the girl's location).

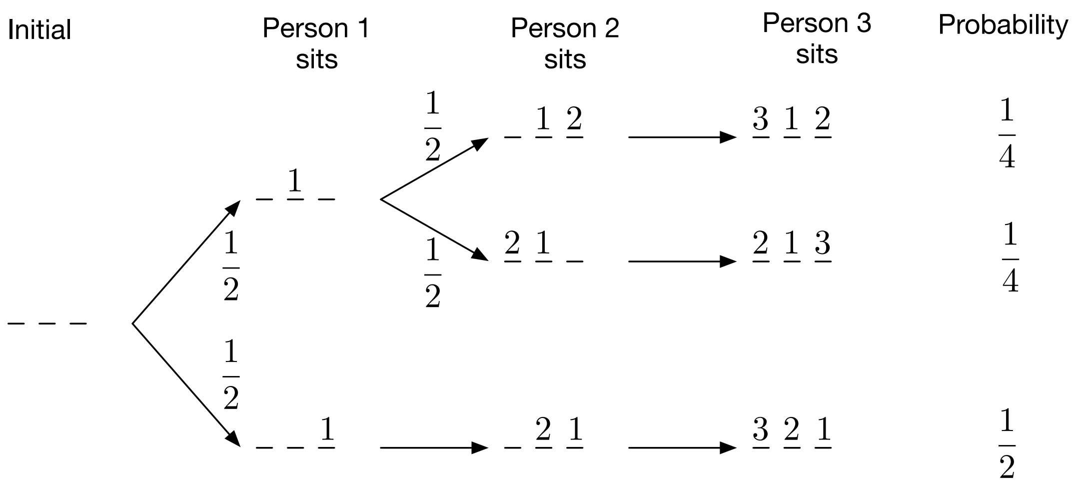

Figure 1: Example for a family of three in problem 7.

7. (3 points) A family of five eats at a table with five chairs, and each family member has an assigned seat. One day the first person decides to sit in a different seat, and selects one of seats 2 through 5 uniformly at random. The rest of the family then file into the room one by one, starting with person 2 and ending with person 5. Each person does the following: if her seat is empty she sits in her seat, but if her seat is not empty she sits in one of the open seats, selected uniformly at random. Please do the following:

   - Let π be a permutation of the set $\{1, \ldots, 5\}$ indicating the seating arrangement, e.g., (21345) indicates person 2 in seat 1, person 1 in seat 2, and persons 3, 4, 5 in their own seats. List the possible permutations and give the probability of each one. These probabilities should sum to one.
   - Let A_k be the event that person k sits in her own seat. Find $p_k = \mathbb{P}(A_k)$ for $k \in \{1, \ldots, 5\}$. These probabilities need *not* sum to one.
   - Let B_k be the event that exactly k people sit in their own seats. Find $q_k = \mathbb{P}(B_k)$ for $k \in \{0, 1, \ldots, 5\}$. These probabilities should sum to one.

   Hint: let each person's seating decision be called a stage, so there are 5 *stages in the seating. Make a* 5*-stage tree, with the root node being the empty table, and the leaf nodes being all possible permutations. The first stage are the four branches from the root, corresponding to the four seating choices of person* 1*. Each of these branches has sub-branches in stage* 2*, corresponding to the seating choices of person* 2*, etc. Use this tree to compute the possible permutations and then compute the probability of each permutation (leaf) by computing the product of the probabilities all the choices required to get to that leaf. The tree for a family of three is shown in Fig. 1.*