Financial Mathematics 33000

Lecture 3

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Multi-period model

Stochastic processes

Multi-period models: Arbitrage and Fundamental Thm

Now allow intermediate trading

Start with an example. Two periods, so three time points $t = 0, 1, 2$.

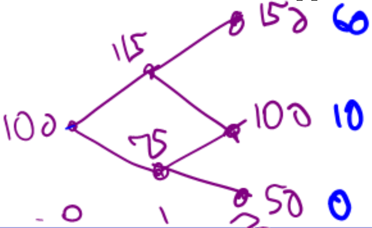
Four outcomes $\Omega = \{UU, UD, DU, DD\}$. Bank acct with $r = 0$.

Let $S_0 = 100$.

Let $S_1(UD) = S_1(UU) = 115$ and $S_1(DD) = S_1(DU) = 75$.

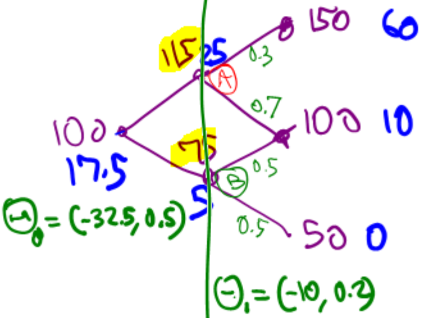
Let $S_2(UU) = 150$, $S_2(UD) = S_2(DU) = 100$, $S_2(DD) = 50$.

Replicate a 90-call with expiry $T = 2$? No way using a static portfolio of bank acct and stock. But suppose we allow trading at time $t = 1$.



$\Theta_1 = (-90, 1)$ (B, S)

Now allow intermediate trading



At time 1, at (A), what's the option price?

Replication: Hold (60-10)/(150-100) = 1 share of stock, and -90 units of bank account
Time-1 value is 1*115-90*1 = 25

Risk-neutral pricing: from (A) the up probability p satisfies 115 = p*150+(1-p)*100
Solve this to get p=0.3.
Option price 0.3*60+0.7*10 = 25

At time 0, what's the option price?

Replication: Hold (25-5)/(115-75)=0.5 shares of stock, and -32.5 units of bank account.
Time-0 value is 0.5*100-32.5*1 = 17.5

Risk-neutral pricing: omitted here.

At time 1, at (B), what's the option price?

Replication: Hold (10-0)/(100-50)=0.2 shares and -10 units of the bank account.
Time-1 value is 0.2*75-10*1 = 5

Risk-neutral pricing: from (B) the up probability q satisfies q*100+(1-q)*50 = 75, thus q=0.5.
Option price 0.5*10+0.5*0 = 5

(B): $(-32.5, 0.5)\cdot(1, 75) \stackrel{?}{=} (-10, 0.2)\cdot(1, 75)$ ✓

Multi-period model

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Stochastic process

A stochastic process is a set of random variables, indexed by time

- Discrete time: the set of time points is countable, for example

$$X_0, X_1, X_2, X_3, \ldots$$

  or

$$X_{t_0}, X_{t_1}, X_{t_2}, X_{t_3}, \ldots$$

  where $t_0 < t_1 < t_2 < \cdots$

- Continuous time: the set of time points is an interval, for example

$$X_t,\ t \geq 0 \qquad \text{or} \qquad X_t,\ t \in [0, T]$$

Some statements that we give below will literally be true for finite sample spaces but ignore technicalities (integrability, measurability) in infinite case.

Random walk

A random walk (started at a nonrandom point S_0) is a stochastic process $S_0, S_1, S_2, \ldots$ such that

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n$$

where $X_1, X_2, \ldots,$ are independent and identically distributed random variables.

- A simple random walk: $S_0 = 0$ and
  $\mathbb{P}(X_n = 1) = p$ and $\mathbb{P}(X_n = -1) = 1 - p$, where $0 < p < 1$.
- Symmetric random walk: simple random walk with $p = 1/2$.

Interview question

$N \sim \text{Geometric}(p)$

$\mathbb{E}N = \frac{1}{p}$

A stock price is currently 1.01. The stock can move only in steps of ± 0.01, and is a symmetric random walk. We will always put a bid (a limit order to buy) 0.01 below the current price, thus 1.00 for now, if it goes down we get filled at 1.00, if it goes to 1.02 the new bid is 1.01. What is the expectation of the price at which we finally buy the stock?

All prices in cents.

If you got filled at time N, you pay $99+N$ cents

$\mathbb{E}(99+N) = 99 + \mathbb{E}N$

≈ 101

Filtration

In multi-period models, we want to represent the revelation of information as time passes.

- A *filtration* $\{\mathcal{F}_t : \ t \geq 0\}$ represents, for each t, all information revealed at or before time t.

