Homework 5

Sergei Tikhonov

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Exercise 1

We need to write the process:

$$Z_t = \exp(W_t^2 - 1)$$

in terms of drift and diffusion components.

(a) Let's start with assuming $X_t = W_t$. Corresponding deterministic function is $f(x) = \exp(x^2 - 1)$. Let's apply Ito's Lemma:

$$\mathrm{d}(\exp(X_t^2 - 1)) = f'_x \mathrm{d}X_t + \frac{1}{2} f''_{xx} (\mathrm{d}X_t)^2 =$$

$$= 2X_t \exp(X_t^2 - 1)\mathrm{d}X_t + \frac{1}{2}(2 \cdot \exp(X_t^2 - 1) + 2X_t \exp(X_t^2 - 1)2X_t)(\mathrm{d}X_t)^2$$

Substituting back $W_t = X_t$:

$$\mathrm{d}(\exp(W_t^2 - 1)) = 2W_t \exp(W_t^2 - 1)\mathrm{d}W_t + (2W_t^2 + 1)\exp(W_t^2 - 1)(\mathrm{d}W_t)^2$$

As a result:

$$\mathrm{d}(\exp(W_t^2 - 1)) = (2W_t^2 + 1)\exp(W_t^2 - 1)\mathrm{d}t + 2W_t \exp(W_t^2 - 1)\mathrm{d}W_t$$

(b) For now, let's assume $X_t = W_t^2 - 1$. We know that $\mathrm{d}X_t = dt + 2W_t \mathrm{d}W_t$ and corresponding deterministic function is $f(x) = \exp(x)$. Let's apply Ito's Lemma again:

$$\mathrm{d}(\exp(X_t)) = f'_x \mathrm{d}X_t + \frac{1}{2} f''_{xx}(\mathrm{d}X_t)^2 = \exp(X_t)\mathrm{d}X_t + \frac{1}{2}\exp(X_t)(\mathrm{d}X_t)^2$$

$$= \exp(W_t^2 - 1)(\mathrm{d}t + 2W_t \mathrm{d}W_t) + \frac{1}{2}\exp(W_t^2 - 1)[(\mathrm{d}t)^2 + 2W_t(\mathrm{d}t)(\mathrm{d}W_t) + 4W_t^2(\mathrm{d}W_t)^2]$$

As a result:

$$\mathrm{d}(\exp(X_t)) = \exp(W_t^2 - 1)(1 + 2W_t^2)\mathrm{d}t + 2W_t \exp(W_t^2 - 1)\mathrm{d}W_t$$

(c) To check whether Z_t is a martingale, one can have a look at drift term. In order for Z_t to be a martingale, it should be equal to 0 with probability 1.

$$\mu_t = \underbrace{\exp(W_t^2 - 1)}_{>0} \underbrace{(1 + 2W_t^2)}_{>0} \neq 0 \quad \text{a.s.}$$

Here we can see that it's not equal to zero, that's why Z_t is not a martingale.

Exercise 2

Let's consider the following stochastic differential equation:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

The corresponding deterministic function is $f(t, x) = \exp(\kappa t)x$. Let's apply Ito's Lemma:

$$\mathrm{d}(\exp(\kappa t)X_t) = f_t'\mathrm{d}t + f_x'\mathrm{d}X_t + \frac{1}{2}f_{tt}''(\mathrm{d}t)^2 + f_{tx}''(\mathrm{d}X_t)(\mathrm{d}t) + \frac{1}{2}f_{xx}''\mathrm{d}X_t$$

$$\mathrm{d}(\exp(\kappa t)X_t) = \kappa\exp(\kappa t)X_t\mathrm{d}t + \exp(\kappa t)\mathrm{d}X_t + \frac{1}{2}\kappa^2\exp(\kappa t)(\mathrm{d}t)^2 + \kappa\exp(\kappa t)(dX_t)(dt) + \frac{1}{2}\cdot 0 \cdot (\mathrm{d}X_t)^2$$

After simplifying the statement above:

$$\mathrm{d}(\exp(\kappa t)X_t) = \kappa\exp(\kappa t)X_t\mathrm{d}t + \exp(\kappa t)(\kappa(\theta - X_t)\mathrm{d}t + \sigma\mathrm{d}W_t)$$

$$\mathrm{d}(\exp(\kappa t)X_t) = \kappa\exp(\kappa t)X_t\mathrm{d}t + \exp(\kappa t)\sigma\mathrm{d}W_t$$

Let's now rewrite the Ito process above using integral notation:

$$\exp(\kappa t)X_t = X_0 + \int_0^t \kappa\theta\exp(\kappa s)\mathrm{d}s + \int_0^t \exp(\kappa s)\sigma\mathrm{d}W_s$$

