Financial Mathematics 33000

Lecture 5

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2022 October 26

Arbitrage in continuous time

Black-Scholes model

B-S formula via replication

Delta, Gamma, Theta

Arbitrage

- Let prices be $\mathcal{F}_t$-adapted Itô processes $(X_t^1, \ldots, X_t^N) =: \mathbf{X}_t$.
- A *portfolio/trading strategy* is an $\mathcal{F}_t$-adapted vector process $\mathbf{\Theta}_t := (\theta_t^1, \ldots, \theta_t^N)$ of quantities held in each asset $1, \ldots, N$.
- Say that the trading strategy is *self-financing* if its value $V_t := \mathbf{\Theta}_t \cdot \mathbf{X}_t$ satisfies (with probability 1) for all t

$$\mathrm{d}V_t = \mathbf{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t, \qquad \text{equivalently } V_t = V_0 + \int_0^t \mathbf{\Theta}_u \cdot \mathrm{d}\mathbf{X}_u$$

- *Arbitrage* is a [admissible] self-financing trading strategy $\mathbf{\Theta}_t$ with

$$V_0 = 0 \quad \text{and both:} \quad \begin{array}{l} \mathsf{P}(V_T \geq 0) = 1 \\ \mathsf{P}(V_T > 0) > 0 \end{array}$$

or

$$V_0 < 0 \quad \text{and} \quad \mathsf{P}(V_T \geq 0) = 1.$$

Replication and hedging

- Definition: a trading strategy $\mathbf{\Theta}$ *replicates* a time-T payoff Y_T if it is self-financing, and its value $V_T = Y_T$ (with probability 1).
- Law of one price: At any time $t < T$, the no-arbitrage price of an asset paying Y_T must be the value of the replicating portfolio.
- To *hedge* a payoff usually means: to [try to] replicate the *negative* of the payoff (or the portion of the payoff attributable to some particular source of risk). For example, to hedge a position that is short one option usually means to [try to] replicate a position that is *long* the option. I say "try to" because "hedge" can mean an approximation to replication – such as superreplication, or broadly speaking, any strategy to reduce some notion of risk.

Motivation for GBM to model a stock price

BM is a natural starting point for model-building

- BM is the $\Delta t \to 0$ limit, in distribution, of a random walk (with zero-mean IID steps, scaled to have variance Δt).

But some problems with W_t or $\alpha t + \beta W_t$ as a model for a stock price:

- BM can go negative, and so can scaled BM with drift.
- If $\mathrm{d}S_t = \alpha \mathrm{d}t + \beta \mathrm{d}W_t$ then each $S_{t+\Delta t} - S_t$ is independent of $\mathcal{F}_t$. A 10+ dollar move is equally likely, whether S_t is at 20 or 100.

For a *GBM* S, the drift and diffusion are *proportional to* S.

- S stays positive.
- Each *log return* $\log \frac{S_{t+\Delta t}}{S_t}$ (or return $\frac{S_{t+\Delta t}}{S_t} - 1$) is indep of $\mathcal{F}_t$. A 10+ *percent* move is equally likely, whether S_t is at 20 or 100.

Black-Scholes model

In continuous time, consider two basic assets:

- Money-market or bank account: each unit has price $B_t = e^{rt}$.

  Equivalently, it has dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t \qquad B_0 = 1$$

- Non-dividend-paying stock: share price S has GBM dynamics

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t \qquad S_0 > 0$$

  where *volatility* $\sigma > 0$ and W is BM, under physical probabilities.

  Think of volatility σ as $\sqrt{\text{Variance of log-returns, per unit time}}$

  "annualized"

Find: time-t price C_t of call which pays $C_T = (S_T - K)^+$ at time T, where $K > 0$.

Replication

- Lecture 6 will use the *martingale/risk-neutral pricing* approach:

  By Fundamental Thm, take risk-neutral $\mathbb{E}$ of discounted payoff.

