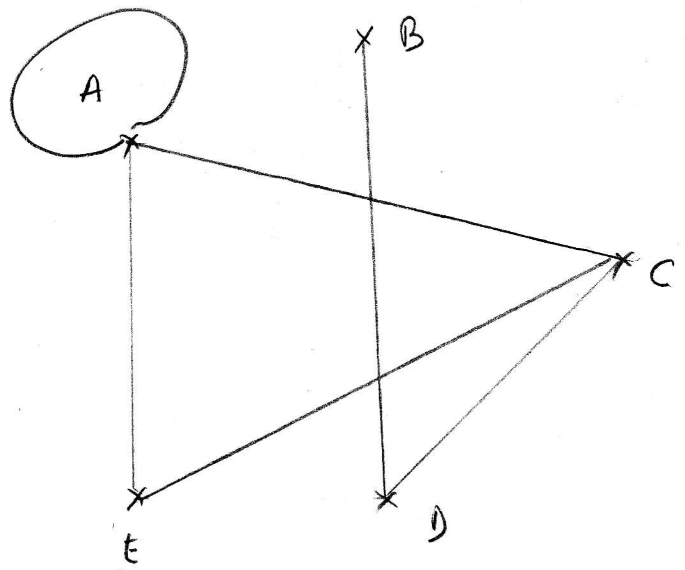
C-432 Introduction to Graph Theory

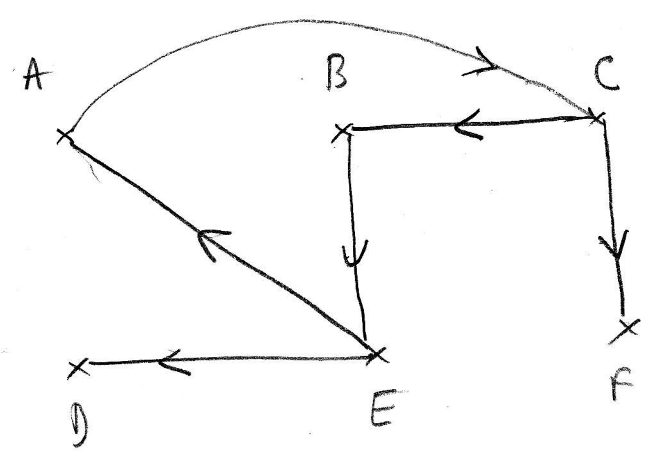
Mathematically a graph is a set of vertices V and a relation defined on V X V

For example let V = {A,B,C,D,E} and the relations are (A,C), (B,D), (A,A), (C,E),(A,E) and (C,D)

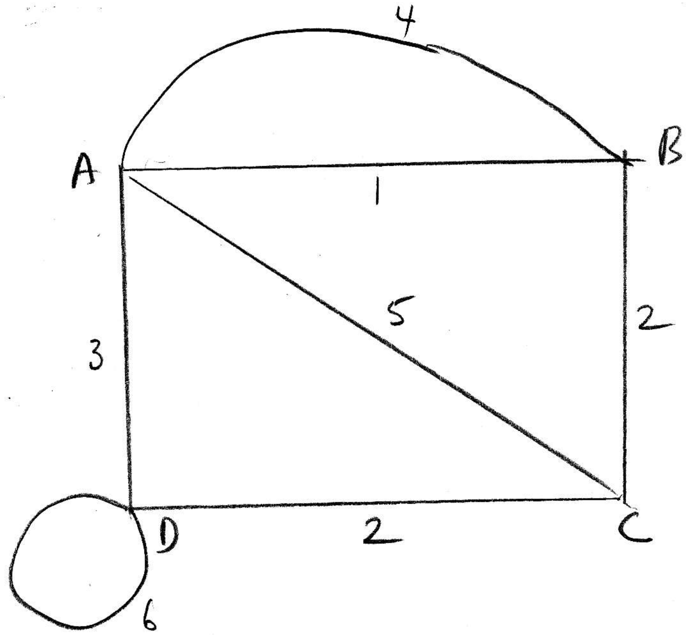
Pictorially the graph can be represented as below where the relations become the edges of the graph.



