ECE 361 Probability for Engineers (Fall, 2016)
Lecture 7b

§4.1 Derived distributions

Let the RV Y be a function $Y = g(X)$ of a continuous RV X. We aim to calculate the PDF of Y, which we call the *derived distribution*. The two steps to doing this are:

Calculation of the PDF of $Y = g(X)$ of a continuous RV X.

1. Calculate the CDF F_Y using:
$$F_Y(y) = \mathbb{P}(g(X) \leq y) = \int_{x:g(x)\leq y} f_X(x)\mathrm{d}x. \tag{1}$$

2. Differentiate to obtain the PDF f_Y via:
$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y). \tag{2}$$

Example. Let X be uniform over $[0,1]$ and $Y = \sqrt{X}$:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\sqrt{X} \leq y) = \mathbb{P}(X \leq y^2) = y^2. \tag{3}$$

Thus:

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = 2y,\ 0 \leq y \leq 1. \tag{4}$$

See Fig. 1.

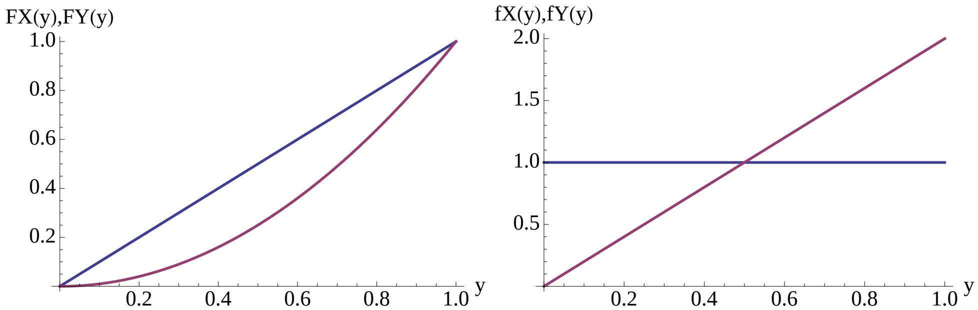

Figure 1: The CDF (left) and the PDF (right) for X (blue) and $Y = \sqrt{X}$.

Example. John Slow drives 180 miles from Boston to New York City at a constant speed, where the speed is uniformly chosen at random between 30 and 60 mph. What is the PDF of the trip duration?

Let $X \sim \mathrm{Uni}(30, 60)$ be the speed and $Y = 180/X$ the duration. The CDF for Y is:

$$\mathbb{P}(Y \leq y) = \mathbb{P}\left(\frac{180}{X} \leq y\right) = \mathbb{P}\left(X \geq \frac{180}{y}\right) = 1 - F_X(180/y),\ 3 \leq y \leq 6. \tag{5}$$

Now:

$$F_X(x) = \frac{x - 30}{30},\ 30 \leq x \leq 60, \tag{6}$$

so that

$$F_Y(y) = 1 - F_X(180/y) = 1 - \frac{180/y - 30}{30} = 2 - 6/y,\ 3 \leq y \leq 6. \tag{7}$$

Now differentiate to obtain:

$$f_Y(y) = \frac{6}{y^2},\ 3 \le y \le 6. \tag{8}$$

See Fig. 2.

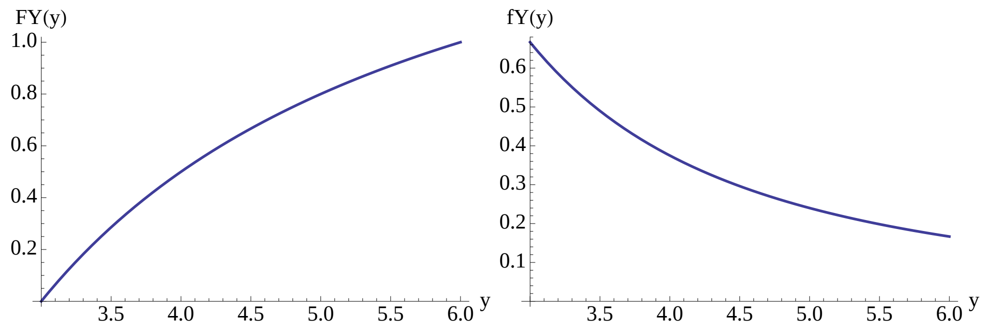

Figure 2: The CDF (left) and the PDF (right) for the trip duration in the example.

