Problem Set 5

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Exercise 1

Let's define simple symmetric random walk:

$$X_1, X_2, ... - \text{iid such that:}$$

$$\mathbb{E}(X_j) = 0 \qquad \mathbb{V}\text{ar}(X_j) = \mathbb{E}(X_j^2) = \sigma^2$$

$$\mathbb{P}\{X_j = a\} = \mathbb{P}\{X_j = -a\} = \frac{1}{2}, \quad a \in \mathbb{R}_+$$

We know that in the discrete case we can check only one-step condition to check whether a stochastic process is a martingale or not (because of the tower rule):

$$\mathbb{E}(M_n|\mathcal{F}_m) = M_m \qquad \Longleftrightarrow \qquad \mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$$

$$\text{for } n > m$$

so we can use the right-hand side rule to determine whether M_n is a martingale.

Also, the condition that:

$$\mathbb{E}(|M_n|) < \infty$$

is satisfied for all the process below because we deal with the Borel functions on simple symmetric random walk with finite expectation:

$$\mathbb{E}(S_n) = \mathbb{E}(S_0 + X_1 + .. + X_n) = 0$$

1.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_n + X_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_n|\mathcal{F}_n) + \mathbb{E}(X_{n+1}|\mathcal{F}_n) = S_n + \mathbb{E}(X_{n+1}) = S_n = M_n$$

This is a martingale, supermartingale and submartingale with respect to $\mathcal{F}_n$.

2.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+1}^2|\mathcal{F}_n) = \mathbb{E}((S_n + X_{n+1})^2|\mathcal{F}_n) = \mathbb{E}(S_n^2 + 2S_nX_{n+1} + X_{n+1}^2|\mathcal{F}_n) =$$

$$S_n^2 + 2S_n\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) = S_n^2 + \sigma^2 > M_n$$

This is a submartingale with respect to $\mathcal{F}_n$.

3.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+1}^3|\mathcal{F}_n) = \mathbb{E}((S_n + X_{n+1})^3|\mathcal{F}_n) = \mathbb{E}(S_n^3 + 3 \cdot S_n^2 \cdot X_{n+1} + 3 \cdot S_n \cdot X_{n+1}^2 + X_{n+1}^3|\mathcal{F}_n) =$$

$$= \mathbb{E}(S_n^3|\mathcal{F}_n) + 3 \cdot S_n^2 \cdot \mathbb{E}(X_{n+1}) + 3 \cdot S_n \cdot \mathbb{E}(X_{n+1}^2) + \mathbb{E}(X_{n+1}^3) = S_n^3 + 3 \cdot S_n \cdot \sigma^2$$

Because the expression depends on the sign of S_n (whether it is positive, negative, or zero), this is not a martingale.

4.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(2^{S_{n+1}}|\mathcal{F}_n) = \mathbb{E}(2^{S_n + X_{n+1}}|\mathcal{F}_n) = 2^{S_n}\mathbb{E}(2^{X_{n+1}}) = M_n \cdot \mathbb{E}(2^{X_{n+1}}) > M_n$$

The last inequality holds because of the following:

$$\mathbb{E}(2^{X_{n+1}}) = 2^a \cdot \frac{1}{2} + 2^{-a} \cdot \frac{1}{2} = \frac{1}{2}(2^a + 2^{-a}) > 1 \qquad \forall a > 0$$

That is why this is a submartingale with respect to $\mathcal{F}_n$.

5.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}\left(\frac{S_{n+1}}{n+1}\middle|\mathcal{F}_n\right) = \frac{1}{n+1}\mathbb{E}(S_{n+1}|\mathcal{F}_n) = \frac{1}{n+1}S_n < M_n$$

This is a supermartingale with respect to $\mathcal{F}_n$.

