Homework 2

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Exercise 1

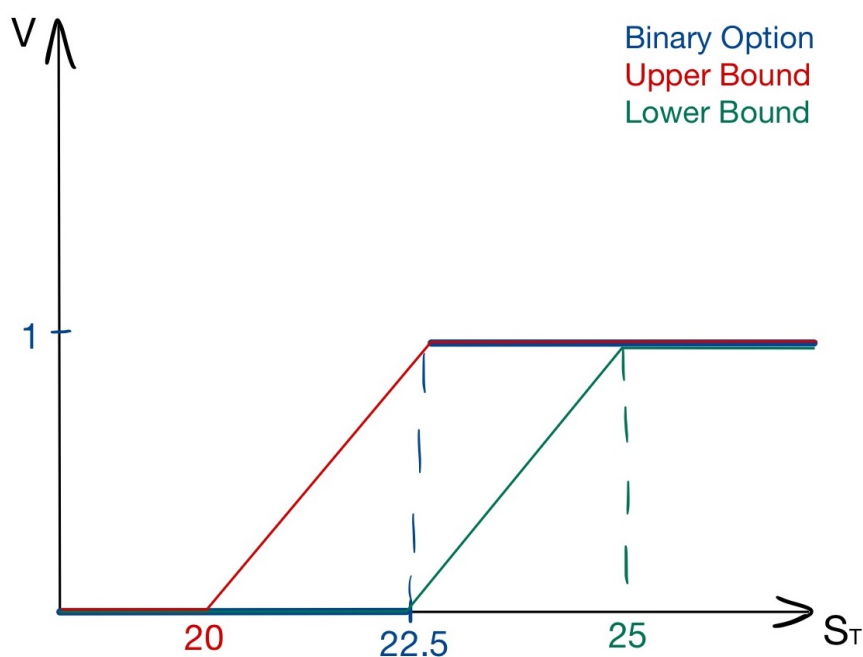(a) Let's construct upper and lower bound for the binary call.

Upper bound:
$$\left(\frac{+1}{22.5 - 20} C(20); \ \frac{-1}{22.5 - 20} C(22.5)\right)$$

Lower bound:
$$\left(\frac{+1}{22.5 - 20} C(22.5); \ \frac{-1}{22.5 - 20} C(25)\right)$$

Graphical representation:

As a result, the bounds are:

$$\frac{4.15}{2.5} - \frac{2.6}{2.5} = 0.62 \leq C_{Bin}(22.5) \leq 0.8 = \frac{6.15}{2.5} - \frac{4.15}{2.5}$$
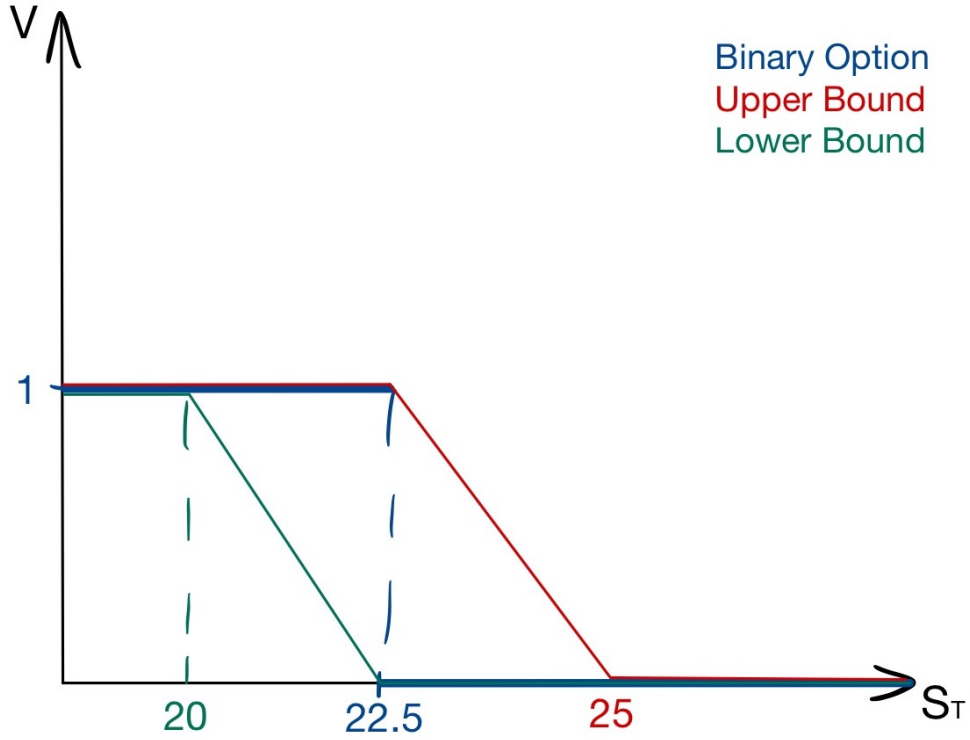
(b) Let's construct upper and lower bound for the binary put.

