ECE 361 Probability for Engineers (Fall, 2016)
Homework 1 Solutions

For each of the following you are to:

1. Write the sample space Ω.
2. Give the probability of each outcome in the sample space.
3. Show the probabilities sum to one.

Please answer the following questions:

1. (3 points) Consider flipping a *fair* coin. The coin is repeatedly flipped until a head occurs, and the outcome of the experiment is the number of times the coin is flipped. *Hint: recall the geometric series $a^0 + a^1 + a^2 + \cdots$ has a sum $1/(1-a)$ for $0 \leq a < 1$.*

   Solution.

   (a) The sample space is $\Omega = \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the natural numbers.

   (b) The outcome n occurs when the coin comes up tails for the first $n-1$ tosses, and then comes up heads on toss n. Let p_n denote the probability that the n tosses are required. Then
$$p_1 = 1/2, \ p_2 = (1/2)(1/2), \ p_3 = (1/2)(1/2)(1/2), \tag{1}$$
   and so $p_n = (1/2)^n$ for each $n \in \mathbb{N}$.

   (c) Observe
$$p_1 + p_2 + \cdots = \sum_{n=1}^{\infty} (1/2)^n = -1 + \sum_{n=0}^{\infty} (1/2)^n = -1 + \frac{1}{1 - 1/2} = 1. \tag{2}$$

2. (3 points) Consider flipping a *fair* coin. The coin is repeatedly flipped until either of the following occurs: *two* successive heads occur (i.e., two heads in a row) or *two* successive tails occur (i.e., two tails in a row). The outcome of the experiment is the number of times the coin is flipped. *Hint: there are many ways to solve this problem, but here is one way. Define the event A_n for those outcomes where strictly more than n flips are required, and B_n for those outcomes where exactly n flips are required. Use logic to characterize A_n and thereby compute $\mathbb{P}(A_n)$. Now A_n^c (where c denotes complement) is the event that n or fewer flips are required. Express A_n^c in terms of B_n and A_{n-1}^c, and use this relation to obtain $p_n = \mathbb{P}(B_n)$.*

   Solution.

   (a) The sample space is $\Omega = \{2, 3, \ldots\}$.

   (b) Let n be the number of times the coin is flipped, and let A_n denote the event that *strictly* more than n flips are required. For example, A_2 is true when the first two flips are either ht or th, A_3 is true when the first three flips are either hth or tht, and in general, A_n is true when the first n flips are either *hthth...* or *ththt....* Each such sequence has probability $(1/2)^n$, and thus $\mathbb{P}(A_n) = (1/2)^{n-1}$. For example:
$$\mathbb{P}(A_2) = 1/2, \ \mathbb{P}(A_3) = 1/4, \ \mathbb{P}(A_4) = 1/8. \tag{3}$$
   Observe A_n^c is the event that n or fewer flips are required, with $\mathbb{P}(A_n^c) = 1 - \mathbb{P}(A_n) = 1 - (1/2)^{n-1}$. For example:
$$\mathbb{P}(A_2^c) = 1/2, \ \mathbb{P}(A_3^c) = 3/4, \ \mathbb{P}(A_4^c) = 7/8. \tag{4}$$

Write B_n for the event that *exactly* n flips are required, and observe $A_n^c = B_n \cup A_{n-1}^c$. In words, the event that n or fewer flips are required (A_n) is the union of the disjoint events that exactly n flips are required (B_n) and that $n-1$ or fewer flips are required (A_{n-1}). As these events are disjoint, we have, with $p_n = \mathbb{P}(B_n)$,

$$\mathbb{P}(A_n^c) = \mathbb{P}(B_n) + \mathbb{P}(A_{n-1}^c), \quad 1-(1/2)^{n-1} = p_n + (1-(1/2)^{n-2}). \tag{5}$$

The solution is $p_n = (1/2)^{n-1}$. For example:

$$p_2 = 1/2, \; p_3 = 1/4, \; p_4 = 1/8. \tag{6}$$

(c) We can check this sums to one:

$$\sum_{n=2}^{\infty} p_n = \sum_{n=2}^{\infty} (1/2)^{n-1} = \sum_{n=1}^{\infty} (1/2)^n = 1. \tag{7}$$

3. (3 points) Two *indistinguishable* coins, each *fair*, are each simultaneously flipped. This process is repeated until both coins show a head, and the outcome of the experiment is the number of times the pair of coins is flipped (not the total number of flipped coins).

