Lecture 16: PGM Representation

PGM

- Mariage of graph theory and probability theory
- Tool of choice for Bayesian statistical learning

Bayesian statistical learning v.s. PGM (aka. "Bayes Nets")

- Bayesian Statistical learning
  - Model joint distribution of X's, Y, and parameters $r.v.$'s
  - Priors: marginals on parameters
  - Training:
    - update prior to posterior using observed data
  - Prediction:
    - output posterior, or some function of it (MAP)
- PGM ("Bayes Nets")
  - Efficient joint representation
    - Independence made explicit
    - Trade-off between expressiveness and need for data, easy to make
    - Easy for Practitioners to model
  - Algorithms to fit parameters, compute marginals, posterior

Joint distribution

- All joint distributions on discrete $r.v.$'s can be represented as table
  - Table assign probability per row
- We can make probabilistic inference from joint on $r.v.$'s
  - Compute any other distributions involving our $r.v.$'s
  - **Bayes rule + marginalisation**
  - Example: Naive Bayes
- Bad: Computational complexity
  - Tables have exponential number of rows in number of $r.v.$'s
  - Therefore -> poor space & time to marginalise
- Ugly: Model complexity
  - Way too flexible
  - Way too many parameters to fit
    - Need lots of data OR will overfit

Independence

- If assume S, T independent, model need 6 params
  - $P(S,T)$ factors to P(S), P(T) -> 2 params
  - $P(L|T,S)$ modelled in full -> 4 params
- For assumption-free model, need 7 params
  - $P(L,T,S)$ modelled in full -> $2^3-1=7$ params
- Independence assumptions

- Can be reasonable in light of domain expertise
- Allow us to **factor** -> Key to tractable models

Factoring Joint Distributions

- Chain Rule: For **any ordering** of $r.v.'s$ can always factor:
  - $P(X_1, X_2, ..., X_k) = \prod_{i=1}^{k} P(X_i|X_{i+1}, ..., X_k)$
- Model's independence assumptions correspond to:
  - Dropping conditioning $r.v.$'s in the factors
  - E.g. Unconditional independence: $P(X_1|X_2) = P(X_1)$
  - E.g. Conditional independence: $P(X_1|X_2, X_3) = P(X_1|X_2)$
    - Given X_2, X_1 and X_3 independent
- Simpler factors: speed up inference and avoid overfitting

Directed PGM

- Nodes -> Random variables
- Edges -> Conditional independence
  - Node table: $P(child|parents)$
  - Child **directly** depends on parents
- **Joint factorisation**
  - $P(X_1, ..., X_k) = \prod_{i=1}^{k} P(X_i|X_j \in parents(X_i))$

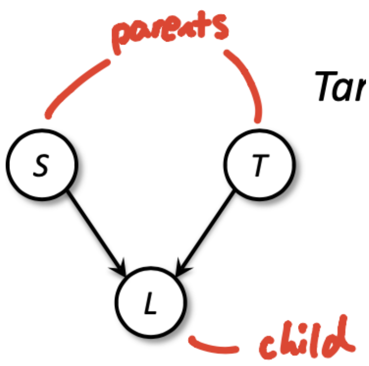
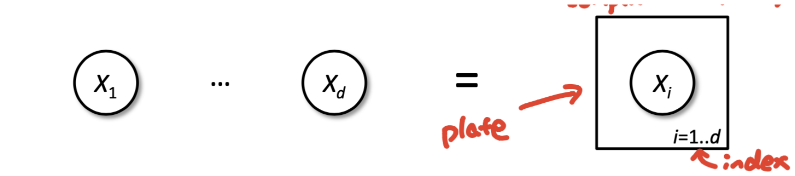
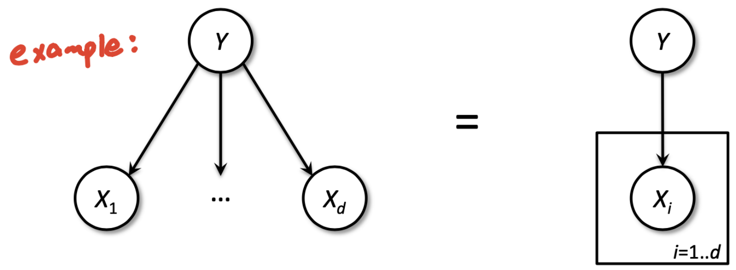


