Financial Mathematics 33000

Lecture 1

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Introduction

General properties of arbitrage-free prices

General properties of forwards and options

FINM 33000 and FINM 32000

I teach this course and its sequel.

- Fall 2022: Option pricing theory
- Winter or Spring 2023: Computational option pricing
- "Option pricing" is meant in a broad sense: the pricing and hedging of options and other financial *derivative contracts*
- A *derivative security* or *derivative contract* is a financial instrument whose payoff is defined in terms of an *underlying* (e.g.: A asset such as a stock or bond. An index. An interest rate.)

GOOG

call option on GOOG

The main idea

I quote from Björk's book:

- A financial derivative is defined *in terms of* some underlying asset which already exists on the market
- A derivative cannot therefore be priced arbitrarily *in relation to* the underlying prices if we want to *avoid mispricing between the derivative and the underlying price.*
- We thus want to price the derivative in a way that is *consistent* with the underlying prices given by the market.
- We are not trying to compute the price of the derivative in some "absolute" sense. The idea instead is to determine the price of the derivative *in terms of the market prices of the underlying assets.*

We assume frictionless markets

We will specify a set of basic tradeable assets and a set of times.

At any such time, each basic tradeable asset has a market price, and we can buy/sell/hold arbitrary quantities at that time, at that price.

In other words, assume *frictionless markets*. In particular,

- No transaction costs: no commissions, no fees, no taxes
- No bid-ask spread. No slippage. No market impact
- No default risk. No counterparty risk
- No margin constraints
- Can hold fractional quantities of assets
- Can sell what you do not own (*sell short* or *go short* or *short*), and hold a negative quantity (a *short* position).

Introduction

General properties of arbitrage-free prices

General properties of forwards and options

Assets

- The market has risks described by a probability measure P.
- It includes N tradeable *assets* with nonrandom time-0 prices

  $$\mathbf{X}_0 := (X_0^1, \ldots, X_0^N)$$

  and random time-T prices ("payoffs")

  $$\mathbf{X}_T := (X_T^1, \ldots, X_T^N)$$

  No distinction between final payment X_T vs. final asset price X_T.
- In this section, no assumptions about which times $t \in (0, T)$ exist in our market. Our general analysis applies to a one-period model (which includes only 0 and T), continuous-time model (which includes all of $t \in [0, T]$), and any intermediate model.

Examples of tradeable assets

- A *zero-coupon bond* or *discount bond* with maturity T:

  Each unit pays at time T a fixed payoff, let us say $Z_T = 1$.
- A non-dividend-paying *stock*: Each unit has time-t price $S_t \geq 0$.

  Can think of stock as a claim on a time-T random payoff $S_T \geq 0$.
- A *bank account* or *money market acct*: Each unit has time-t price

$$B_t := \exp\left(\int_0^t r_u \mathrm{d}u\right). \quad \text{If } r \text{ is constant}, B_t = e^{rt}$$

  for some (possibly random) r_u, called the time-u *instantaneous spot rate* of interest or *short rate*. Note: B solves the diff eq

$$\frac{\mathrm{d}B_t}{\mathrm{d}t} = r_t B_t \qquad \text{with } B_0 = 1.$$

  Can think of bank account as having time-T payoff $\exp(\int_0^T r_t \mathrm{d}t)$.

Static portfolios

- A *static portfolio* is a vector of quantities

$$\boldsymbol{\Theta} := (\theta^1, \ldots, \theta^N)$$

where each θ is nonrandom and constant in time.

Each θ^n denotes the number of units of asset n, for $n = 1, \ldots, N$.

If $\theta^n > 0$ we say the portfolio is *long* asset n.

If $\theta^n < 0$ we say the portfolio is *short* asset n.

- The time-t *value* of portfolio $\boldsymbol{\Theta}$ is

$$V_t := \boldsymbol{\Theta} \cdot \mathbf{X}_t = \theta^1 X_t^1 + \cdots + \theta^N X_t^N$$

If we are dealing with multiple portfolios, we may give V a superscript to indicate which portfolio.