Solution.

(a) The sample space is $\Omega = \mathbb{N}$.

(b) The outcome n occurs when either of the two coins comes up tails for each of the first $n-1$ tosses, and then comes up heads on toss n. Let p_n denote the probability that the n tosses are required. The outcomes of each pair of coin flips are $\{hh, ht, tt\}$, with probabilities $1/4, 1/2, 1/4$, respectively, and so the probability of "success" in any stage is $1/4$ (as in the previous problem):

$$p_1 = 1/4, \; p_2 = (3/4)(1/4), \; p_3 = (3/4)(3/4)(1/4), \tag{8}$$

Thus $p_n = (3/4)^{n-1}(1/4)$ for each $n \in \mathbb{N}$.

(c) Observe

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1} \left(\frac{1}{4}\right) = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n} = \frac{1}{4} \times \frac{1}{1-3/4} = 1. \tag{9}$$

4. (3 points) Two *distinguishable* coins, each *fair*, are each simultaneously flipped. This process is repeated until both coins show a head, and the outcome of the experiment is the number of times the pair of coins is flipped (not the total number of flipped coins).

Solution.

(a) The sample space is $\Omega = \mathbb{N}$.

(b) The outcome n occurs when either of the two coins comes up tails for each of the first $n-1$ tosses, and then comes up heads on toss n. Let p_n denote the probability that the n tosses are required. The outcomes of each pair of coin flips are $\{hh, ht, th, tt\}$, each with probability $1/4$, and so the probability of "success" in any stage is $1/4$:

$$p_1 = 1/4, \; p_2 = (3/4)(1/4), \; p_3 = (3/4)(3/4)(1/4), \tag{10}$$

Thus $p_n = (3/4)^{n-1}(1/4)$ for each $n \in \mathbb{N}$.

(c) Observe

$$\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n-1}\left(\frac{1}{4}\right) = \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n-1} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n} = \frac{1}{4}\times\frac{1}{1-3/4} = 1. \tag{11}$$

There is no difference from the perspective of the number of coin flips required whether the coins are distinguishable or not: the only conceptual change is the sample space for the coin flips being $\{hh, ht, th, tt\}$ vs. $\{hh, ht, tt\}$.

5. (3 points) Consider flipping a *biased* coin, where $p = \mathbb{P}(\text{heads})$, for some $p \in (0,1)$ (the open interval of real numbers between 0 and 1). The coin is repeatedly flipped until a head occurs, and the outcome of the experiment is the number of times the coin is flipped.

Solution.

(a) The sample space is $\Omega = \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the natural numbers.

(b) The outcome n occurs when the coin comes up tails for the first $n-1$ tosses, and then comes up heads on toss n. Let p_n denote the probability that the n tosses are required. Then

$$p_1 = p,\ p_2 = (1-p)p,\ p_3 = (1-p)(1-p)p, \tag{12}$$

and so $p_n = (1-p)^{n-1}p$ for each $n \in \mathbb{N}$.

(c) Observe

$$\sum_{n=1}^{\infty}(1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^{n} = p\frac{1}{1-(1-p)} = 1. \tag{13}$$

6. (3 points) Consider flipping a *biased* coin, where $p = \mathbb{P}(\text{heads})$, for some $p \in (0,1)$ (the open interval of real numbers between 0 and 1). The coin is repeatedly flipped until either of the following occurs: *two* successive heads occur (i.e., two heads in a row) or *two* successive tails occur (i.e., two tails in a row). The outcome of the experiment is the number of times the coin is flipped. *Hint: the same approach of events A_n, A_n^c, B_n from the hint for problem 2 applies here, but the probabilities of A_n now depend upon p. Carefully consider n even vs. n odd.*

Solution.