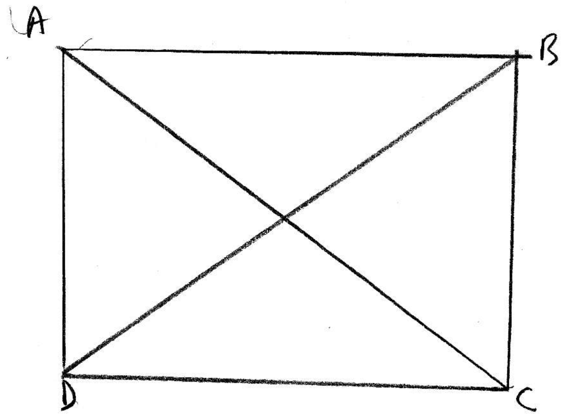
Graphs may be undirected as above (edges can be traversed an either direction) or directed as below (where an edge can be traversed only in one direction denoted by the arrow)



Graphs can be weighted if each edge has a weight (or cost of traversing) associated with it as shown below: (in certain applications it is assumed that in an unweighted graph each edge has a weight of 1 by default thus converting every graph into a weighted graph)



A circuit is a path which starts at some vertex, traverses a series of edges and eventually returns to the starting vertex. (In a directed graph the traversal of any edge can only be in the direction pointed by the arrow). In an undirected graph, a circuit provides two paths between any two vertices on the circuit

