Problem Set 2

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Exercise 1

1. Let's calculate the following probability:

$$q_n = \mathbb{P}\{S_{2n} = 0\} =$$

Notice that S_{2n} is a random variable with binomial distribution, because we have a sequence of $2n$ independent experiments with two outcomes: $+1$ and -1. For this random variable to be zero, there should be n "+1" and n 1" among $2n$ trials.

$$= \binom{2n}{n} p^n (1-p)^{2n-n} = \frac{(2n)!}{n!n!} p^n (1-p)^n$$

Let's use Stirling's formula to deal with factorial:

$$\frac{\sqrt{2\pi}(2n)^{2n+\frac{1}{2}}e^{-2n}}{\sqrt{2\pi}(n)^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}(n)^{n+\frac{1}{2}}e^{-n}} p^n (1-p)^n = \frac{2^{2n}}{\sqrt{\pi n}} p^n (1-p)^n = \frac{1}{\sqrt{\pi n}} (4p(1-p))^n$$

2. We will check the convergence of the following sum:

$$\sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} (4p(1-p))^n \leq$$

Let's apply the Comparison test:

$$\leq \sum_{n=1}^{\infty} (4p(1-p))^n =$$

Since $0 < 4p(1-p) < 1$ we will use the formula for the sum of geometric series:

$$= \frac{1}{1 - 4p(1-p)} < \infty$$

The larger series is convergent so the smaller series must also be convergent.

To make a conclusion about the fact the random walk doesn't return to the origin infinitely often, we will return to our reasoning during the lecture. Let V be the total number of visits to the origin and I_n is the indicator function of whether the walker is at the origin at time:

$$V = \sum_{n=1}^{\infty} I_n$$

Using the fact that the expectation of an indicator random variable is just the probability of that event:

$$\mathbb{E}(V) = \sum_{n=1}^{\infty} \mathbb{P}\{S_n = 0\} = \sum_{n=1}^{\infty} q_n < \infty$$

As a result, the random walk does not return to the origin infinitely often.

Exercise 2

We will consider the symmetric random walk:

$$\mathbb{P}\{X_j = 1\} = \frac{1}{2} \qquad \mathbb{P}\{X_j = -1\} = \frac{1}{2}$$

And the asymmetric random walk from the Exercise 1, $0.5 < p < 1$:

$$\mathbb{P}\{X_j = 1\} = 1 - p \qquad \mathbb{P}\{X_j = -1\} = p$$

We will start with calculating $\mathbb{E}(X_1)$, $\mathbb{V}\mathrm{ar}(X_1)$, $\mathbb{E}(S_n)$, and $\mathbb{V}\mathrm{ar}(S_n)$:

$$\mathbb{E}(X_1) = 1 \cdot (1-p) + (-1) \cdot p = 1 - 2p$$

$$\mathbb{V}\mathrm{ar}(X_1) = \mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2 = 1 - (1-2p)^2 = 4(1-p)p$$

$$\mu_n = \mathbb{E}(S_n) = \mathbb{E}(X_1 + ... + X_n) = \mathbb{E}(X_1) + ... + \mathbb{E}(X_n) = n\mathbb{E}(X_1) = n(1-2p)$$

$$\sigma_n^2 = \mathbb{V}\mathrm{ar}(S_n) = \mathbb{V}\mathrm{ar}(X_1 + ... + X_n) = \mathbb{V}\mathrm{ar}(X_1) + ... + \mathbb{V}\mathrm{ar}(X_n) = n\,\mathbb{V}\mathrm{ar}(X_1) = 4n(1-p)p$$

According to the Central Limit Theorem, as $n \to \infty$ the random variable $Z = \frac{S_n - \mu_n}{\sigma_n} \sim N(0,1)$. Let's consider the cases with different parameter p. Specifically, we will start with symmetric random walk, so $p = \frac{1}{2}$:

$$\mu_n = \mathbb{E}(S_n) = n(1 - 2 \cdot \frac{1}{2}) = 0$$

$$\sigma_n^2 = \mathbb{V}\mathrm{ar}(S_n) = n$$

Then the probability is:

$$\lim_{n\to\infty} \mathbb{P}\left\{S_n < \frac{2}{3}\sqrt{n}\right\} = \lim_{n\to\infty} \mathbb{P}\left\{\frac{S_n}{\sqrt{n}} < \frac{2}{3}\right\} = \mathbb{P}\left\{Z < \frac{2}{3}\right\} = \Phi\left(\frac{2}{3}\right) \approx 0.7454$$

