ECE 361 Probability for Engineers (Fall, 2016)
Homework Solutions 6

Please answer the following questions:

1. (2 points) Let $Z \sim N(0,1)$ be a standard normal RV, with CDF $\Phi(z) = \mathbb{P}(Z \leq z)$.

   - For $z > 0$, give an expression for $\mathbb{P}(|Z| > z)$ in terms of z and $\Phi(\cdot)$.
     Solution. Observe

$$\begin{aligned} \mathbb{P}(|Z| > z) &= \mathbb{P}(Z < -z \cup Z > +z) \\ &= \mathbb{P}(Z < -z) + \mathbb{P}(Z > +z) \\ &= 2\mathbb{P}(Z > z) \\ &= 2(1 - \mathbb{P}(Z \leq z)) \\ &= 2(1 - \Phi(z)). \end{aligned} \tag{1}$$

     The first step is definition of absolute value, the second is due to the fact that events $\{Z < -z\}$ and $\{Z > +z\}$ are disjoint, the third is by the symmetry of the normal distribution around 0, and the fourth and fifth are by definition of the CDF.

   - Compute $\mathbb{P}(|Z| > k)$ for $k \in [5]$.
     Solution. We compute

k	$\Phi(k)$	$\mathbb{P}(\lvert Z\rvert > k)$
1	0.84134475	0.31731051
2	0.97724987	0.04550026
3	0.99865010	0.00269980
4	0.99996833	0.00006334
5	0.99999971	0.00000057

(2)

2. (2 points) Let $X \sim N(\mu, \sigma)$ be a normal RV for a given (μ, σ) pair. Let $Z \sim N(0,1)$ be a standard normal RV, with CDF $\Phi(z) = \mathbb{P}(Z \leq z)$. Fix $x > 0$ and find $\mathbb{P}(|X| > x)$. Your answer should be in terms of x, μ, σ and $\Phi(\cdot)$. *Hint: standardize X.*

   Solution. Using the hint:

$$\begin{aligned} \mathbb{P}(|X| > x) &= \mathbb{P}(X < -x \cup X > x) \\ &= \mathbb{P}(X < -x) + \mathbb{P}(X > x) \\ &= \mathbb{P}\left(\frac{X-\mu}{\sigma} < -\frac{x-\mu}{\sigma}\right) + \mathbb{P}\left(\frac{X-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \\ &= \mathbb{P}\left(Z < -\frac{x-\mu}{\sigma}\right) + \mathbb{P}\left(Z > \frac{x-\mu}{\sigma}\right) \\ &= \mathbb{P}\left(Z > \frac{x-\mu}{\sigma}\right) + \mathbb{P}\left(Z > \frac{x-\mu}{\sigma}\right) \\ &= 2\mathbb{P}\left(Z > \frac{x-\mu}{\sigma}\right) \\ &= 2\left(1 - \mathbb{P}\left(Z \leq \frac{x-\mu}{\sigma}\right)\right) \\ &= 2\left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right) \end{aligned} \tag{3}$$

3. (3 points) Let $X \sim N(\mu_x, \sigma_x)$ and $Y \sim N(\mu_y, \sigma_y)$ be independent normal RVs. Find the mean and the variance of W, where:

- $W = aX + bY$
  Solution.

$$\begin{aligned} \mathbb{E}[W] &= a\mathbb{E}[X] + b\mathbb{E}[Y] = a\mu_x + b\mu_y \\ \mathrm{Var}(W) &= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) = a^2\sigma_x^2 + b^2\sigma_y^2 \end{aligned} \tag{4}$$

- $W = aX - bY$
  Solution.

$$\begin{aligned} \mathbb{E}[W] &= a\mathbb{E}[X] - b\mathbb{E}[Y] = a\mu_x - b\mu_y \\ \mathrm{Var}(W) &= a^2\mathrm{Var}(X) + (-b)^2\mathrm{Var}(Y) = a^2\sigma_x^2 + b^2\sigma_y^2 \end{aligned} \tag{5}$$

- $W = aXY$
  Solution.

$$\begin{aligned} \mathbb{E}[W] &= a\mathbb{E}[X]\mathbb{E}[Y] = a\mu_x\mu_y \\ \mathbb{E}[W^2] &= a^2\mathbb{E}[X^2]\mathbb{E}[Y^2] = a^2(\mathrm{Var}(X) + \mathbb{E}[X]^2)(\mathrm{Var}(Y) + \mathbb{E}[Y]^2) = a^2(\sigma_x^2 + \mu_x^2)(\sigma_y^2 + \mu_y^2) \\ \mathrm{Var}(Z) &= \mathbb{E}[W^2] - \mathbb{E}[W]^2 = a^2(\sigma_x^2 + \mu_x^2)(\sigma_y^2 + \mu_y^2) - (a\mu_x\mu_y)^2 \end{aligned} \tag{6}$$

4. (3 points) Let $X \sim \mathrm{Uni}([0,1])$ and $Y \sim \mathrm{Uni}([0,1])$ be independent RVs, and consider the pair (X,Y) as a random point on the (x,y) plane, in fact in the unit box $[0,1]^2$. Find the PDF and CDF for each of the RVs given below. Plot the PDF and CDF for each RV.

