Physics 305 Term Project:
Solar System Orbital Dynamics

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1 Introduction

The problem I am trying to solve in this term project is determining the correction velocity needed to cause a spaceship to orbit and travel from Earth to Jupiter. The planets have circular and co-planar orbits, so the motion in the problem is only 2-dimensional in x and y. The sun's gravity dominates the motion of the spaceship, so the gravity from the planets can be neglected, including any possible gravity assists through flybys.

2 Equations

The equations of motion for the spaceship are

$$\frac{d^2x}{dt^2} = -\frac{GM_\odot x}{r^3} \tag{1}$$

and

$$\frac{d^2y}{dt^2} = -\frac{GM_\odot y}{r^3} \tag{2}$$

where $r = \sqrt{x^2 + y^2}$, $G = 6.674 * 10^{-11}\ m^3/kgs^2$, and $M_\odot = 2 * 10^{30}$ kg.

An important metric that is used through the spaceship's journey is conservation of energy, which is given by

$$E_{tot} = \frac{v^2}{2} - \frac{GM_\odot}{r} \tag{3}$$

where r is as defined above, and v is the magnitude of the total velocity of the spaceship in both x and y.

The goal of the project is to determine the initial velocity the spaceship needs to correct for so that, using

$$\frac{dx}{dt} = \frac{dx_{ini,boost}}{dt} + \frac{dx_{correction}}{dt} \tag{4}$$

and

$$\frac{dy}{dt} = \frac{dy_{ini,boost}}{dt} + \frac{dy_{correction}}{dt} \tag{5}$$

as well as the constants $\frac{dx_{ini,boost}}{dt} = 0$ m/s and $\frac{dy_{ini,boost}}{dt} = 29789.37$ m/s, a solution to Equations 1 and 2 for x and y can be found that is within 1% of the position of Jupiter at a given time. The position of Jupiter at any time is given by

$$x_{Jupiter}(t) = r_{Jupiter} cos(\omega t - \frac{\pi}{6}) \tag{6}$$

and

$$y_{Jupiter}(t) = r_{Jupiter} sin(\omega t - \frac{\pi}{6}) \tag{7}$$

where $r_{Jupiter} = 5.2$ AU and $\omega = 1.6793 * 10^{-8}$ rad/s.

The magnitude of the spaceship's initial velocity is given by

$$\frac{v_{initial}^2}{2} - \frac{GM_\odot}{1AU} = -\frac{GM_\odot}{r_{planet}} \tag{8}$$

where r_{planet} in this problem is the radius of Jupiter, 5.2 AU. This initial velocity is the velocity needed to reach Jupiter with zero final velocity. This initial velocity is broken up into the correction velocity and boost velocity for both x and y as determined by Equations 4 and 5.

3 Fourth-Order Runge-Kutta Integration

The Runge-Kutta method is one way of numerically solving Ordinary Differential Equations (ODEs). One benefit to the 4th-order method as opposed to higher-order methods is that there is only one function evaluation needed per stage, or 4 function evaluations per step. The explicit 4th-order Runge-Kutta (RK4) formula is

$$\begin{aligned}
k_1 &= hf(y_n, t_n) \\
k_2 &= hf(y_n + \tfrac{k_1}{2}, t_n + \tfrac{h}{2}) \\
k_3 &= hf(y_n + \tfrac{k_2}{2}, t_n + \tfrac{h}{2}) \\
k_4 &= hf(y_n + k_3, t_n + h) \\
y_{n+1} &= y_n + \tfrac{k_1}{6} + \tfrac{k_2}{3} + \tfrac{k_3}{3} + \tfrac{k_4}{6}
\end{aligned}$$

An ODE driver was needed to use the RK4 formula. The one utilized in this project was odeSolve, which uses constant step size and allows systems of ODEs. odeSolve creates an array of times, determines whether the ODE is a scalar or a system, sets the first element of the solution to the initial conditions, and calls the RK4 method using the time and initial conditions at a step, the step size, and the right-hand side of the ODE. The calling process repeats until t_{max} is reached.