Now, we will consider the case when $0.5 < p < 1$:

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{S_n - n(1-2p)}{2\sqrt{n(1-p)p}} < \frac{\frac{2}{3}\sqrt{n} - n(1-2p)}{2\sqrt{n(1-p)p}}\right\} = \lim_{n\to\infty} \mathbb{P}\left\{Z < \underbrace{\frac{\frac{2}{3}\sqrt{n} - n(1-2p)}{2\sqrt{n(1-p)p}}}_{L(n)}\right\} =$$

$$= \lim_{n\to\infty} \int_{-\infty}^{L(n)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$$

Taking into account $0.5 < p < 1$, we will calculate the limit above:

$$\lim_{n\to\infty} L(n) = \lim_{n\to\infty} \frac{-n(1-2p) + \frac{2}{3}\sqrt{n}}{2\sqrt{n(1-p)p}} = \lim_{n\to\infty} \frac{-n(1-2p)}{2\sqrt{n(1-p)p}} = +\infty$$

For now, we will use the property that the area under the entire curve is equal to 1:

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

This result is expected, because the random walk is asymmetric with, in our case, $P\{X_j = -1\} > \mathbb{P}\{X_j = 1\}$, so as $n \to \infty$, S_n will eventually decrease and will be less than some positive value.

Exercise 3

Suppose we have the simple symmetric random walk:

$$S_n = S_0 + X_1 + ... + X_n \qquad\qquad S_0 = 0$$

Also, let's say that N - number of steps until we have an upswing time.

In order to solve this problem, I will apply "first step analysis". The idea is to notice that each time after $X_j = -1$ we start looking for 5 consecutive ”+1” again but continue our counter with these additional steps. In this situation, we can simply use the law of total expectation to calculate the expected number of steps until we have an upswing time.

The law of total expectation:

$$\mathbb{E}(N) = \mathbb{E}(N|\text{”}-1\text{”})\cdot\mathbb{P}\{\text{”}-1\text{”}\} + \mathbb{E}(N|\text{”}+1,-1\text{”})\cdot\mathbb{P}\{\text{”}+1,-1\text{”}\} + \mathbb{E}(N|\text{”}+1,+1,-1\text{”})\cdot\mathbb{P}\{\text{”}+1,+1,-1\text{”}\} +$$

$$+\mathbb{E}(N|\text{”}+1,+1,+1,-1\text{”})\cdot\mathbb{P}\{\text{”}+1,+1,+1,-1\text{”}\} + \mathbb{E}(N|\text{”}+1,+1,+1,+1,-1\text{”})\cdot\mathbb{P}\{\text{”}+1,+1,+1,+1,-1\text{”}\} +$$

$$+\mathbb{E}(N|\text{”}+1,+1,+1,+1,+1\text{”})\cdot\mathbb{P}\{\text{”}+1,+1,+1,+1,+1\text{”}\}$$

Suppose we already observed N steps. For example, let's consider the case when we observe two consecutive ”+1” and then ”−1”: we start looking for 5 consecutive ”+1” again but have $N + 3$ steps (two for ”+1” and one for ”−1”). That is why $\mathbb{E}(N|\text{”}+1+1-1\text{”}) = \mathbb{E}(N) + 3$. Using this observation, let's substitute conditional expected values:

$$\mathbb{E}(N) = (\mathbb{E}(N)+1)\frac{1}{2} + (\mathbb{E}(N)+2)\frac{1}{4} + (\mathbb{E}(N)+3)\frac{1}{8} + (\mathbb{E}(N)+4)\frac{1}{16} + (\mathbb{E}(N)+5)\frac{1}{32} + 5\frac{1}{32}$$

$$\frac{1}{32}\mathbb{E}(N) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{5}{32} = \frac{62}{32}$$

$$\mathbb{E}(N) = 62$$