Arbitrage: common-language definition

Arbitrage is a combination of transactions that tries to profit from price inconsistencies. Examples:

- A stock is being bid at venue A for a higher price than it is being offered at venue B. Buy it at site B, sell it at site A.
- Asset F is a combination of assets G and H, but is priced lower than the sum of the constituent prices. Buy F, sell G, sell H.
- A combination of assets is underpriced or overpriced relative to a "fair" or "predicted" value from a statistical model.
  Buy or sell that combination.

In common language, "arbitrage" may involve risk of loss.

Arbitrage: mathematical definition

A static portfolio $\boldsymbol{\Theta}$ is a "type 1" *arbitrage* if its value V satisfies

$$V_0 = 0 \quad \text{and both:} \quad \begin{array}{l} \mathsf{P}(V_T \geq 0) = 1 \\ \mathsf{P}(V_T > 0) > 0 \end{array}$$

(Zero initial investment, and no risk of loss, some chance of gain.)

A static portfolio $\boldsymbol{\Theta}$ is a "type 2" *arbitrage* if its value V satisfies

$$V_0 < 0 \quad \text{and} \quad \mathsf{P}(V_T \geq 0) = 1$$

(Initially receive a credit . . . which you will definitely not repay.)

A static portfolio $\boldsymbol{\Theta}$ is an *arbitrage* if it's either a type 1 or type 2 arbitrage.

Arbitrage

- Prices which admit arbitrage are, in some sense, incorrect.

  Existence of arbitrage is a severe form of the inconsistency and mispricing that we want to avoid.
- Assume no arbitrage, unless otherwise indicated.

  Thus, when we try to price some asset, we are looking for an *arbitrage-free* price.
- Some authors define arbitrage without “type 2”.

  The distinction between our definition (type 1 or type 2) and their definition (type 1 only) is essentially immaterial, because:

  If there exists an asset whose price is always nonnegative and not always zero, then type 1 arb exists whenever type 2 arb exists.

Examples

A portfolio is *not* an arbitrage if its value satisfies:

- $V_0 = 0$, and $\mathsf{P}(V_T = 50) = 0.99$, $\mathsf{P}(V_T = -5) = 0.01$.

  If there's any chance of loss, then it's not an arbitrage.

- $V_0 = 1$, and $V_T = 2$ with probability 1.

  By definition, $V_0 = 1$ implies the portfolio is not an arbitrage.

  Initial investment is required to buy this portfolio.

- $V_0 = -2$, and $V_T = -1$ with probability 1.

  This is not an arb because $V_T = -1$. Receiving 2 initially, then later paying only 1, does *not* necessarily lock in a gain. Because, without assumptions about interest rates, we don't know whether the initial 2 can be parked in an asset worth at least 1 at time T.

Example

- Suppose that assets F, G, H satisfy
  - $F_T \geq G_T + H_T$ with probability 1.
  - $F_0 < G_0 + H_0$

  120 50 80

  Then

  $$(1 \text{ unit of } F, -1 \text{ unit of } G, -1 \text{ unit of } H)$$

  is a type-2 arbitrage. Get \$10 at beginning..... never pay back
- Example: $F_T = G_T + H_T$ where F=bicycle, G=wheels, H=frame

Type 1: (1 unit of F, -1 unit G, -1 unit H, 10 units of B)

Pay \$0 at beginning ...... Get $10e^{rT}$ at end

Absent arbitrage, prices satisfy consistency conditions

Suppose portfolio $\boldsymbol{\Theta}^a$ *superreplicates* portfolio $\boldsymbol{\Theta}^b$, which means that $\mathsf{P}(V_T^a \geq V_T^b) = 1$. Then $V_0^a \geq V_0^b$, otherwise arbitrage exists.

Proof.

If instead $V_0^a < V_0^b$, then construct portfolio $\boldsymbol{\Theta} := \boldsymbol{\Theta}^a - \boldsymbol{\Theta}^b$.

(In other words, go *long* $\boldsymbol{\Theta}^a$ and *short* $\boldsymbol{\Theta}^b$.)

Its time-0 value is $V_0 = \boldsymbol{\Theta} \cdot \mathbf{X}_0 = \boldsymbol{\Theta}^a \cdot \mathbf{X}_0 - \boldsymbol{\Theta}^b \cdot \mathbf{X}_0 = V_0^a - V_0^b < 0$.