- Lecture 5 will price options using *replication*, two ways:

  First: an intuitive derivation, by replicating B using C and S

  Then: a careful proof, by replicating C using S and B

Fischer Black, Myron Scholes, Robert Merton

Plan of intuitive derivation: Replicate B using C and S

- Construct risk-free (= zero $\mathrm{d}W$ term) portfolio of (C, S).
- If self-financing, then the portfolio value's drift must be proportional, at rate r, or else there is arbitrage of portfolio vs B.
- On the other hand, if $C_t = C(S_t, t)$ for some smooth function C, then Itô says that the portfolio value's drift can be expressed in terms of C's partial derivatives.
- Therefore $C(S, t)$ satisfies a PDE.
- Solve this PDE to obtain formula for C.

Construct a risk-free portfolio

- Use (1 option, $-a_t$ share), choosing a_t to cancel the option risk.

  Portfolio value is

$$V_t = C_t - a_t S_t.$$

- So some authors claim that

$$\mathrm{d}V_t = \mathrm{d}C_t - a_t \mathrm{d}S_t.$$

  But the product rule says that

$$\mathrm{d}(a_t S_t) = a_t \mathrm{d}S_t + S_t \mathrm{d}a_t + (\mathrm{d}a_t)(\mathrm{d}S_t),$$

  so it's not true that $\mathrm{d}(a_t S_t) = a_t \mathrm{d}S_t$. Ignoring this point ...

Construct a risk-free portfolio

- Assume $C_t = C(S_t, t)$ where C is some smooth function. By Itô ,

$$\mathrm{d}V_t = \frac{\partial C}{\partial t}\mathrm{d}t + \frac{\partial C}{\partial S}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mathrm{d}S_t)^2 - a_t \mathrm{d}S_t$$

where C and its partials are evaluated at (S_t, t).

- Now make these cancel by choosing $a_t := \frac{\partial C}{\partial S}(S_t, t)$. Then

$$\mathrm{d}V_t = \frac{\partial C}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mathrm{d}S_t)^2 = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2\right)\mathrm{d}t$$

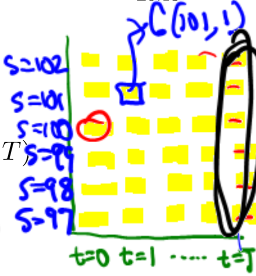
- On the other hand, V_t is the value of a risk-free portfolio, so

$$\mathrm{d}V_t = rV_t\mathrm{d}t = r\left(C_t - S_t\frac{\partial C}{\partial S}\right)\mathrm{d}t$$

- Comparing right-hand sides,

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 = rC$$

The Black-Scholes PDE and formula



- So want C to solve a PDE for $(S,t) \in [0,\infty) \times (0,T)$

$$\boxed{\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC}$$

with terminal condition $C(S,T) = (S-K)^+$ given by the payoff.

- Solution: the *Black-Scholes formula*. For $t < T$,

$$\boxed{C^{BS}(S,t) := e^{-r(T-t)}(FN(d_1) - KN(d_2))}$$

where N is the standard normal CDF, and $F := Se^{r(T-t)}$ and

Φ

$$\boxed{d_{1,2} := d_{+,-} := \frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}}$$

and $C^{BS}(S,T) := (S-K)^+ = \lim_{t\to T} C^{BS}(S,t)$.

Can directly check: C^{BS} solves PDE. (How to *find* C^{BS}? Later.)

How *not* to do stochastic calculus

What about the claim that $\mathrm{d}(C_t - a_t S_t) = \mathrm{d}C_t - a_t \mathrm{d}S_t$?

Bogus justifications:

- The share holdings a_t are "instantaneously constant."

  Nonsense. In fact a_t is changing (and, moreover, changing so fast that we needed to introduce Itô calculus).

- Portfolio of (1 option, $-a_t$ shares) is "self-financing"

  It's not. In fact there's no way to vary this portfolio's share holdings without outside funding. (The option position does not provide any funding, because it is fixed at 1 unit).

The intuitive derivation is helpful (and can be improved), but is not a proof. Let's actually give a proof now.

Black-Scholes formula: Careful proof

- Plan: replicate 1 option using a portfolio of (S, B).
- Let $C^{BS}(S,t)$ be the B-S formula.

