PHYS 332 Computer Project:
Field due to a Charged Sphere with a Tiny Hole

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1 Introduction

The goal of this project is to examine the differences that occur between a spherical charged shell and a spherical charged shell which has uncharged hole in the top. §2 will deal with the analytic calculation of the electric field, §3 will deal with the the numerical calculation of the electric field, §4 will show the field with and without the sphere graphically and provide interpretation of the results, and §5 will talk about lessons learned and consequences of the calculation.

2 Analytic Calculation of the Electric Field

The setup for the geometry of the problem is shown in Figure 1.

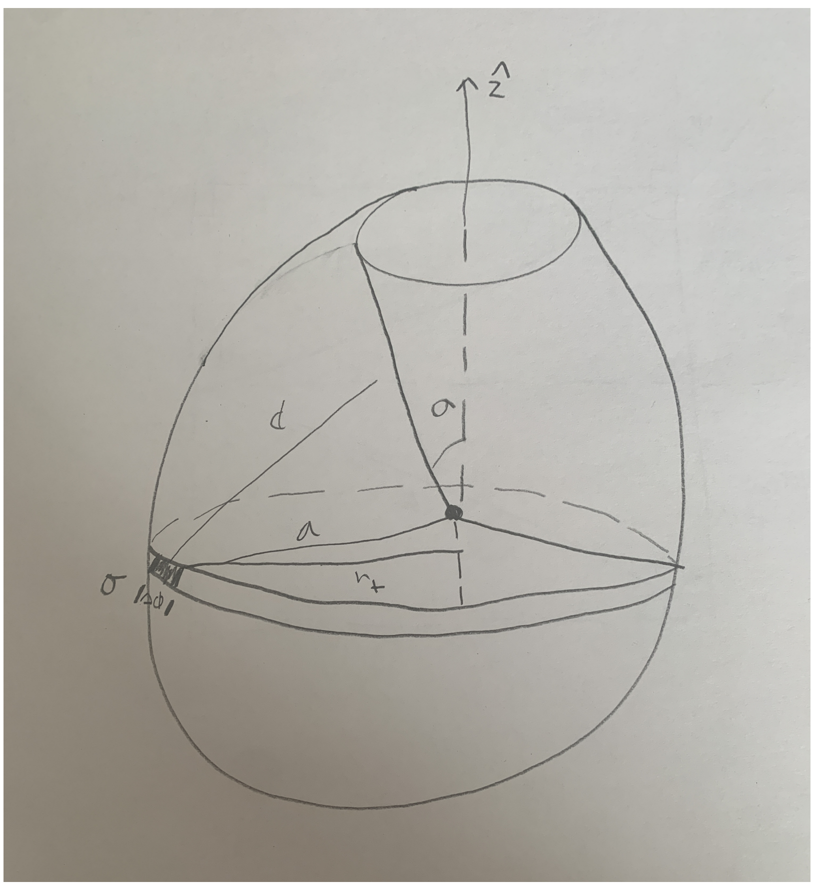

Figure 1: Diagram to illustrate the geometry of the sphere with a hole in it.

From this setup, we can obtain an initial expression for the z-component of the electric field as

$$\Delta E_z = \frac{1}{4\pi\epsilon_0}\frac{\Delta q}{d^2} cos\Psi \tag{1}$$

where Δq is the charge element, d is labeled in Figure 1, and $cos\Psi$ is solved for using Figure 2.

To find Δq, we use the equation

$$\Delta q = \sigma a^2 sin(\theta) d\theta d\phi \tag{2}$$

where σ is the charge density, a is the radius of the sphere, θ is used in Figure 2 and is the angle measured down from the +z-axis to the charge element, and ϕ goes around the sphere from 0 to 2π, which will be substituted into the final expression for the electric field.

Using the geometry drawn in Figure 2, we can obtain d and $cos\Psi$.

