Financial Mathematics 33000

Lecture 7

Roger Lee

2022 November 30

Pricing and hedging

The ingredients of a derivatives pricing/hedging problem/solution:

- **Contract** to be priced/hedged
- **Dynamics** of underlying
- **Solution** approach
  - Replication or Expectation
  - Analytical or Computational

Replication or Expectation

Other Payoffs

Other Dynamics: Forward prices

Other Dynamics: Dividend-paying stocks

Analytical or Computational

Replication and Expectation paths to solutions

L5

Underlying dynamics (example: GBM in the B-S case)

replication

Contract (example: call or put)

Risk-neutral pricing

Fundamental Thm / Risk-Neutral pricing

PDE with terminal condition

$\mathbb{E}$(contract's discounted payout)

solve PDE

Calculate $\mathbb{E}$

Pricing formula

Take $\frac{\partial}{\partial S}$

How many shares to hold in replicating portfolio at time 0

$C = e^{-r(T-t)}[FN(d_1) - KN(d_2)] = SN(d_1) - Ke^{-r(T-t)}N(d_2)$

Same formula, multiple interpretations

if $R_{grow} = r$

Black-Scholes $\boxed{N(d_2)}$ at time t, where $d_2 = \frac{\log(S_t e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}$:

- $N(d_2)$ is the risk-neutral probability of $S_T > K$.
- $e^{-r(T-t)}N(d_2)$ is the value of a binary call.
- $e^{-r(T-t)}N(d_2)$ is $-\partial C/\partial K$, where C is vanilla call value.
- $-Ke^{-r(T-t)}N(d_2)$ is value of vanilla-call replicator's B holdings.

Black-Scholes $\boxed{N(d_1)}$ at time t, where $d_1 = \frac{\log(S_t e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$:

skip

- $N(d_1)$ is the *share-measure* probability of $S_T > K$. It's the time-t price, in shares, of an asset that pays 1 share if $S_T > K$.
- $S_t N(d_1)$ is the value of an asset-or-nothing call.
- $N(d_1)$ is $\partial C/\partial S$, the delta of a vanilla call.
- $S_t N(d_1)$ is value of vanilla-call replicator's share holdings.

PDE can come from probabilistic approach too

Recall: under $\mathbb{P}$, "every tradeable asset's proportional drift rate is r".

- Apply this to S (where $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$) to get

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}\tilde{W}_t$$

  where $\tilde{W}$ is a $\mathbb{P}$-BM with $\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \lambda_t \mathrm{d}t$.

  The drift changes (to rS_t), but the *volatility does not.*

- Apply this to the option price C, assuming $C_t = C(S_t, t)$. By Itô

$$\mathrm{d}C_t = \frac{\partial C}{\partial t}\mathrm{d}t + \frac{\partial C}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mathrm{d}S)^2.$$

  Equate the drift of C to rC:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

  This is the B-S PDE. Terminal condition: $C(S,T) = (S-K)^+$

Replication or Expectation

Other Payoffs

Other Dynamics: Forward prices

Other Dynamics: Dividend-paying stocks

Analytical or Computational

Pricing general payoff functions of S_T

To find time-t price:

- PDE approach: To price instead an option paying $f(S_T)$, use the PDE that comes from the dynamics, changing only the terminal condition to $C(S,T) = f(S)$.
- Expectations approach: Calculate

$$e^{-r(T-t)} \int_0^\infty f(s)p(s)\mathrm{d}s$$

where p is the time-t conditional probability density of S_T. Or,

$$e^{-r(T-t)} \int_{-\infty}^\infty f(e^x)p_L(x)\mathrm{d}x$$

where p_L is the time-t conditional probability density of $\log S_T$.

Hedging general payoff functions of S_T

- The replication argument showed: if $C(S,t)$ is a function that satisfies the B-S PDE with terminal condition $C(S,T) = f(S)$, then a portfolio of $\partial C/\partial S$ shares and $(C - S_t \cdot \partial C/\partial S)/B_t$ units of the bank account replicates a $f(S_T)$ payoff, and self-finances.
- So to hedge a contract on $f(S_T)$, we can use PDE *or* risk-neutral $\mathbb{E}$, to find the option pricing function C which satisfies the B-S PDE. We can then calculate $\partial C/\partial S$ to find the delta hedge.