Example: in the previous model,

- $\mathcal{F}_1$ is the information about whether the first step was U or D.
- $\mathcal{F}_2$ is the information about whether the first two steps were UU, UD, DU, or DD.

Filtrations

We want to represent the revelation of information as time passes.

- A *filtration* $\{\mathcal{F}_t : \ t \geq 0\}$ represents, for each t, all information revealed at or before time t.

Example: Flip a coin at times $1, 2, 3$.

- Sample space {`HHH`, `HHT`, `HTH`, `HTT`, `THH`, `THT`, `TTH`, `TTT`}
- $\mathcal{F}_1$ is the information available at time 1, namely whether the first flip was `H` or `T`.
- $\mathcal{F}_2$ is the information available at time 2, namely whether the first two flips were `HH`, `HT`, `TH`, or `TT`.
- $\mathcal{F}_3$ is the time-3 information, namely whether the first three flips were `HHH`, `HTH`, `THH`, `TTH`, `HHT`, `HTT`, `THT`, `TTT`

Filtrations

More precisely: at each time, the sample space is partitioned into "information sets". At that time, you know which info set you are in, but not which outcome will happen within the info set.

- The time-1 information sets are

$$\{\mathtt{HHH}, \mathtt{HTH}, \mathtt{HHT}, \mathtt{HTT}\}, \{\mathtt{THH}, \mathtt{TTH}, \mathtt{THT}, \mathtt{TTT}\}$$

  and $\mathcal{F}_1$ is the set of those information sets (and their unions).

- The time-2 *information sets* are

$$\{\mathtt{HHH}, \mathtt{HHT}\}, \{\mathtt{HTH}, \mathtt{HTT}\}, \{\mathtt{THH}, \mathtt{THT}\}, \{\mathtt{TTH}, \mathtt{TTT}\}$$

  and $\mathcal{F}_2$ is the set of those information sets (and their unions).

- Time-3 info sets: $\{\mathtt{HHH}\}, \{\mathtt{HHT}\}, \{\mathtt{HTH}\}, \{\mathtt{HTT}\}, \{\mathtt{THH}\}, \{\mathtt{THT}\}, \{\mathtt{TTH}\}, \{\mathtt{TTT}\}$

Conditional Expectations

- You can take expectations conditional on the information available at a given time.

$$\mathbb{E}(X|\mathcal{F}_t) \quad \text{also written as} \quad \mathbb{E}_t(X)$$

is defined to be the random variable whose value on each of the information sets A in $\mathcal{F}_t$ is $\mathbb{E}(X|A)$.

- For example, let X be the number of heads in the 3 flips. Then

$$\mathbb{E}_1(X) = \mathbb{E}(X|\mathcal{F}_1) = \begin{cases} 2 & \text{on } \{\mathtt{HHH}, \mathtt{HTH}, \mathtt{HHT}, \mathtt{HTT}\}, \\ 1 & \text{on } \{\mathtt{THH}, \mathtt{TTH}, \mathtt{THT}, \mathtt{TTT}\} \end{cases}$$

Conditional expectations

Again the notation

$$\mathbb{E}_t X := \mathbb{E}(X|\mathcal{F}_t)$$

means the conditional expectation of X, given the time-t information.

Some properties of conditional expectation:

For [integrable] random variables X, Y,

- "Taking out what's known":

  If X is measurable wrt $\mathcal{F}_t$ then $\mathbb{E}_t(XY) = X\mathbb{E}_t Y$.
- If X is independent of $\mathcal{F}_t$, then $\mathbb{E}_t X = \mathbb{E}X$.
- "Law of iterated expectations": If $s < t$ then $\mathbb{E}_s(\mathbb{E}_t X) = \mathbb{E}_s X$.

Let's assume that $\mathcal{F}_0$ is trivial. So $\mathbb{E}_0$ is the same thing as $\mathbb{E}$.

Adapted processes

A stochastic process Y is *adapted* to $\{\mathcal{F}_t\}$, if Y_t is $\mathcal{F}_t$-measurable for each t, meaning the value of Y_t is determined by the information in $\mathcal{F}_t$.

This means that Y_t is constant on each information set of $\mathcal{F}_t$.

For instance, in option pricing theory, we:

- Construct our models so that asset prices X_t are adapted to $\mathcal{F}_t$.

  Interpretation: At time t the market has revealed the price X_t.
- Define our trading strategies to require that the quantities θ_t be adapted to $\mathcal{F}_t$.

  Interpretation: Allow trading, but determined only by what has been revealed, not by future outcomes.