Its time-T value is $V_T = V_T^a - V_T^b \geq 0$ with probability 1.

Hence $\boldsymbol{\Theta}$ is an arbitrage. $\square$

In this proof, we used a general technique for constructing arbitrage:

- Go long what is cheap (undervalued), and short what is rich (overvalued). In other words: buy low, sell high.

The law of one price

Likewise, if $\boldsymbol{\Theta}^a$ *subreplicates* $\boldsymbol{\Theta}^b$, meaning $\mathsf{P}(V_T^a \leq V_T^b) = 1$, then $V_0^a \leq V_0^b$. By combining the two inequalities, therefore,

$$\boxed{\text{If } \mathsf{P}(V_T^a = V_T^b) = 1, \text{ then } V_0^a = V_0^b.}$$

In other words, if $\boldsymbol{\Theta}^a$ *replicates* $\boldsymbol{\Theta}^b$, then $V_0^a = V_0^b$.

- This is the *law of one price*. Any two static portfolios with identical future payouts must have identical current prices.
- "You can summarize the essence of quantitative finance," according to Emanuel Derman, as follows:
  "If you want to know the value of a security, use the price of another security [or *portfolio* of securities] that's as similar to it as possible."

Price vs Value vs Payoff

- Time-t *price* = how much it costs to buy/sell something at time t. (Exceptions: "forward price", "futures price")
- Time-t *value* = how much it *should* cost to buy/sell something

Meaning of "should" depends on the context. In this course, the only notion of "should" is that arbitrage should not exist. So for us, "value" is what it costs to buy/sell something, in the absence of arbitrage. But since we have a standing assumption of no-arbitrage, we really have no distinction between price and value, *unless* we are in a situation where arbitrage exists (e.g. HW: "find an arbitrage").

- Payoff = Payout = how much a contract pays = Value of the contract *at expiration* (assuming single payment)

Introduction

General properties of arbitrage-free prices

General properties of forwards and options

Discount bond: valuation

Consider a discount bond Z maturing at T, and a bank account B.

If interest rate r_t is non-random, then

$B_T = 1.10$

$$Z_0 = 1/B_T$$

$Z_0 = \frac{1}{1.1} \approx 0.91$

Therefore $Z_0 = e^{-\int_0^T r_t \mathrm{d}t} = e^{-rT}$ if r is constant.

$B_T = e^{rT} \Rightarrow Z_0 = \frac{1}{B_T} = e^{-rT}$

Proof.

A portfolio consisting of $1/B_T$ units of the bank account has time-T value $(1/B_T) \times B_T = 1$, which is identical to $Z_T = 1$.

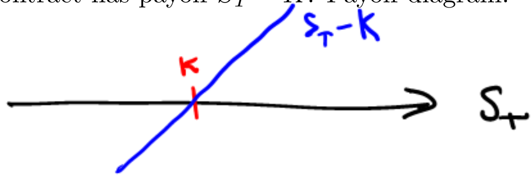
- In particular, if r is constant, then 1 unit of bank is identical to e^{rT} bonds, and 1 bond is identical to e^{-rT} units of the bank acct.

So the portfolios must have equal time-0 values: $Z_0 = (1/B_T) \times 1$. □

Forward contract: definition

Consider a random variable S_T whose value is revealed at time T.

- A *forward contract* on S_T with maturity / delivery date T and nonrandom delivery price K obligates the holder to, at time T, pay K and receive S_T (dollars if "cash" settled. If "physical" settlement, you get an asset, whose time-T price we denote S_T.)
- So the forward contract has payoff $S_T - K$. Payoff diagram:



- Forward contract is an example of a *derivative* – a security whose payout is contractually related to some *underlying* variable.

Forward contract: valuation

Consider a forward contract on a non-dividend-paying stock S, with delivery date T and any delivery price K. Then the time-0 value of the forward contract is $S_0 - KZ_0$.

Proof.

The portfolio

maturity T

$Z_T = 1$

$$\mathbf{\Theta} = (1 \text{ share}, \ -K \text{ bonds})$$

has time-T value $V_T = \mathbf{\Theta}_T \cdot \mathbf{X}_T = 1 \times S_T - K \times 1 = S_T - K$.

