ECE 361 Probability for Engineers (Fall, 2016)
Homework 5

Assigned:	Thursday October 27, 2016	
Due:	Tuesday November 8, 2016	**(at the beginning of class 11am)**
Returned:	Monday November 14, 2016	**(in recitation)**

Your full name ______________________________

Your Drexel student ID ______________________________

ECE 361 homework policies:

1. There will be eight homework assignments, roughly one per week, but check the Course Calendar for exact dates.
2. Each homework will count equally towards your overall homework grade, although the lowest homework will be dropped.
3. Homework must be done INDIVIDUALLY. Although you are free to discuss course content in general with your classmates and peers, you are expected to NOT discuss particulars about homework problems with them or anyone else.
4. Similarly, although you are encouraged to read external sources (online or offline), it is expected that your homework handed in reflects YOUR work, and not work that you found elsewhere.
5. Homework will always be due at the BEGINNING of lecture. One of the Teaching Assistants will be in the lecture hall at the beginning of class to collect the assignments. At 11:10am (ten minutes into the start of lecture) the TA will leave the lecture with the assignments.
6. No submissions will be accepted after that time. No exceptions.
7. Homework will be returned to you during recitations.
8. Homework assignments should have your full NAME, your STUDENT ID, the ASSIGNMENT NUMBER, should be STAPLED, and should be written CLEARLY, with your final ANSWER clearly indicated, and all supporting WORK provided. Sloppy, unclear, or illegible work will not be graded.

Please answer the following questions:

1. (5 points) Let X be a continuous RV with support $\mathcal{X} = [0,1]$ and PDF $f_X(x) = cx(1-x)$ for $x \in \mathcal{X}$ and some constant c.
   - Give the constant c such that f_X is a valid PDF.
   - Give the CDF $F_X(x)$.
   - Find $\mathbb{E}[X]$.
   - Find $\mathrm{Var}(X)$.
   - Plot $f_X(x)$ vs. x and $F_X(x)$ vs. x over $x \in \mathcal{X}$ on the same plot. Indicate $\mathbb{E}[X]$ on the x-axis.
2. (5 points) Let X be a continuous RV with support $\mathcal{X} = (0,\infty)$ and PDF $f_X(x) = cx^2\mathrm{e}^{-x}$ for $x \in \mathcal{X}$ and some constant c.
   - Give the constant c such that f_X is a valid PDF.
   - Give the CDF $F_X(x)$.
   - Find $\mathbb{E}[X]$.
   - Find $\mathrm{Var}(X)$.
   - Plot $f_X(x)$ vs. x and $F_X(x)$ vs. x over $x \in (0,10]$ on the same plot. Indicate $\mathbb{E}[X]$ on the x-axis.
3. (5 points) Let X be a continuous RV with support $\mathcal{X} = (-\infty,+\infty)$ and PDF $f_X(x) = c(1+x^2)^{-1}$ for $x \in \mathcal{X}$ and some constant c.
   - Give the constant c such that f_X is a valid PDF.
   - Give the CDF $F_X(x)$.
   - Plot $f_X(x)$ vs. x and $F_X(x)$ vs. x over $x \in [-5,+5]$ on the same plot.
4. (5 points) The CDF F_X of a continuous RV X may be used to *generate* random variates from that distribution through a technique called *CDF inversion*. The basic idea is to solve the equation $u = F_X(x)$ for x, say $x = F_X^{-1}(u)$, and to then generate $F_X^{-1}(U)$, for $U \sim \mathrm{Uni}([0,1])$ a continuous uniform RV. The justification for this process is the fact that the event $\{X \leq x\}$ is equivalent to the event $\{g(X) \leq g(x)\}$ for any function $g : \mathcal{X} \to \mathbb{R}$ that is *increasing*, i.e., if $x < y$ then $g(x) < g(y)$. Recall *all* CDFs are *nondecreasing*, i.e., $x < y$ implies $F_X(x) \leq F_X(y)$ (note the nonstrict inequality), but some are in fact (strictly) *increasing*. If $F_X(x)$ is increasing then observe, with $Y = F_X^{-1}(U)$:

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(F_X^{-1}(U) \leq x) = \mathbb{P}(F_X(F_X^{-1}(U)) \leq F_X(x)) = \mathbb{P}(U \leq F_X(x)) = F_X(x). \tag{1}$$

   Here, we have used two key facts. First, if $F_X(x)$ is increasing with inverse $F_X^{-1}(u)$ then $F_X(F_X^{-1}(u)) = u$ (this is the defining property of a function and its inverse). Second, if $U \sim \mathrm{Uni}([0,1])$ then $F_U(u) = \mathbb{P}(U \leq u) = u$ for $u \in [0,1]$. The above proof shows that the CDF for the RV $Y = F_X^{-1}(U)$ equals the CDF $F_X(x)$, i.e., the procedure of generating a *uniform* RV $U \sim \mathrm{Uni}([0,1])$ and computing $Y = F_X^{-1}(U)$ yields a RV with distribution $F_X(x)$, as desired.