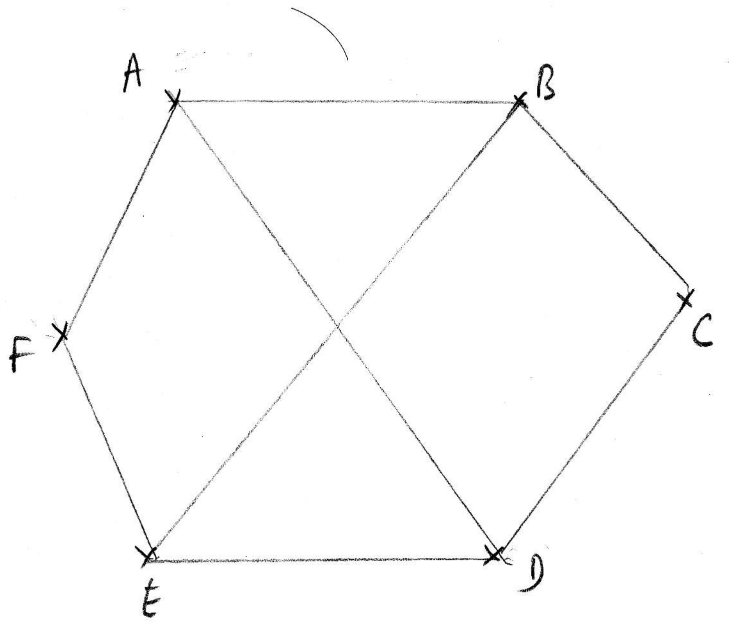


Some circuits are: A – B – C – A

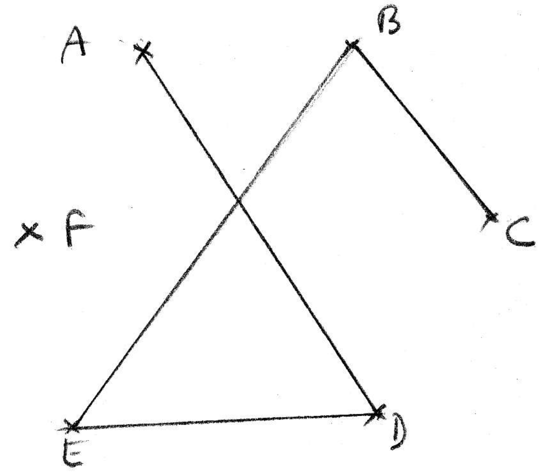
D – A – B – C – D

B – D – A – C – B

A tree is a graph without any circuits as shown below

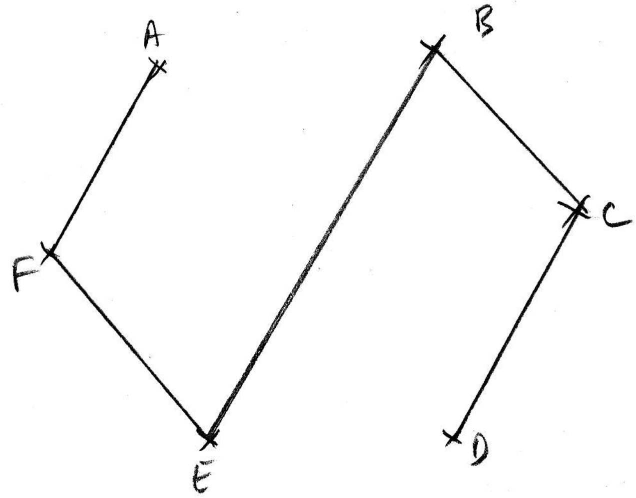


Graph G



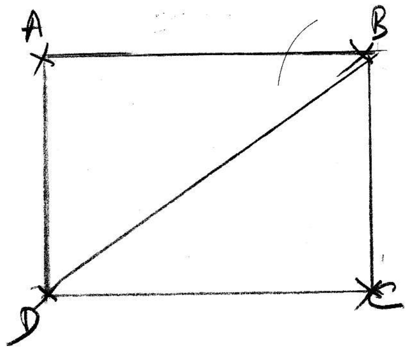
A tree in G
(F is disconnected)

A spanning tree of a graph is a tree which provides connectivity for all its vertices. The first example is a tree formed from the graph G , but it is not a spanning tree. The second example is a spanning tree



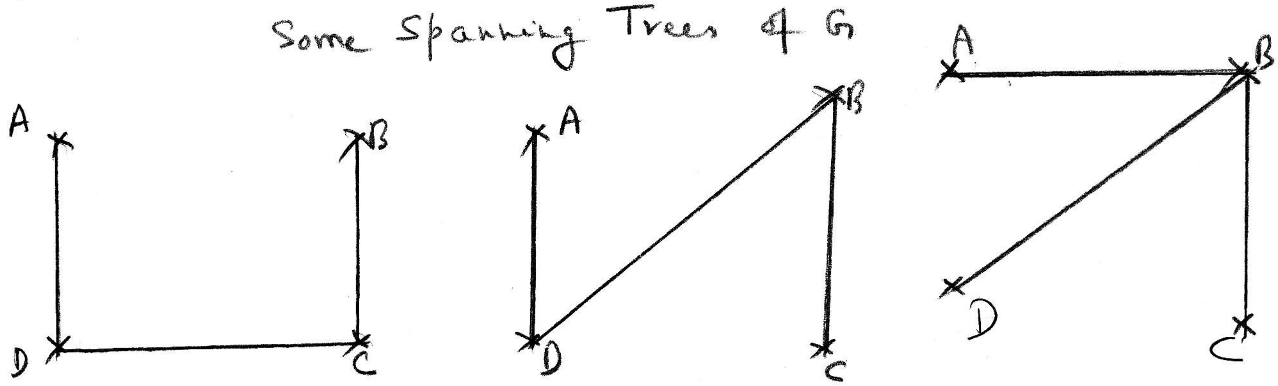
A spanning tree provides full connectivity to all vertices

A given graph in general can have many spanning trees; the spanning tree is not unique. The example below shows several spanning trees (there are more) of the graph G