Forward prices

The time-t *forward price* for time-T delivery of some underlying X_T (not necessarily a tradeable asset) is defined to be the particular level of K^* such that the forward contract on X_T with delivery price K^* and delivery date T has time-t value 0. "fair strike"

Assuming only frictionless markets, no arbitrage, and non-random interest rates, the time-t forward price F_t

- satisfies $F_t = K^*$ where $e^{-r(T-t)}\mathbb{E}_t(X_T - K^*) = 0$, therefore

$$\boxed{F_t = \mathbb{E}_t X_T}$$

  and F is a martingale. In particular, $F_T = X_T$. Therefore, if F follows GBM, can price options on F_T, and therefore options on X_T, using C^{BS} formula with underlying $= F_t$ and $R_{grow} = 0$.

Forward prices

- F_t also equals the time-t *futures price* for time-T delivery of X. (A futures contract is not the same thing as a forward contract. However, futures prices = forward prices, when interest rates are non-random. You are not required to learn about futures contracts/prices for this class.)
- F_t satisfies a generalized put-call parity that does not assume the underlying X is tradeable:

$$C_t - P_t = e^{-r(T-t)}(F_t - K)$$

if the call, put, and forward contract all have strike (delivery price) K and expiry T. Proof: replicate the K-strike forward contract with the F_t-strike forward plus $F_t - K$ bonds.

Replication or Expectation

Other Payoffs

Other Dynamics: Forward prices

Other Dynamics: Dividend-paying stocks

Analytical or Computational

What is a tradeable asset?

Common language standpoint: something you can buy/sell.

Mathematical standpoint: We have already defined tradeable assets.

- A tradeable asset is just a member of the vector $\mathbf{X}$ of adapted stochastic processes (representing the market's tradeable assets).
- Each trading/portfolio strategy $\boldsymbol{\Theta}$ in the assets $\mathbf{X}$ is allowed to change at any times in some designated set of trading times. At all t, the portfolio's time-t value is defined to be $V_t := \boldsymbol{\Theta}_t \cdot \mathbf{X}_t$.
- The strategy $\boldsymbol{\Theta}$ in assets $\mathbf{X}$ is defined to be self-financing if

$$\mathrm{d}V_t = \boldsymbol{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t$$

as you recall. These definitions already incorporate "tradeability".

What is a tradeable asset?

Embedded within these mathematical definitions are requirements that can be labelled as [frictionless] “tradeability”:

- The ability to buy and hold arbitrary quantities at prices $\mathbf{X}$. This includes negative quantities.
- The definition that declares “self-financing (no deposits / no withdrawals)” to be equivalent to “value changes are fully attributable to asset price changes” assumes that portfolio values
  - Are not allowed to change due to transaction costs.
  - Are not allowed to change due to dividends or storage costs.

What is a tradeable asset?

Financial modeling standpoint:

- A tradeable asset is an object which satisfies [to whatever extent the financial modeler demands] the "tradeability" requirements implicit in our mathematical definitions about trading strategies.

In particular, a tradeable asset X has the following properties (or at least can be modelled as having the following properties):

- Is available to be bought or sold frictionlessly at all designated times t, at price X_t
- Can be held in arbitrary quantities (including negative), without receiving any dividends, nor incurring any costs.

Examples

X is *not* the price process of a tradeable asset for $t \in [0, T]$ if:

(ignoring trivial/exceptional circumstances)

- X_t is the time-t price of a dividend paying stock
- X_t is the forward price of a stock (dividend-paying or not)
- $X_t = S_t^2$ where S_t is a stock that follows B-S dynamics
- $X_t = (S_t - K)^+$ where S_t is a stock that follows B-S dynamics
- X_t is an interest rate
- X_t is the S&P 500 index
- X_t is the time-t temperature in this room

Examples

X *is* the price process of a tradeable asset for $t \in [0, T]$ if

- X_t is the time-t price of a non-dividend paying stock
- X_t is the time-t price of a dividend-paying stock, *together* with all of its re-invested dividend payments since time 0.

X could be the price process of a tradeable asset for $t \in [0, T]$ if

- X_t is the time-t value of contract which pays (only) at time T the quantity V_T, where V_T is *any* random variable (not necessarily an asset price) revealed at time $T > t$.

  Ex: X_t is the time-t value of an option on a time-T interest rate.

  Ex: X_t is the time-t value of a contract paying S_T^2 at time T.

Examples

X could be the price process of a tradeable asset for $t \in [0, T]$ if

- X_t is the time-t value of contract which pays (only) at time T the quantity V_T, where V_T is *any* random variable revealed at time T.
  Ex: X_t is the time-t value of an option on a time-T interest rate.
  Ex: X_t is the time-t value of a contract paying S_T^2 at time T.