  We are not assuming that $C^{BS}(S_t,t)$ is the option price;

  that will be the *conclusion*.
- Let's hold

  $$a_t := \frac{\partial C^{BS}}{\partial S}(S_t,t) \text{ shares}, \quad b_t := \frac{C^{BS}(S_t,t) - a_t S_t}{B_t} \text{ bank acct units}$$

  Portfolio value is then

  $$V_t = a_t S_t + b_t B_t = a_t S_t + (C^{BS}(S_t,t) - a_t S_t) = C^{BS}(S_t,t)$$

Black-Scholes formula: Careful proof

- In particular, the final portfolio value is $C^{BS}(S_T, T) = (S_T - K)^+$
- And the portfolio self-finances, because

$$
\begin{aligned}
\mathrm{d}V_t = \mathrm{d}C^{BS}(S_t, t) &= \left(\frac{\partial C^{BS}}{\partial t} + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial S^2}\sigma^2 S_t^2\right)\mathrm{d}t + \frac{\partial C^{BS}}{\partial S}\mathrm{d}S_t \\
&= r\left(C^{BS} - S_t\frac{\partial C^{BS}}{\partial S}\right)\mathrm{d}t + \frac{\partial C^{BS}}{\partial S}\mathrm{d}S_t \\
&= a_t \mathrm{d}S_t + r b_t B_t \mathrm{d}t = a_t \mathrm{d}S_t + b_t \mathrm{d}B_t
\end{aligned}
$$

$= (a,b)\cdot d(S,B)$

because C^{BS} solves the PDE.

$= \Theta \cdot dX$

- So the portfolio replicates the option.

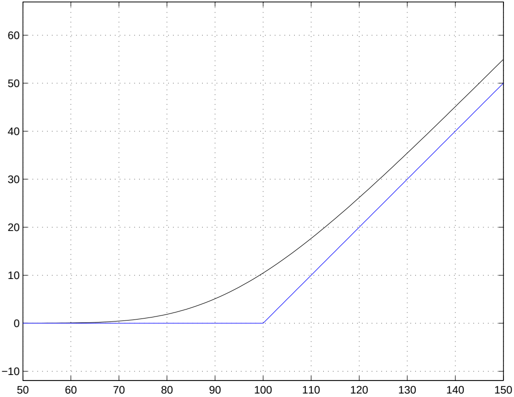
Conclusion: at any time $t < T$, the unique no-arb price of the option equals the portfolio value, which is $C^{BS}(S_t, t)$.

Call price vs S

Let $K = 100$, $T - t = 1$, $\sigma = 0.2$, $r = 0.05$.

Call price $C^{BS}(S_t)$ and intrinsic value $:= (S_t - K)^+$

plotted against S_t

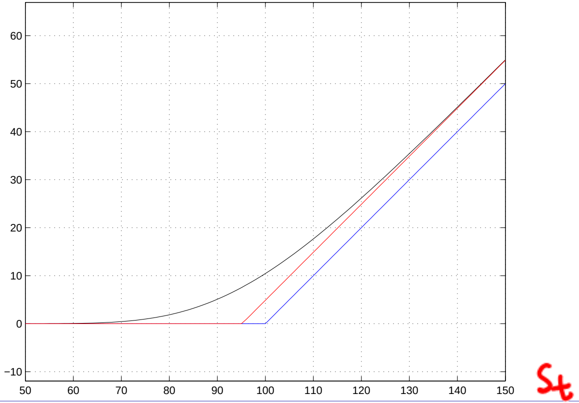


S_t

Call price vs S

Let $K = 100$, $T - t = 1$, $\sigma = 0.2$, $r = 0.05$.

Call price $C^{BS}(S_t)$ and intrinsic value $:= (S_t - K)^+$ and lower bound $(S_t - Ke^{-r(T-t)})^+$, plotted against S_t



Replication and linearity

Recall: in one-period binomial model, we replicated by holding $(c_u - c_d)/(s_u - s_d)$ shares, matching the slope of the payoff function.

In one-period three-state model, we could not replicate with a static portfolio of $\{$bond, stock$\}$, unless the option payoff is linear in S.