Example. Let $Y = X^2$ where X is an RV with known PDF. Then:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}),\ y \ge 0. \tag{9}$$

Differentiation gives:

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right). \tag{10}$$

The linear case

Let $Y = aX + b$. Suppose $a > 0$. Then:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(aX + b \le y) = \mathbb{P}\left(X \le \frac{y-b}{a} \right) = F_X\left(\frac{y-b}{a} \right) \tag{11}$$

and

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a} \right). \tag{12}$$

The case for $a < 0$ is similar. We find:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a} \right). \tag{13}$$

Example. Let $X \sim \mathrm{Exp}(\lambda)$ and $Y = aX + b$. Then:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a} \right) = \begin{cases} \frac{\lambda}{|a|} \mathrm{e}^{-\lambda \frac{y-b}{a}}, & \frac{y-b}{a} \ge 0 \\ 0, & \text{else} \end{cases} \tag{14}$$

Example. Let $X \sim N(\mu, \sigma)$ and $Y = aX + b$. Then:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a} \right) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}(|a|\sigma)} \mathrm{e}^{-\frac{(y-(a\mu+b))^2}{2(|a|\sigma)^2}}. \tag{15}$$

Thus $Y \sim N(a\mu + b, |a|\sigma)$. We can obtain the same result a different way. Recall that for $X \sim N(\mu, \sigma)$ we have $\mathbb{E}[X] = \mu$ and $\mathrm{var}(X) = \sigma^2$. Further, recall that for $Y = aX + b$ we have $\mathbb{E}[Y] = a\mathbb{E}[X] + b$ and $\mathrm{var}(Y) = a^2\sigma^2$ (for any X, not just normal). Further recall that linear functions of normal RVs are normal RVs, so we know $Y = aX + b$ is normal. Combining, we must conclude that $Y \sim N(a\mu + b, |a|\sigma)$.

The monotonic case

A strictly monotonic function g is of one of two types: i) increasing ($x < y \Rightarrow g(x) < g(y)$), or ii) decreasing ($x < y \Rightarrow g(x) > g(y)$). We will restrict our attention in this subsection to continuous strictly monotonic functions, which have an inverse function: $y = g(x)$ iff $x = h(y)$. Examples:

$$g(x) = \frac{c}{x}, h(y) = \frac{c}{y}, \quad g(x) = ax + b, h(y) = \frac{y-b}{a}, \quad g(x) = \mathrm{e}^{ax}, h(y) = \frac{1}{a}\log y. \tag{16}$$

The main result is that we can directly compute the PDF of $Y = g(X)$ for strictly monotonic g.

PDF for a strictly monotonic function of a continuous RV. Let g be strictly monotonic with inverse h, i.e., $y = g(x)$ iff $x = h(y)$. Then

$$f_Y(y) = f_X(h(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} h(y) \right|. \tag{17}$$

The proof is straightforward. For increasing g

$$F_Y(y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq h(y)) = F_X(h(y)) \ \Rightarrow f_Y(y) = f_X(h(y)) \frac{\mathrm{d}}{\mathrm{d}y} h(y). \tag{18}$$

The proof for decreasing g is similar. See Fig. 4.3.

Example. Let $Y = X^2$ for $X \sim \mathrm{Uni}[0,1]$. Thus g is strictly increasing with inverse $h(y) = \sqrt{y}$ and:

$$f_Y(y) = f_X(h(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} h(y) \right| = 1 \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}}, \ y \in [0,1]. \tag{19}$$

References

[1] *Introduction to Probability, 2nd Edition* by Dimitri P. Bertsekas and John N. Tsitsiklis, Athina Scientific Press, 2008.