- Recall that if $Z_1, \dots, Z_n$ are independent $N(0,1)$ then

$$X = \sum_{k=1}^{n} Z_k^2 \sim \chi_n^2$$

is a chi square with n degrees of freedom.

- Recall that if M is an $p \times n$ matrix whose columns are independent and all have a $N_p(0, \Sigma)$ distribution, then

$$\underbrace{\mathcal{Y}}_{p \times p} = MM^T \sim W_p(\Sigma, n) . \tag{1}$$

generalisation of χ^2 dist (multiple dim)

2. Show that if $X \sim N_p(\mu, \Sigma)$ and Σ is invertible, then

$$Y = (X - \mu)^T \Sigma^{-1} (X - \mu) \sim \chi_p^2 . \tag{2}$$

$$X \sim N_p(\mu, \Sigma)$$

$$X - \mu \sim N_p(0, \Sigma)$$

$$Z = \Sigma^{-1/2}(X-\mu) \sim N_p(0, I_p) \quad \text{(normalise)}$$

$$\begin{aligned} Y &= (X-\mu)^T \Sigma^{-1} (X-\mu) \\ &= \left[(X-\mu)^T (\Sigma^{-1/2})^T\right]\left[\Sigma^{-1/2}(X-\mu)\right] \\ &= Z^T Z \\ &= \sum_{i=1}^{p} Z_i^2 \sim \chi_p^2 \end{aligned}$$

$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_p \end{bmatrix}$, $Z_i \sim N(0,1)$, Z_i's independent

3. Let $p = 1$ and take $\Sigma = \sigma^2$, a number. Show that $\mathcal{Y}$ defined above is $W_1(\sigma^2, n)$, is also equal to σ^2 times a χ_n^2.

If $M: p \times n$ with independent cols, each col $\sim N_p(0, \Sigma)$

$\Rightarrow y = MM^T \sim W_p(\Sigma, n)$

Now $p = 1$ & $\Sigma = \sigma^2$

$\Rightarrow y = MM^T$ where $M: 1 \times n$ matrix, cols independent with $N(0, \sigma^2)$ distribution

$\Rightarrow y \sim W_1(\sigma^2, n)$

$$M = [M_1, \dots, M_n]$$

$M_i \sim N(0, \sigma^2)$

$$\mathcal{Y} = MM^T \sim W_1(\sigma^2, n)$$

$$= [M_1, \dots, M_n] \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} M_i^2$$

$\left(\frac{1}{\sigma} M_i \sim N(0,1) \right)$

$$= \sigma^2 \sum_{i=1}^{n} \left(\frac{1}{\sigma} M_i \right)^2$$

$$= \sigma^2 \cdot \underbrace{\sum_{i=1}^{n} Z_i^2}_{\sim \chi^2_n}$$

4. Show that if $\mathcal{Y}$ is defined at (1) and B is a $q \times p$ matrix then

$$B\mathcal{Y}B^T \sim W_q(B\Sigma B^T, n) .$$

If $\underset{p \times p}{\mathcal{Y}} = MM^T \sim W_p(\Sigma, n)$ & $B : q \times p$ $(M : p \times n)$

Now, $\underbrace{B\mathcal{Y}B^T}_{q \times q} = BMM^TB^T$

$$= (BM)(M^TB^T)$$

$$= NN^T \qquad (N = BM)$$

We have cols of $M \sim N_p(0, \Sigma)$, $M : p \times n$

$$\Rightarrow N = \underbrace{BM}_{q \times n} = \begin{bmatrix} B_1^T \\ \vdots \\ B_q^T \end{bmatrix} [M_1 \cdots M_n]$$

cols of M

rows of B

$$= \begin{bmatrix} B_1^T M_1 & B_1^T M_2 & \cdots & B_1^T M_n \\ \vdots & \vdots & & \vdots \\ B_q^T M_1 & B_q^T M_2 & \cdots & B_q^T M_n \end{bmatrix}$$

$$= [\underbrace{BM_1}_{\substack{(q\times p)\times(p\times 1)\\ = q\times 1}} \cdots BM_n] \quad \text{(where } BM_j \sim N_q(0, B\Sigma B^T))$$

constant matrix

normal dist

M_j & M_k (cols) are independent $\Leftrightarrow$ $Cov(M_j, M_k) = O_{p\times p}$

$\Rightarrow BM_j$ & BM_k independent since $Cov(BM_j, BM_k) = B\,O_{p\times p}B^T = O_{q\times q}$

Thus, $N = BM = [BM_1, \ldots, BM_n]$ is $q\times n$ matrix whose cols are independent $N_q(0, B\Sigma B^T)$

$\Rightarrow ByB^T = NN^T \sim W_q(B\Sigma B^T, n)$

5. Show thatif $\mathcal{Y}$ is defined at (1) and a is a $p\times 1$ vector such that $a^T\Sigma a \neq 0$, then

$$a^T\mathcal{Y}a/a^T\Sigma a \sim \chi^2_n.$$

$y = MM^T \sim W_p(\Sigma, n)$ (cols of $M \sim N_p(0,\Sigma)$, $M: p\times n$)

$a^Tya = a^TMM^Ta$

$= NN^T$ (where $\underbrace{N}_{1\times n} = \underbrace{a^T}_{1\times p}\underbrace{M}_{p\times n}$, cols of $N \sim N_p(0, \underbrace{a^T\Sigma a}_{\sigma^2}))$

(from Q3) $a^Tya \sim \underbrace{W_1}_{(p=1)}(\underbrace{a^T\Sigma a}_{\sigma^2}, n) = \underbrace{a^T\Sigma a}_{\sigma^2}\cdot \chi^2_n$

$$\Rightarrow \frac{a^Tya}{a^2\Sigma a} \sim \chi^2_n$$