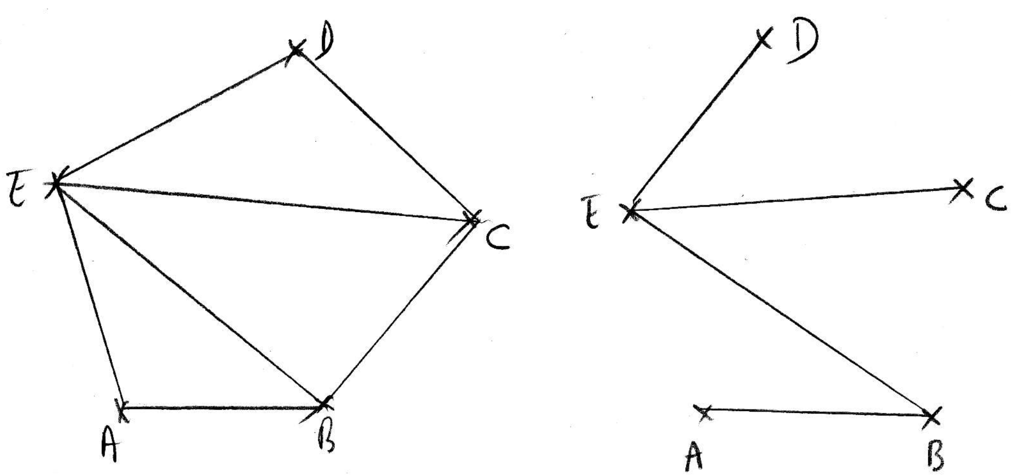


Graph G

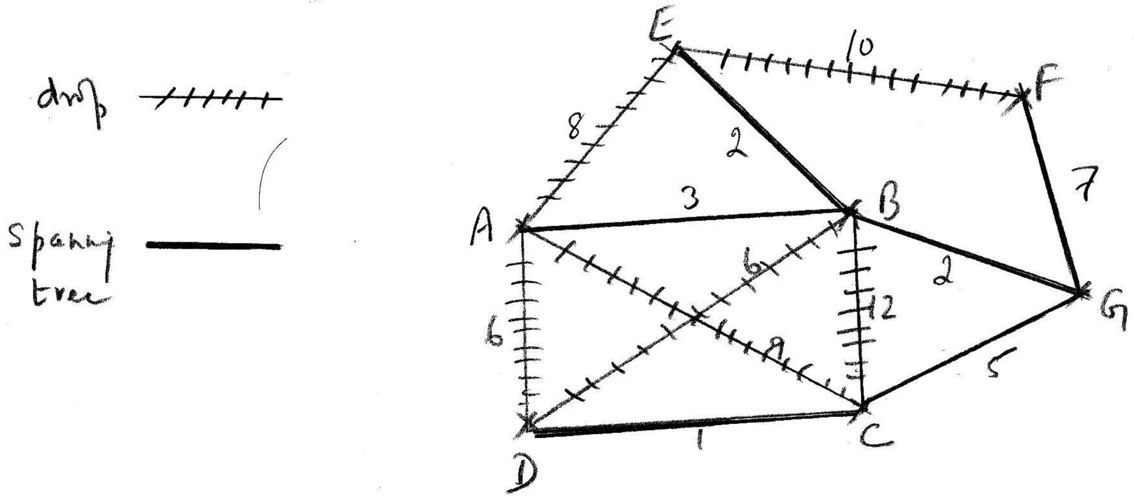
Some Spanning Trees of G



It can be proved that any spanning tree of a graph with n vertices has n-1 edges. If the tree has more than n-1 edges circuits will form and then it is no longer a tree. If the tree has less than n-1 edges, ~~all the~~ one or more vertices will not have connectivity with some other vertices and it is no longer a spanning tree. The example below is a graph with 5 vertices and any spanning tree has 4 edges. It is obvious that adding or removing any edge from the spanning tree either creates a circuit or makes the tree disconnected.