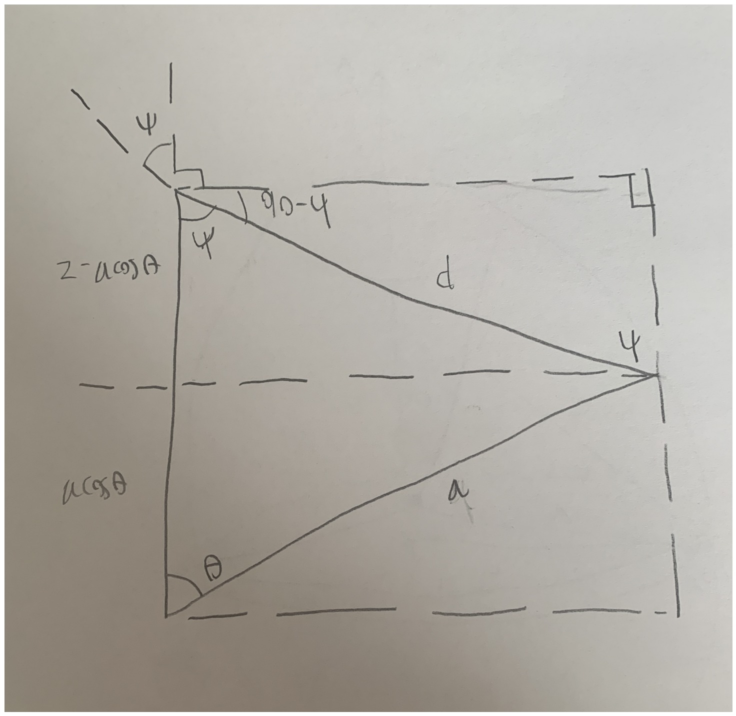

Figure 2: Diagram to find $cos\Psi$ using known values from Figure 1

Using the law of cosines for the large triangle, the value for d is given by

$$d^2 = z^2 + a^2 - 2azcos(\theta) \tag{3}$$

where z is the distance from the center of the sphere to where the field point is measured.

To obtain $cos\Psi$, we break up the larger triangle into two smaller triangles. Using the triangle just above the line splitting the large triangle, the value of $cos\Psi$ is given by

$$cos\Psi = \frac{z - acos\theta}{d} \tag{4}$$

Putting Equations 2, 3, and 4 into Equation 1, and making the substitutions that $d\phi = 2\pi$ around the entire sphere and that $d = \sqrt{z^2 + a^2 - 2azcos(\theta)}$ from Equation 3 into Equation 4 results in

$$\Delta E_z = \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma(a^2 sin(\theta))}{(z^2 + a^2 - 2azcos(\theta))^{3/2}} (z - acos\theta) d\theta \tag{5}$$

To obtain the z-component of the electric field, we integrate from $\theta = \alpha$ to $\theta = \pi$,

$$E_z = \int_\alpha^\pi \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma(a^2 sin(\theta))}{(z^2 + a^2 - 2azcos(\theta))^{3/2}} (z - acos\theta) d\theta \tag{6}$$

move constants outside the integral,

$$E_z = \frac{2\pi\sigma a^2}{4\pi\epsilon_0}\int_\alpha^\pi \frac{sin(\theta)(z - acos\theta)}{(z^2 + a^2 - 2azcos(\theta))^{3/2}} d\theta \tag{7}$$

and finally obtain the expression for the electric field by evaluating at the limits of integration

$$E_z = \frac{\sigma a^2}{2\epsilon_0}\left[\frac{1}{z^2} - \frac{a - zcos(\alpha)}{z^2\sqrt{-2azcos(\alpha) + z^2 + a^2}}\right] \tag{8}$$

3 Numerical Calculation of the Electric Field

Using the analytical solution from Equation 8, a simple program was created in Python to evaluate the value of the integral. The code evaluates the field at $z = 0.01na$, where n is an integer from 0 to 500 and adds the z-value and the respective value of the field at that z to lists for use in graphing. To allow for easier evaluation of the function, the charge density σ was set to 5, the radius of the sphere, a, was set to 1, and the evaluation of the integral was started at $\alpha = 1°$ because there was no charge density between $\alpha = 0°$ and $\alpha = 1°$ due to the hole.

```
import matplotlib.pyplot as plt
import numpy as np
import scipy.constants as constant

eps_0 = constant.epsilon_0

def Electric_Field (sigma, a, alpha, z):
    return ((sigma*a**2)/(2*eps_0))*((1/z**2)-((a-z*np.cos(alpha))/(z**2*np.sqrt((-2*a*z*
    np.cos(alpha))+z**2+a**2))))

field_list = []
z_list = []
for n in range (1, 501):
    field = Electric_Field(5, 1, 0.01745329, z=0.01*n*1) #alpha is in radians
    field_list.append(field)
    z_list.append(0.01*n*1)
```

4 Graphing and Interpreting Results

Both the function from Equation 8 in §2 and the points obtained in §3 were plotted on top of each other. All graphs were plotted with the logarithm of the electric field to make the shape of the curve clearer.