(a) The sample space is $\Omega = \{2, 3, \ldots\}$.

(b) We use the events A_n, A_n^c, B_n from problem 2. If n is even then the two length-n sequences *htht*... and *thth*... each have $n/2$ heads and $n/2$ tails, and thus the probability of each is $p^{n/2}\bar{p}^{n/2}$ (recall $\bar{p} \equiv 1-p$). If n is odd then one sequence has $(n-1)/2$ heads and $(n+1)/2$ tails, and vice-versa for the other sequence. Thus:

$$\mathbb{P}(A_n) = \left\{\begin{array}{ll} 2p^{n/2}\bar{p}^{n/2}, & n \in \{2,4,6,\ldots,\} \\ p^{(n-1)/2}\bar{p}^{(n+1)/2} + p^{(n+1)/2}\bar{p}^{(n-1)/2}, & n \in \{3,5,7,\ldots,\} \end{array}\right. \tag{14}$$

which may be written more succinctly as

$$\mathbb{P}(A_n) = \left\{\begin{array}{ll} 2q^{n/2}, & n \in \{2,4,6,\ldots,\} \\ q^{(n-1)/2}, & n \in \{3,5,7,\ldots,\} \end{array}\right. \tag{15}$$

for $q \equiv p\bar{p}$. For example:

$$\mathbb{P}(A_2) = 2q,\ \mathbb{P}(A_3) = q,\ \mathbb{P}(A_4) = 2q^2. \tag{16}$$

Then, recalling B_n and that $p_n = \mathbb{P}(B_n)$,

$$p_n = \left\{\begin{array}{ll} (1-2q)q^{\frac{n}{2}-1}, & n \in \{2,4,6,\ldots,\} \\ q^{\frac{n-1}{2}}, & n \in \{3,5,7,\ldots,\} \end{array}\right. \tag{17}$$

(c) We can check this sums to one:

$$\sum_{n \in \{2,4,\ldots\}} p_n = (1-2q)(q^0 + q^1 + q^2 + \cdots) = \frac{1-2q}{1-q}, \tag{18}$$

and

$$\sum_{n \in \{3,5,\ldots\}} p_n = q + q^2 + q^3 + \cdots = \frac{q}{1-q}. \tag{19}$$

Their sum is

$$\frac{1-2q}{1-q} + \frac{q}{1-q} = 1. \tag{20}$$

7. (3 points) Consider three *biased coins*. Coin 1 has bias p, coin 2 has bias q, and coin 3 has bias r, for $0 < p < q < r < 1$. You are to pick two of the three coins. Once picked, you then flip the two coins simultaneously. If the coins show the same face, you win. If the coins show different faces, you lose. The outcome of the experiment is win or lose. Give the probability of winning for each of the three choices (p,q), (p,r), and (q,r).

Solution. The winning probabilities are immediate ($\bar{x} \equiv 1-x$):

$$\begin{array}{lll} \text{pick } (p,q) & \Rightarrow \mathbb{P}(\text{win}) = & pq + \bar{p}\bar{q} \\ \text{pick } (p,r) & \Rightarrow \mathbb{P}(\text{win}) = & pr + \bar{p}\bar{r} \\ \text{pick } (q,r) & \Rightarrow \mathbb{P}(\text{win}) = & qr + \bar{q}\bar{r} \end{array} \tag{21}$$

8. (3 points) Consider three *biased coins*. Coin 1 has bias p, coin 2 has bias q, and coin 3 has bias r, for $0 < p < q < r < 1$. There are three players and each player takes one of the three coins. The three players simultaneously toss their coins. If all three coins show the same face then there is no winner. If the coin faces disagree, however, the player with the unique face wins and the two players with the shared face lose. The outcome of the experiment is the winner of the game.