Martingales

[redacted]. we say M_t is a *martingale* with respect to a filtration $\{\mathcal{F}_t\}$ [if $\mathcal{F}_t$ unspecified, then assume filtration consisting of history of M] if: M_t is adapted to $\{\mathcal{F}_t\}$, and for all t and all T with $0 \leq t < T$,

$$\boxed{\mathbb{E}_t M_T = M_t}$$

with probability 1. Interpretations:

- "Today's expectation of tomorrow's level is today's level"
- No "drift". No "trend".

$$\mathbb{E}_t(M_T - M_t) = 0$$

Martingales

- Let S be a simple random walk. Is S a martingale? Only if $p=\frac{1}{2}$
- Let X_t be the number of heads in flips 1 through t.

  Is X a martingale? No. Unless $P(H)=0$. $\mathbb{E}_t(X_T - X_t) = \frac{1}{2}(T-t) \neq 0$.
- Let $X_0 = 0$. Let $\mathsf{P}(X_1 = 1) = \mathsf{P}(X_1 = -1) = 0.5$.

  Let $X_t = tX_1$ for $t = 2, 3, 4, \ldots$.

  Is $\mathbb{E}X_T = X_0$ for all T? Is X a martingale?

Optional stopping theorem

- If M is a martingale and $S \leq T$ are bounded stopping times then

$$\mathbb{E}_S M_T = M_S$$

- Still true if S, T are unbounded, provided that M is *uniformly integrable* (UI). Sufficient conditions for UI (either):
  There exists Y such that $\mathbb{E}|Y| < \infty$ and for all t, $|M_t| \leq Y$.
  There exists Z such that $\mathbb{E}|Z| < \infty$ and for all t, $M_t = \mathbb{E}_t Z$.
- Still true if M_n is discrete sum of IID and $\mathbb{E}T < \infty$.

In general $\mathbb{E}M_T = M_0$ can fail: Let $M_t = \exp(W_t - t/2)$.
Let $S = 0$ and let T be the first time that M hits 0.1. (Such time exists with probability 1, because $\log M_t$ is Normal with mean $-t/2$ and standard deviation $\sqrt{t}$.) Then $\mathbb{E}M_T = 0.1 \neq M_0 = 1$.

Optional stopping theorem: Random walk

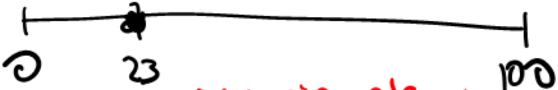
A drunken guy is trying to cross a 100ft bridge. He has crossed 23ft. The probabilities of going one step forward and one step backward are equal. The steps have equal length. Find the probability p he ends up at the bridge's other side before returning to the bridge's starting point.

Solution:

Let T be his exit time from [either end of] the interval $(0, 100)$.

His time-t position M_t is a bounded (hence UI) martingale.

Then ...



assuming step size 1

$$23 = M_0 = E_0 M_T = p\,100 + (1-p)\,0 \Rightarrow p = 23\%.$$

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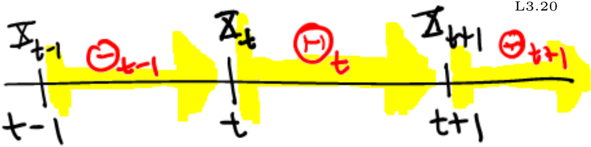
Trading strategy



A *trading strategy* on $t = 0, 1, \ldots, T$ is a sequence $\mathbf{\Theta}_t$ adapted to $\mathcal{F}_t$.

Let us agree to view $\mathbf{\Theta}_t$ as the vector of quantities of the tradeable assets held *after* all time-t trading at prices $\mathbf{X}_t$.

Say that the trading strategy is *self-financing* if for all $t > 0$,

$$\mathbf{\Theta}_{t-1} \cdot \mathbf{X}_t = \mathbf{\Theta}_t \cdot \mathbf{X}_t$$

with probability 1.

This implies that the change in portfolio value from time t to $t+1$ is

$$V_{t+1} - V_t = \mathbf{\Theta}_{t+1} \cdot \mathbf{X}_{t+1} - \mathbf{\Theta}_t \cdot \mathbf{X}_t = \mathbf{\Theta}_t \cdot \mathbf{X}_{t+1} - \mathbf{\Theta}_t \cdot \mathbf{X}_t = \mathbf{\Theta}_t \cdot (\mathbf{X}_{t+1} - \mathbf{X}_t)$$

So the change in value is fully attributable to gains and losses in asset prices.

Total P&L

Note that we can sum from $t = 0$ to $t = T - 1$

$$V_T - V_0 = \sum_{t=0}^{T-1} \boldsymbol{\Theta}_t \cdot (\mathbf{X}_{t+1} - \mathbf{X}_t)$$

This looks like a discrete version of the stochastic integral

$$\int_0^T \boldsymbol{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t$$

Idea: P&L from a self-financing trading strategy is a stochastic integral, namely the integral of quantity with respect to price.

