Homework 3

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Exercise 1

(a) Let's write the system of equations in which the first equation shows that a stock is a martingale, the second equation shows that an option is a martingale, the third equation that is normalizing condition (for strictly positive p_d, p_m, p_u):

$$\begin{cases} S_0 = \frac{1}{1.2}(60 \cdot p_d + 120 \cdot p_m + 240 \cdot p_u) \\ C_0 = \frac{1}{1.2}(30 \cdot p_d + 30 \cdot p_m + 0 \cdot p_u) \\ p_d + p_m + p_u = 1 \end{cases}$$

$$\begin{cases} 145 = 50 \cdot p_d + 100 \cdot p_m + 200 \cdot p_u \\ 11 = 30 \cdot p_d + 20 \cdot p_m \\ p_d + p_m + p_u = 1 \end{cases} \qquad \begin{cases} 29 = 10 \cdot p_d + 20 \cdot p_m + 40 \cdot p_u \\ 11 = 30 \cdot p_d + 20 \cdot p_m \\ p_d + p_m + p_u = 1 \end{cases}$$

The solution of the system of equations is:

$$\begin{pmatrix} p_d & p_m & p_u \end{pmatrix} = \begin{pmatrix} \frac{6}{10} & \frac{1}{10} & \frac{3}{10} \end{pmatrix}$$

Since there exists a unique equivalent martingale measure, according to the Second Fundamental Theorem, the market $\{B, S, C\}$ is complete.

(b) Let's try to replicate the payoff X_T using $\{B, S, C\}$:

$$\begin{cases} \alpha \cdot 240 + \beta \cdot 1.2 + \gamma \cdot 0 = 120 \\ \alpha \cdot 120 + \beta \cdot 1.2 + \gamma \cdot 24 = 60 \\ \alpha \cdot 60 + \beta \cdot 1.2 + \gamma \cdot 36 = 0 \end{cases}$$

The only solution of the aforementioned system of equations is:

$$\Theta = \begin{pmatrix} 1 & -100 & 2 \end{pmatrix}$$

So, we've shown that it is possible to replicate the payoff X_T by proving that:

$$\begin{pmatrix} 120 \\ 60 \\ 0 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 240 \\ 120 \\ 60 \end{pmatrix}, \begin{pmatrix} 1.2 \\ 1.2 \\ 1,2 \end{pmatrix}, \begin{pmatrix} 0 \\ 30 \\ 30 \end{pmatrix} \right\}$$

(c) Let's start with replicating portfolio. Applying the Law of One Price (LOOP), we can find no-arbitrage price of contract X_0:

$$X_0 = 145 \cdot 1 + 1 \cdot (-100) + 2 \cdot 10 = 65$$

For now, let's use the pricing probabilities:

$$X_0 = \frac{1}{1.2} \left(\frac{6}{10} \cdot 120 + \frac{1}{10} \cdot 60 + \frac{3}{10} \cdot 0 \right) = 65$$

The prices are equal, so we are happy!

Exercise 2

(a) Let's write the system of equations in which the first equation shows that a stock is a martingale, the second equation shows that an option is a martingale, the third equation that is normalizing condition (for strictly positive p_d, p_m, p_u):

$$\begin{cases} S_0 = \frac{1}{1.2}(60 \cdot p_d + 120 \cdot p_m + 240 \cdot p_u) \\ C_0 = \frac{1}{1.2}(36 \cdot p_d + 24 \cdot p_m + 0 \cdot p_u) \\ p_d + p_m + p_u = 1 \end{cases}$$

$$\begin{cases} 145 = 50 \cdot p_d + 100 \cdot p_m + 200 \cdot p_u \\ 11 = 20 \cdot p_m + 30 \cdot p_d \\ p_d + p_m + p_u = 1 \end{cases} \qquad \begin{cases} 29 = 10 \cdot p_d + 20 \cdot p_m + 40 \cdot p_u \\ 11 = 20 \cdot p_m + 30 \cdot p_d \\ p_d + p_m + p_u = 1 \end{cases}$$

The aforementioned system does not have solutions.

Since there is no an equivalent martingale measure, according to the Second Fundamental Theorem, the market $\{B, S, C\}$ is incomplete.