Upper bound:

$$\left(+1Z; \; \frac{-1}{22.5 - 20} C(22.5); \; \frac{+1}{22.5 - 20} C(25)\right)$$

Lower bound:

$$\left(+1Z; \; \frac{-1}{22.5 - 20} C(20); \; \frac{+1}{22.5 - 20} C(22.5)\right)$$

As a result, the bounds are:

$$0.95 + \frac{4.15}{2.5} - \frac{6.15}{2.5} = 0.15 \leq P_{Bin}(22.5) \leq 0.33 = 0.95 - \frac{4.15}{2.5} + \frac{2.6}{2.5}$$

(c) We have the contract with the following payoff:

$$\begin{cases} 2.5 & S_T \leq 25 \\ S_T - 22.5 & S_T > 25 \end{cases}$$

In this case, we can completely replicate the contract by the following portfolio:

$$\left(+2.5Z; \; \frac{-1}{22.5 - 20}C(20); \; \frac{+1}{22.5 - 20}C(22.5)\right)$$

As a result, the price of the contract is:

$$G_0 = 2 \cdot 0.95 + 2.6 - 4.15 = 0.35$$

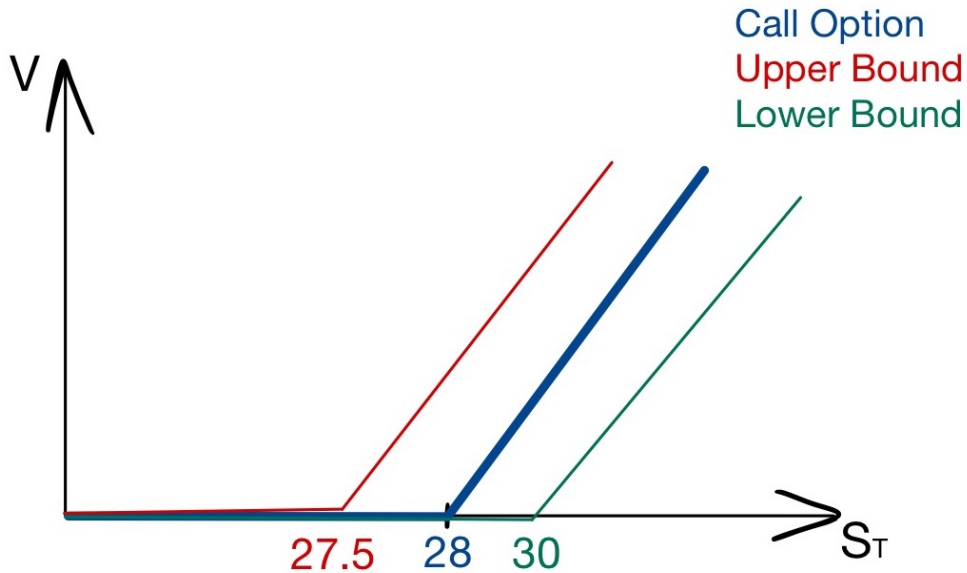(d) Finally, we have vanilla call option with strike 28.

