

# Homework 5

Sergei Tikhonov

October 29th 2023

## Exercise 1

We need to write the process:

$$Z_t = \exp(W_t^2 - 1)$$

in terms of drift and diffusion components.

(a) Let's start with assuming  $X_t = W_t$ . Corresponding deterministic function is  $f(x) = \exp(x^2 - 1)$ . Let's apply Ito's Lemma:

$$\begin{aligned} d(\exp(X_t^2 - 1)) &= f'_x dX_t + \frac{1}{2} f''_{xx} (dX_t)^2 = \\ &= 2X_t \exp(X_t^2 - 1) dX_t + \frac{1}{2} (2 \cdot \exp(X_t^2 - 1) + 2X_t \exp(X_t^2 - 1) 2X_t) (dX_t)^2 \end{aligned}$$

Substituting back  $W_t = X_t$ :

$$d(\exp(W_t^2 - 1)) = 2W_t \exp(W_t^2 - 1) dW_t + (2W_t^2 + 1) \exp(W_t^2 - 1) (dW_t)^2$$

As a result:

$$d(\exp(W_t^2 - 1)) = (2W_t^2 + 1) \exp(W_t^2 - 1) dt + 2W_t \exp(W_t^2 - 1) dW_t$$

(b) For now, let's assume  $X_t = W_t^2 - 1$ . We know that  $dX_t = dt + 2W_t dW_t$  and corresponding deterministic function is  $f(x) = \exp(x)$ . Let's apply Ito's Lemma again:

$$\begin{aligned} d(\exp(X_t)) &= f'_x dX_t + \frac{1}{2} f''_{xx} (dX_t)^2 = \exp(X_t) dX_t + \frac{1}{2} \exp(X_t) (dX_t)^2 \\ &= \exp(W_t^2 - 1) (dt + 2W_t dW_t) + \frac{1}{2} \exp(W_t^2 - 1) [(dt)^2 + 2W_t (dt)(dW_t) + 4W_t^2 (dW_t)^2] \end{aligned}$$

As a result:

$$d(\exp(X_t)) = \exp(W_t^2 - 1) (1 + 2W_t^2) dt + 2W_t \exp(W_t^2 - 1) dW_t$$

(c) To check whether  $Z_t$  is a martingale, one can have a look at drift term. In order for  $Z_t$  to be a

martingale,  $\mu_t$  should be equal to 0 with probability 1.

$$\mu_t = \underbrace{\exp(W_t^2 - 1)}_{>0} \underbrace{(1 + 2W_t^2)}_{>0} \neq 0 \quad \text{a.s.}$$

Here we can see that it's not equal to zero, that's why  $Z_t$  is not a martingale.

## Exercise 2

Let's consider the following stochastic differential equation:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

The corresponding deterministic function is  $f(t, x) = \exp(\kappa t)x$ . Let's apply Ito's Lemma:

$$d(\exp(\kappa t)X_t) = f'_t dt + f'_x dX_t + \frac{1}{2}f''_{tt}(dt)^2 + f''_{tx}(dX_t)(dt) + \frac{1}{2}f''_{xx}dX_t^2$$

$$d(\exp(\kappa t)X_t) = \kappa \exp(\kappa t)X_t dt + \exp(\kappa t)dX_t + \frac{1}{2}\kappa^2 \exp(\kappa t)(dt)^2 + \kappa \exp(\kappa t)(dX_t)(dt) + \frac{1}{2} \cdot 0 \cdot (dX_t)^2$$

After simplifying the statement above:

$$d(\exp(\kappa t)X_t) = \kappa \exp(\kappa t)X_t dt + \exp(\kappa t)(\kappa(\theta - X_t)dt + \sigma dW_t)$$

$$d(\exp(\kappa t)X_t) = \kappa \exp(\kappa t)X_t dt + \exp(\kappa t)\sigma dW_t$$

Let's now rewrite the Ito process above using integral notation:

$$\exp(\kappa t)X_t = X_0 + \int_0^t \kappa \theta \exp(\kappa s) ds + \int_0^t \exp(\kappa s) \sigma dW_s$$

