Homework 2

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Exercise 1

(a) Let's construct upper and lower bound for the binary call.

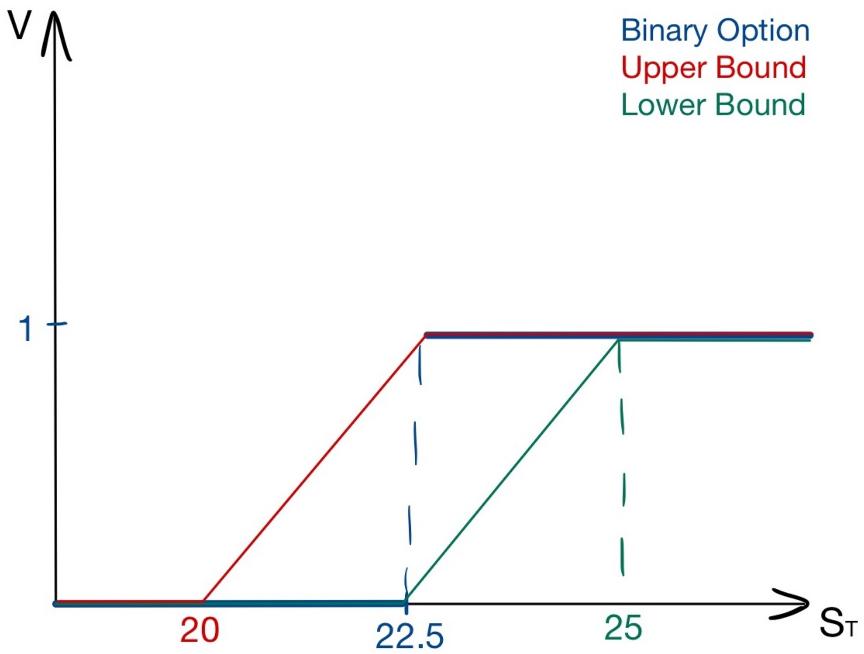
Upper bound:

$$\left(\frac{+1}{22.5-20}\,C(20);\ \frac{-1}{22.5-20}\,C(22.5)\right)$$

Lower bound:

$$\left(\frac{+1}{22.5-20}\,C(22.5);\ \frac{-1}{22.5-20}\,C(25)\right)$$

Graphical representation:



As a result, the bounds are:

$$\frac{4.15}{2.5} - \frac{2.6}{2.5} = 0.62 \leq C_{Bin}(22.5) \leq 0.8 = \frac{6.15}{2.5} - \frac{4.15}{2.5}$$

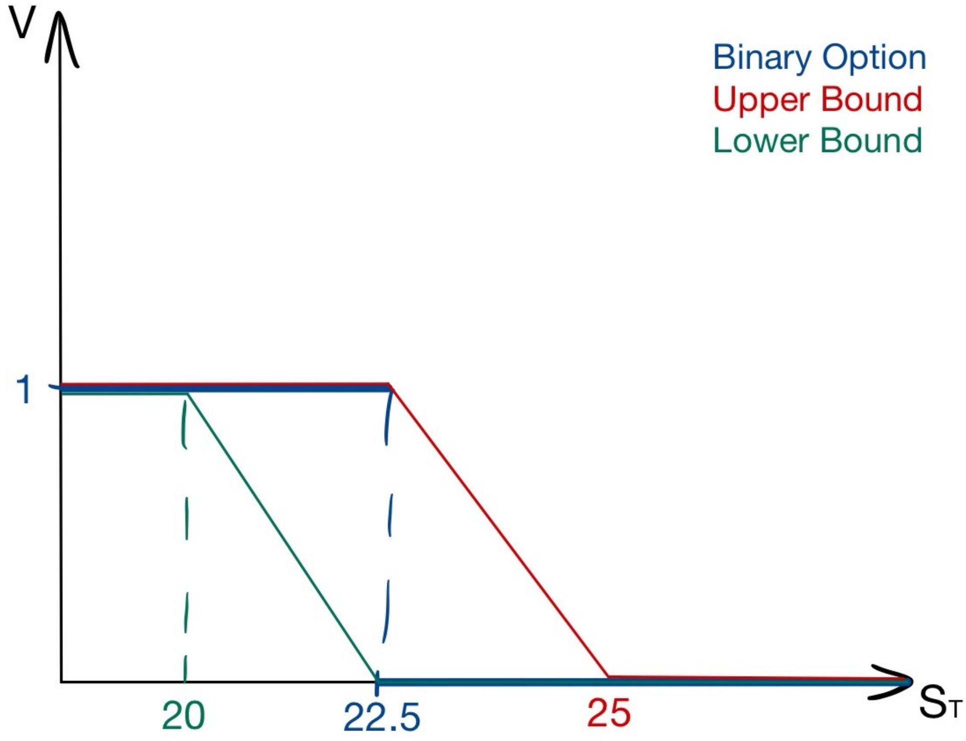
(b) Let's construct upper and lower bound for the binary put.