To achieve replication, we could introduce additional hedging assets, or we could go to a *multi-period* model.

Replication and linearity in continuous time

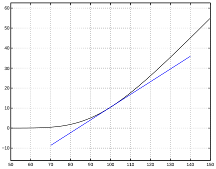
With extra nodes, the option value becomes "locally" linear in S.

At time t, match the slope, wrt $S_{t+\Delta t}$, of the possible values of $C_{t+\Delta t}$.

Slope changes in time, but that's ok; just rebalance the portfolio.

Continuous time:

At time t match the slope, wrt $S_{t+\mathrm{d}t}$, of the possible values of $C_{t+\mathrm{d}t}$



Arbitrage in continuous time

Black-Scholes model

B-S formula via replication

Delta, Gamma, Theta

Sensitivities or "Greeks": Delta, Gamma, Theta

Definition:

Suppose an asset or portfolio has time-t value $C_t = C(S_t, t)$.

- Its *delta*, at time-t, is $\frac{\partial C}{\partial S}(S_t, t)$.
- Its *gamma*, at time-t, is $\frac{\partial^2 C}{\partial S^2}(S_t, t)$.
- Its *theta*, at time-t, is $\frac{\partial C}{\partial t}(S_t, t)$.
- These definitions do not assume that C is a call price, and do not assume the Black-Scholes model.

In the remaining L5 slides, to get specific formulas, we do assume Black Scholes (L5.7).

Delta

For a call, in the B-S model, at time t,

$$\text{Delta} := \frac{\partial C^{BS}}{\partial S} = N(d_1) + S_t N'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} = \boxed{N(d_1)}$$

recalling that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Interpretations of delta:

- slope of option price C^{BS}, plotted as a function of S
- how much the option price moves, per unit move in S
- number of shares of S needed to replicate one option

This allows us to view the B-S call price

$$C^{BS}(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

as the value of the replicating portfolio, which consists of the value in the shares, and the value in the bank account.

Gamma

For a call, in the B-S model, at time t,

$$\text{Gamma} := \frac{\partial^2 C^{BS}}{\partial S^2} = \frac{\partial}{\partial S} N(d_1) = N'(d_1)\frac{\partial d_1}{\partial S} = \boxed{\frac{N'(d_1)}{S_t \sigma \sqrt{T-t}}}$$

Interpretations:

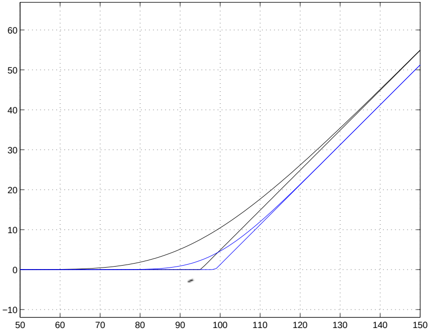
- convexity of C^{BS} wrt S
- how much the Delta moves, per unit move in S
- how much rebalancing of the replicating portfolio is needed, per unit move in S

Delta and gamma are also defined for portfolios. For N assets having time-t deltas $\mathbf{\Delta}_t \in \mathbb{R}^N$ and gammas $\mathbf{\Gamma}_t \in \mathbb{R}^N$, the portfolio $\mathbf{A}_t \in \mathbb{R}^N$ has time-t delta $\mathbf{A}_t \cdot \mathbf{\Delta}_t$ and gamma $\mathbf{A}_t \cdot \mathbf{\Gamma}_t$.

Call price

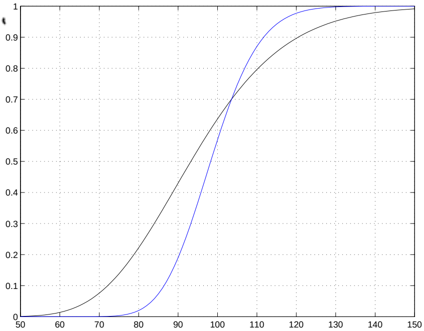
Call price $C^{BS}(S_t)$ and lower bound, plotted against S_t,

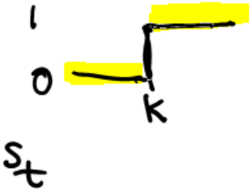
for $T - t = 1$, and $T - t = 0.25$.



