Financial Mathematics 33000

Lecture 2

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One period, two states

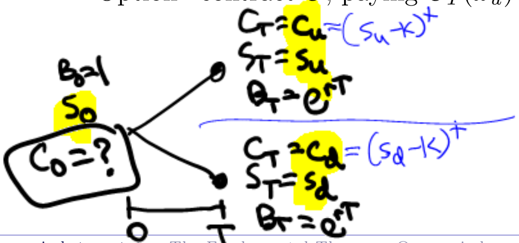
The Fundamental Theorem

One-period, more discrete states

Fundamental Theorem, again

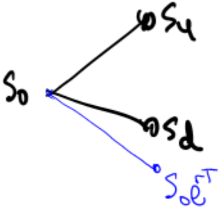
Binomial model specification

- Times 0 and T. No intermediate trading; all portfolios are static.
- Up and down state $\{\omega_u, \omega_d\}$ at time T, each with probability > 0.
- Bank account: Each unit has time-t value $B_t = e^{rt}$ for $t = 0, T$.
- Stock S: Let S_T take values $S_T(\omega_u) = s_u$ and $S_T(\omega_d) = s_d$ in the up and down state respectively, where $s_u > s_d$.
- "Option" contract C, paying $C_T(\omega_u) = c_u$ and $C_T(\omega_d) = c_d$.



Exercises: Find an arbitrage

- Suppose that $S_0 e^{rT} \leq s_d$. Find an arbitrage.
- Suppose that $S_0 e^{rT} \geq s_u$. Find an arbitrage.



(+1 unit of S,
$-S_0$ units of B)

Exercises: Find an arbitrage

Skip

- **Brexit:** In June 2016, voters in the UK voted to *remain* in or *leave* the European Union. In May 2016, bookmakers posted 4:1 odds against leaving. Thus an asset S that pays $s_u = 1$ GBP if "leave", and $s_d = 0$ GBP if "remain", costs $S_0 = 0.2$ GBP in May. Bookmakers also accepted bets in EUR *with the same odds* – despite the fact that GBP was expected to weaken if "leave". GBP/EUR is 1.3 in May, and assume it goes to 1.1 if "leave". Model this as a second asset S^* that pays $1/1.1$ GBP if "leave", and 0 if "remain", and costs $S_0^* = 0.2/1.3$ GBP in May.

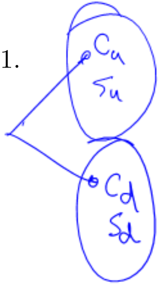
  Find an arbitrage.

  `http://theconversation.com/how-to-beat-the-bookies-with-a-brexit-bet-60009`

Option pricing via replication

Given $S_0, s_u, s_d, c_u, c_d, r$, find arbitrage-free time-0 option price C_0.

Solution: Construct portfolio (α, β) of (bank acct, stock) that replicates the option. Want $\mathsf{P}(\text{time-}T \text{ portfolio value} = C_T) = 1$.



$$\alpha e^{rT} + \beta s_u = c_u$$

$$\alpha e^{rT} + \beta s_d = c_d$$

Solve for α and β:

$$\beta = \frac{c_u - c_d}{s_u - s_d} \qquad \text{and} \qquad \alpha = e^{-rT}(c_d - \beta s_d)$$

By no-arb, time-0 option value = time-0 portfolio value. Conclude:

$$C_0 = \alpha + \beta S_0.$$

Value of replicating portfolio

Rewrite, collecting c_u and c_d terms:

$$C_0 = \alpha + \beta S_0 = e^{-rT}(c_d - \beta s_d + \beta S_0 e^{rT}) = e^{-rT}\left[c_d + \frac{c_u - c_d}{s_u - s_d}(S_0 e^{rT} - s_d)\right]$$

$$\text{Therefore } \boxed{C_0 = e^{-rT}(p_u c_u + p_d c_d)}$$

where

$$p_u := \frac{S_0 e^{rT} - s_d}{s_u - s_d}, \qquad p_d := \frac{s_u - S_0 e^{rT}}{s_u - s_d} = 1 - p_u$$

Two special cases of (c_u, c_d) are $(1, 0)$ and $(0, 1)$:

- Let an "up-contract" pay $U_T(\omega_u) = 1$ and $U_T(\omega_d) = 0$.