Upper bound:

$$\left(+1Z;\ \frac{-1}{22.5-20}\,C(22.5);\ \frac{+1}{22.5-20}\,C(25)\right)$$

Lower bound:

$$\left(+1Z;\ \frac{-1}{22.5-20}\,C(20);\ \frac{+1}{22.5-20}\,C(22.5)\right)$$



As a result, the bounds are:

$$0.95+\frac{4.15}{2.5}-\frac{6.15}{2.5}=0.15\leq P_{Bin}(22.5)\leq 0.33=0.95-\frac{4.15}{2.5}+\frac{2.6}{2.5}$$

(c) We have the contract with the following payoff:

$$\begin{cases} 2.5 & S_T \leq 25 \\ S_T-22.5 & S_T>25 \end{cases}$$

In this case, we can completely replicate the contract by the following portfolio:

$$(\ +2.5Z;\ C(25)\ )$$

As a result, the price of the contract is:

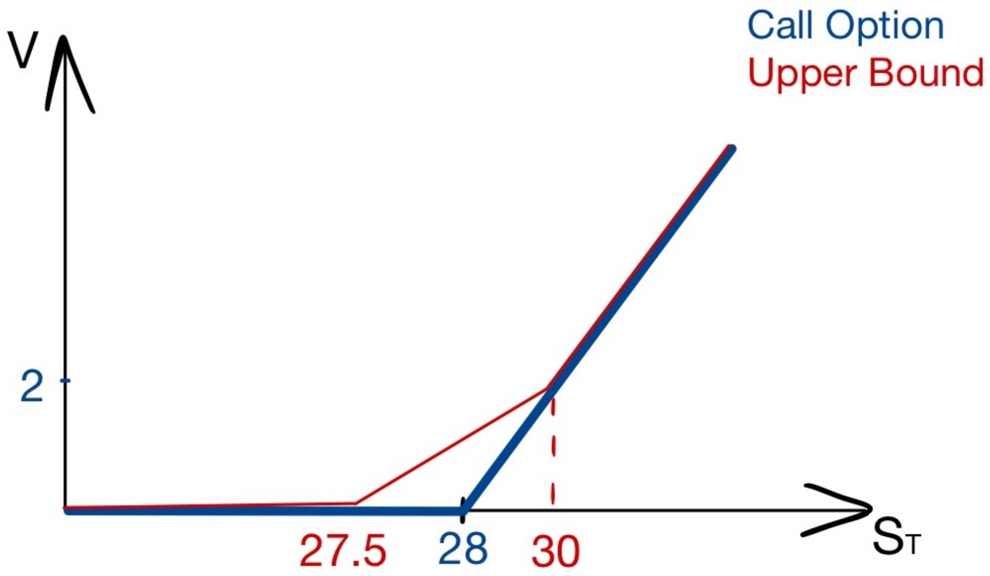
$$G_0=2\cdot 0.95+2.6=4.5$$

(d) Finally, we have vanilla call option with strike 28. Let's recall the fact that we can use call options to adjust convexity, namely let's consider the portfolio:

$$\left(\frac{30-28}{30-27.5}C(27.5);\ \frac{28-27.5}{30-27.5}C(30)\right)$$

If $S_T < K_1 = 27.5$ both options are out of money, we will not exercise both options since they are worthless.
If $27.5 = K_1 \leq S_T \leq K_2 = 30$, we will exercise only $C(27.5)$ option.
If $K_2 > 30$, we will exercise both options since they are both in money.
This portfolio *superreplicates* the option with strike $K = 28$. As a result, the upper bound is:

$$C_0(28) \leq \frac{(30-28)C(27.5)+(28-27.5)C(27.5)}{30-27.5} = \frac{2\cdot 1.5 + 0.5\cdot 0.8}{2.5} = 1.36$$