As a result (here I also substituted  $s$  to  $t$  just for notation purposes):

$$X_T = \exp(-\kappa T)X_0 + \int_0^T \kappa \theta \exp(\kappa(t - T)) dt + \int_0^T \exp(\kappa(t - T)) \sigma dW_t$$

(b) Here we can calculate the Riemann integral at time moment  $T$ :

$$X_T = \exp(-\kappa T)X_0 + \theta(1 - \exp(-\kappa T)) + \int_0^T \exp(\kappa(t - T)) \sigma dW_t$$

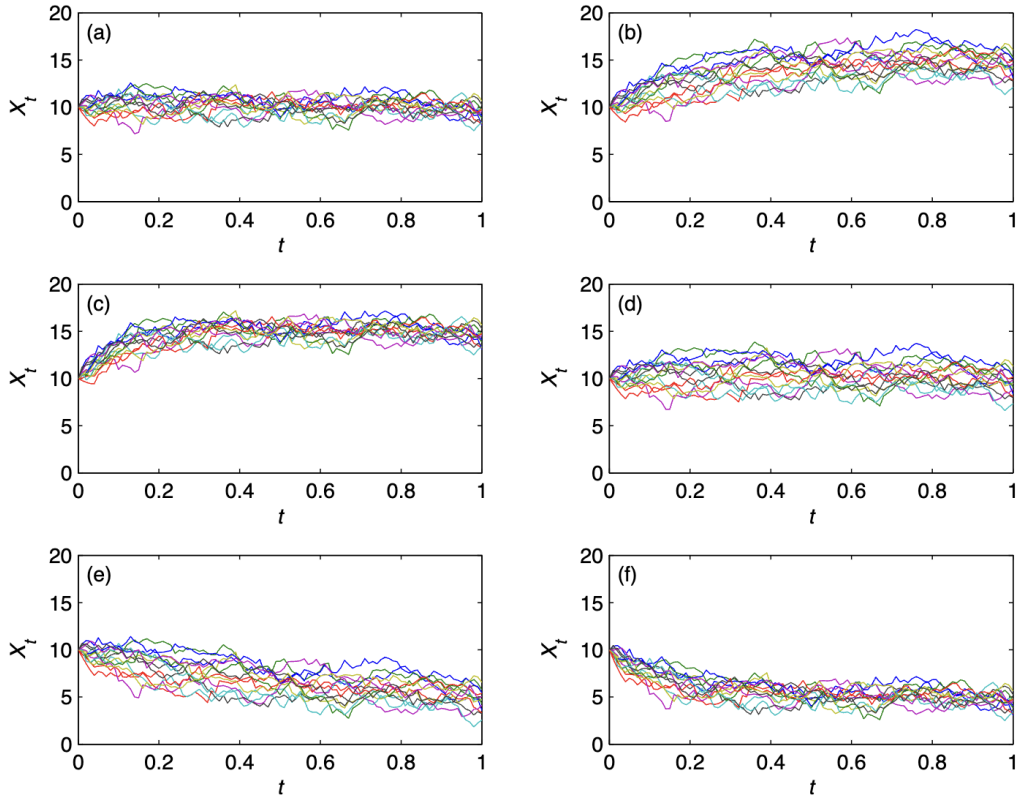
By definition, expected value of diffusion part is equal to zero. Then the expected value of  $X_T$  is:

$$\mathbb{E}(X_T) = \exp(-\kappa T)X_0 + \theta(1 - \exp(-\kappa T))$$

Using the property Ito isometry, let's calculate the variance of  $X_T$ :

$$\begin{aligned}\mathbb{V}\text{ar}(X_T) &= \mathbb{V}\text{ar}\left(\int_0^T \exp(\kappa(t-T))\sigma dW_t\right) = \int_0^T \sigma^2 \exp(2\kappa(t-T))dt = \\ &= \frac{\sigma^2}{2\kappa}(1 - \exp(-2\kappa T))\end{aligned}$$

### Exercise 3



(a)  $\theta = 10, k = 8$

(d)  $\theta = 10, k = 3$

(b)  $\theta = 15, k = 3$

(e)  $\theta = 5, k = 3$

(c)  $\theta = 15, k = 8$

(f)  $\theta = 5, k = 8$