  Then time-0 up-contract value $= e^{-rT} p_u$.

- Let a "down-contract" pay $D_T(\omega_u) = 0$ and $D_T(\omega_d) = 1$.

  Then time-0 down-contract value $= e^{-rT} p_d$.

Understanding the pricing formula as a decomposition

Result $C_0 = e^{-rT}(p_u c_u + p_d c_d)$ can be understood as a decomposition.

- Example: A contract that pays 5 in the up state and 3 in the down state decomposes into 5 up-contracts plus 3 down-contracts.

$\langle^{5}_{3}$ = 5 units of $\langle^{1}_{0}$ + 3 units of $\langle^{0}_{1}$

So $C_T = 5U_T + 3D_T$ hence $C_0 = 5U_0 + 3D_0 = e^{-rT}(5p_u + 3p_d)$.

- More generally, payment C_T of (c_u, c_d) in (up,down) states decomposes as

$$C_T = c_u U_T + c_d D_T$$

which has time-0 value

$$C_0 = c_u U_0 + c_d D_0 = c_u \times e^{-rT} p_u + c_d \times e^{-rT} p_d$$

Interpreting the pricing formula as an expectation

Result $C_0 = e^{-rT}(p_u c_u + p_d c_d)$ can be understood also as expectation:

$$C_0 = e^{-rT}\mathbb{E}C_T$$

or equivalently: $C_0/B_0 = \mathbb{E}(C_T/B_T)$ where $B_0 = 1$ and $B_T = e^{rT}$

What is the meaning of $\mathbb{E}$?

- $\mathbb{E}$ is expectation wrt the measure $\mathbb{P}$ that assigns probability p_u to up-move, and probability p_d to down-move.
- Note that p_d, p_u are > 0 and < 1, or else arbitrage exists. So $\mathbb{P}$ is indeed a probability measure.
- But $\mathbb{P}$ does not represent actual physical probabilities.

Two probability measures

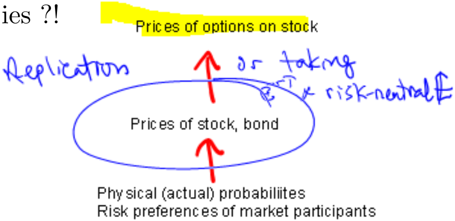
- P is called the *actual* or *physical* probability measure.

  It has *no direct relevance here*:

  Given the specification of this model, we do not care about the value of P(up) for the purpose of pricing.

- $\mathbb{P}$ is called a *risk-neutral measure* or *martingale measure*. Some authors denote as "$\mathbb{Q}$". Important in derivatives pricing.

- Irrelevance of physical probabilities ?!



The [First] Fundamental Theorem of Asset Pricing

$$\text{No arbitrage} \iff \begin{array}{l} \text{there exists a probability measure } \mathbb{P}, \\ \text{equivalent to } \mathsf{P}, \text{such that the discounted prices} \\ \text{of all tradeable assets are martingales wrt } \mathbb{P}. \end{array}$$

Definitions:

- $\mathbb{P}$ *equivalent to* P means: for any event A, $\mathbb{P}(A) = 0$ iff $\mathsf{P}(A) = 0$.
- In this one-period model, M_t is a *martingale* means: $M_0 = \mathbb{E}M_T$.
  (Today's level equals today's expectation of tomorrow's level)
- *Discounted* price means price X divided by bank acct price: X/B

Thus, to say that the discounted price X/B is a martingale here means that $X_0/B_0 = \mathbb{E}(X_T/B_T)$; equivalently $X_0 = e^{-rT}\mathbb{E}X_T$.

