Homework 1

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Exercise 1

(a) We construct the portfolio of assets: take a loan at a low interest rate and put it at a high one.

$$\Theta = (-1 \text{ B}, +1 \text{ B}^*)$$

Then the initial value of the portfolio:

$$V_0 = -B_0 = B_0^* = -1 + 1 = 0$$

And the final value of the portfolio:

$$V_T = B_T^* - B_T = e^{r^*T} - e^{rT} > 0 \quad \text{since } r^* > r \text{ and exp - monotonically increasing function}$$

As a result, we have **type 1 arbitrage**: zero initial value, no risk of loss, some chance of gain (in our particular case, we always receive a positive gain).

(b) Let's construct the portfolio of assets:

$$\Theta = \left(-1 \ S, +1 \ C, +\frac{99.5}{0.9} \ Z\right)$$

Then the initial value:

$$V_0 = -100 + 0.5 + 0.9 \cdot \frac{99.5}{0.9} = 0$$

We have two different cases for the realizations of S_T. Firstly, $S_T \geq 110$:

$$V_T = S_T - K - S_T + Z_T = -110 + 110.(5) = 0.(5) > 0$$

Secondly, $S_T < 110$:

$$V_T = 0 - S_T + Z_T = -S_T + 110.(5) > 0.(5) > 0$$

As a result, we have **type 1 arbitrage**: zero initial value, no risk of loss, some chance of gain (in our particular case, we always receive a positive gain).

(c) In this part, let's have a look at payoffs our contracts have:

$$G_T = \begin{cases} S_T & S_T \leq 110 \\ 110 & S_T > 110 \end{cases} \qquad C_T = \begin{cases} 0 & S_T \leq 110 \\ S_T - 110 & S_T > 110 \end{cases}$$

We can replicate the contract G using the stock S and the call option C. Let's long the stock and short the call option:

$$\begin{cases} S_T - 0 = S_T & S_T \leq 110 \\ S_T - S_T + 110 = 110 & S_T > 110 \end{cases}$$

The strategy matches the payoffs of G! What about initial values?

$$S_0 - C_0 = 100 - 20 = 80 < 85 = G_0$$

So buy low, sell high. Let's consider the portfolio of assets:

$$\Theta = (+1\ S, -1\ G, -1\ C)$$

The the initial value of the portfolio:

$$V_0 = S_0 - G_0 - C_0 = 100 - 85 - 20 = -5 < 0$$

The there are two possible cases. Let's start with $S_T > 110$:

$$V_T = S_T - (S_T - 110) - 110 = 0$$

And $S_T \leq 110$:

$$V_T = S_T - 0 - S_T = 0$$

As a result, we have **type 2 arbitrage**: we initially receive a credit which we will definitely not repay.

(d) In this section, we can check the upper bounds of call options. In general:

$$(S_0 - KZ_0)^+ \leq C_0(K_1) - C_0(K_2) \leq (K_2 - K_1)Z_0$$

The first and second strikes:

$$C_0(20) - C_0(22.5) = 6.4 - 3.1 = 3.3 \not\leq 2.25 = (22.5 - 20) \cdot 0.9 = (K_2 - K_1)Z_0$$

The first and third strikes:

$$C_0(20) - C_0(25) = 6.4 - 1 = 5.4 \not\leq 4.5 = (25 - 20) \cdot 0.9 = (K_3 - K_1)Z_0$$

Let's consider the portfolio of assets:

$$\Theta = \left(-1\ C(20), +1\ C(22.5), +1\ C(25), +\frac{2.3}{0.9}\ Z\right)$$

Then the initial value of the portfolio:

$$V_0 = -6.4 + 3.1 + 1 + 0.9 \cdot \frac{2.3}{0.9} = 0$$

Suppose $S_T < 20$:

$$V_T = 0 + 0 + 0 + \frac{2.3}{0.9} > 0$$

$S_T \in [20, 22.5)$:

$$V_T = 20 - S_T + \frac{2.3}{0.9} = 22.(5) - S_T > 0$$

$S_T \in [22.5, 25)$:

$$V_T = 20 - S_T + S_T - 22.5 + \frac{2.3}{0.9} = 0.0(5) > 0$$

$S_t \geq 25$:

$$V_T = 20 - S_T + S_T - 22.5 + S_T - 25 + \frac{2.3}{0.9} = \underbrace{S_T - 25}_{\geq 0} + \underbrace{2.(5) - 2.5}_{>0} > 0$$

As a result, we have **type 1 arbitrage**: zero initial value, no risk of loss, some chance of gain (in our particular case, we always receive a positive gain).

(e) Let's consider the following portfolio. Since Θ is not unique, the weights are adjusted to have a beautiful values:

$$\Theta = \left(-40\ X, +1\ F, +\frac{18}{0.9} Z\right)$$

The initial value of the portfolio:

$$V_0 = -10 + 0.9 \cdot \frac{18}{0.9} - 40 \cdot 0.2 = 8 - 8 = 0$$

Then the final payoff:

$$V_T = S_T - 100 - 40 \cdot 2 \cdot \log\left(\frac{S_T}{100}\right) + 18$$

For a second, let's assume that S_T is a deterministic value and $V_T(S_T)$ is a deterministic function. The First Order condition:

$$V_T'(\omega) = 1 - 80 \cdot \frac{100}{S_T(\omega)} \cdot \frac{1}{100} = \frac{S_T(\omega) - 80}{S_T(\omega)} = 0$$