   We will verify this claim by constructing in this exercise a large collection of independent and identically distributed samples $(X_1,\ldots,X_N)$, with each $X_i \sim F_X$ for $i \in [N]$, and computing the *empirical distribution* for this sample. This exercise is our first foray into *statistics*. First, sort the N samples such that $X_1 < X_2 \ldots < X_N$; note it is safe to assume a strict inequality since the CDF is for a continuous RV and there is zero probability of two samples taking the same value. Second, assign to point i the value i/N for $i \in [N]$. Now connect the points with the continuous function $\tilde{F}_X(x)$ which takes value:

$$\tilde{F}_X(x) = \sum_{i=1}^{N+1} \frac{i-1}{N} \mathbf{1}(x_{i-1} < x \leq x_i). \tag{2}$$

where $x_0 = -\infty$ and $x_{N+1} = +\infty$. Here, the notation $\mathbf{1}(A)$ equals 1 if A is true and 0 if A is false, and is called an *indicator function*. Thus $\tilde{F}_X(x)$ is a sum of indicators and precisely one of these N indicators is true for each $x \in \mathbb{R}$.

Consider the *exponential* distribution $F_X(x) = 1 - \mathrm{e}^{-\lambda x}$ for $x \in \mathcal{X} = [0, +\infty)$, with parameter $\lambda > 0$. Observe $F_X(x)$ is strictly increasing and therefore we can use the CDF inversion technique to generate exponential variates. Please do the following:

- Find $F_X^{-1}(u)$ for $F_X(x) = 1 - \mathrm{e}^{-\lambda x}$ for arbitrary $\lambda > 0$.
- Fix $\lambda = 1$ and generate $N = 25$ independent and identically distributed variates $X_1, \ldots, X_N$ with $X_i \sim F_X^{-1}(U)$ for $U \sim \mathrm{Uni}([0,1])$ using your computer's random number generator. E.g., Matlab's `rand` or Mathematica's `Random` or Python's `random`.
- Sort these values so $X_1 < \cdots < X_N$ and plot the empirical distribution $\tilde{F}_X$ along with the given distribution $F_X(x) = 1 - \mathrm{e}^{-\lambda x}$ for $\lambda = 1$ over $x \in [0, 6]$.
- Repeat the above steps for $\lambda = 2$, plotting over $x \in [0, 3]$.

5. (5 points) Consider a pair of independent and identically distributed uniform RVs (X, Y) with $X \sim \mathrm{Uni}([0,1])$ and $Y \sim \mathrm{Uni}([0,1])$. Think of each random pair (X, Y) as a random point on the unit square $[0,1] \times [0,1]$ on the (x, y)-plane. Let the RV $Z = \sqrt{X^2 + Y^2}$ be the random distance of that random point from the origin. Please do the following:

   - Find the support $\mathcal{Z}$ for the RV Z.
   - Find the CDF $F_Z(z)$ for the RV Z.
   - Find the PDF $f_Z(z)$ for the RV Z.
   - Find $\mathbb{E}[Z]$.
   - Plot the CDF $F_Z(z)$ and the PDF $f_Z(z)$ vs. $z \in \mathcal{Z}$. Label the point $\mathbb{E}[Z]$ on the z-axis.

6. (5 points) Consider a pair of independent and identically distributed uniform RVs (X, Y) with $X \sim \mathrm{Uni}([-1,+1])$ and $Y \sim \mathrm{Uni}([-1,+1])$. Think of each random pair (X, Y) as a random point on the unit square $[-1,+1] \times [-1,+1]$ on the (x, y)-plane. Let the RV $W = \mathbf{1}(\sqrt{X^2 + Y^2} \leq 1)$ be the indicator that the random point is at a distance of one or less from the origin, i.e., $W = 1$ if $\sqrt{X^2 + Y^2} \leq 1$ and $W = 0$ else. Please do the following:

   - Prove that if $\mathbf{1}(A)$ is the function that equals 1 if A is true and 0 if A is false then $\mathbb{E}[\mathbf{1}(A)] = \mathbb{P}(A)$ for any event A. In words, the expected value of an indicator RV for an event A is the probability of A.
   - Find $\mathbb{E}[W]$.
   - Generate a sequence of $N = 1000$ random (X, Y) values $((X_i, Y_i), i \in [N])$ with each (X_i, Y_i) as above, and compute W_i for each one, as above, yielding a random binary string $(W_i, i \in [N])$. For each $n \in [N]$ define the sequence $(f(n), n \in [N])$, with $f(n) = \frac{1}{n} \sum_{i=1}^{n} W_i$ the fraction of ones in the first n positions. Plot $f(n)$ vs. $n \in [N]$.
   - Guess from your plot the true value of $\lim_{n \to \infty} f(n)$.