ECE 361 Probability for Engineers (Fall, 2016)
Lecture 6a

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§3.2 Cumulative distribution functions

Example. The maximum of several *discrete uniform* RVs. Let $X = \max\{X_1, \ldots, X_k\}$ where $X_i \sim \mathrm{Uni}([n])$ for $i \in [k]$. Find the PMF for X, p_X. We first find the CDF F_X for $j \in [n]$:

$$F_X(j) = \mathbb{P}(X \le j) = \mathbb{P}(\max\{X_1, \ldots, X_k\} \le j) = \mathbb{P}(X_1 \le j, \ldots, X_k \le j) = \mathbb{P}(X_1 \le j) \cdots \mathbb{P}(X_k \le j) = F_{X_1}(j)^k. \quad (1)$$

Thus $F_X(j) = (j/n)^k$ and the PMF is:

$$p_X(j) = F_X(j) - F_X(j-1) = \left(\frac{j}{n}\right)^k - \left(\frac{j-1}{n}\right)^k, \ j \in [n]. \quad (2)$$

Now consider the same argument for *continuous uniform* RVs. Let $X = \max\{X_1, \ldots, X_k\}$ where $X_i \sim \mathrm{Uni}([0,t])$ for $i \in [k]$. Find the PDF for X, f_X. We first find the CDF F_X for $s \in [0,t]$:

$$F_X(s) = \mathbb{P}(X \le s) = \mathbb{P}(\max\{X_1, \ldots, X_k\} \le s) = \mathbb{P}(X_1 \le s, \ldots, X_k \le s) = \mathbb{P}(X_1 \le s) \cdots \mathbb{P}(X_k \le s) = F_{X_1}(s)^k \quad (3)$$

Thus $F_X(s) = (s/t)^k$ and the PDF is:

$$f_X(s) = \frac{\mathrm{d}}{\mathrm{d}s} F_X(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{t}\right)^k = \frac{k s^{k-1}}{t^k} = \frac{k}{s}\left(\frac{s}{t}\right)^k, \ s \in [0,t]. \quad (4)$$

The geometric and exponential CDFs

Recall the geometric PMF ($X \sim \mathrm{Geo}(p)$) and exponential PDF ($Y \sim \mathrm{Exp}(\lambda)$):

$$\begin{aligned} p_X(n) &= (1-p)^{n-1} p, \ n \in \mathbb{N} \\ f_Y(y) &= \lambda \mathrm{e}^{-\lambda y}, \ y \in \mathbb{R}_+ \end{aligned} \quad (5)$$

We find their CDFs by summing and integrating, respectively:

$$\begin{aligned} F_X(k) &= \sum_{l=1}^{k} p_X(l) = \sum_{l=1}^{k} (1-p)^{l-1} p = 1 - (1-p)^k, \ k \in \mathbb{N} \\ F_Y(y) &= \int_0^y f_Y(t)\mathrm{d}t = \int_0^y \lambda \mathrm{e}^{-\lambda t}\mathrm{d}t = 1 - \mathrm{e}^{-\lambda y}, \ y \in \mathbb{R}_+ \end{aligned} \quad (6)$$

If we fix λ for Y and p for X and choose $\delta = -\log(1-p)/\lambda$ then one can show $F_Y(n\delta) = F_X(n)$ for each n. A plot of the CDFs nearly coincide, see Fig. 3.8.

Example

Two towns, A and B are 50 miles apart. Car 1 leaves town A and car 2 leaves town B independently of one another. The departure time of each car is random and uniformly distributed between 12 and 1 o'clock. The speed of each car is 100 miles per hour and they travel directly to meet one another. Denote by X the distance between the meeting point and A. Find F_X, the CDF for X. *Hints: i)* let T_A, T_B be independent uniform RVs on $[0,1]$, and let $Y = T_A - T_B$ be their difference; *ii)* find the CDF for Y with support $\mathcal{Y} = [-1,1]$ noting the two cases $y \in [-1,0]$ and $y \in [0,1]$; *iii)* find an expression for the RV X as a linear function of Y provided $|Y| < 1/2$, and consider X for $Y > 1/2$ and $Y < -1/2$ separately.

Solution. Let T_A, T_B be the random times that the two cars leave their respective times, where $T_A \sim \mathrm{Uni}[0,1]$ and $T_B \sim \mathrm{Uni}[0,1]$. For $|T_A - T_B| \ge 1/2$ the cars will meet at one of the towns: for $T_A - T_B > 1/2$ they meet at A while for

$T_B - T_A > 1/2$ they meet at B. For all other pairs (T_A, T_B) they will meet on the road. Let T be the time they meet, which is characterized by the fact that the sum of the distances they have traveled must equal 50:

$$100(T - T_A) + 100(T - T_B) = 50 \Leftrightarrow T = \frac{2(T_A + T_B) + 1}{4}. \tag{7}$$

The random distance from A, say X, is $X = 100(T - T_A) = 25 + 50(T_B - T_A)$ when $|T_B - T_A| < 1/2$. Combining:

$$X = \begin{cases} 50, & T_B - T_A > 1/2 \\ 0, & T_B - T_A < -1/2 \\ 25 + 50(T_B - T_A), & |T_B - T_A| < 1/2 \end{cases} \tag{8}$$