6.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+2}S_{n+1}|\mathcal{F}_n) = \mathbb{E}((S_n + X_{n+1} + X_{n+2})(S_n + X_{n+1})|\mathcal{F}_n) =$$

$$= \mathbb{E}(S_n^2 + S_n \cdot X_{n+1} + S_n \cdot X_{n+1} + X_{n+1}^2 + S_n \cdot X_{n+2} + X_{n+2} \cdot X_{n+1}|\mathcal{F}_n) =$$

$$= S_n^2 + 2 \cdot S_n \cdot \mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) + S_n \cdot \mathbb{E}(X_{n+2}) + \mathbb{E}(X_{n+2}X_{n+1}) = S_n^2 + \sigma^2 + \mathbb{E}(X_{n+2})\mathbb{E}(X_{n+1}) = S_n^2 + \sigma^2$$

This is not a martingale.

7.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(X_0X_1 + X_1X_2 + ... + X_{n-1}X_n + X_nX_{n+1}|\mathcal{F}_n) =$$

$$= X_0X_1 + X_1X_2 + ... + X_{n-1}X_n + X_n \cdot \mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_0X_1 + X_1X_2 + ... + X_{n-1}X_n = M_n$$

This is a martingale, supermartingale and submartingale with respect to $\mathcal{F}_n$.

Exercise 2

R_n, G_n - number of red and green balls at time moment n, respectively.

$$R_0 = 1,\ G_0 = 1 \quad a.s.$$

The share of red balls M_n at time moment n:

$$M_n = \frac{R_n}{R_n + G_n} = \frac{R_n}{n+2}$$

What we need to prove is:

$$\mathbb{P}\left\{M_n = \frac{k}{n+2}\right\} = \frac{1}{n+1} \qquad k = 1, 2, ..., n+1$$

Let's use the method of mathematical induction:
Base case: $n = 0$

$$\mathbb{P}\left\{M_0 = \frac{k}{n+2}\right\} = \mathbb{P}\left\{M_0 = \frac{1}{2}\right\} = 1 = \frac{1}{0+1} \qquad \text{since } M_0 = \frac{R_0}{R_0 + G_0} = \frac{1}{1+1} = \frac{1}{2} \quad a.s.$$

Inductive step: Assume it holds for n, that is:

$$\mathbb{P}\left\{M_n = \frac{k}{n+2}\right\} = \frac{1}{n+1} \qquad k = 1, 2, ..., n+1$$

Then for $n := m+1$ (just for notation reasons, in order not to confuse n here and n in inductive assumption):

$$\mathbb{P}\left\{M_{m+1} = \frac{k}{m+3}\right\} =$$

Using the fact that M_n is a non-homogeneous time Markov Chain, let's calculate the total probability for $k = 1, 2, ..., m$:

$$= \mathbb{P}\left\{M_m = \frac{k}{m+2}\right\} \cdot \mathbb{P}\left\{\text{"red drawn"}\right\} + \mathbb{P}\left\{M_m = \frac{k+1}{m+2}\right\} \cdot \mathbb{P}\left\{\text{"green drawn"}\right\} =$$

Using the induction assumption and substituting the probabilities of drawing red or green ball:

$$\frac{1}{m+1} \cdot \frac{k}{m+2} + \frac{1}{m+1} \cdot \frac{m+2-k-1}{m+2} = \frac{1}{m+2}$$

Since $n = m+1$ as it was introduced before, the inductive step is succesfully proved and M_n has uniform distribution on the set:

$$\left\{\frac{1}{n+2}, ..., \frac{n+1}{n+2}\right\}$$

2. The code for simulations starts from the next page.

My simulations are in line with the logic above: the share of read balls has uniform distribution. Specifically, we do simulation for big n, so we can see the convergence to continuous random variable $U[0,1]$.

```python
import numpy as np
import matplotlib.pyplot as plt

def simulate_Polya_urn():
    iterations = 4000
    draws = []

    draws.append('R')
    draws.append('G')

    for n in range(iterations):
        red_probability = draws.count('R') / len(draws)
        green_probability = draws.count('G') / len(draws)
        new_ball = np.random.choice(a = ['R', 'G'], p =
[red_probability, green_probability])
        draws.append(new_ball)
        if n == 2000:
            red = draws.count('R') / len(draws)
            green = draws.count('G') / len(draws)
            print('After 2000 iterations:')
            print(f'The share of red balls is {red}')
            print(f'the share of white balls is {green}')

    red = draws.count('R') / len(draws)
    green = draws.count('G') / len(draws)
    print('After 4000 iterations:')
    print(f'The share of red balls is {red}')
    print(f'the share of white balls is {green}\n')

    return draws

for epoch in range(30):
    print(f'Epoch number is {epoch}')
    draws = simulate_Polya_urn()
```

```
Epoch number is 0
After 2000 iterations:
The share of red balls is 0.13879181228157764
the share of white balls is 0.8612081877184223
After 4000 iterations:
The share of red balls is 0.14292853573213393
the share of white balls is 0.8570714642678661

Epoch number is 1
After 2000 iterations:
The share of red balls is 0.326510234648028
the share of white balls is 0.673489765351972
After 4000 iterations:
The share of red balls is 0.31859070464767614
the share of white balls is 0.6814092953523239
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Epoch number is 2
After 2000 iterations:
The share of red balls is 0.7668497254118821
the share of white balls is 0.23315027458811782
After 4000 iterations:
The share of red balls is 0.7661169415292354
the share of white balls is 0.23388305847076463

Epoch number is 3
After 2000 iterations:
The share of red balls is 0.3914128806789815
the share of white balls is 0.6085871193210185
After 4000 iterations:
The share of red balls is 0.40429785107446276
the share of white balls is 0.5957021489255372

Epoch number is 4
After 2000 iterations:
The share of red balls is 0.5976035946080879
the share of white balls is 0.4023964053919121
After 4000 iterations:
The share of red balls is 0.5939530234882558
the share of white balls is 0.40604697651174415

Epoch number is 5
After 2000 iterations:
The share of red balls is 0.17174238642036946
the share of white balls is 0.8282576135796306
After 4000 iterations:
The share of red balls is 0.16841579210394803
the share of white balls is 0.8315842078960519

Epoch number is 6
After 2000 iterations:
The share of red balls is 0.36994508237643536
the share of white balls is 0.6300549176235647
After 4000 iterations:
The share of red balls is 0.3663168415792104
the share of white balls is 0.6336831584207896

Epoch number is 7
After 2000 iterations:
The share of red balls is 0.24413379930104842
the share of white balls is 0.7558662006989516
After 4000 iterations:
The share of red balls is 0.23813093453273362
the share of white balls is 0.7618690654672664

Epoch number is 8
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After 2000 iterations:
The share of red balls is 0.7878182725911134
the share of white balls is 0.21218172740888666
After 4000 iterations:
The share of red balls is 0.7788605697151424
the share of white balls is 0.22113943028485758

Epoch number is 9
After 2000 iterations:
The share of red balls is 0.6949575636545182
the share of white balls is 0.3050424363454818
After 4000 iterations:
The share of red balls is 0.6816591704147926
the share of white balls is 0.3183408295852074

Epoch number is 10
After 2000 iterations:
The share of red balls is 0.7728407388916625
the share of white balls is 0.2271592611083375
After 4000 iterations:
The share of red balls is 0.7741129435282359
the share of white balls is 0.22588705647176413

Epoch number is 11
After 2000 iterations:
The share of red balls is 0.671992011982027
the share of white balls is 0.32800798801797304
After 4000 iterations:
The share of red balls is 0.6764117941029485
the share of white balls is 0.32358820589705145

Epoch number is 12
After 2000 iterations:
The share of red balls is 0.042935596605092365
the share of white balls is 0.9570644033949076