The forward contract also has time-T value $S_T - K$.

So the time-0 value of the forward contract must equal the time-0 value of the replicating portfolio, which is

$$V_0 = \mathbf{\Theta}_0 \cdot \mathbf{X}_0 = (1, -K) \cdot (S_0, Z_0) = 1 \times S_0 - K \times Z_0.$$

skip

Forward price

The *forward price* F_0 which sets at time 0 for delivery at time T is the delivery price such that the forward contract has zero value at time 0.

- A forward price is not the same thing as the value of a forward contract.
- A forward contract on a no-dividend stock S has time-0 value

$$S_0 - KZ_0.$$

Choice of K that makes value zero is S_0/Z_0.

$$\boxed{\text{Thus } F_0 = S_0/Z_0\text{. If } r \text{ is constant, then } F_0 = S_0 e^{rT}.}$$

This does not depend on the dynamics of S.

skip

Forward price example

- If $r = 0.02$ and the share price today is $S_0 = 600$, and you and I want to enter costlessly today into a contract for time-1 delivery of S in exchange for a delivery price to be paid at time-1, the only arbitrage-free way to set that delivery price is $600 \times e^{0.02} \approx 612$.

Even if bullish, it'd be wrong for me to agree to pay, say, 650.

- Your portfolio $(-650 \text{ bond}, 1 \text{ share}, -1 \text{ forward contract})$ is an arbitrage because $V_0 = -650e^{-0.02} + 600 < 0$ and $V_T = 0$.
- Another arb: $(-600 \text{ bank}, 1 \text{ share}, -1 \text{ forward contract})$ because $V_0 = 0$ and $V_1 = -600e^{0.02} + S_1 - (S_1 - 650) > 0$.

  In other words, you sell me the contract, borrow 600 to buy the share today. At time 1, deliver the share, collect 650, repay 612.

Affine payoff

More generally, consider the following "affine" (or "linear") contract on a non-dividend-paying stock S. The contract pays, by definition,

$$a + bS_T$$

where a and b are constants. Then its time-0 value is

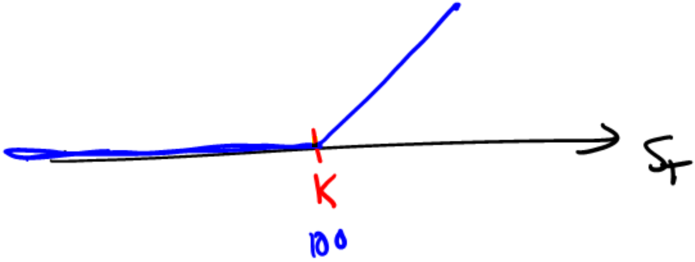
$$aZ_0 + bS_0$$

because it is replicated by

(a units of bond, b units of stock)

Call option: definition

A (European-style) *call* option with strike K and expiry T on an underlying process S, gives the holder the right, but not obligation, at time T, to pay K and receive S_T (dollars, or asset worth S_T dollars). So call has payoff $(S_T - K)^+$, where $x^+ := \max(x, 0)$. Payoff diagram:



At time $t \leq T$, the call option is said to be *in the money* if $S_t > K$, *at the money* if $S_t = K$, *out of the money* if $S_t < K$.

Uses of call options

Why would you use a call option? Examples:

- Suppose you are bullish on the underlying.

  Buying the call costs $x\%$ of the stock price, while, potentially, participating in $y\%$ of the stock gains, where $x < y$.

  So, compared to buying stock, buying a call can limit your downside, and/or increase your leverage.
- Suppose you own the underlying. Selling a call ("call writing") trades away some upside, in exchange for current income.
- Suppose you think the options market is overpricing the call.

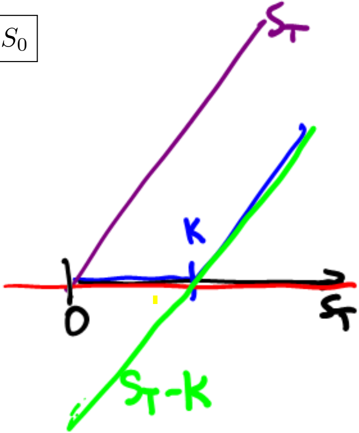
  Profit by selling the call for more than what it costs to replicate.