Call delta

Call delta $= N(d_1)$, plotted against S_t, for $T - t = 1$, $T - t = 0.25$.

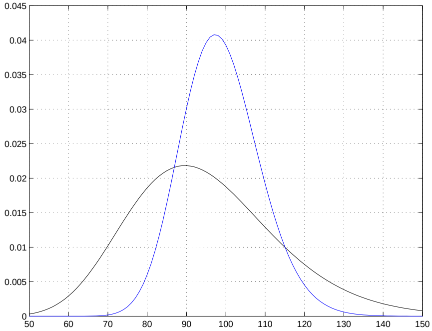




Delta of a call is strictly between 0 and 1.

Call gamma

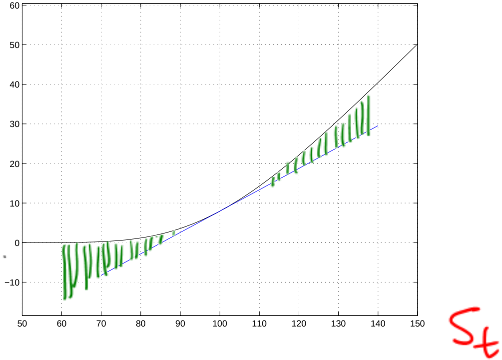
Call gamma plotted against S_t, for $T - t = 1$, and $T - t = 0.25$.



Gamma of a call is positive.

Discrete rebalancing

At time t, go long 1 call, short $\partial C/\partial S$ shares. Allocate the proceeds into the bank. Don't immediately rebalance. Let $r = 0$ and $S_t = 100$.

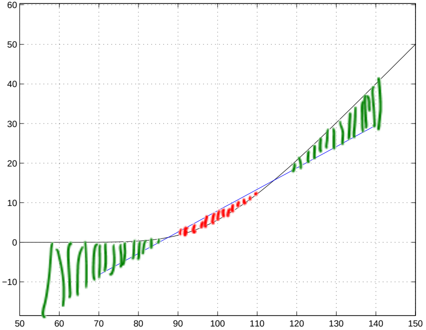


Black curve (call value) minus blue line (shares + bank) = profit due to move in S. Always net positive profit?

Discrete rebalancing

Wait until $t+\delta t$

At time t, go long 1 call, short $\partial C/\partial S$ shares. Allocate the proceeds into the bank. Don't immediately rebalance. Let $r = 0$ and $S_t = 100$.



$S_{t+\delta t}$

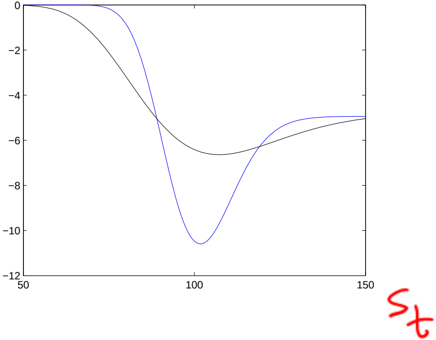
Black curve (call value) minus blue line (shares + bank) = profit due to move in S. Always net positive profit? No, because of timedecay

Call theta

For a call, in the B-S model, at time t,

$$\text{Theta} = \frac{\partial C^{BS}}{\partial t} = \frac{-S_t N'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

Call's theta, plotted against S_t, for $T-t=1$ and $T-t=0.25$.



Greeks related to each other

BS PDE links theta, gamma, delta, and option price

$$\boxed{\Theta + rS\Delta + \frac{1}{2}\Gamma\sigma^2 S^2 = rC}$$

("Option" can have general time-T payoff, not necessarily call/put.)