Solution. Let the players $1,2,3$ denote the holders of coins with probabilities p,q,r, respectively. Make the outcome table, with $\bar{x} \equiv 1-x$ and 0 denoting no winner:

ω	$\mathbb{P}(\omega)$	winner
hhh	pqr	0
hht	$pq\bar{r}$	3
hth	$p\bar{q}r$	2
htt	$p\bar{q}\bar{r}$	1
thh	$\bar{p}qr$	1
tht	$\bar{p}q\bar{r}$	2
tth	$\bar{p}\bar{q}r$	3
ttt	$\bar{p}\bar{q}\bar{r}$	0

(22)

From here it is clear that the winning probabilities (p_0, p_1, p_2, p_3) for the three players (with p_0 the probability no one wins):

$$\begin{array}{rcl} p_0(p,q,r) & = & pqr + \bar{p}\bar{q}\bar{r} \\ p_1(p,q,r) & = & p\bar{q}\bar{r} + \bar{p}qr \\ p_2(p,q,r) & = & p\bar{q}r + \bar{p}q\bar{r} \\ p_3(p,q,r) & = & pq\bar{r} + \bar{p}\bar{q}r \end{array} \tag{23}$$

9. **Extra credit (3 points).** In question 7. above, which selection of coins gives the best probability of winning? That is, given (p, q, r) with $0 < p < q < r < 1$, give a *very simple* expression to pick the pair of coins that maximizes the chance of winning. Credit only for the correct expression in its most simple form.

   Solution. Define $f_{t,u} = tu + \bar{t}\bar{u}$ (with $\bar{x} \equiv 1 - x$) as the probability of winning with biases (t, u), and $\Delta f_{t,u,v} = f_{t,u} - \max\{f_{t,v}, f_{u,v}\}$ as the winning probability improvement (possibly negative) of (t, u) over (t, v) and (u, v). With this notation:

$$\begin{aligned} (p,q) \text{ best} \quad &\Leftrightarrow \quad f_{p,q,r} \geq 0 \\ (q,r) \text{ best} \quad &\Leftrightarrow \quad f_{q,r,p} \geq 0 \\ (p,r) \text{ best} \quad &\Leftrightarrow \quad f_{p,r,q} \geq 0 \end{aligned} \tag{24}$$

   Simplifying these three expressions results in the answer:

$$q \begin{cases} < 1/2, & (p,q) \text{ best} \\ = 1/2, & (p,q) \text{ or } (q,r) \text{ best} \\ > 1/2, & (q,r) \text{ best} \end{cases} \tag{25}$$

   Observe (p, r) are never the best choice, as is intuitive.

10. **Extra credit (3 points).** In question 8. above, suppose the biases of the three coins (p, q, r) are known to you and your two competitors. Suppose the coins are picked as follows: player 1 picks one of the three coins, player 2 picks one of the two remaining coins, and player 3 picks the remaining coin. As a function of (p, q, r), give a *very simple* expression for the coin player 1 should pick to maximize her chance of winning. Credit only for the correct expression in its most simple form.

    Solution. Recall p_1, p_2, p_3 from the solution of question 8. Define

$$\begin{aligned} \Delta_1(p,q,r) &= p_1(p,q,r) - \max\{p_2(p,q,r), p_3(p,q,r)\} \\ \Delta_2(p,q,r) &= p_2(p,q,r) - \max\{p_1(p,q,r), p_3(p,q,r)\} \\ \Delta_3(p,q,r) &= p_3(p,q,r) - \max\{p_1(p,q,r), p_2(p,q,r)\} \end{aligned} \tag{26}$$

    as the difference in winning probability (possibly negative) of coin i, for $i \in \{1, 2, 3\}$ above the winning probabilities of the other two coins. Simplifying the expressions $\Delta_i(p, q, r) \geq 0$ for $i \in \{1, 2, 3\}$ yields

$$q \begin{cases} < 1/2, & \text{coin with bias } r \text{ best} \\ = 1/2, & \text{both coins with biases } p, r \text{ best} \\ > 1/2, & \text{coin with bias } p \text{ best} \end{cases} \tag{27}$$

    Observe the coin with bias q is never the best choice, as is intuitive.