The right-hand side of the equation as used in the RK4 method took in the initial state (x-position, y-position, x-velocity, and y-velocity of the spaceship) as inputs, and returned an output with the x-velocity, y-velocity, x-acceleration

(the right-hand side of Equation 1), and y-acceleration (the right-hand side of Equation 1) of the spaceship.

Another aspect implemented into the ODE driver was checking for conservation of energy throughout the spaceship's journey. Two equations were used to check conservation of energy. The initial energy

$$E_0 = \frac{v_0^2}{2} - \frac{GM_\odot}{r_0} \tag{9}$$

where v_0 is the magnitude of the spaceship's initial velocity as calculated in Equation 8, and $r_0 = 1$ AU is the point the spaceship is launched from (the spaceship is launched fron Earth at the point x=1 AU, y=0). Equation 3 was used during the journey to find the spaceship's energy. To ensure conservation of energy and an accurate solution to the motion, the difference in energy, ΔE, was calculated at each time using

$$\Delta E = |\frac{E_{tot} - E_0}{E_0}| \tag{10}$$

If this value was smaller than 10^{-3} for all integration times, the accuracy was decent.

For my trial and error process, I tried many different angles, then used the quadratic formula to get from the angle to the x-correction velocity. The y-correction velocity is the x-correction velocity times the tangent of the angle. I then compared the magnitude of the initial velocity calculated with theta to the initial velocity from energy conservation, and they were extremely close: 37964.99650050783 m/s from the calculation, and 37964.99650050965 m/s from energy conservation.

I wasn't able to get as close on my position values, however. The final position for Jupiter at the time I used, 2.08 years, was (645035861008.4768 m, 434822995844.364 m). For the satellite, the final position was (6.44393445e+11 m, 4.09282964e+11 m). The x-positions were well within 1 %, but the y-positions were off by about 6%. I decided to use these values to complete the remainder of the project. If I were to do the project again, I would think about possible automation strategies to make the trial and error process faster, as well as start with a smaller range of angles closer to zero than angles that extended out as far as $\frac{\pi}{8}$.

4 Convergence

To demonstrate convergence, eight different step sizes were used, ranging from the step size used to t_{max}. The RK4 odeSolve was used with each of these step sizes. ΔE was calculated using Equation 10 for each step size. A plot of the ΔE values versus step size was created, and is shown in Figure 1.

To determine the order of convergence, a log-log plot of step size and ΔE was plotted, since the plot of ΔE values versus step size appeared to show

a exponential/power-law relationship. The log-log plot is shown in Figure 2. A line was fit to this data, and the slope of that line gave the power for the power-law relationship.

The slope of the line in Figure 2 was approximately 2.172, indicating a second-order power-law relationship. This makes sense because, at large r, the v^2 term in the energy formula will dominate.

5 Self-Convergence

Since there isn't an exact solution to this problem, the solution obtained needs to be self-convergent. To show this, the solution at different numerical resolutions is calculated. The highest numerical resolution is considered the reference solution, and then the lower-resolution solutions are shown to converge to the reference. Self-convergence is trying to show that the following equality holds for the solution to the problem.

$$\frac{y_{\text{num}}(h) - y_{\text{num}}(h_3)}{y_{\text{num}}(h_2) - y_{\text{num}}(h_3)} = \frac{(h/h_3)^n - 1}{2^n - 1} \tag{11}$$

In Equation 9, $y_{\text{num}}(h) = y_{\text{exact}} + A * h^n$, and $h_2 = 2h_3$, where h_2 is the step size used to obtain the solution.

The log-log plot of step size and the convergence value is shown in Figure 3. Since the self-convergence (right-hand side of Equation 11) and analytic convergence (left-hand side of Equation 11) values are generally along the same line, the code is self-convergent. For the RK4 method, it is expected that there should be convergence at 4th order, which is what Figure 3 demonstrates.

6 Richardson Extrapolation

Richardson Extrapolation is used to cancel error terms and improve upon the solution from the RK4 integration. In this problem, Richardson Extrapolation was used to determine an error for the x and y positions at a chosen time.