Exercise 2

(a) We need to prove the following formula:

$$f(s) = f(K_*) + f'(K_*)(s-K_*) + \int_0^{K_*} f''(K)(K-s)^+dK + \int_{K_*}^{\infty} f''(K)(s-K)^+dK$$

Suppose $s > K_* > 0$. Then:

$$f(s) = f(K_*) + f'(K_*)(s - K_*) + \underbrace{\int_0^{K_*} f''(K)(K - s)^+ dK}_{=0,\text{ since } s>K_*} + \int_{K_*}^{\infty} f''(K)(s - K)^+ dK$$

Let's consider the right integral under the aforementioned condition ($s > K_* > 0$):

$$\int_{K_*}^{\infty} f''(K)(s - K)^+ dK = \int_{K_*}^{s} f''(K)(s - K) dK =$$

For now, let's apply integration by parts:

$$= (s - K)f'(K)\Big|_{K_*}^{s} - \int_{K_*}^{s} (-1)f'(K)dK = -(s - K_*)f'(K_*) + f(s) - f(K_*)$$

Substituting the integral in the initial formula:

$$f(s) = f(K_*) + f'(K_*)(s - K_*) - (s - K_*)f'(K_*) + f(s) - f(K_*)$$

All the terms eliminate each other, so the formula holds.
Suppose $K_* > s > 0$. Then:

$$f(s) = f(K_*) + f'(K_*)(s - K_*) + \int_0^{K_*} f''(K)(K - s)^+ dK + \underbrace{\int_{K_*}^{\infty} f''(K)(S - K)^+ dK}_{=0,\text{ since } s<K_*}$$

Let's consider the right integral under the aforementioned condition ($K_* > s > 0$):

$$\int_0^{K_*} f''(K)(K - s)^+ dK = \int_s^{K_*} f''(K)(K - s)^+ dK =$$

For now, let's apply integration by parts:

$$= (K-s)f'(K)\Big|_s^{K_*} - \int_s^{K_*} f'(K)dK = (K_*-s)f'(K_*) - f(K_*) + f(s) = -(s-K_*)f'(K_*) - f(K_*) + f(s)$$

Substituting the integral in the initial formula:

$$f(s) = f(K_*) + f'(K_*)(s - K_*) - (s - K_*)f'(K_*) - f(K_*) + f(s)$$

All the terms eliminate each other, so the formula holds.
As a result, we prove the formula for both possible cases, so the formula holds.
(b) Let's consider the integral for put options from the formula above. Let us represent the integral

as a Riemann sum:

$$\int_0^{K_*} f''(K)(K-s)^+ dK = \sum_{i=1}^{n} f''(K_i)(K_i - s)^+ \Delta K_i$$

Using the formula for "log contracts" :

$$f''(K) = (-2\log K_*)'' = +\frac{2}{K_*^2}$$

Since the strikes are at all the positive integer multiples of 5:

$$K = 5m \qquad\qquad m \in \mathbb{N}$$

Then:

$$\Delta K_i = K_i - K_{i-1} = 5m - 5(m-1) = 5$$

Let's substitute the aforementioned expressions to the Riemann sum:

$$\sum_{i=1}^{392} f''(K_i)(K_i - s)^+ \Delta K_i = \sum_{i=1}^{392} \frac{2}{K_i^2}(K_i - s)^+ 5 =$$

$$= \frac{2}{K_1^2}5(K_1 - s) + ... + \frac{2}{K_{390}^2}5(K_{390} - s) + \frac{2}{K_{391}^2}5(K_{391} - s) + \frac{2}{K_{392}^2}5(K_{392} - s)$$

We need to calculate the number of put options at K_{390} (corresponding to the strike 1950):

$$\frac{2}{K_{390}^2}5 = \frac{10}{1950^2} = 0.00000262...$$