As a result (here I also substituted s to t just for notation purposes):

$$X_T = \exp(-\kappa T)X_0 + \int_0^T \kappa\theta\exp(\kappa(t - T))\mathrm{d}t + \int_0^T \exp(\kappa(t - T))\sigma\mathrm{d}W_t$$

(b) Here we can calculate the Riemann integral at time moment T:

$$X_T = \exp(-\kappa T)X_0 + \theta(1 - \exp(-\kappa T)) + \int_0^T \exp(\kappa(t - T))\sigma\mathrm{d}W_t$$

By definition, expected value of diffusion part is equal to zero. Then the expected value of X_T is:

$$\mathbb{E}(X_T) = \exp(-\kappa T)X_0 + \theta(1 - \exp(-\kappa T))$$

Using the property Ito isometry, let's calculate the variance of X_T:

$$\mathbb{E}(X_T^2) = (\exp(-\kappa T)X_0 + \theta(1-\exp(-\kappa T)))^2 + \mathbb{E}\left(\int_0^T \exp(\kappa(t-T))\sigma \mathrm{d}W_t\right)^2 =$$

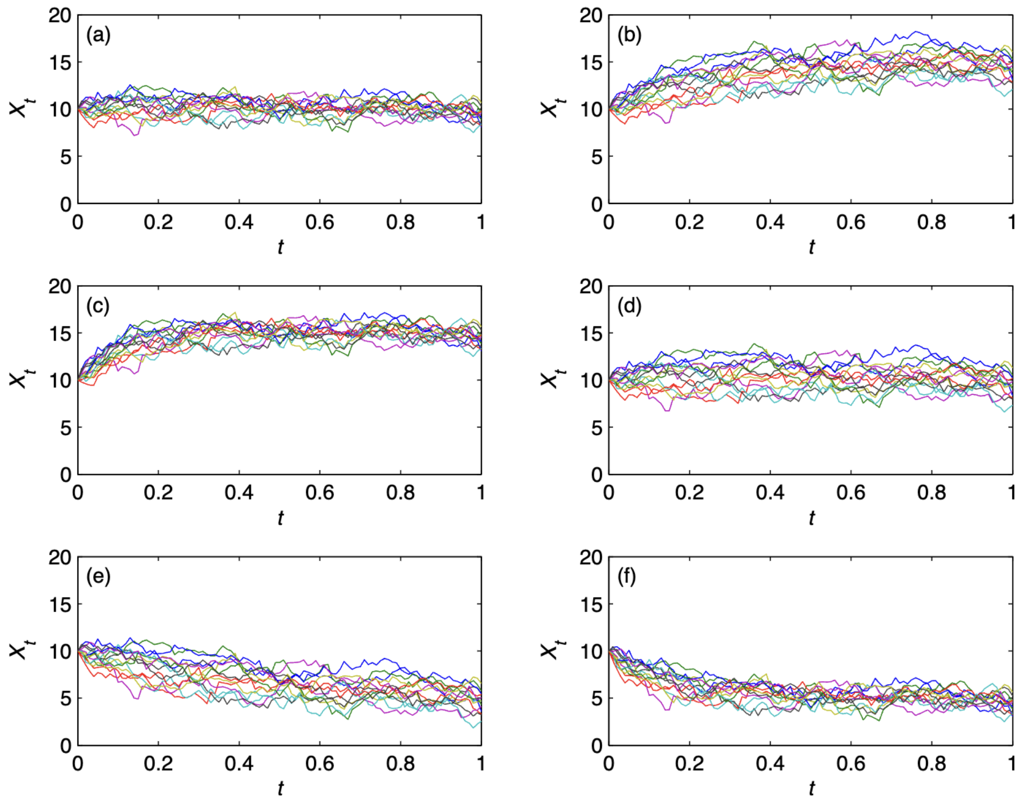
$$= (\exp(-\kappa T)X_0 + \theta(1-\exp(-\kappa T)))^2 + \int_0^T \sigma^2 \exp(2\kappa(t-T))\mathrm{d}t =$$

$$= (\exp(-\kappa T)X_0 + \theta(1-\exp(-\kappa T)))^2 + \frac{\sigma^2}{2\kappa}(1-\exp(-2\kappa T))$$

As a result:

$$\mathbb{V}\mathrm{ar}(X_T) = \mathbb{E}(X_T^2) - (\mathbb{E}(X_T))^2 = \frac{\sigma^2}{2\kappa}(1-\exp(-2\kappa T))$$

Exercise 3



$$(\mathrm{a})\theta = 10,\ k = 8 \qquad (\mathrm{d})\theta = 10,\ k = 3$$

$$(\mathrm{b})\theta = 15,\ k = 3 \qquad (\mathrm{e})\theta = 5,\ k = 3$$

$$(\mathrm{c})\theta = 15,\ k = 8 \qquad (\mathrm{f})\theta = 5,\ k = 8$$