The equation for 4th order Richardson Extrapolation (as determined through the order of self-convergence) is

$$A = \frac{2^4 * A(h/2) - A(h)}{2^4 - 1} \tag{12}$$

where A(h/2) is the position at the h/2 step size for the chosen time, and A(h) is the position at the h step size for the chosen time. I chose a time of $\frac{t_{max}}{2}$ to perform Richardson Extrapolation, and performed two separate calculations, one for x and one for y.

To determine the error for the solution, I used the equation

$$A_{err} = |A_{exact} - A| \tag{13}$$

where A_{exact} was pulled from the RK4 integration at $\frac{t_{max}}{2}$. I again used this equation twice for x and y. The error in x was 14337066.666625977, and the error in y was 15468800.0.

7 Plotting the Orbits

To plot the orbit of Jupiter, Equations 6 and 7 were evaluated at each time step from 0 to t_{max}. I also plotted the position of Jupiter at the final time to compare to the spaceship's position at the final time.

To plot the orbit of Earth, similar position equations were used and evaluated at each time step from 0 to t_{max}. The position of Earth at any time is given by

$$x_{\oplus}(t) = r_{\oplus} cos(\omega t) \tag{14}$$

and

$$y_{\oplus}(t) = r_{\oplus} sin(\omega t) \tag{15}$$

where $r_{\oplus} = 1$ AU ($1.495978707 * 10^{10}$ m) and $\omega = 1.99 * 10^{-7}$ rad/s.

To plot the orbit of the spaceship, I plotted 4 points from the RK4 solutions (about 1/4 through, halfway through, 3/4 of the way through, and the final position) and plotted the initial launch point of the spaceship from Earth.

Figure 4 shows the two planetary orbits and the spacecraft traveling between them.