In particular, if $r = 0$ then

$$\Theta = -\frac{1}{2}\Gamma\sigma^2 S^2$$

Discrete rebalancing and gamma

A discretely delta-hedged position that is long gamma (meaning gamma > 0. For example: long call, short shares):

- has net profit if $|\Delta S|$ is large enough to overcome time decay
- has net loss if $|\Delta S|$ is too small, relative to time decay

A discretely delta-hedged position that is short gamma (meaning gamma < 0. For example: short call, long shares):

- has net loss if $|\Delta S|$ is too large, relative to time decay
- has net profit if $|\Delta S|$ is small enough, relative to time decay

So such positions are sensitive to “realized volatility”.

Dynamics of hedge

Gamma of stock is $\frac{\partial^2}{\partial S^2} S = 0$

Delta of stock is $\frac{\partial S}{\partial S} = 1$

- You have an option position, and want to trade shares to maintain delta-neutrality (delta=0).

gamma>0 gamma<0

- For which kind of options position – long gamma or short gamma – do you buy S on dips, and sell S on rallies?

Long gamma: $\Gamma > 0$

When $S\uparrow$ the $\Delta\uparrow$. To maintain Δ-neutrality, sell stock.

When $S\downarrow$ the $\Delta\downarrow$. To maintain Δ-neutrality, buy stock.

Implied Volatility

Given a time-t price C for a given call option (K, T) on an underlying S_t assuming interest rate r, the **implied volatility** is the σ such that

$$C = C^{BS}(S_t, t, K, T, r, \sigma)$$

where C^{BS} is the Black-Scholes formula.

- This exists and is unique (if C is within arbitrage bounds).
- The bigger the dollar price C, the bigger the implied vol σ_I
- Gives us another way quoting an option price. Instead of saying the option is trading at \$`x.xx`, can say it's trading at `yy`% vol.
- We will say much more about implied volatility next quarter

Realized Volatility

Realized variance of S, sampled at interval Δt, from time 0 to time T can be defined, using log-returns by letting $t_n = n\Delta t$ and $T = t_N$ and

$$RVar = \frac{1}{T}\sum_{n=0}^{N-1}\left(\log\frac{S_{t_{n+1}}}{S_{t_n}}\right)^2$$

Alternatively could use simple returns, letting $\Delta S = S_{t_{n+1}} - S_{t_n}$ and

$$RVar = \frac{1}{T}\sum_{n=0}^{N-1}\left(\frac{\Delta S}{S_{t_n}}\right)^2$$

Realized volatility of S is

$$RVol = \sqrt{RVar}$$

If S follows GBM with instantaneous volatility σ, then $RVol \to \sigma$ as $\Delta t \to 0$.

PnL from Gamma Scalping

Let $r = 0$. You buy a call, paying $C^{BS}(\sigma_I)$, where σ_I is implied vol.

Delta-hedge it at intervals Δt. In what cases would you profit/lose?

By Taylor, $C(S + \Delta S) \approx C(S) + (\Delta S)C'(S) + \frac{1}{2}(\Delta S)^2 C''(S)$, $+ (\Delta t)\frac{\partial C}{\partial t}$

$t + \Delta t$

So your profit from t to $t + \Delta t$ is approximately

$$\begin{aligned} \frac{1}{2}\Gamma \times (\Delta S)^2 + \Theta \times \Delta t &= \frac{1}{2}\Gamma S^2 \left(\frac{\Delta S}{S}\right)^2 - \frac{1}{2}\Gamma \sigma_I^2 S^2 \Delta t \\ &= \frac{1}{2}\Gamma S^2 \left(\left(\frac{\Delta S}{S}\right)^2 - \sigma_I^2 \Delta t\right) \end{aligned}$$

Total profit from time 0 to T is

$$\sum_{n=0}^{N-1} \frac{1}{2}\Gamma_{t_n} S_{t_n}^2 \left(\left(\frac{\Delta S}{S_{t_n}}\right)^2 - \sigma_I^2 \Delta t\right)$$

Ignoring the ΓS^2, this would imply that you profit if $\boxed{RVol > \sigma_I}$.

Conclusion

Working under Black-Scholes dynamics,

- Today we priced options using *replication*, and we examined the behavior of the replicating portfolio.
- Next time we will price options using *martingale methods*: Apply Fundamental Thm, and take expectation of discounted payoff.