# Problem Set 6

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September 13th 2023

## Exercise 1

1. $T = \min\{n : S_n < 0\}$ is $\mathcal{F}_n$ measurable because at time moment $n$ we have enough information $X_1, ..., X_n$ to calculate $S_n = X_1 + ... + X_n$. This is why $T$ is a stopping time.

2. $T = \min\{n : \frac{S_n}{n} > S_1\}$ is $\mathcal{F}_n$ measurable because at time moment $n$ we have enough information $X_1, ..., X_n$ to calculate $S_n = X_1 + ... + X_n$ and divide it for $n$. This is why $T$ is a stopping time.

3. $T = \min\{n : S_{n+1} > S_n\}$ is **not** $\mathcal{F}_n$ measurable because at time moment $n$ we don't have enough information $X_1, ..., X_n$ to calculate $S_{n+1} = X_1 + ... + X_{n+1}$. This is why $T$ is not a stopping time.

4. $\tau = \min\{m : S_m \geq 4\}$ and $T = \min\{n > m : S_n \leq -5\}$.
$\tau$ is $\mathcal{F}_n$ measurable because $n > m$, so we have enough information $X_1, ..., X_m$ to calculate $S_m = X_1 + ... + X_m$ (in fact, $\mathcal{F}_m \subset \mathcal{F}_n$).

$T$ is also $\mathcal{F}_n$ measurable since we have enough information $X_1, ..., X_n$ to calculate $S_n = X_1 + ... + X_n$. So $T$ is a stopping time.

Obviously, $\tau$ is also a stopping time with respect to $\mathcal{F}_m$, because it's the very same situation as in the part 1.

## Exercise 2

1. Let's just check whether it is a martingale, submartingale, or supermartingale just using the definition:

$$\mathbb{E}(S_n|\mathcal{F}_m) = \mathbb{E}(S_m + X_{m+1} + ... + X_{m+n}) = S_m + (n - m) \cdot \mathbb{E}(X_n) =$$

Let's calculate the expected value of $X_j$:

$$\mathbb{E}(X_j) = 1 \cdot q + (1 - q) \cdot (-1) = 2q - 1$$

As a result:

$$= S_m + (2q - 1)(n - m) = S_m + (2q - 1)(n - m)$$

Since $n > m$ and $2q - 1 > 0$, this is a submartingale.

2. Let's find $r$ such that $M_n$ is a martingale:

$$\mathbb{E}(M_n|\mathcal{F}_m) = \mathbb{E}(S_n + rn|\mathcal{F}_m) = rn + \mathbb{E}(S_m|\mathcal{F}_m) = rn + S_m + (2q-1)(n-m) =$$

In order to be a martingale:

$$r = -(2q-1)$$

So:

$$= S_m - (2q-1)m = M_m$$

3. Let's prove that $M_n$ is a martingale by definition:

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(\theta^{S_n+X_{n+1}}|\mathcal{F}_n) = \theta^{S_n}\mathbb{E}(\theta^{X_{n+1}}|\mathcal{F}_n) = \theta^{S_n}(\theta^1 \cdot q + \theta^{-1} \cdot (1-q)) =$$

$$= \theta^{S_n}\left(\frac{1-q}{q} \cdot q + \frac{q}{1-q} \cdot (1-q)\right) = \theta^{S_n} = M_n$$

4. Let's consider the stopping time:

$$T = \min\{j : S_j = b \text{ or } S_j = -a\}$$

Since $S_n$ is not a martingale, we can use $M_n$ from the very previous part.

Let's check the conditions of the Optional Sampling Theorem:
$M_{n\wedge T}$ is definitely bounded:

$$\theta^{-a} \leq M_{n\wedge T} \leq \theta^b$$

For this two-side bounded stopping time, it is $\mathbb{P}\{T < \infty\} = 1$ as was shown in the lecture. To show it, we can majorize it by geometric random variable $M$, for which we know that $\mathbb{P}\{M = \infty\} = 0$. Let's apply the Optional Sampling Theorem (III):

$$\mathbb{E}(M_T) = \mathbb{E}(M_0)$$

$$\mathbb{P}\{S_T = b\} \cdot \theta^b + (1 - \mathbb{P}\{S_T = b\}) \cdot \theta^{-a} = 1$$