Plate notation

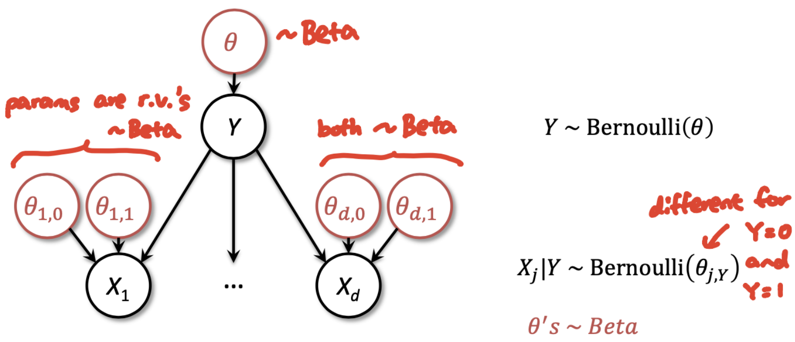
- Short-hand for repeats
- Simplifying growing complicated PGM





PGM: frequentist v.s. Bayesian

- PGM -> joints
- Bayesian add: node per param

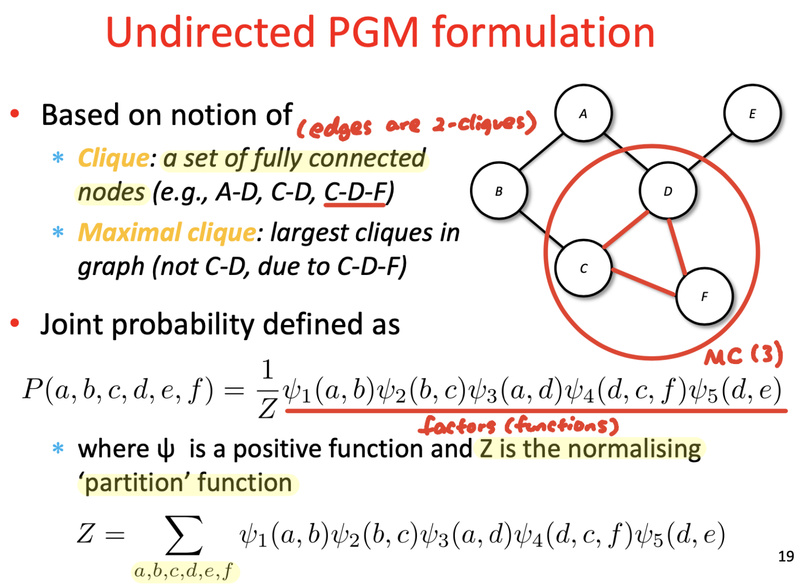


Undirected PGMs

- Parameterised by **arbitrary positive valued functions** of the variables and **global normalisation**
    - Aka. Markov Random Field
- Undirected v.s. Directed PGM
    - Undirected:
        - Graph with undirected edges
        - Probability:
            - Each node a $r.v.$
            - Each **clique $\mathbf{C}$ has "factor"**:
                - $\psi_C(X_j : j \in C) \geq 0$
            - Joint $\propto$ product of factors
    - Directed:
        - Graph with directed edges
        - Probability:
            - Each node a $r.v.$
            - Each **node has conditional probability**:
                - $p(X_i|X_j \in parents(X_i))$
            - Joint = product of conditional probabilities
    - Key difference = **normalisation**
        - $\propto$ in undirected PGM