Minimal spanning tree(s) can be defined in a weighted graph. In such a tree, the weights of all edges which are part of the spanning tree is minimized among all spanning trees of the graph. To form a minimal spanning tree, start with the graph and drop edges in the descending order of their weights, always taking care not to lose connectivity, until exactly n-1 edges are left. An example is shown below



G has 7 vertices and 12 edges

Spanning tree has 7 - 1 = 6 edges

6 edges need to be dropped in decreasing order of cost without losing connectivity to any vertex

BC	12	drop
EF	10	drop
AC	9	drop
AE	8	drop
✓ FG	7	cannot drop - no connectivity to F
AD	6	drop
BD	6	drop
✓ CG	5	6 edges left - no more can be dropped
✓ AB	3	
✓ BE	2	
✓ BG	2	
✓ CD	1	

Cost of minimal spanning tree = 7 + 5 + 3 + 2 + 2 + 1
= 20

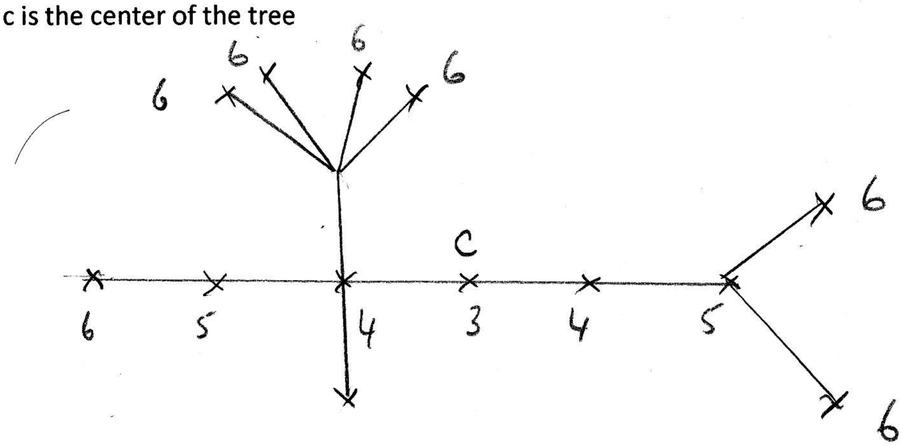
Eccentricity of a Vertex

It is defined as the maximum distance from that vertex to any other vertex in the tree

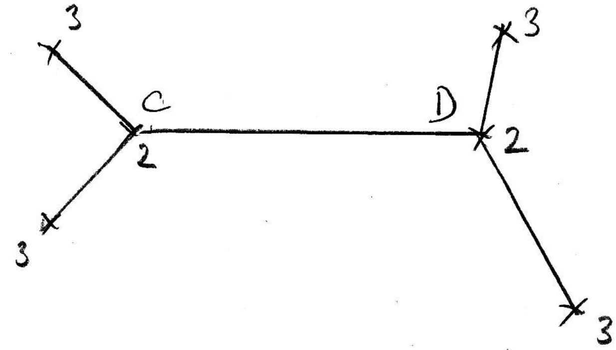
$E(v) = \max(v, v_i), v_i \varepsilon T$

A vertex with minimum eccentricity is called a "center" of the tree

In the example below the numbers shown are the eccentricities of each vertex. Since c minimizes the eccentricity, c is the center of the tree



It can be proven that every tree has either one or two centers. If there are two centers, they are necessarily adjacent. The tree below has two centers C and D each with eccentricity 2



The “radius” of a tree is the eccentricity of the center

The “diameter” of a tree is the maximum distance between any two of its vertices

The radius is not necessarily half the diameter.

For example on the prepage the first tree has a diameter of 6 and radius of 3

Whereas the second tree has a diameter of 3 and radius of 2