Then:

$$S_T^* = 100$$

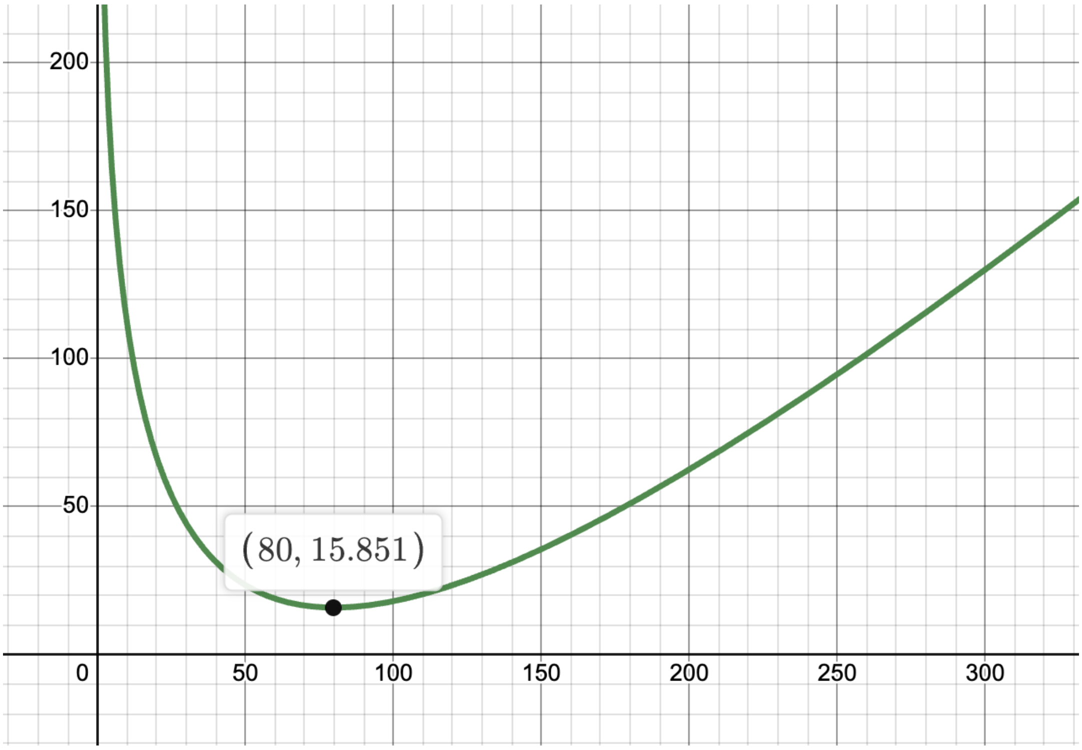
Check the second order condition:

$$V_T'' = \frac{80}{S_T^2} > 0 \quad \Rightarrow \quad \min$$

That is:

$$V_T^* = 80 - 100 - 80 \log\left(\frac{80}{100}\right) + 18 \approx 15.851 > 0$$

The graphical representation:



Exercise 2

Let's consider the portfolio assets:

$$\Theta = (+1 \text{ US.Biden}, -1 \text{ PA.Biden})$$

Then the value of the portfolio:

$$V_0 = 1 \cdot 0.83 - 1 \cdot 0.84 = -0.01 < 0$$

Suppose Biden won an election and lost PA:

$$V_T = \theta_{\text{US.Biden}} \cdot X_T^{\text{US.Biden}} + \theta_{\text{PA.Biden}} \cdot X_T^{\text{PA.Biden}} = 1 \cdot 1 - 1 \cdot 0 = 1$$

And Biden won an election and win PA:

$$V_T = \theta_{\text{US.Biden}} \cdot X_T^{\text{US.Biden}} + \theta_{\text{PA.Biden}} \cdot X_T^{\text{PA.Biden}} = 1 \cdot 1 - 1 \cdot 1 = 0$$

And Trump won an election:

$$V_T = \theta_{\text{US.Biden}} \cdot X_T^{\text{US.Biden}} + \theta_{\text{PA.Biden}} \cdot X_T^{\text{PA.Biden}} = 1 \cdot 0 - 1 \cdot 0 = 0$$

As a result, we have **type 2 arbitrage**: we initially receive a credit which we will definitely not repay.

2. Let's consider the portfolio assets:

$$\Theta = (+1 \text{ US.Biden}, +1 \text{ AZ.Trump}, +1 \text{ GA.Trump}, +1 \text{ PA.Trump}, +1 \text{ GA.Biden}, +1 \text{ AZ.Biden}, -2.99 \text{ B})$$

Then the value of the portfolio:

$$V_0 = 0.83 + 0.2 + 0.44 + 0.16 + 0.56 + 0.8 - 2.99 = 0$$

Suppose Biden won the election in all 3 states:

$$V_T = 1 + 1 + 1 - 2.99 = 3 - 2.99 = 0.01$$

Biden won the election in GA and AZ (so Trump won PA):

$$V_T = 1 + 1 + 1 + 1 - 2.99 = 1.01$$

Biden won the election in GA and PA **OR** in AZ and PA (the payoffs are the same):

$$V_T = 1 + 1 + 1 + 1 - 2.99 = 1.01$$

And Biden won the election in exactly one state (the result is indifferent in what particular state, because Trump won in other states):

$$V_T = 4 - 2.99 = 1.01$$

Trump won the election iff he won in each of these 3 states:

$$V_T = 3 - 2.99 = 0.01$$

As a result, we have **type 1 arbitrage**: zero initial value, no risk of loss, some chance of gain (in our particular case, we always receive a positive gain).