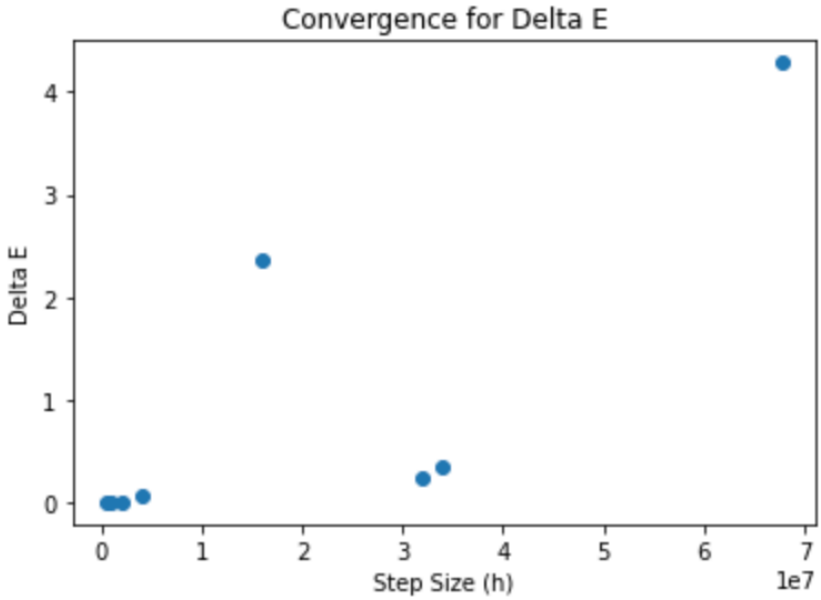

Figure 1: Initial Convergence Plot

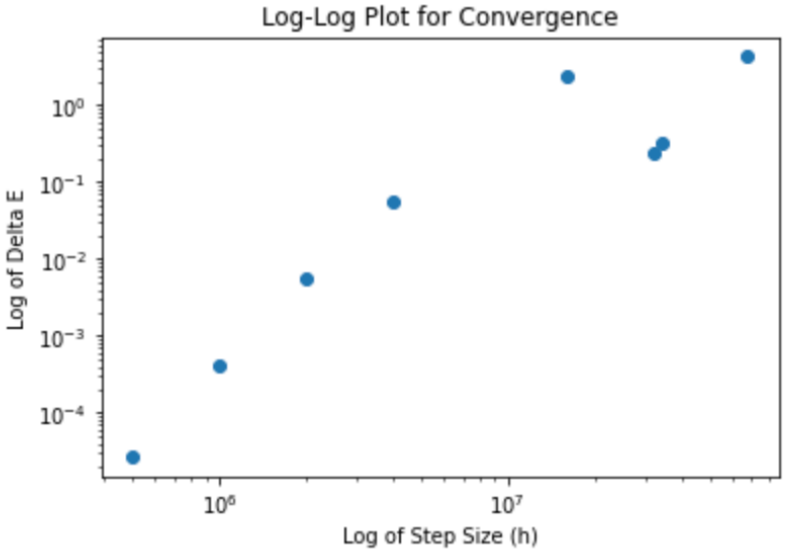

Figure 2: Log-Log Convergence Plot to Show Power-Law Relationship

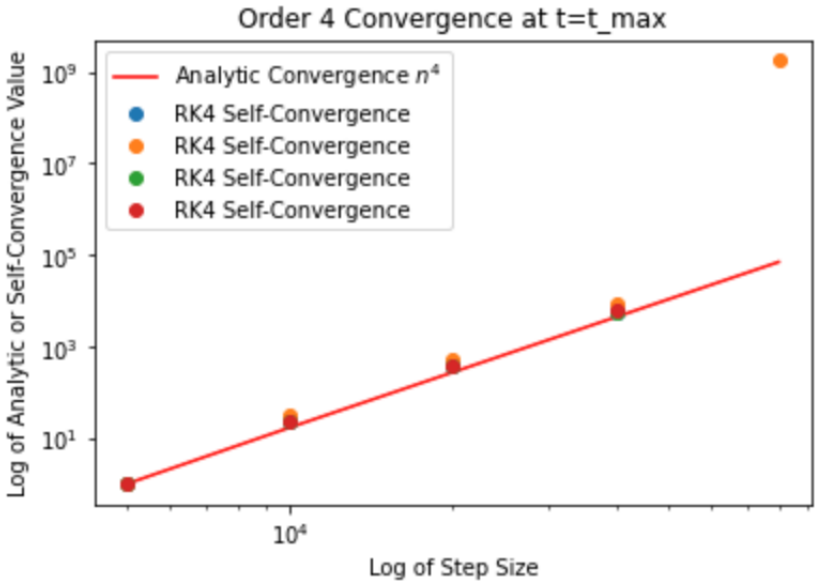

Figure 3: Plot of Self-Convergence

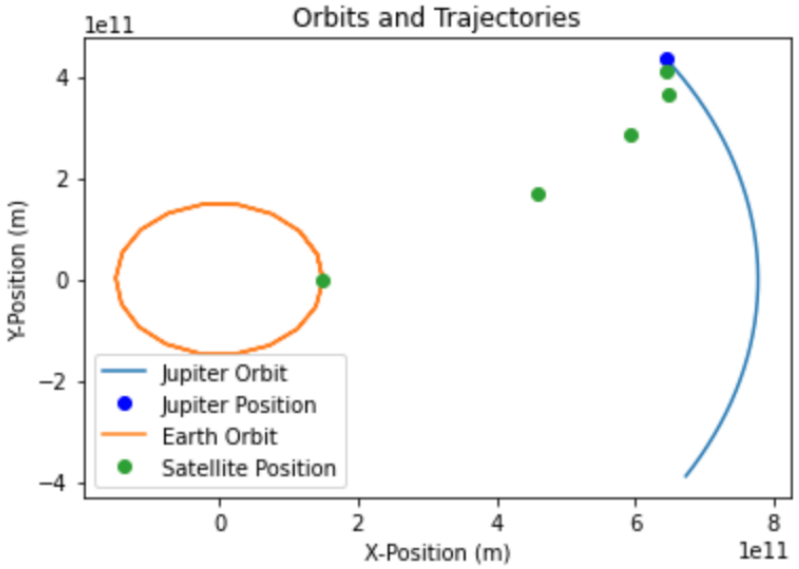

Figure 4: Plot of the Orbits of Earth, Jupiter, and the Spaceship