  Or profit by selling the call outright, if you have directional views.

Call option: bounds wrt underlying

The time-0 price C_0 of a call on a no-dividend stock S satisfies

$$\boxed{(S_0 - KZ_0)^+ \leq C_0 \leq S_0}$$



Proof.

See payoff diagram:

- Call payoff dominates payoff of forward with delivery price K.
- Call payoff dominates a zero payoff.
- Call payoff is dominated by the stock.

Hence $C_0 \geq S_0 - KZ_0$ and $C_0 \geq 0$ and $C_0 \leq S_0$ □

Call option: bounds wrt other calls

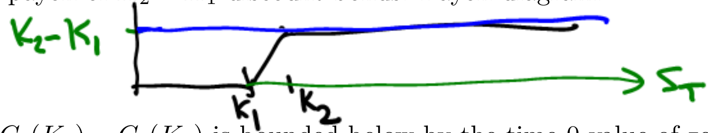
The time-0 call prices $C_0(K_1)$ and $C_0(K_2)$, for strikes $K_1 < K_2$ (with same expiry, on same underlying) satisfy

$$\boxed{0 \le C_0(K_1) - C_0(K_2) \le (K_2 - K_1)Z_0}$$

Proof.

Consider a bull *call spread*, long the K_1 call, short the K_2 call.

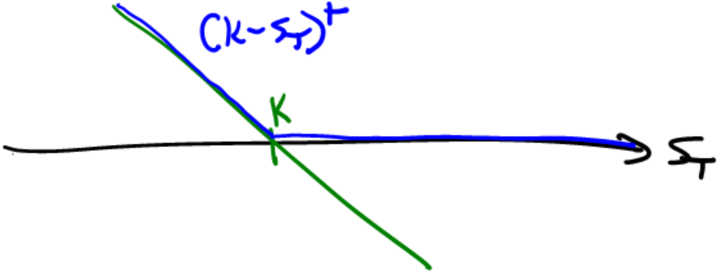
The call spread payoff dominates the zero payoff, but is dominated by the payoff of $K_2 - K_1$ discount bonds. Payoff diagram:



So $C_0(K_1) - C_0(K_2)$ is bounded below by the time-0 value of zero, above by the time-0 value of $K_2 - K_1$ bonds. □

Put option: definition

A (European-style) *put* option with strike K and expiry T on an underlying process S, gives the holder the right, but not obligation, at time T to pay S_T (dollars, or asset worth S_T dollars) and receive K. So put has payoff $(K - S_T)^+$. Payoff diagram:



At time $t \leq T$, the put option is said to be *in the money* if $S_t < K$, *at the money* if $S_t = K$, *out of the money* if $S_t > K$.

Uses of put options

Why would you use a put option? Examples:

- Suppose you are bearish on the underlying.

  Buying a put limits your potential loss to the cost of the option.

  (Shorting stock exposes you to unlimited loss.)

- Suppose you own the underlying. Buying a put protects you against the underlying going below K. It's insurance.

- Suppose you think the options market is overpricing the put.

  Profit by selling the put for more than what it costs to replicate.

  Or sell the put outright, if you have a directional view.

Put-call parity

Let $P_0(K,T)$ and $C_0(K,T)$ be time-0 prices of a European put and call, with identical (K,T), on a no-dividend stock S. Let $Z_0(T)$ be the time-0 price of a T-maturity discount bond. Then

$$C_0(K,T) = P_0(K,T) + S_0 - KZ_0(T)$$

Proof.

Payoff diagram:

Payoffs are equal, hence prices at earlier date are equal. □

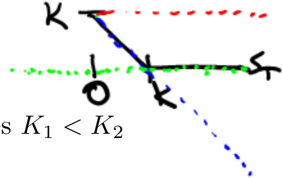
When was put-call parity discovered?