It remains to find the CDF of the difference of two uniform RVs. Let $Y = T_B - T_A$ with support $\mathcal{Y} = [-1, 1]$. Then, using Fig. 1 the two cases are $y \in [-1, 0]$ and $y \in [0, 1]$. For $y \in [0, 1]$:

$$\begin{aligned} F_Y(y) &= 1 - \mathbb{P}(Y > y) = 1 - \mathbb{P}(T_B - T_A > y) = 1 - \int_0^{1-y} \mathbb{P}(T_B - T_A > y | T_A = t) f_{T_A}(t) \mathrm{d}t \\ &= 1 - \int_0^{1-y} (1 - F_{T_B}(t + y)) f_{T_A}(t) \mathrm{d}t. \end{aligned} \tag{9}$$

Using the distributions for T_A and T_B we obtain:

$$F_Y(y) = 1 - \int_0^{1-y} (1 - (t + y)) 1 \mathrm{d}t = \frac{1}{2} + y - \frac{y^2}{2}. \tag{10}$$

For $y \in [-1, 0]$:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(T_B - T_A \le y) = \int_{-y}^{1} \mathbb{P}(T_B - T_A \le y | T_A = t) f_{T_A}(t) \mathrm{d}t = \int_{-y}^{1} F_{T_B}(t + y) f_{T_A}(t) \mathrm{d}t. \tag{11}$$

Using the distributions for T_A and T_B we obtain:

$$F_Y(y) = \int_{-y}^{1} (t + y) 1 \mathrm{d}t = \frac{1}{2} + y + \frac{y^2}{2}. \tag{12}$$

Combining, the CDF for Y is:

$$F_Y(y) = \begin{cases} \frac{1}{2} + y + \frac{y^2}{2}, & -1 \le y \le 0 \\ \frac{1}{2} + y - \frac{y^2}{2}, & 0 \le y \le 1 \end{cases} \tag{13}$$

We now express X in terms of Y:

$$X = \begin{cases} 0, & Y > 1/2 \\ 50, & Y < -1/2 \\ 25 + 50Y, & |Y| < 1/2 \end{cases} \tag{14}$$

For the boundary cases:

$$\begin{aligned} \mathbb{P}(X = 0) &= \mathbb{P}(Y > 1/2) = 1 - F_Y(1/2) = 1 - \left(\frac{1}{2} + \frac{1}{2} - \frac{(1/2)^2}{2} \right) = 1/8 \\ \mathbb{P}(X = 50) &= \mathbb{P}(Y < -1/2) = F_Y(-1/2) = \frac{1}{2} + (-1/2) + \frac{(-1/2)^2}{2} = 1/8 \end{aligned} \tag{15}$$

For the remaining cases:

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(25 + 50Y \le x) = \mathbb{P}(Y \le (x - 25)/50) = \begin{cases} \frac{1}{2} \left(\frac{25 + x}{50} \right)^2, & 0 < x \le 25 \\ \frac{1}{2} \left(\frac{-x^2 + 150x - 625}{50^2} \right), & 25 < x < 50. \end{cases} \tag{16}$$

All together:

$$F_X(x) = \begin{cases} \frac{1}{8}, & x = 0 \\ \frac{1}{2} \left(\frac{25 + x}{50} \right)^2, & 0 < x \le 25 \\ \frac{1}{2} \left(\frac{-x^2 + 150x - 625}{50^2} \right), & 25 < x < 50 \\ \frac{1}{8}, & x = 50 \end{cases} \tag{17}$$

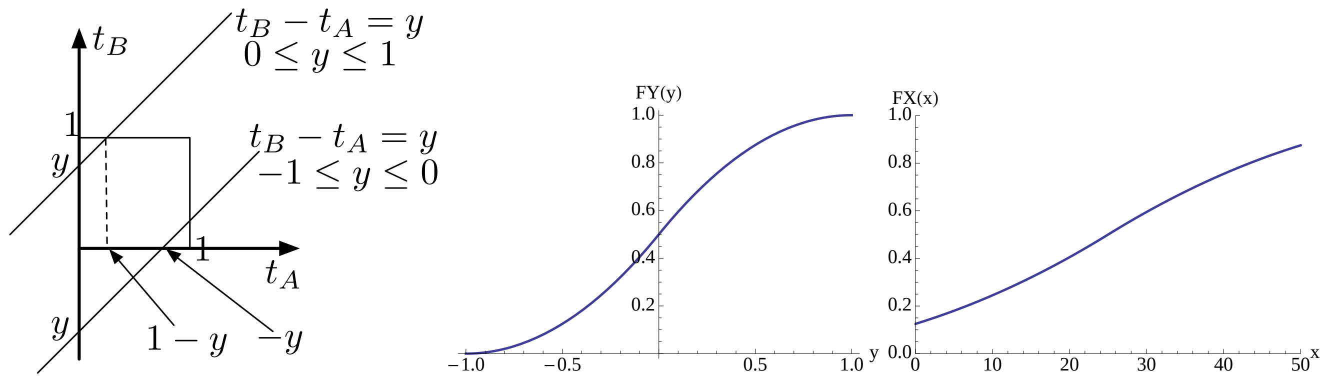

Figure 1: **Left:** the difference of two uniform RVs $Y = T_B - T_A$ for $T_A, T_B \sim \mathrm{Uni}[0,1]$ has support $\mathcal{Y} = [-1,1]$, and the CDF is found by considering the two cases $y \in [-1,0]$ and $y \in [0,1]$. **Middle:** the CDF for Y. **Right:** the CDF for X for $0 < x < 50$. Note the CDF has $F_X(0) = 1/8$ and $F_X(50) = 1/8$.

References

[1] *Introduction to Probability, 2nd Edition* by Dimitri P. Bertsekas and John N. Tsitsiklis, Athina Scientific Press, 2008.