- $Z = \min(X,Y)$
  Solution. The support for Z is $[0,1]$. We give two derivations. The first derivation, for $z \in [0,1]$ is:

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \le z) \\ &= \mathbb{P}(\min(X,Y) \le z) \\ &= \int_0^1 \mathbb{P}(\min(X,Y) \le z | X = x) f_X(x) \mathrm{d}x \\ &= \int_0^1 \mathbb{P}(\min(x,Y) \le z) f_X(x) \mathrm{d}x \\ &= \int_0^z \mathbb{P}(\min(x,Y) \le z) f_X(x) \mathrm{d}x + \int_z^1 \mathbb{P}(\min(x,Y) \le z) f_X(x) \mathrm{d}x \\ &= \int_0^z 1 f_X(x) \mathrm{d}x + \int_z^1 \mathbb{P}(Y \le z) f_X(x) \mathrm{d}x \\ &= \int_0^z f_X(x) \mathrm{d}x + \int_z^1 z f_X(x) \mathrm{d}x \\ &= F_X(z) + z\bar{F}_X(z) \\ &= z + z(1-z) \\ &= z(2-z) \end{aligned} \tag{7}$$

where we have used the total probability theorem, conditioning on X, and have used the notation $\bar{F}_X(x) = 1 - F_X(x)$ to denote the complementary CDF (CCDF) of X, i.e., $\bar{F}_X(x) = \mathbb{P}(X > x)$.

An alternate derivation, significantly simpler, is as follows:

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \leq z) \\ &= 1 - \mathbb{P}(Z > z) \\ &= 1 - \mathbb{P}(\min(X,Y) > z) \\ &= 1 - \mathbb{P}(X > z, Y > z) \\ &= 1 - \mathbb{P}(X > z)\mathbb{P}(Y > z) \\ &= 1 - \bar{F}_X(z)\bar{F}_Y(z) \\ &= 1 - (1-z)^2 \\ &= 1 - (1 - 2z + z^2) \\ &= z(2-z). \end{aligned} \tag{8}$$

This proof exploits the fact that the event $\{\min(X,Y) > z\}$ is equivalent to the event $\{X > z, Y > z\}$. Differentiating gives the PDF $f_Z(z) = 2(1-z)$. In summary:

$$F_Z(z) = \left\{ \begin{array}{ll} 0, & z < 0 \\ z(2-z), & 0 \leq z \leq 1 \\ 1, & z > 1 \end{array} \right. , \; f_Z(z) = \left\{ \begin{array}{ll} 0, & z < 0 \\ 2(1-z), & 0 \leq z \leq 1 \\ 0, & z > 1 \end{array} \right. . \tag{9}$$

The PDF and CDF are shown below.

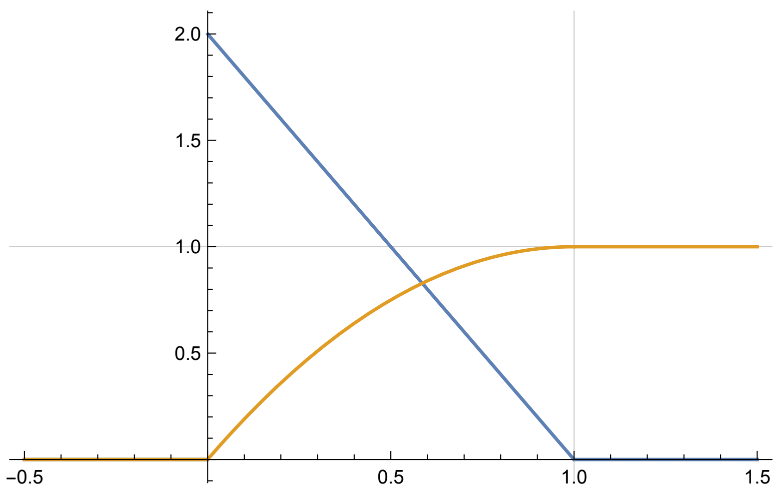

Figure 1: PDF and CDF for $Z = \min(X,Y)$ in Problem 4 (a).