Proof of Fundamental Theorem

Let's prove it in the case of the one-period binomial model, with an arbitrary number of assets, including a stock and a bank account. (True much more generally, but need technical assumptions)

- Proof that No arb $\Rightarrow$ existence of martingale measure $\mathbb{P}$:

  We proved this L2.6-L2.9. The measure $\mathbb{P}$ is, explicitly: $\mathbb{P}(\text{up}) = p_u$ and $\mathbb{P}(\text{down}) = 1 - p_u$, with p_u specified in L2.7. We need to check that $\mathbb{P}$ is a probability measure ($0 \leq p_u \leq 1$), and indeed an *equivalent* probability measure ($0 < p_u < 1$), which follows from no-arbitrage (how?). With respect to that $\mathbb{P}$,

  $$X_0/B_0 = \mathbb{E}(X_T/B_T)$$

  for all assets X, by L2.9.

Proof of Fundamental Theorem

- Proof that Existence of martingale measure $\mathbb{P} \Rightarrow$ No type-1 arb:
  Consider any static portfolio $\mathbf{\Theta}$ of assets $\mathbf{X}$. Each asset price $X_0^n = e^{-rT}\mathbb{E}X_T^n$. Multiply by quantity θ^n, then $\sum$ across assets:

  $$\mathbf{\Theta} \cdot \mathbf{X}_0 = e^{-rT}\mathbb{E}(\mathbf{\Theta} \cdot \mathbf{X}_T).$$

  So discounted *portfolio* value is also martingale: $V_0 = e^{-rT}\mathbb{E}V_T$.
  If $V_0 \neq 0$, then not arb; we're done. So take $V_0 = 0 \Rightarrow \mathbb{E}V_T = 0$.
  If $\mathbb{P}(V_T < 0) \neq 0$ then not arb; done. So take $\mathbb{P}(V_T < 0) = 0$.
  Then $\mathbb{P}(V_T > 0) = 0$, because a nonnegative, zero-expectation, random variable must vanish with probability 1. (Reason: if $\mathbb{P}(V_T > 0) > 0$, then $\mathbb{P}(V_T > \varepsilon) > 0$ for some $\varepsilon > 0$, hence $\mathbb{E}V_T \geq \varepsilon\mathbb{P}(V_T > \varepsilon) > 0$). Conclusion: $\mathbf{\Theta}$ is not a (type-1) arb. $\square$

Option pricing via the Fundamental Theorem

An alternative to pricing via replication is to use Fundamental Thm:

Basic asset prices ⇒ risk-neutral probabilities ⇒ option price

(1) Apply Fundamental Thm to S to infer risk-neutral probabilities:

$$S_0 = e^{-rT}\mathbb{E}S_T = e^{-rT}[p_u s_u + (1-p_u)s_d].$$

Solve to obtain

$$p_u = \frac{S_0 e^{rT} - s_d}{s_u - s_d}.$$

(2) Now use p_u to price the option:

$$C_0 = e^{-rT}\mathbb{E}C_T = e^{-rT}[p_u c_u + (1-p_u)c_d]$$

Two techniques for derivatives pricing:

Build replicating portfolio, or Find $\mathbb{E}$(discounted payoff).

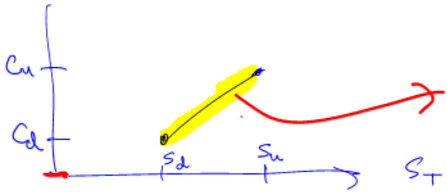
Fundamental Theorem of Asset Pricing

Rigorous justifications in general settings are achieved by 1981, by Michael Harrison, Stanley Pliska, and David Kreps.



Replication in two-state model

Recall: we replicated the option using $\beta := (c_u - c_d)/(s_u - s_d)$ shares.