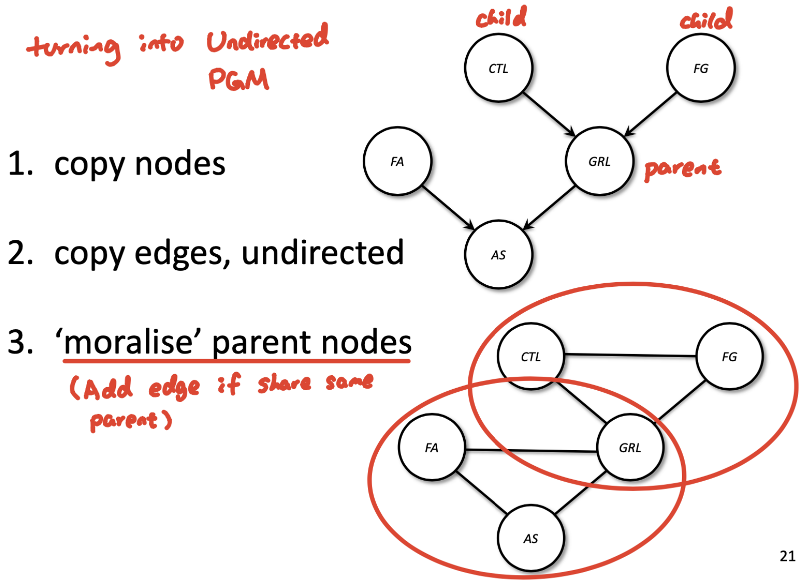
Undirected PGM formulation

- Based on notion of:
  - Clique: a set of **fully connecte**d nodes
  - Maximal clique: largest cliques in graph
- Joint probability defined as:
  - (Product of all the cliques)
  - $P(a,b,c,d,e,f) = \frac{1}{Z}\psi_1(a,b)\psi_2(b,c)\psi_3(a,d)\psi_4(d,c,f)\psi_5(d,e)$
  - where ψ is a **positive function**
  - and Z is the **normalising** "partition" function
    - $Z = \sum_{a,b,c,d,e,f} \psi_1(a,b)\psi_2(b,c)\psi_3(a,d)\psi_4(d,c,f)\psi_5(d,e)$



Directed to undirected

- Directed PGM formulated as:
  - $P(X_1, \ldots, X_k) = \prod_{i=1}^{k} P(X_i | X_{\pi_i})$
  - where π indexes parents
- Equivalent to U-PGM with
  - each **conditional probability** term is **included** in one factor function, ψ_c:
    - clique structure links **groups of variables**
    - normalisation term trivial, Z = 1
- Turning D-PGM to U-PGM:



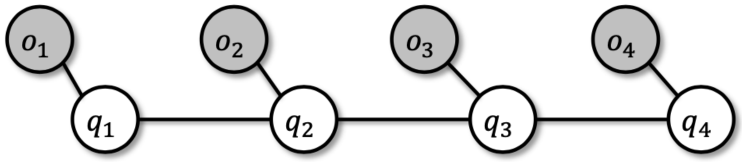
Why U-PGM

- Pros:
    - **Generalisation** of D-PGM
    - Simpler means of modelling without the need for per-factor normalisation
    - General inference algorithms use U-PGM representation
        - (Support both types of PGM)
- Cons:
    - (Slightly) weaker independence
    - Calculating global normalisation term (Z) intractable in general

PGM examples

- Hidden Markov Model (HMM)
    - **Directed**
    - Sequential observed outputs from hidden states
    - States: ejections & transitions
    - 2 assumptions:
        - Markov assumption
        - Output independence assumption
    - Applications:
        - NLP
        - Speech recognition
        - Biological sequences
        - Computer vision
    - Fundamental tasks (corresponding):
        - HMM:
            - Evaluation: determine likelihood $P(O|\mu)$
                - O: observation sequence
                - μ: HMM
            - Decoding: determine most probable hidden state Q
            - Learning: learn parameters A, B, Π
        - PGM:

        - Probabilistic inference
        - MAP point estimate
        - Statistical inference
- Kalman filter
    - Same with continuous Gaussian $r.v.$'s
- Conditional Random Field (CRF)
    - **Undirected**
    - Same model applied to sequences
        - Observed outputs are words, speech, etc.
        - States are tags: part-of-speech, alignment, etc.
    - Discriminative: model $P(Q|O)$
        - v.s. HMM's which are generative $P(Q,O)$
        - undirected PGM more general and expressive