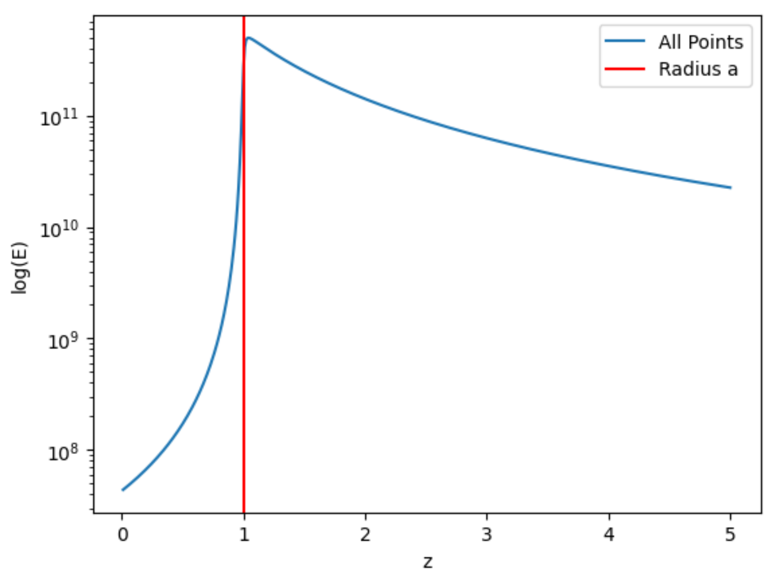

Figure 3: Final graph showing all the points from Section 3 and a vertical line denoting where the sphere's radius is in terms of z.

Based on the graph from Figure 3, we see a shift in the shape of the graph before the radius line versus after the radius line. The change happens when the field point is slightly outside the radius of the sphere, which would make sense since that is when the field point is moving from inside the sphere through the uncharged portion. This will be better illustrated in Figure 7 when we compare the sphere with the hole and the sphere without the hole.

The following three figures show the graph from Figure 3 but zoomed into three regions: z=0 to z=a (Figure 4), z=a to z=2a (Figure 5), and z > 2a (Figure 6). The function from §2 is plotted in gray, the points obtained in §3 are plotted in blue, and the radius of the sphere is plotted in red.

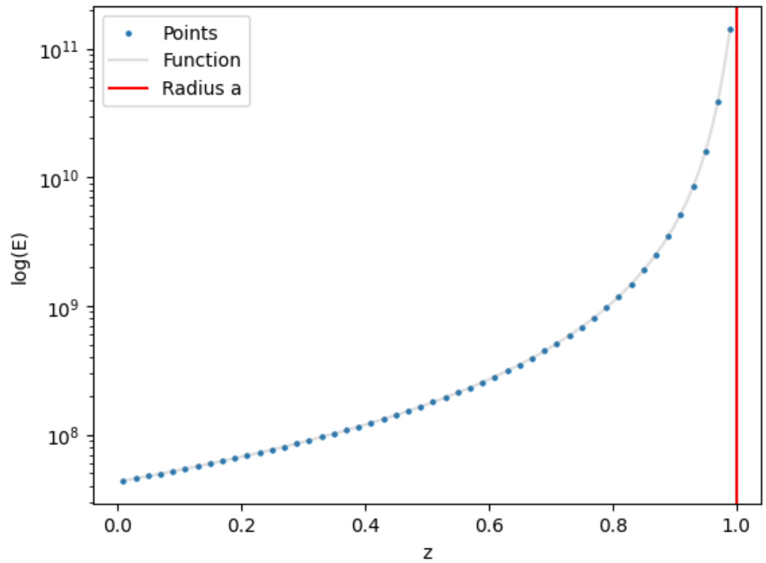

Figure 4: Zooming in on the electric field within the sphere.

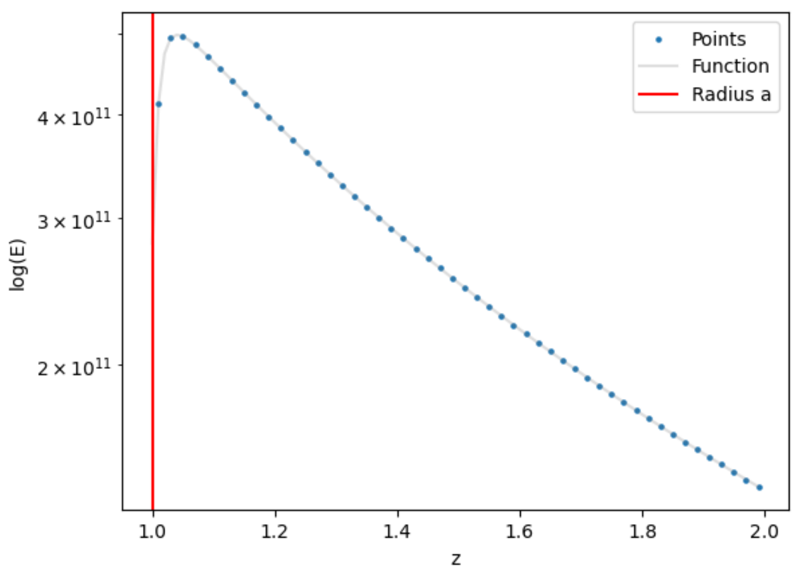

Figure 5: Zooming in on the electric field from the sphere's radius to twice the sphere's radius above it.