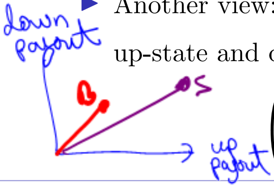


$\text{slope} = \dfrac{c_u - c_d}{s_u - s_d}$

- Match the slope by choosing the appropriate number of shares.

  Match the level using the appropriate number of bank acct units.

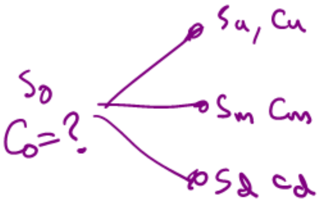
- Another view: For each asset, write its payoff as a vector of up-state and down-state payoffs. Replication possible because



$$\begin{pmatrix} c_u \\ c_d \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} e^{rT} \\ e^{rT} \end{pmatrix}, \begin{pmatrix} s_u \\ s_d \end{pmatrix} \right\}$$

A three-state model

- Times 0 and T. No intermediate trading; all portfolios are static.
- Up, middle, down state at time T, each with positive probability
- Bank account: Each unit has time-t value $B_t = e^{rt}$, for $t = 0, T$.
- Stock S: Let S_T take values $s_u > s_m > s_d$ in up, mid, down states respectively.
- Option C: Let C_T take values c_u, c_m, c_d in up, mid, down states.



Replication in three-state model

Example: Let $r = 0$, let $S_0 = 100$, $s_u = 130, s_m = 100, s_d = 80$.

Consider a 90-call: $c_u = 40, c_m = 10, c_d = 0$. Can we replicate it?

Answer:

Can replicate option on the upside by holding 1 share of S

Can replicate option on the downside by holding 0.5 shares of S

But can't simultaneously replicate both risks.

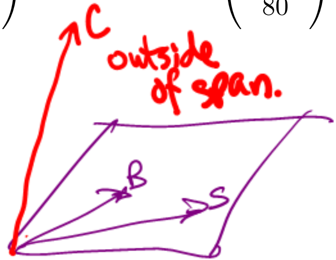
Replication and spanning

Another view: Write payoffs as vectors

$$\text{Bank acct payoff} \begin{pmatrix} \text{up-payoff} \\ \text{mid-payoff} \\ \text{down-payoff} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad \text{Stock payoff} \begin{pmatrix} 130 \\ 100 \\ 80 \end{pmatrix}$$

And the 90-call payoff is

$$\begin{pmatrix} 40 \\ 10 \\ 0 \end{pmatrix} \notin \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 130 \\ 100 \\ 80 \end{pmatrix} \right\}$$



Complete markets

Market is said to be *complete* if every random variable Y_T can be replicated, meaning there exists $\mathbf{\Theta}$ such that $\mathsf{P}(\mathbf{\Theta} \cdot \mathbf{X}_T = Y_T) = 1$, where $\mathbf{\Theta} := (\theta^1, \ldots, \theta^N)$ and $\mathbf{X}_T(\omega) := (X_T^1(\omega), \ldots, X_T^N(\omega))$.

Examples:

- The market of $\{\text{bank acct}, \text{stock}\}$ in the two-state model is complete. We were able to solve for $\Theta = (\alpha, \beta)$.
- The market of $\{\text{bank acct}, \text{stock}\}$ in the three-state model is *incomplete*, because the 90-call payoff could not be replicated.

The martingale measures in this example

Probability measures on this space can be specified by probabilities (p_u, p_m, p_d). They form a martingale measure iff they are an equivalent probability measure such that A/B is MG for each asset A.

The first condition is that p_u, p_m, p_d are positive (> 0), and

$$p_u + p_m + p_d = 1.$$

The second condition is that $\mathbb{E}(S_T/B_T) = S_0/B_0$. Equivalently,

$$130p_u + 100p_m + 80p_d = 100.$$

This system has infinitely many solutions. Two examples:

$$(p_u, p_m, p_d) = (0.20, 0.50, 0.30) \quad \text{and} \quad (p_u, p_m, p_d) = (0.30, 0.25, 0.45)$$

· $(40, 10, 0) = 13$ · $(40, 10, 0) = 14.5$

Martingale measure exists but is not unique.