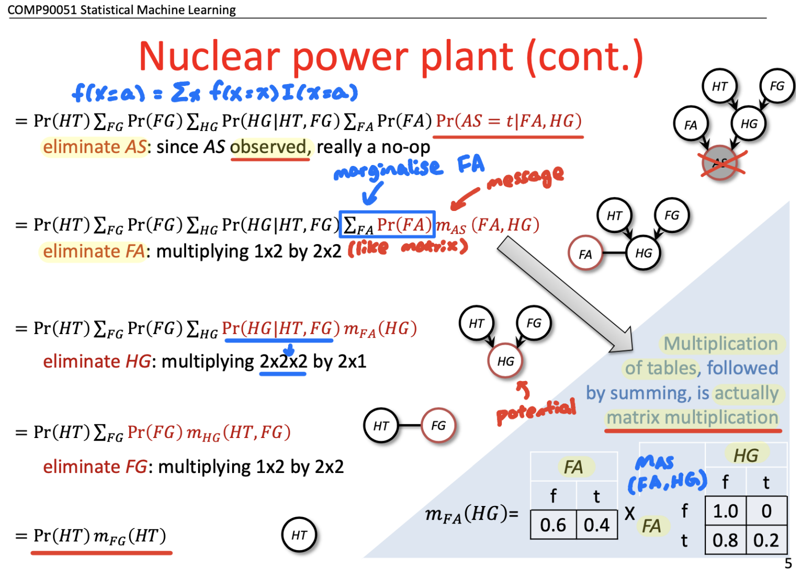


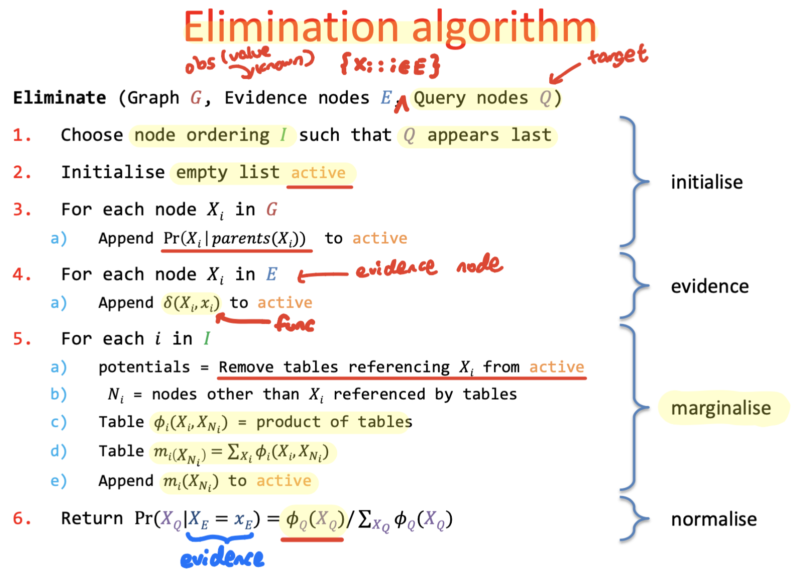
Lecture 17: PGM Probabilistic and Statistical Inference

Probabilistic inference on PGMs

- Computing marginal and conditional distributions from the joint of a PGM
    - Using **Bayes rule and marginalisation**
- Joint + Bayes rule + marginalisation -> anything !
- Example:
    - $P(HT|AS=t) = \frac{P(HT,AS=t)}{P(AS=t)} = \frac{\Sigma_{FG,HG,FA} P(AS=t,FA,HG,FG,HT)}{\Sigma_{FG,HG,FA,HT'} P(AS=t,FA,HG,FG,HT')}$
    - HT' means All values of HT
    - Numerator: $\Sigma_{FG,HG,FA} P(HT)P(HG|HT,FG)P(FG)P(AS=t|FA,HG)P(FA)$
        - Can distribute the sums as far down as possible: $P(HT)\Sigma_{FG}P(FG)\Sigma_H GP(HG|HT,FG)\Sigma_F AP(FA)P(AS=t|FA,HG)$

Elimination algorithm





Statistical inference on PGMs

- Learning (tables / params) from data
  - Fitting probability tables to observations
  - E.g. as a frequentist; a Bayesian would just use probabilistic inference to updat prior to posterior
- Probabilistic inference
  - **Computing other distributions** from joint
  - Elimination, sampling algorithms
- Statistical inference
  - Learn **parameters** from data

1. Fully-observed case (easy)
   - MLE -> maximise full joint (likelihood)
     - $\underset{\theta\in\Theta}{\arg\max} \prod_{i=1}^{n}\prod_j p(X^j = x_i^j | X^{parents(j)} = x_i^{parents(j)})$
   - Decomposes easily, leads to **counts-based estimates**
     - Maximise log-likelihood instead; become sum of logs
     - Big maximisation of all params together, **decouples into small independent problems**

- Example:
  - Training a naive bayes classifier
  - (Counting -> probabilities)

2. Presense of **unobserved** variables
   - Latent, or unobserved variables
   - MLE:
     - Maximise likelihood of observed data only
     - Marginalise full joint to get desired "partial" joint
       - This won't decouple
   - Solution: Use **EM algorithm** !