Why "could be"?

- Maybe X is not available in the particular market, and cannot be synthesized from what is available. (But note that in derivatives pricing the typical question is: If the contract that pays V_T *would* be made available, what price would it have? So we typically treat as tradeable the derivative that we propose to introduce.)

Contrast two examples

Let S the price process of a tradeable asset.

Can the process X_t be the price process of a tradeable asset …

- If $X_T = S_T^2$?

  Yes (if that contract is available, or if you make it available).

  But you cannot dictate that $X_t = S_t^2$ at earlier times t.

- If $X_t = S_t^2$ for all $t \in [0, T]$?

  No. Because arbitrage would arise (except in trivial cases).

Including dividend-paying stocks in our portfolios

Two approaches:

- Enlarge the mathematical theory to allow assets to have dividend/consumption streams. Need to (re)define concepts, such as "self-financing portfolio".
- Keep the mathematical theory as it is. But to apply it to some object, you must bundle the object *together* with whatever dividend/consumption stream it generates. This *bundle/package* can be considered a tradeable asset.

We take the second approach.

Stock paying continuous dividends

Let's price an option which pays $f(S_T)$ at time T, where S is the price of a stock that pays dividends to stockholders at a constant yield q.

- This means that if Q_t denotes the total dollar amount of dividend paid during $[0, t]$ by one share, then $\mathrm{d}Q_t = qS_t\mathrm{d}t$.

So S is not tradeable, but it makes sense to consider as tradeable:

- A contract which pays S_T at time T.
- Equivalently, this is a zero-delivery-price forward contract on S_T.
- Equivalently, this is a bundle/package, starting at time 0 with $A_0 := e^{-qT}$ shares, pooled with all reinvested dividends. Thus

$$\mathrm{d}A_t = A_t\mathrm{d}Q_t/S_t = A_t(qS_t\mathrm{d}t)/S_t = qA_t\mathrm{d}t$$

  So at time t the bundle has $A_t = e^{-q(T-t)}$ shares. Note $A_T = 1$.

Dynamics

- Let X_t be the value of this bundle/package. We have

$$X_t = e^{-q(T-t)} S_t.$$

Note that $X_T = S_T$; so the payoff $f(S_T)$ is identical to $f(X_T)$.

- Now assume Black-Scholes dynamics for S:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

where W is P-BM. Then $\log X_t = -q(T-t) + \log S_t$, hence

$$\mathrm{d}\log X_t = q\mathrm{d}t + \mathrm{d}\log S_t = (\mu + q - \sigma^2/2)\mathrm{d}t + \sigma \mathrm{d}W_t$$

hence

$$\mathrm{d}X_t = (\mu + q) X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t.$$

The replication approach

skip

By the same replication arguments that we have already seen (L5), the option price must be $\tilde{C}(X_t, t)$ where $\tilde{C}$ satisfies the B-S PDE

$$\frac{\partial \tilde{C}}{\partial t} + rX\frac{\partial \tilde{C}}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 \tilde{C}}{\partial X^2} = r\tilde{C}$$

with $\tilde{C}(X, T) = f(X)$. Let $C(S, t) := \tilde{C}(e^{-q(T-t)}S,\ t)$. Then

$$\begin{aligned}
\frac{\partial C}{\partial S} &= e^{-q(T-t)}\frac{\partial \tilde{C}}{\partial X} \\
\frac{\partial^2 C}{\partial S^2} &= e^{-2q(T-t)}\frac{\partial^2 \tilde{C}}{\partial X^2} \\
\frac{\partial C}{\partial t} &= \frac{\partial \tilde{C}}{\partial t} + qe^{-q(T-t)}S\frac{\partial \tilde{C}}{\partial X}
\end{aligned}$$

so $X\frac{\partial \tilde{C}}{\partial X} = S\frac{\partial C}{\partial S}$, and $X^2\frac{\partial^2 \tilde{C}}{\partial X^2} = S^2\frac{\partial^2 C}{\partial S^2}$, and $\frac{\partial \tilde{C}}{\partial t} = \frac{\partial C}{\partial t} - qS\frac{\partial C}{\partial S}$, where the LHS of these three equations are evaluated at $X = e^{-q(T-t)}S$.

Solution

Hence

$$\frac{\partial C}{\partial t} + (r-q)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

with terminal condition $C(S,T) = f(S)$.