(b) Let's try to replicate the payoff X_T using $\{B, S, C\}$:

$$\begin{cases} \alpha \cdot 240 + \beta \cdot 1.2 + \gamma \cdot 0 = 120 \\ \alpha \cdot 120 + \beta \cdot 1.2 + \gamma \cdot 24 = 60 \\ \alpha \cdot 60 + \beta \cdot 1.2 + \gamma \cdot 36 = 0 \end{cases}$$

The aforementioned system does not have solutions.

So, we've shown that it is not possible to replicate the payoff X_T by proving that:

$$\begin{pmatrix} 120 \\ 60 \\ 0 \end{pmatrix} \notin \text{span} \left\{ \begin{pmatrix} 240 \\ 120 \\ 60 \end{pmatrix}, \begin{pmatrix} 1.2 \\ 1.2 \\ 1,2 \end{pmatrix}, \begin{pmatrix} 0 \\ 24 \\ 36 \end{pmatrix} \right\}$$

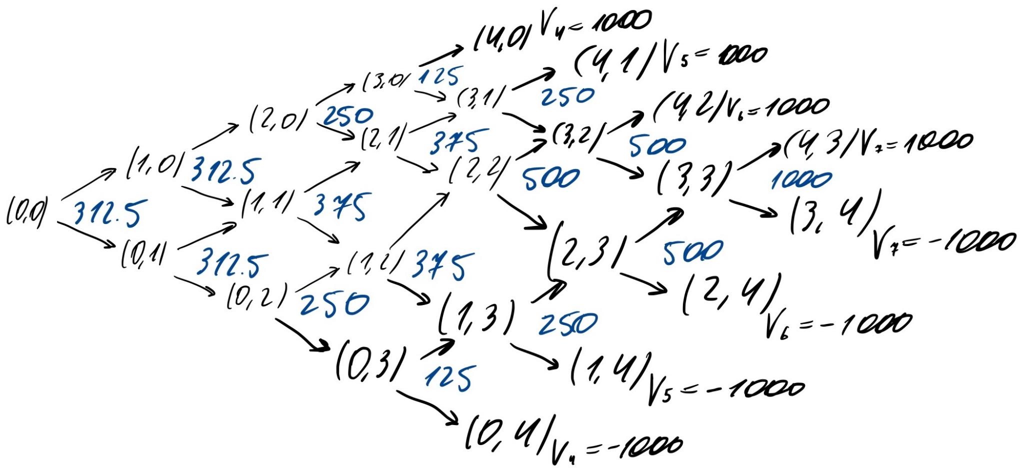
(c) The answer to (b) is no, so we cannot use replication and risk neutral probabilities to find X_0.

Exercise 3

We will work with the contact "White Sox wins the nth game":

$$W = \begin{cases} +1 & \text{White Sox wins the nth game: } W_u \\ -1 & \text{White Sox loses the nth game: } W_d \end{cases}$$

The blue color indicates a bet at each state:



How to come up with these numbers? Let's start from the state (3-3):

$$\begin{cases} \alpha B_t + \beta W_u = 1000 \\ \alpha B_t + \beta W_d = -1000 \end{cases}$$

After subtracting the second equations from the first one:

$$\beta(W_u - W_d) = 2000 \qquad \beta = \frac{2000}{W_u - W_d} = \frac{2000}{2} = 1000 \qquad \alpha = 0$$

We've obtained the expected result: in the last game we bet on win and either win \$1000 or lose \$1000. What about other states? We can continue this procedure iteratively for each node, and we can quickly notice that each previous bet is equal to the average of two following bets.

For additional clarification, let's consider the case when the capper bets with negative balance.

Let's, for example, consider the game (0-1):

$$\begin{cases} \alpha B + \beta W_u = 0 \\ \alpha B + \beta W_d = -625 \end{cases}$$

The solution is:

$$\beta = \frac{625}{2} = 312.5 \qquad \alpha = -312.5$$

Here the capper uses a banking account, which works under zero interest rate. If the capper lost a series, he would owe \$1000 to casino. Otherwise he would acquire \$1000.

As a result, the bet at the time period 0 is equal to 312.5.