Homework 6

Sergei Tikhonov

November 16th 2023

Exercise 1

(a) According to this proposal, the employee will obtain the following profit at $t = 1$:

$$(S_1 - \min(30, 0.75 \cdot S_1)) \frac{3000}{\min(30, 0.75 \cdot S_1)}$$

Let's simplify it:

$$V_1 = 3000 \cdot \left(\frac{S_1}{\min(30, 0.75 \cdot S_1)} - 1 \right) = 3000 \left(\frac{4}{3} \cdot \max\left(1, \frac{S_1}{40}\right) - 1 \right) = 1000 + 100 \cdot \max(0, S_1 - 40)$$

After simplifying the expression above, we can clearly see that this payoff is a combination of 1000 units of banking account and 100 unites of call option with strike $K = 40$. Then time $t = 0$ value using the Black-Scholes model:

$$V_0 = 1000 \cdot e^{-0.05} + 100 \cdot C(40) = 1000 \cdot e^{-0.05} + 100 \cdot (S_0 \cdot N(d_1) - K \cdot e^{-0.05} \cdot N(d_2))$$

$$= 951.23 + 100 \cdot 4.48 = 1399.2$$

(b) In the previous section, we stated that the payoff received at $t = 1$ is the combination of 1000 units of banking account and 100 units of call option (long). In order to perfectly hedge this position, one can short 100 call options with strike $K = 40$. As a result, $V_1 = 1000$ no matter what S_1 will be. Since we can just short 100 units of call options at $t = 0$ and hence perfectly hedge, we don't need rebalancing.

(c) For now, we cannot use call and put options to hedge perfectly that risk. However, we can replicate the payoff of the aforementioned call option using delta hedging. The position in stocks at $t = 0$:

$$a_0 = 100 \cdot \frac{\partial C}{\partial S} = 100 \cdot N(d_1) = 100 \cdot 0.63 = 63$$

For implementing delta hedging, we need to rebalance (at each t) the position over the course of the year.

Exercise 2

The dynamics of stocks and bonds:

$$\mathrm{d}B_t = rB_t\mathrm{d}t \qquad\qquad \mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t$$

(a) According to the task, we have the following portfolio, where a_t and b_t are number of stocks and bonds:

$$L_t = a_t \cdot S_t + b_t \cdot B_t$$

Substituting the number of stocks as $a_t = \beta\frac{L_t}{S_t}$, the number of shares is:

$$b_t = \frac{L_t - a_t \cdot S_t}{B_t} = \frac{L_t(1-\beta)}{e^{rt}}$$

(b) According to the task, the portfolio is self-financing:

$$\mathrm{d}L_t = a_t\mathrm{d}S_t + b_t\mathrm{d}B_t$$

Then, let's substitute $\mathrm{d}S_t$ and $\mathrm{d}B_t$, as well as a_t and b_t:

$$\mathrm{d}L_t = a_t(\mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t) + b_t rB_t\mathrm{d}t$$

$$\mathrm{d}L_t = \mu \cdot \beta \cdot L_t\mathrm{d}t + \sigma \cdot \beta\mathrm{d}W_t + L_t \cdot r \cdot (1-\beta)\mathrm{d}t$$

This is also Geometric Brownian Motion, because it satisfies the corresponding stochastic differential equation:

$$\mathrm{d}L_t = L_t(\mu \cdot \beta + (1-\beta) \cdot r)\mathrm{d}t + L_t \cdot \sigma \cdot \beta\mathrm{d}W_t$$

The drift term is:

$$\mu \cdot \beta + (1-\beta) \cdot r$$

The volatility term is:

$$\sigma \cdot \beta$$

Exercise 3

(a) Put-call parity or the time-t prices of a call and put on S, where the call and put have the same strike K and same expiry T:

$$C(t,S) = P(t,S) + S_t - Ke^{-r(T-t)}$$

(b) We need to prove the formula of put option price:

$$P(t,S) = Ke^{-r(T-t)} \cdot N(-d_2) - S_t \cdot N(-d_1)$$

From the lecture notes, we know that the price of call option is:

$$C(t, S) = S_t \cdot N(d_1) - e^{-r(T-t)} \cdot K \cdot N(d_2)$$

Let's substitute the price of call option to the put-call parity stated above:

$$C(t, S) = S_t \cdot N(d_1) - e^{-r(T-t)} \cdot K \cdot N(d_2) + S_t - Ke^{-r(T-t)}$$

$$P(t, S) = -S_t \cdot (-N(d_1) + 1) + K \cdot e^{-r(T-t)} \cdot (1 - N(d_2))$$

Using the property of standard normal distribution:

$$P(t, S) = K \cdot e^{-r(T-t)} \cdot N(-d_2) - S_t \cdot N(-d_1)$$

(c) Let's derive the delta of put option by taking derivative with respect to S of put-call parity:

$$\frac{\mathrm{d}}{\mathrm{d}S} C(t, S) = \frac{\mathrm{d}}{\mathrm{d}S} \left(P(t, S) + S_t - K \cdot e^{-r(T-t)} \right)$$

$$C'(t, S) = P'(t, S) + 1$$

From the lecture notes, we know that the delta of call option is:

$$\text{delta}_{\text{call}} = N(d_1)$$

Then the delta of put option:

$$\text{delta}_{\text{put}} = N(d_1) - 1$$

(d) For the gamma of put option, let's continue taking derivatives of put-call parity:

$$\frac{\mathrm{d}}{\mathrm{d}S} C'(t, S) = \frac{\mathrm{d}}{\mathrm{d}S} \left(P'(t, S) + 1 \right)$$

$$C''(t, S) = P''(t, S)$$

From the lecture notes, we know that the gamma of call option is:

$$\text{gamma}_{\text{call}} = \frac{N'(d_1)}{S_t \cdot \sigma \cdot \sqrt{T - t}}$$

The gamma of put option is also:

$$\text{gamma}_{\text{put}} = \frac{N'(d_1)}{S_t \cdot \sigma \cdot \sqrt{T - t}}$$