The [first] fundamental theorem

The first fundamental theorem still holds in the multiple-state setting with an arbitrary number of assets, regardless of completeness.

$$\boxed{\text{No arb} \iff \exists \text{ equivalent martingale measure } \mathbb{P}}$$

Proof of "$\Leftarrow$" is like in binomial model, but "$\Rightarrow$" is harder.

Let's just give some intuition ...

One period, two states

The Fundamental Theorem

One-period, more discrete states

Fundamental Theorem, again

Why can't we price by taking the payoff's expectation using actual probabilities

Because people are not risk-neutral.

- Your \$10000 car has a actual 1% chance of being unrecoverably stolen this year. You may be willing to pay a lot more than \$100 to insure against this. Not because you are irrational, but because you are risk-averse.
- Consider a physically 50/50 coin flip worth \$1 million or nothing. You might rationally refuse to pay more than \$400K for this coin flip. Because each dollar in the bad state may be more precious than a dollar in the good state.

What is the risk-neutral probability of an event

skip

It's the price of a one-unit payout contingent on the event.

Consider an event G. Consider an asset which pays:

$$\text{1 bank acct unit if } G \text{ occurs, otherwise } 0.$$

Let p_G denote the time-0 price of this "G" asset, in units of B.

What can we say about p_G?

Answer: The following are consequences of no-arbitrage.

- If $\mathsf{P}(G) = 0$ then $p_G = 0$.
- If $\mathsf{P}(G) > 0$ then $p_G > 0$.
- If $\mathsf{P}(G) = 1$ then $p_G = 1$.

Likewise for an asset contingent on some event H.

What is the risk-neutral probability of an event

skip

Consider disjoint events G and H. Consider an asset which pays:

1 bank acct unit if $G \cup H$ occurs, 0 otherwise.

Let $p_{G \cup H}$ denote the time-0 price of this asset, in units of B.

Then $p_{G \cup H} =?$

Answer: Replicate this $G \cup H$ asset

by holding 1 unit of the G asset and 1 unit of the H asset.

- By law of one price, $p_{G \cup H} = p_G + p_H$.

These prices p satisfy the definition of a probability measure.

So define the risk-neutral probability of an event to be the *price* of an asset which pays: 1 bank acct unit if the event occurs, else 0.

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Why can we price by taking $\mathbb{P}$-expectations

Suppose we have J possible outcomes $\{\omega_1, \ldots, \omega_J\}$. Suppose an asset pays Y units of B at time T, where Y is a random variable.

Thus the payment is $Y(\omega_j)$ bank acct units, if jth outcome occurs.

What's the time-0 price of the asset which pays Y?

Answer: Replicate it by holding, for $j = 1, \ldots, J$,

$Y(\omega_j)$ units of a basic asset which pays: 1 if ω_j occurs, 0 otherwise

Replicating portfolio's time-0 value, in units of the bank account, is

$$\sum_{j=1}^{J} \text{Quantity} \times \text{Price} = \sum_{j=1}^{J} Y(\omega_j) p_{\omega_j}$$

This is the expectation of Y with respect to risk-neutral probabilities!

Why can we price by taking $\mathbb{P}$-expectations

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- Let X_t be the time-t value in dollars of an asset which pays X_T dollars at time T.
- Then X_t/B_t is its time-t value, and X_T/B_T is the payout, expressed in units of the bank account.
- So, according the the previous page,

$$\frac{X_0}{B_0} = \mathbb{E}\frac{X_T}{B_T}$$

where $\mathbb{E}$ denotes risk neutral expectation.

- (What's missing from this proof? Need to show it works even if the "basic" assets don't exist.)

Summary: why can we price by taking $\mathbb{P}$-expectations

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Because the following actions result in identical calculations:

- Pricing:

  Take a payoff, decompose into a portfolio of 0/1 "Arrow-Debreu" assets, and sum the quantity times price of each asset.