- Suppose we have a call, $f(S) := (S-K)^+$.

  The PDE solution is

  $$C(S,t) = C^{BS}(S,t,K,T,r-q,r,\sigma) = e^{-r(T-t)}[FN(d_1) - KN(d_2)],$$

  where $F := Se^{(r-q)(T-t)}$ and

  $$d_{1,2} := d_{\pm} := \frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}.$$

The risk-neutral pricing / martingale approach

Now we can do either

- Write the PDE by setting the risk-neutral drift of $C(S_t, t)$ equal to rC, and solve the PDE.
- Or find the $\mathbb{P}_t$-distribution, and solve for $\mathbb{E}_t$ of discounted payout. At time t,

$$\log S_T \sim \text{Normal}(\log S_t + (r - q - \sigma^2/2)(T - t),\ \sigma^2(T - t)).$$

Therefore, for a call option, by L6.4,

$$C_t = e^{-r(T-t)}\mathbb{E}_t(S_T - K)^+ = C^{BS}(S_t, t, K, T, r - q, r, \sigma)$$

which agrees with the PDE solution.

The risk-neutral pricing / expectations approach

Need to change to risk-neutral measure. X is tradeable.
And under physical measure, X is GBM with volatility sigma.
So, under risk-neutral measure, X is GBM but with drift rXdt.
Volatility is still sigma.

Now we can do either

- Write the PDE by setting the risk-neutral drift of $C(S_t, t)$ equal to rC, recovering L6.21; and solve the PDE.
- Or find the $\mathbb{P}_t$-distribution, and solve for $\mathbb{E}_t$ of discounted payout.

$$dX_t = rX_t\,dt + \sigma X_t\,d\tilde{W}_t$$

$$\Rightarrow dS_t = (r-q)S_t\,dt + \sigma S_t\,d\tilde{W}_t$$

Drifts	physical	risk-neutral
X	$\mu+q$	r
S	μ	$r-q$

At time t,

$$\log S_T \sim \text{Normal}(\log S_t + (r - q - \sigma^2/2)(T-t),\ \sigma^2(T-t)).$$

Therefore, for a call option, by L6.4,

$$C_t = e^{-r(T-t)}\mathbb{E}_t(S_T - K)^+ = C^{BS}(S_t, t, K, T, r-q, r, \sigma)$$

which agrees with the PDE solution.

Reasonable to assume continuously-paid div yields?

Depends on the underlying.

- FX: Yes usually.

  When the underlying is a foreign currency, the "dividend yield" is the interest rate paid by the foreign-currency-denominated bank account. Can model these interest payments as a continuous dividend.

- Single stocks: No.

  Dividend payments are discrete, not continuous. If a stock has ex-dividend dates on the 15th of Feb, May, Aug, Nov, with a dividend yield of 4% annual, then taking $q = 0.04$ is wrong if the pricing date = Mar 1 and expiry = Apr 20. No dividends!

Reasonable to assume continuously-paid div yields?

- Single stocks: No.

  Moreover, if we want to account for early exercise (single-stock options are usually "American-style"), then dividend timing matters (e.g., to determine whether to exercise an American call, necessary to check the day immediately before ex-div date).

  skip

- Stock index: Maybe.

  ~ 400 companies, each making up to 4 div payments annually, could be modeled as a continuous (but time-varying) yield. But rather than forecasting q to compute $F_t = S_t e^{(r-q)(T-t)}$, consider estimating $F_t = \mathbb{E}_t S_T$ directly, using put-call parity, or futures.

How to model discrete dividends?

A simple approach:

- European option on S_T is a European option on forward price F_T because $F_T = S_T$
- Model the forward price as GBM with drift 0.
- Solve for time-0 option price in terms of F_0. Can rewrite in terms of S_0.

Replication or Expectation

Other Payoffs

Other Dynamics: Forward prices

Other Dynamics: Dividend-paying stocks

Analytical or Computational

A look ahead

This quarter has focused on analytic solutions. But we do not have simple exact formulas

- when the dynamics of the risk factors are too complicated.

  For example, simple exact formulas are less common:

  …when working with higher-dimensional models,

  …or when outside the class of Gaussian models (BM/GBM)
- when the contract to be priced/hedged is too complicated.

  For example, due to early-exercise or path dependency.

Then we use numerical methods (trees, finite differences, Monte Carlo, Fourier, and reinforcement learning) to evaluate $\mathbb{E}$ or solve PDE.

Next quarter! Spring.