Upper bound:
$$(+1\ C(27.5))$$

Lower bound:
$$(+1\ C(30))$$

The graphical representation:

As a result, the bounds are:
$$0.8 \leq C_0(28) \leq 1.5$$

Exercise 2

(a) We need to prove the following formula:

$$f(s) = f(K_*) + f'(K_*)(s - K_*) + \int_0^{K_*} f''(K)(K - s)^+ dK + \int_{K_*}^{\infty} f''(K)(s - K)^+ dK$$

Suppose $s > K_* > 0$. Then:

$$f(s) = f(K_*) + f'(K_*)(s - K_*) + \underbrace{\int_0^{K_*} f''(K)(K - s)^+ dK}_{=0,\ \text{since}\ s > K_*} + \int_{K_*}^{\infty} f''(K)(s - K)^+ dK$$

Let's consider the right integral under the aforementioned condition ($s > K_* > 0$):

$$\int_{K_*}^{\infty} f''(K)(s - K)^+ dK = \int_{K_*}^{s} f''(K)(s - K)dK =$$

For now, let's apply integration by parts:

$$= (s-K)f'(K)\Big|_{K_*}^{s} - \int_{K_*}^{s}(-1)f'(K)dK = -(s-K_*)f'(K_*) + f(s) - f(K_*)$$

Substituting the integral in the initial formula:

$$f(s) = f(K_*) + f'(K_*)(s-K_*) - (s-K_*)f'(K_*) + f(s) - f(K_*)$$

All the terms eliminate each other, so the formula holds.

Suppose $K_* > s > 0$. Then:

$$f(s) = f(K_*) + f'(K_*)(s-K_*) + \int_{0}^{K_*} f''(K)(K-s)^+ dK + \underbrace{\int_{K_*}^{\infty} f''(K)(S-K)^+ dK}_{=0,\ \text{since } s < K_*}$$

Let's consider the right integral under the aforementioned condition ($K_* > s > 0$):

$$\int_{0}^{K_*} f''(K)(K-s)^+ dK = \int_{s}^{K_*} f''(K)(K-s)^+ dK =$$

For now, let's apply integration by parts:

$$= (K-s)f'(K)\Big|_{s}^{K_*} - \int_{s}^{K_*} f'(K)dK = (K_*-s)f'(K_*) - f(K_*) + f(s) = -(s-K_*)f'(K_*) - f(K_*) + f(s)$$

Substituting the integral in the initial formula:

$$f(s) = f(K_*) + f'(K_*)(s-K_*) - (s-K_*)f'(K_*) - f(K_*) + f(s)$$

All the terms eliminate each other, so the formula holds.

As a result, we prove the formula for both possible cases, so the formula holds.

 (b) Let's consider the integral for put options from the formula above. Let us represent the integral as a Riemann sum:

$$\int_{0}^{K_*} f''(K)(K-s)^+ dK = \sum_{i=1}^{n} f''(K_i)(K_i - s)^+ \Delta K_i$$

Using the formula for "log contracts" :

$$f''(K) = (-2\log K_*)'' = +\frac{2}{K_*^2}$$

Since the strikes are at all the positive integer multiples of 5:

$$K = 5m \qquad m \in \mathbb{N}$$

Then:
$$\Delta K_i = K_i - K_{i-1} = 5m - 5(m-1) = 5$$

Let's substitute the aforementioned expressions to the Riemann sum:

$$\sum_{i=1}^{392} f''(K_i)(K_i - s)^+ \Delta K_i = \sum_{i=1}^{392} \frac{2}{K_i^2}(K_i - s)5 =$$

$$= \frac{2}{K_1^2}5(K_1 - s) + \dots + \frac{2}{K_{390}^2}5(K_{390} - s) + \frac{2}{K_{391}^2}5(K_{391} - s) + \frac{2}{K_{392}^2}5(K_{392} - s)$$

We need to calculate the number of put options at K_{390} (corresponding to the strike 1950):

$$\frac{2}{K_{390}^2}5 = \frac{10}{1950^2} = 0.00000262...$$