- $Z = \max(X,Y)$

  Solution. The support for Z is $[0,1]$. For $z \in [0,1]$:

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \leq z) \\ &= \mathbb{P}(\max(X,Y) \leq z) \\ &= \mathbb{P}(X \leq z, Y \leq z) \\ &= F_X(z)F_Y(z) \\ &= z^2. \end{aligned} \tag{10}$$

Differentiating gives $f_Z(z) = 2z$. In summary:

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ z^2, & 0 \leq z \leq 1 \\ 1, & z > 1 \end{cases}, \quad f_Z(z) = \begin{cases} 0, & z < 0 \\ 2z, & 0 \leq z \leq 1 \\ 0, & z > 1 \end{cases}. \tag{11}$$

The PDF and CDF are shown below.

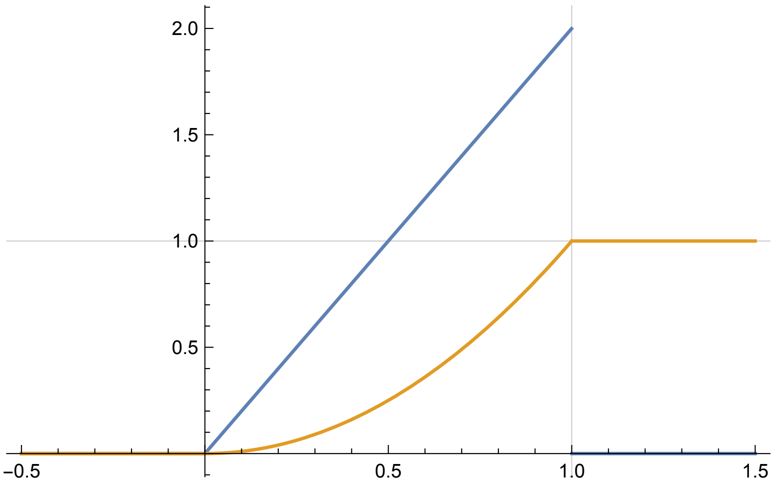

Figure 2: PDF and CDF for $Z = \max(X, Y)$ in Problem 4 (b).

- $Z = X + Y$. *Hint: observe the support is* $[0, 2]$, *and consider the two cases* $z \in [0, 1]$ *and* $z \in [1, 2]$ *separately. In each case you may use the total probability theorem, conditioning on the value of* x, *and then split the resulting integral over* $x \in [0, 1]$ *into two sub-intervals, with the split determined by the value of* z. *Pay close attention to the boundaries, e.g.,* $\mathbb{P}(X \leq x) = 0$ *for* $x < 0$ *and* $\mathbb{P}(X \leq x) = 1$ *for* $x > 1$. *Check that your final answer is a valid CDF: zero for* $z < 0$, *one for* $z > 2$, *and nondecreasing for* $z \in [0, 2]$.

Solution. The support is $[0,2]$. For $z \in [0,1]$:

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) \\
&= \mathbb{P}(X+Y \leq z) \\
&= \int_0^1 \mathbb{P}(X+Y \leq z | X = x) f_X(x) \mathrm{d}x \\
&= \int_0^1 \mathbb{P}(x+Y \leq z) f_X(x) \mathrm{d}x \\
&= \int_0^1 \mathbb{P}(Y \leq z-x) f_X(x) \mathrm{d}x \\
&= \int_0^z \mathbb{P}(Y \leq z-x) f_X(x) \mathrm{d}x + \int_z^1 \mathbb{P}(Y \leq z-x) f_X(x) \mathrm{d}x \\
&= \int_0^z (z-x) f_X(x) \mathrm{d}x + \int_z^1 0 f_X(x) \mathrm{d}x \\
&= z \int_0^z f_X(x) \mathrm{d}x - \int_0^z x f_X(x) \mathrm{d}x \\
&= z \int_0^z 1 \mathrm{d}x - \int_0^z x \mathrm{d}x \\
&= z^2 - \frac{1}{2} z^2 \\
&= \frac{1}{2} z^2
\end{aligned} \tag{12}$$

For $z \in [1,2]$:

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) \\
&= \mathbb{P}(X+Y \leq z) \\
&= \int_0^1 \mathbb{P}(X+Y \leq z | X = x) f_X(x) \mathrm{d}x \\
&= \int_0^1 \mathbb{P}(x+Y \leq z) f_X(x) \mathrm{d}x \\
&= \int_0^1 \mathbb{P}(Y \leq z-x) f_X(x) \mathrm{d}x \\
&= \int_0^{z-1} \mathbb{P}(Y \leq z-x) f_X(x) \mathrm{d}x + \int_{z-1}^1 \mathbb{P}(Y \leq z-x) f_X(x) \mathrm{d}x \\
&= \int_0^{z-1} 1 f_X(x) \mathrm{d}x + \int_{z-1}^1 (z-x) f_X(x) \mathrm{d}x \\
&= \int_0^{z-1} 1 \mathrm{d}x + \int_{z-1}^1 (z-x) \mathrm{d}x \\
&= \int_0^{z-1} 1 \mathrm{d}x + z \int_{z-1}^1 1 \mathrm{d}x - \int_{z-1}^1 x \mathrm{d}x \\
&= (z-1) + z(1-(z-1)) - \frac{1}{2}\left(1^2 - (z-1)^2\right) \\
&= 2z - 1 - \frac{1}{2} z^2
\end{aligned} \tag{13}$$

Combining:

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \frac{1}{2}z^2, & 0 \le z \le 1 \\ 2z - 1 - \frac{1}{2}z^2, & 1 < z \le 2 \\ 1, & z > 2 \end{cases}, \quad f_Z(z) = \begin{cases} 0, & z < 0 \\ z, & 0 \le z \le 1 \\ 2 - z, & 1 < z \le 2 \\ 0, & z > 1 \end{cases}. \tag{14}$$

The PDF and CDF are shown below.

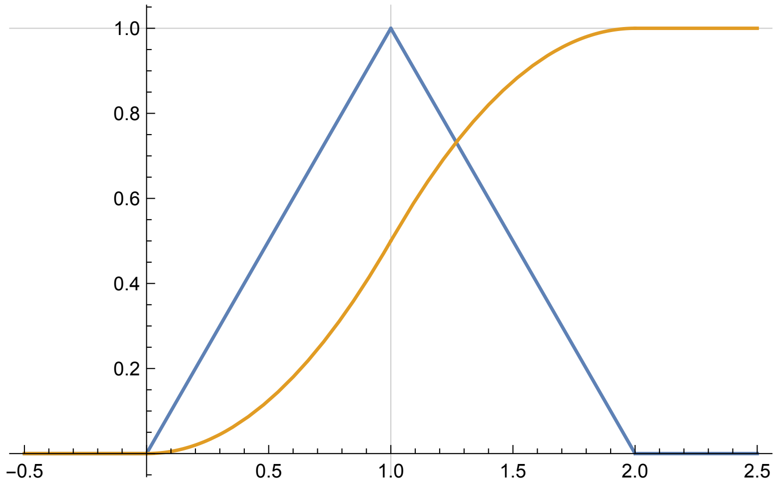

Figure 3: PDF and CDF for $Z = X + Y$ in Problem 4 (c).

5. (2 points) Let $(X_1, \ldots, X_N)$ be independent and identically distributed RVs, with $X_n \sim N(0,1)$ for each $n \in [N]$.

- Define $U = \min(X_1, \ldots, X_N)$. Find the PDF and CDF of U. Plot the PDF and CDF for $N = 1, 2, 5, 10$.
Solution. First find the CCDF:

$$\begin{aligned} \bar{F}_U(u) &= \mathbb{P}(U > u) \\ &= \mathbb{P}(\min(X_1, \ldots, X_N) > u) \\ &= \mathbb{P}(X_1 > u, \ldots, X_N > u) \\ &= \mathbb{P}(X_1 > u) \cdots \mathbb{P}(X_N > u) \\ &= (1 - \mathbb{P}(X_1 \le u)) \cdots (1 - \mathbb{P}(X_N \le u)) \\ &= (1 - \Phi(u))^N. \end{aligned} \tag{15}$$

Thus $F_U(u) = 1 - (1 - \Phi(u))^N$ and $f_U(u) = N(1 - \Phi(u))^{N-1}\phi(u)$, for $\phi(u)$ the standard normal PDF. The PDF and CDF are shown below.
- Define $V = \max(X_1, \ldots, X_N)$. Find the PDF and CDF of V. Plot the PDF and CDF for $N = 1, 2, 5, 10$.
Solution.

$$\begin{aligned} F_V(v) &= \mathbb{P}(V \le v) \\ &= \mathbb{P}(\max(X_1, \ldots, X_N) \le v) \\ &= \mathbb{P}(X_1 \le v, \ldots, X_N \le v) \\ &= \mathbb{P}(X_1 \le v) \cdots \mathbb{P}(X_N \le v) \\ &= \mathbb{P}(X_1 \le v) \cdots \mathbb{P}(X_N \le v) \\ &= \Phi(v)^N. \end{aligned} \tag{16}$$