After 4000 iterations:
The share of red balls is 0.04997501249375312
the share of white balls is 0.9500249875062469

Epoch number is 13
After 2000 iterations:
The share of red balls is 0.873689465801298
the share of white balls is 0.12631053419870195
After 4000 iterations:
The share of red balls is 0.8648175912043978
the share of white balls is 0.1351824087956022

Epoch number is 14
After 2000 iterations:
The share of red balls is 0.4328507239141288
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the share of white balls is 0.5671492760858712
After 4000 iterations:
The share of red balls is 0.4337831084457771
the share of white balls is 0.5662168915542228

Epoch number is 15
After 2000 iterations:
The share of red balls is 0.15327009485771342
the share of white balls is 0.8467299051422865
After 4000 iterations:
The share of red balls is 0.1489255372313843
the share of white balls is 0.8510744627686156

Epoch number is 16
After 2000 iterations:
The share of red balls is 0.7294058911632552
the share of white balls is 0.2705941088367449
After 4000 iterations:
The share of red balls is 0.724887556221889
the share of white balls is 0.27511244377811095

Epoch number is 17
After 2000 iterations:
The share of red balls is 0.07338991512730904
the share of white balls is 0.9266100848726909
After 4000 iterations:
The share of red balls is 0.06321839080459771
the share of white balls is 0.9367816091954023

Epoch number is 18
After 2000 iterations:
The share of red balls is 0.21517723414877685
the share of white balls is 0.7848227658512231
After 4000 iterations:
The share of red balls is 0.2178910544727636
the share of white balls is 0.7821089455272364

Epoch number is 19
After 2000 iterations:
The share of red balls is 0.7329006490264603
the share of white balls is 0.2670993509735397
After 4000 iterations:
The share of red balls is 0.7298850574712644
the share of white balls is 0.27011494252873564

Epoch number is 20
After 2000 iterations:
The share of red balls is 0.1907139291063405
the share of white balls is 0.8092860708936596
After 4000 iterations:
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The share of red balls is 0.1934032983508246
the share of white balls is 0.8065967016491754

Epoch number is 21
After 2000 iterations:
The share of red balls is 0.8047928107838243
the share of white balls is 0.19520718921617575
After 4000 iterations:
The share of red balls is 0.8030984507746127
the share of white balls is 0.1969015492253873

Epoch number is 22
After 2000 iterations:
The share of red balls is 0.3165252121817274
the share of white balls is 0.6834747878182726
After 4000 iterations:
The share of red balls is 0.3183408295852074
the share of white balls is 0.6816591704147926

Epoch number is 23
After 2000 iterations:
The share of red balls is 0.5386919620569146
the share of white balls is 0.4613080379430854
After 4000 iterations:
The share of red balls is 0.5464767616191905
the share of white balls is 0.4535232383808096

Epoch number is 24
After 2000 iterations:
The share of red balls is 0.7194208686969545
the share of white balls is 0.28057913130304546
After 4000 iterations:
The share of red balls is 0.7178910544727636
the share of white balls is 0.28210894552723637

Epoch number is 25
After 2000 iterations:
The share of red balls is 0.9191213180229656
the share of white balls is 0.08087868197703445
After 4000 iterations:
The share of red balls is 0.9250374812593704
the share of white balls is 0.07496251874062969

Epoch number is 26
After 2000 iterations:
The share of red balls is 0.417873190214678
the share of white balls is 0.582126809785322
After 4000 iterations:
The share of red balls is 0.4107946026986507
the share of white balls is 0.5892053973013494
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Epoch number is 27
After 2000 iterations:
The share of red balls is 0.26959560659011483
the share of white balls is 0.7304043934098852
After 4000 iterations:
The share of red balls is 0.27136431784107945
the share of white balls is 0.7286356821589205

Epoch number is 28
After 2000 iterations:
The share of red balls is 0.3594608087868198
the share of white balls is 0.6405391912131803
After 4000 iterations:
The share of red balls is 0.3750624687656172
the share of white balls is 0.6249375312343828

Epoch number is 29
After 2000 iterations:
The share of red balls is 0.8062905641537693
the share of white balls is 0.19370943584623065
After 4000 iterations:
The share of red balls is 0.8160919540229885
the share of white balls is 0.1839080459770115
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