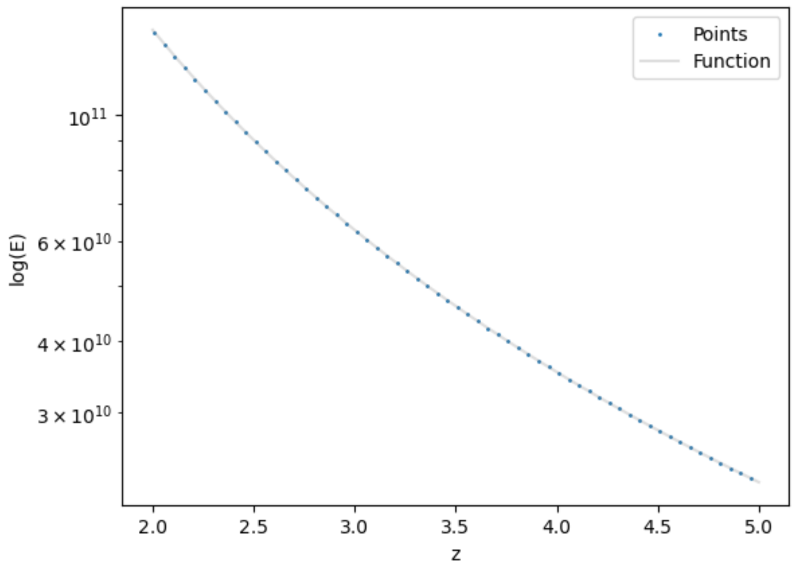

Figure 6: Zooming in on the electric field from twice the sphere's radius above it to five times the sphere's radius above it.

Figure 7 shows the difference between the charged sphere with a hole that we examine in this project and a charged sphere that does not have a hole, whose electric field is modeled by the equation

$$E_z = \frac{\sigma 2\pi a^2 sin(\theta)}{4\pi\epsilon_0(-2azcos(\alpha) + z^2 + a^2)} \tag{9}$$

where α begins at $0°$ since there is a charge density across the entire sphere. The difference between the two is most apparent slightly outside the radius of the sphere since that difference is charge is small but will have a large impact on field points outside the top of the sphere.

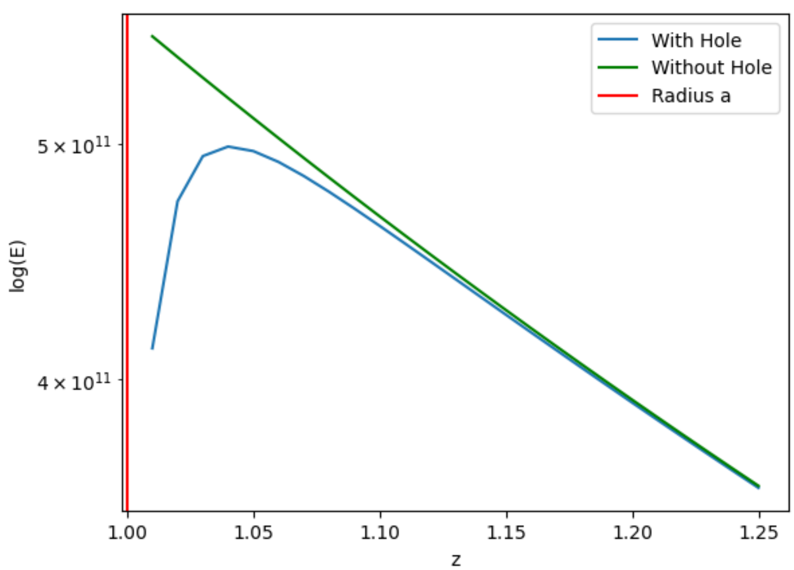

Figure 7: Comparing the electric field for the charged sphere with the hole versus a charged sphere without a hole.

5 Conclusions

In conclusion, this project delved into the differences between a spherical charged shell and a similar shell with an uncharged hole. Through an analytic derivation of the electric field and numerical evaluation, we were able to learn about the nature of electrostatic fields in such configurations. This project highlighted the impact of the absence of charge in a localized region, emphasizing the influence of the uncharged portion on field distributions in surrounding space.