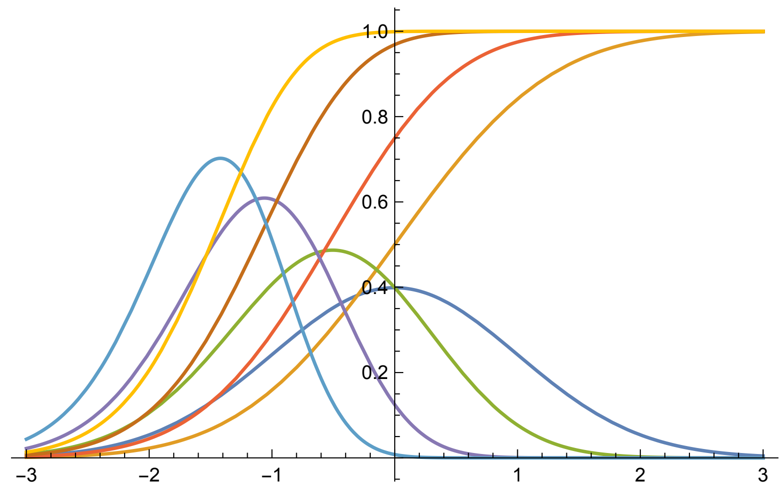

Figure 4: PDF and CDF for $U = \min(X_1, \ldots, X_N)$ in Problem 5 (a).

Thus $f_V(v) = N\Phi(v)^{N-1}\phi(v)$, for $\phi(v)$ the standard normal PDF. The PDF and CDF are shown below.

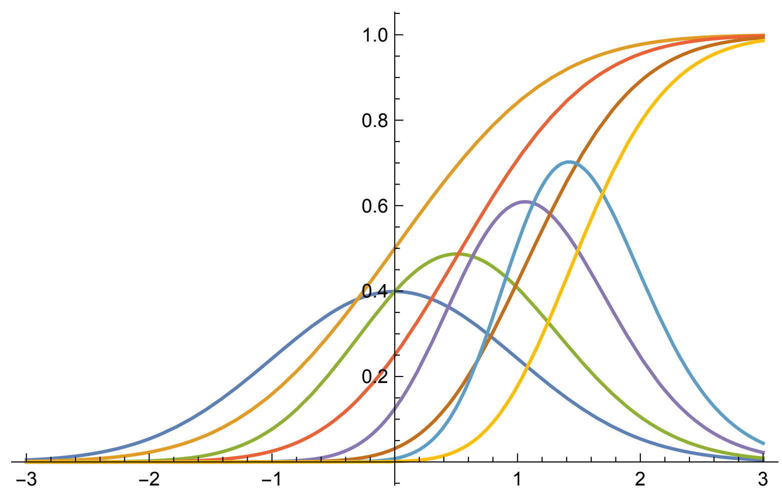

Figure 5: PDF and CDF for $V = \max(X_1, \ldots, X_N)$ in Problem 5 (b).

6. (2 points) Let $(X_1, \ldots, X_N)$ be independent RVs, with $X_n \sim \exp(\lambda_n)$ for each $n \in [N]$, i.e., $F_{X_n}(x) = 1 - \mathrm{e}^{-\lambda_n x}$ for $x \geq 0$. Assume the parameters $(\lambda_1, \ldots, \lambda_N)$ obey $\lambda_n > 0$ for each $n \in [N]$. Define $Y = \min(X_1, \ldots, X_N)$. Find the PDF and CDF of Y.

Solution.

$$\begin{aligned}
\bar{F}_Y(y) &= \mathbb{P}(Y > y) \\
&= \mathbb{P}(\min(X_1, \ldots, X_N) > y) \\
&= \mathbb{P}(X_1 > y, \ldots X_N > y) \\
&= \mathbb{P}(X_1 > y) \cdots \mathbb{P}(X_N > y) \\
&= \mathrm{e}^{-\lambda_1 y} \cdots \mathrm{e}^{-\lambda_N y} \\
&= \mathrm{e}^{-(\lambda_1 + \cdots + \lambda_N) y}.
\end{aligned} \tag{17}$$

Thus $F_Y(y) = 1 - \mathrm{e}^{-(\lambda_1 + \cdots + \lambda_N)y}$, and thus we see that $Y \sim \exp(\lambda)$, where $\lambda \equiv \lambda_1 + \cdots \lambda_N$. The PDF is $f_Y(y) = \lambda \mathrm{e}^{-\lambda y}$, for $y \geq 0$. In words, the minimum of a collection of N independent exponentially distributed RVs is itself an exponentially distributed RV, with parameter equal to the sum of the parameters in the collection.