- It's in *Confusion de Confusiones* (1688) by José de la Vega

- Put-call parity is far older, and more fundamental, than any particular model e.g. Black-Scholes (1973)

Put option: bounds wrt underlying, and wrt other puts

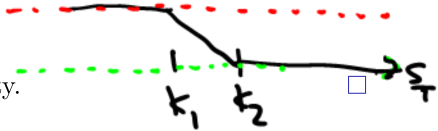
The time-0 price of a put on a non-dividend-paying stock S satisfies



$$\boxed{(KZ_0 - S_0)^+ \leq P_0 \leq KZ_0.}$$

The time-0 put prices $P_0(K_1)$ and $P_0(K_1)$, for strikes $K_1 < K_2$ (with same expiry, on same underlying) satisfy

$$\boxed{0 \leq P_0(K_2) - P_0(K_1) \leq (K_2 - K_1)Z_0.}$$

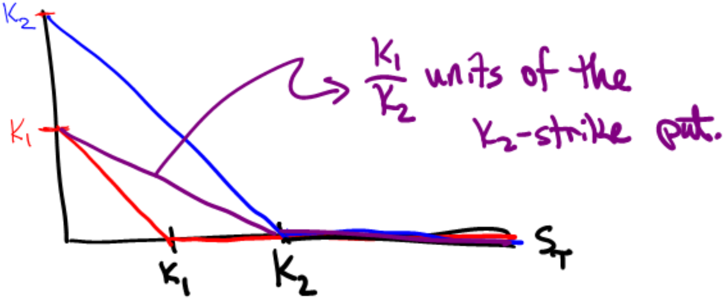


Proof.

Compare payoffs. Or use put-call parity. □

Put option: bounds wrt other puts, revisited

If $K_1 < K_2$ then $P_0(K_1) \leq P_0(K_2)$. Proof by comparing payoffs:

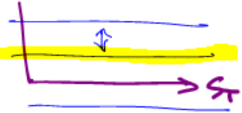


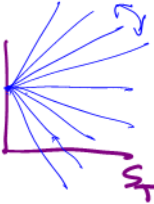
Better yet,

$$P_0(K_1) \leq \frac{K_1}{K_2} P_0(K_2)$$

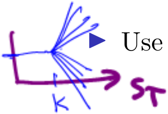
General payoffs

Using static positions in T-expiry bonds, forwards, calls, and puts on S, we can replicate or bound (superreplicate, subreplicate) general functions of S_T.

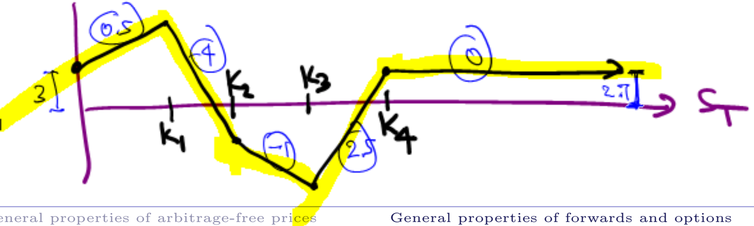




- Use bonds to adjust level.
- Use forwards (or perhaps S itself) to adjust slope.
- Use calls (or puts) to adjust convexity and concavity.



3 units of bond,
+0.5 units of stock S
-4.5 units of K1-strike call
+3 units of K2-strike call
+3.5 units of K3-strike call
-2.5 units of K4-strike call



GOOG option quotes

2.7 units of bond
-2.5 units of K4-strike put
+3.5 units of K3-strike put
+3 units of K2-strike put
-4.5 units of K1-strike put

GOOG US $ ↑555.40 +.50 Q555.20/555.37Q 1x1
At 12:19 d Vol 1,878,854 O 559.62Q H 562.00P L 552.95P Val 1.048B

GOOG US Equity | 95) Templates | 96) Actions | 97) Expiry | Option Monitor: Option Monitor

GOOGLE INC-C ↑555.4001 .5001 .0901% 555.20 / 555.3701 Hi 562.00 Lo 552.95 Volm 1878854 HV .00

☐Calc Mode Center 555.00 Strikes 19 Exch US Composit 92) Next Earnings(EM) 04/16/14 C