Arbitrage in a multi-period model

Arbitrage is a self-financing trading strategy $\mathbf{\Theta}_t$ whose value $V_t := \mathbf{\Theta}_t \cdot \mathbf{X}_t$ satisfies

$$V_0 = 0 \quad \text{and both:} \quad \begin{array}{l} \mathsf{P}(V_T \geq 0) = 1 \\ \mathsf{P}(V_T > 0) > 0 \end{array}$$

or

$$V_0 < 0 \quad \text{and} \quad \mathsf{P}(V_T \geq 0) = 1$$

Note that static portfolios are a special case of self-financing trading strategies, so the previous definition is consistent with this one.

This definition extends the notion of arbitrage beyond static strategies, to self-financing ones.

Properties of arbitrage-free prices

In the absence of arbitrage, the following properties hold:

- If Θ_t^a and Θ_t^b are self-financing, and if $\mathsf{P}(V_T^a \geq V_T^b) = 1$, then $V_0^a \geq V_0^b$, or else arbitrage would exist, namely $\Theta_t^a - \Theta_t^b$.
- "Law of one price": If Θ_t^a and Θ_t^b are self-financing, and if $\mathsf{P}(V_T^a = V_T^b) = 1$, then $V_0^a = V_0^b$, or else arbitrage would exist.

Martingales

The time-t *conditional expectation* of a random variable Y, written

$$\mathbb{E}_t Y \text{ or } \mathbb{E}(Y|\mathcal{F}_t)$$

is another random variable: the expectation of Y, conditional on the information that has been revealed up to and including time t.

- This means: if $\mathcal{F}_t$ is generated by a finite partition, then on each information set R, define $\mathbb{E}(Y|\mathcal{F}_t)$ on info set R to be $\mathbb{E}(Y|R)$.

(Ignoring integrability conditions) we say that M_t is a *martingale* with respect to filtration $\mathcal{F}_t$, if M_t is adapted to $\mathcal{F}_t$ and for all $t < T$, with probability 1,

$$\mathbb{E}_t M_T = M_t$$

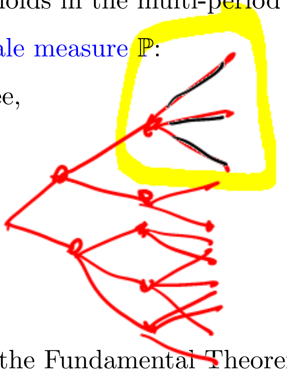
"Today's expectation of tomorrow's level equals today's level."

Fundamental Theorem of Asset Pricing

The fundamental theorem still holds in the multi-period case.

No arb $\Rightarrow \exists$ equivalent martingale measure $\mathbb{P}$:

Idea: In each one-period sub-tree,



apply the one-period version of the Fundamental Theorem, and get risk-neutral probabilities of each single step. Then multiply the probabilities of the individual steps to get a probability measure on all paths. Can verify that this is a martingale measure.

Fundamental Theorem of Asset Pricing: multi-period

$\exists$ equivalent martingale measure $\mathbb{P} \Rightarrow$ No arb:

We are given that $\tilde{\mathbf{X}}_t := B_t^{-1}\mathbf{X}_t$ is a vector of martingales under $\mathbb{P}$.

For self-financing $\boldsymbol{\Theta}$, can show that $\boldsymbol{\Theta}_t \cdot \tilde{\mathbf{X}}_t$ is also a martingale.

(Idea: Can show that martingales have zero drift; here this means

$$\mathbb{E}_t(M_{t+1} - M_t) = 0$$

hence the intuition that a martingale is a wealth process generated by playing zero-expectation games. Varying the amounts θ_t^n bet, across games n and times t, still makes collectively a zero-expectation game.)

The martingale property implies $V_0/B_0 = \mathbb{E}(V_T/B_T)$, so the reasoning we gave in the one-period case again shows that no arbitrage exists.

(You can't risklessly make money by playing zero-expectation games!)

Incomplete market

In an incomplete market (example: the one-period "trinomial" model with 3 possible outcomes and two assets B, S):

- From a replication standpoint: Some payoffs can't be replicated. (In the trinomial example, the only replicable payoffs are linear combinations of 1 and S_T, or equivalently, affine functions of S_T). For payoffs having no replicating portfolio, no-arbitrage alone may not be able to determine a unique price for the payoff.
- From the martingale / risk-neutral valuation standpoint: There are many martingale measures consistent with the prices of the basic assets. Different martingale measures can give different valuations for a derivative asset's payoff.

Completing a market

If we change the assumptions of the model, then we may be able to complete the market, and thus price all payoffs using no-arbitrage.

- Could change the assumptions by adding more basic assets
  In the trinomial example, we could complete the market by adding a third asset outside the span of B and S.
- Could change the assumptions by adding more trading opportunities.
  In the trinomial example, we could complete the market by allowing trading of B and S at one intermediate time point.

Next we will build models with infinitely many outcomes. Hope to replicate general payoffs by trading B and S *continuously* in time.