- Taking a $\mathbb{P}$-expectation:

  Take a random variable, decompose into its possible realizations, and sum the level times $\mathbb{P}$-probability of each realization.

(All "prices" are relative to a designated asset, e.g. the bank account)

To summarize: risk-neutral pricing works because risk-neutral probabilities *are* prices (L2.28). So taking a risk-neutral expectation does the same calculation as pricing by replication (L2.29).

How are actual and risk-neutral probabilities related

- Risk-neutral probabilities $\mathbb{P}$ depend on actual probabilities P combined with the market's risk preferences
- The measures $\mathbb{P}$ and P are "equivalent" meaning that they agree on all events that have probability 0 (or probability 1).
- Again in the discrete setting with outcomes $\{\omega_1, \ldots, \omega_n\}$ each of nonzero probability, the relationship between the risk-neutral measure $\mathbb{P}$ and the actual measure P can expressed by the "likelihood ratio" or "Radon-Nikodym derivative"

$$\frac{\mathbb{P}(\omega)}{\mathsf{P}(\omega)}$$

  It's typically bigger in "bad" states ω, smaller in "good" states ω.

The second fundamental theorem of asset pricing

An arbitrage-free market is complete iff there exists a *unique* martingale measure (MM).

Complete $\Rightarrow$ uniqueness: Assume one-period, J-states $\{\omega_1, \ldots, \omega_J\}$ with > 0 probabilities, N assets including a bank acct with value e^{rt}. For each $j = 1, \ldots, J$, define the *Arrow-Debreu* payoff

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$$A_T^j(\omega) = \begin{cases} 1 & \text{for } \omega = \omega_j \\ 0 & \text{for } \omega \neq \omega_j \end{cases}$$

By completeness, A_T^j has a replicating portfolio $\mathbf{\Theta}^j$ hence a unique arbitrage-free time-0 price $\mathbf{\Theta}^j \cdot \mathbf{X}_0$. So for *any* martingale measure $\mathbb{P}^*$, we have $e^{-rT}\mathbb{E}^* A_T^j = \mathbf{\Theta}^j \cdot \mathbf{X}_0$, where $\mathbb{E}^*$ means expectation wrt $\mathbb{P}^*$. So $\mathbb{P}^*(\{\omega_j\}) = \mathbf{\Theta}^j \cdot \mathbf{X}_0 e^{rT}$ is *unique* MM probability of $\{\omega_j\}$.

The second fundamental theorem of asset pricing: Proof

Uniqueness $\Rightarrow$ complete: By no-arb, there exists a MM $\mathbb{P}$. Suppose market is incomplete. Let's construct a MM $\mathbb{P}^*$ different from $\mathbb{P}$.

Look for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_J)$ such that an equivalent MM is formed by

$$\mathbb{P}^*(\{\omega_j\}) := \mathbb{P}(\{\omega_j\}) + \varepsilon_j \qquad j = 1, \ldots, J.$$

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Want ε such that $\mathbb{E}^*$ still prices all assets correctly. Equivalently, want

$$e^{-rT} \sum_j \varepsilon_j X_T^n(\omega_j) = 0 \qquad n = 1, \ldots, N.$$

Equivalently, want ε orthogonal to each $\mathbf{x}^n \in \mathbb{R}^J$ that represents X^n.

By incompleteness $\operatorname{span}\{\mathbf{x}^1, \ldots, \mathbf{x}^N\} \neq \mathbb{R}^J$, so by Gram-Schmidt, there exists a unit vector $\mathbf{v}$ orthogonal to $\mathbf{x}^1, \ldots, \mathbf{x}^N$.

So $\mathbb{P}^*$ is another equivalent MM, where $\varepsilon := \frac{\min_j |\mathbb{P}(\{\omega_j\})|}{2} \mathbf{v}$. $\square$