81) Center Strike 82) Calls/Puts 83) Calls 84) Puts 85) Term Structure

Calls								Strike	Puts							
Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt		Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt
19 Apr 14 (10d); CSize 100; IDiv .77; R .12; IFwd 556.55								19	19 Apr 14 (10d); CSize 100; IDiv .77; R .12; IFwd 556.55							
1) GOOG 4 C510	48.20	50.80	50.00	58.97	.84	2	34	510.00	58) GOOG 4 P510	3.20	3.60	3.08	55.30	-.14	69	203
2) GOOG 4 C515	43.70	46.80	43.00y	57.94	.82		37	515.00	59) GOOG 4 P515	4.10	4.40	4.24	55.20	-.17	28	304
3) GOOG 4 C520	39.90	40.60	44.40	53.44	.81	11	35	520.00	60) GOOG 4 P520	4.90	5.30	4.90	54.18	-.20	13	485
4) GOOG 4 C525	36.20	37.70	40.00	55.38	.76	1	58	525.00	61) GOOG 4 P525	6.00	6.40	6.14	53.70	-.23	27	163
5) GOOG 4 C530	32.40	33.50	34.00	54.21	.73	1	76	530.00	62) GOOG 4 P530	7.10	7.70	7.05	52.96	-.27	14	183
6) GOOG 4 C535	28.90	29.80	30.56	53.31	.69	45	73	535.00	63) GOOG 4 P535	8.70	9.10	7.80	52.84	-.30	45	348
7) GOOG 4 C540	25.60	26.50	27.52	53.30	.65	12	214	540.00	64) GOOG 4 P540	10.40	10.80	9.91	52.60	-.35	26	249
8) GOOG 4 C545	22.50	23.30	23.70	52.81	.61	18	168	545.00	65) GOOG 4 P545	12.10	12.70	12.30	51.82	-.39	17	148
9) GOOG 4 C550	19.60	20.20	20.00	52.13	.56	168	358	550.00	66) GOOG 4 P550	14.30	14.80	13.80	51.63	-.44	84	155
10) GOOG 4 C555	17.00	17.50	17.95	51.84	.52	192	256	555.00	67) GOOG 4 P555	16.60	17.10	16.50	51.18	-.48	41	159
11) GOOG 4 C560	14.60	15.00	15.41	51.29	.47	472	694	560.00	68) GOOG 4 P560	19.20	19.70	18.61	51.09	-.53	101	113
12) GOOG 4 C565	12.40	13.00	13.14	51.28	.43	115	321	565.00	69) GOOG 4 P565	21.80	22.50	20.50	50.64	-.58	1	29
13) GOOG 4 C570	10.50	10.70	10.51	50.74	.38	154	652	570.00	70) GOOG 4 P570	24.90	25.70	23.82	50.26	-.62	1	51
14) GOOG 4 C575	8.80	9.20	9.20	50.26	.34	244	317	575.00	71) GOOG 4 P575	28.00	28.80	31.60y	49.87	-.67		8
15) GOOG 4 C580	7.30	7.80	7.50	50.28	.30	124	431	580.00	72) GOOG 4 P580	31.40	32.30	29.10	49.19	-.71	11	47
16) GOOG 4 C585	5.90	6.30	6.43	50.02	.25	28	211	585.00	73) GOOG 4 P585	34.80	36.00	38.00y	48.78	-.75		11
17) GOOG 4 C590	4.80	5.30	5.07	50.20	.22	71	333	590.00	74) GOOG 4 P590	37.70	40.20	36.23	46.97	-.80	8	21
18) GOOG 4 C595	3.90	4.40	4.15	50.05	.19	91	136	595.00	75) GOOG 4 P595	41.60	44.10	40.26	44.85	-.84	8	13
19) GOOG 4 C600	3.20	3.60	3.39	49.94	.16	80	837	600.00	76) GOOG 4 P600	45.90	48.30	63.95y	44.80	-.87		35
17 May 14 (38d); CSize 100; IDiv .23; R .16; IFwd 556.62								19	17 May 14 (38d); CSize 100; IDiv .23; R .16; IFwd 556.62							
20) GOOG 5 C510	52.10	54.80	51.30y	36.10	.79		15	510.00	77) GOOG 5 P510	6.70	7.20	5.80	33.61	-.20	6	323

93) Default color legend Zoom 90%