ECE 361 Probability for Engineers (Fall, 2016)
Lecture 5b

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§3.1 Continuous RVs and their PDFs

Expectation

The expectation of a continuous RV is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x \tag{1}$$

This definition should be expected given the corresponding definition for discrete RVs – recall that integration is the limiting process of summation. Similarly, for any function g:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \mathrm{d}x. \tag{2}$$

The variance of X is

$$\mathrm{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f_X(x) \mathrm{d}x. \tag{3}$$

As with discrete RVs:

$$0 \leq \mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \tag{4}$$

For linear functions $Y = aX + b$:

$$\mathbb{E}[Y] = a\mathbb{E}[X] + b, \quad \mathrm{var}(Y) = a^2 \mathrm{var}(X). \tag{5}$$

Example. Mean and variance of a uniform RV. Let $X \sim \mathrm{Uni}[a, b]$. The mean is:

$$\mathbb{E}[X] = \int_a^b x \frac{1}{b-a} \mathrm{d}x = \frac{1}{b-a} \frac{1}{2} x^2 \bigg|_a^b = \frac{a+b}{2}. \tag{6}$$

The second moment is:

$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{b-a} \mathrm{d}x = \frac{1}{b-a} \frac{1}{3} x^3 \bigg|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}. \tag{7}$$

The variance is:

$$\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}. \tag{8}$$

Exponential RV

An exponential RV has PDF

$$f_X(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases} \tag{9}$$

Note this is a legitimate PDF since it integrates to one:

$$\int_0^{\infty} f_X(x) \mathrm{d}x = \int_0^{\infty} \lambda \mathrm{e}^{-\lambda x} \mathrm{d}x = -\mathrm{e}^{-\lambda x} \big|_0^{\infty} = 1 \tag{10}$$

We find the probability that X exceeds some value $a \in \mathbb{R}_+$:

$$\mathbb{P}(X \geq a) = \int_a^{\infty} f_X(x) \mathrm{d}x = -\mathrm{e}^{-\lambda x} \big|_a^{\infty} = \mathrm{e}^{-\lambda a}. \tag{11}$$

Thus the probability that $X > a$ decays exponentially in a at rate λ, hence the name for this distribution. We can find the mean and variance:

$$\mathbb{E}[X] = \frac{1}{\lambda}, \quad \mathrm{var}(X) = \frac{1}{\lambda^2}. \tag{12}$$

To compute $\mathbb{E}[X]$ we require integration by parts:

$$\mathbb{E}[X] = \int_0^\infty x\lambda \mathrm{e}^{-\lambda x} \mathrm{d}x = \left(-x\mathrm{e}^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty \mathrm{e}^{-\lambda x}\mathrm{d}x = 0 - \left.\frac{\mathrm{e}^{-\lambda x}}{\lambda}\right|_0^\infty = \frac{1}{\lambda}. \tag{13}$$

Similarly for $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \int_0^\infty x^2\lambda \mathrm{e}^{-\lambda x} \mathrm{d}x = \left(-x^2\mathrm{e}^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty 2x\mathrm{e}^{-\lambda x}\mathrm{d}x = 0 + \frac{2}{\lambda}\mathbb{E}[X] = \frac{2}{\lambda^2}. \tag{14}$$

Combining these gives the variance:

$$\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}. \tag{15}$$

I will write $X \sim \mathrm{Exp}(\lambda)$ to denote an exponential RV with rate λ.

§3.2 Cumulative distribution functions

In Chapter 2 we used the probability mass function (PMF) to characterize discrete RVs and in §3.1 we used the probability density function (PDF) to characterize continuous RVs. We now introduce a third concept, the cumulative distribution function (CDF) that applies to both types of RVs. If X is an RV we say $X \sim F_X$ to mean X is a RV with CDF F_X, where:

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} \sum_{k \leq x} p_X(k), & \text{discrete} \\ \int_{-\infty}^x f_X(t)\mathrm{d}t, & \text{continuous} \end{cases} \tag{16}$$

Note the helpful illustrations comparing PMFs, PDFs, and CDFs in Fig. 3.6 and 3.7. Key properties of CDFs include:

- To obtain the CDF from the PMF for a discrete RV we compute $F_X(x) = \sum_{k \leq x} p_X(k)$ for each $x \in \mathcal{X}$. To obtain the CDF from the PDF for a continuous RV we compute $F_X(x) = \int_{-\infty}^x f_X(t)\mathrm{d}t$ for each x.
- To obtain the PMF from the CDF for a discrete RV we compute $p_X(k) = F_X(k) - F_X(k-1)$ at each k. To obtain the PDF from the CDF for a continuous RV we compute $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_X(x)$.
- Also note the CDF is monotone non-decreasing in x: if $x \leq y$ then $F_X(x) \leq F_X(y)$. This is a simple consequence of the observation that $(-\infty, x] \subset (-\infty, x+\delta]$ and thus $\mathbb{P}(X \in (-\infty, x]) \leq \mathbb{P}(X \in (-\infty, x+\delta])$, i.e., $F_X(x) \leq F_X(x+\delta)$.
- Finally, note the limits:

$$\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to \infty} F_X(x) = 1. \tag{17}$$

References

[1] *Introduction to Probability, 2nd Edition* by Dimitri P. Bertsekas and John N. Tsitsiklis, Athina Scientific Press, 2008.