Homework 4

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Exercise 1

(a) Suppose C_t is value of call option and V_t - value or replicating portfolio:

$$V_t = \Theta_t \cdot X_t = \begin{cases} 0 \cdot S_t + 0 \cdot 1 & \text{if } S_t \leq K \\ S_t \cdot 1 - K \cdot 1 & \text{if } S_t > K \end{cases}$$

The proof provided in the task shows that since this trading strategy matches the call payoff with probability 1, and since $V_0 = 0 \cdot 100 + 0 \cdot 1 = 0$, it means that $C_0 = V_0 = 0$ (the price at time moment 0 of the call option is equal to 0).

I will show both intuitive and mathematical approaches two emphasizing the flaw.

Intuitive Approach: At time moment 0, option is indeed "out of money" but since we work in multiperiod model, we can notice that it is possible to reach "in the money" if stock price grows enough. In the proof, it turns out the price of such an option is 0 at time moment 0, so we see the presence of arbitrage.

Mathematical Approach: let's elaborate the idea of arbitrage designed above. Indeed, if $V_0 = 0$ then by definition there is an 1st type arbitrage:

$$V_0 = 0 \quad \text{and} \quad \begin{cases} \mathbb{P}\{V_T \geq 0\} = 1 \\ \mathbb{P}\{V_T > 0\} > 0 \end{cases}$$

Since arbitrage exists, the Law of One Price is violated. It means that:

$$\mathbb{P}\{V_T = C_T\} = 1 \nRightarrow V_0 = C_0$$

That's why even though $V_0 = 0$, it doesn't follow that $C_0 = 0$.

(b) We will start with finding risk neutral probabilities. Let's, for example, consider the node in which $S_{11} = 111$. What we have is the following condition:

$$S_{11} = q \cdot S_{12}(u) + (1 - q) \cdot S_{12}(d)$$

$$111 = q \cdot 112 + (1 - q) \cdot 110$$

The only solution is $q = \frac{1}{2}$. Since the stochastic process for the stock S_t at each time moment goes either $+1$ or -1, we will have the very same solution q for all nodes. Just for clarification, let's show a few option prices. For the previous node:

$$C_{11} = \frac{1}{2} \cdot 7 + \frac{1}{2} \cdot 5 = 6$$

For out of money option at time $t = 11$ when $S_{12}(u) = 104$ and $S_{12}(d) = 102$:

$$C_{11} = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$$

In this situation, the call option will never reach positive state, so its price is 0.

Let's then apply the formula that time 0 price of option is discounted expected value of future price under $\mathbb{Q}$ measure:

$$C_0 = \frac{1}{(1+r)^T} \mathbb{E}_{\mathbb{Q}}(C_T)$$

Taking into account that $r = 0$, let's write the formula:

$$C_0 = \mathbb{E}_{\mathbb{Q}}(C_T) = \sum_{i=0}^{n=12} \binom{12}{i} q^i (1-q)^{12-i} \max(S_T(i) - K, 0)$$

where i denotes the number of up movements, thus $S_T(i)$ is stock price at time T corresponding to i up movements.

Since we have random walk that starts from $S_0 = 0$, at $T = 12$ we can reach only even price. Hence, the explicit formula is:

$$C_0 = \binom{12}{12} \cdot \left(\frac{1}{2}\right)^{12} \cdot 112 + \binom{12}{11} \cdot \left(\frac{1}{2}\right)^{11} \cdot \left(1 - \left(\frac{1}{2}\right)\right) \cdot 110 + \binom{12}{10} \cdot \left(\frac{1}{2}\right)^{10} \cdot \left(1 - \left(\frac{1}{2}\right)\right)^2 \cdot 108 +$$

$$+ \binom{12}{9} \cdot \left(\frac{1}{2}\right)^9 \cdot \left(1 - \left(\frac{1}{2}\right)\right)^3 \cdot 106 =$$

$$= \left(\frac{1}{2}\right)^{12} \left(\binom{12}{12} \cdot 112 + \binom{12}{11} \cdot 110 + \binom{12}{10} \cdot 108 \right)$$

Exercise 2

Let's solve this task using cents. It means that:

$$S_0 = 216¢$$

(a) We have the following probabilities for $S_t = S_{t-1} + X_t$:

$$\mathbb{P}\{X_t = 1\} = 1 - \mathbb{P}\{X_t = -1\} = 0.5$$

Let's specify a stopping time for the task:

$$T = \min\{S_T = 200 \text{ or } S_T = 300\}$$

Then we need to find the following probability:

$$\mathbb{P}\{S_T = 300\}$$

To start with, let's prove that S_t is a martingale:

$$\mathbb{E}(S_{t+1}|\mathcal{F}_t) = \mathbb{E}(S_t + X_{t+1}|\mathcal{F}_t) = S_t + \mathbb{E}(X_{t+1}) = S_t$$

Let's apply the Optional Sampling Theorem (note that $S_{t\wedge T}$ is bounded between 200 and 300):

$$\mathbb{E}(S_T) = \mathbb{E}(S_0) = 216$$

$$\mathbb{E}(S_T) = \mathbb{P}\{S_T = 300\} \cdot 300 + \mathbb{P}\{S_T = 200\} \cdot 200 = 216$$

$$\mathbb{P}\{S_T = 300\} \cdot 300 + (1 - \mathbb{P}\{S_T = 300\}) \cdot 200 = 216$$

$$\mathbb{P}\{S_T = 300\} = \frac{16}{100} = 0.16$$

(b) For now, we have the following probabilities for $S_t = S_{t-1} + X_t$:

$$\mathbb{P}\{X_t = 1\} = 1 - \mathbb{P}\{X_t = -1\} = u > \frac{1}{2}$$

We need to find $0 < A < 1$ such that M_t is a martingale.

$$\mathbb{E}(M_{t+1}|\mathcal{F}_t) = \mathbb{E}(A^{S_{t+1}}|\mathcal{F}_t) = A^{S_t}\mathbb{E}(A^{X_{t+1}}|\mathcal{F}_t) = A^{S_t}(u \cdot A + (1-u) \cdot A^{-1}) = A^{S_t}$$

We need to solve the following equation considering u as a parameter:

$$u \cdot A + (1-u) \cdot A^{-1} = 1$$

This is a quadratic equation:

$$u \cdot A^2 - A + 1 - u = 0$$

The discriminant is equal to:

$$D = 1 - 4 \cdot u \cdot (1-u) = (1-2u)^2$$

The roots of the equation are:

$$A_{1,2} = \frac{1 \pm (1-2u)}{2u}$$

$$A = \frac{1-u}{u} \in (0,1) \quad \text{since } u > \frac{1}{2}$$

or

$$A = 1 \notin (0,1)$$

As a result:

$$A = \frac{1-u}{u}$$

(c) We can apply the Optional Sampling Theorem again, but this time for the martingale M_t. This martingale is also bounded: $A^{200} \leq M_{t\wedge T} \leq A^{300}$:

$$\mathbb{E}(M_T) = \mathbb{E}(M_0) = A^{216}$$

$$\mathbb{E}(S_T) = \mathbb{P}\{S_T = 300\} \cdot A^{300} + \mathbb{P}\{S_T = 200\} \cdot A^{200} = A^{216}$$

As a result:

$$\mathbb{P}\{S_T = 300\} = \frac{A^{216} - A^{200}}{A^{300} - A^{200}} = \frac{A^{200} - A^{216}}{A^{200} - A^{300}}$$