As a result:

$$\mathbb{P}\{S_T = b\} = \frac{1-\theta^{-a}}{\theta^b - \theta^{-a}}$$

5. Let's see what will be in the case when a goes to infinity:

$$T_b = \lim_{a\to+\infty} T_{a,b} = \lim_{a\to+\infty} \min\{j : S_j = b \text{ or } S_j = -a\}$$

$$\mathbb{P}\{T_b = \infty\} = \lim_{a\to+\infty} \mathbb{P}\{S_{T_{a,b}} = -a\} = \lim_{a\to+\infty}\left(1 - \frac{1-\theta^{-a}}{\theta^b - \theta^{-a}}\right) =$$

$$= \lim_{a\to+\infty} \frac{\theta^b - 1}{\theta^b - \theta^{-a}} = \frac{\theta^b - 1}{\theta^b}$$

As a result:

$$\mathbb{P}\{T_b < \infty\} = 1 - \mathbb{P}\{T_b = \infty\} = 1 - \frac{\theta^b - 1}{\theta^b} = \frac{1}{\theta^b}$$

# Exercise 3

1. Let's find $q$ such that $Y_n$ is a martingale:

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}(e^{S_{n+1}}|\mathcal{F}_n) = \mathbb{E}(e^{S_n + X_{n+1}}|\mathcal{F}_n) = e^{S_n}\mathbb{E}(e^{X_{n+1}}) =$$

We will calculate expected value of $e^{X_j}$:

$$\mathbb{E}(e^{X_j}) = e \cdot q + e^{-1} \cdot (1 - q)$$

Then:

$$= e^{S_n}(e \cdot q + e^{-1} \cdot (1 - q)) = Y_n(e \cdot q + e^{-1} \cdot (1 - q))$$

In order for $Y_n$ to be a martingale:

$$e \cdot q + e^{-1} \cdot (1 - q) = 1$$

$$q = \frac{1}{e + 1}$$

2. Since $S_n$ is a discrete stochastic process, the first time when $Y_j > 100$ is $Y_j = e^5$. Because $e^4 < 100$, while $e^5 > 100$:

$$T = \min\{j : Y_j > 100\} = \min\{j : Y_j = e^5\}$$

Let's use the very same trick as in the exercise 2:

$$T_a = \min\{j : Y_j = e^5 \text{ or } Y_j = e^{-a}\} \qquad a > 0$$

That is:

$$T = \lim_{a \to +\infty} \min\{j : Y_j = e^5 \text{ or } Y_j = e^{-a}\} = \min\{j : Y_j = e^5\}$$

For now, let's consider the following probability:

$$\mathbb{P}\{T = \infty\} = \lim_{a \to +\infty} \mathbb{P}\{T_a = -a\} = \lim_{a \to +\infty} \frac{e^5 - 1}{e^5 - e^{-a}} = \frac{e^5 - 1}{e^5}$$

As a result:

$$\mathbb{P}\{T < \infty\} = 1 - \mathbb{P}\{T = \infty\} = 1 - \frac{e^5 - 1}{e^5} = \frac{1}{e^5}$$

3. We can see that this expression is just a moment generating function for binomial distribution

(with Bernouli trials such that $X_j = 1$ with $q$ and $X_j = -1$ with $1 - q$) at the point $t = 2$:

$$\mathbb{E}(Y_n^2) = \mathbb{E}\left(e^{2S_n}\right) =$$

However, let's not use the additional knowledge and just derive everything:

$$= \mathbb{E}\left(e^{2(X_1+X_2+...+X_n)}\right) =$$

Using independence and identical distribution of $X_1, X_2, ..., X_n$:

$$= \mathbb{E}\left(e^{2X_1}\right) \cdot \mathbb{E}\left(e^{2X_2}\right) \cdot ... \cdot \mathbb{E}\left(e^{2X_n}\right) = \left(\mathbb{E}(e^{2X_1})\right)^n =$$

Let's calculate the expected value:

$$\mathbb{E}(e^{2X_1}) = e^{2\cdot 1} \cdot q + e^{2\cdot(-1)} \cdot (1 - q) = \frac{e^3 + 1}{e^2 + e}$$

So the result is:

$$= \Bigg(\underbrace{\frac{e^3 + 1}{e^2 + e}}_{>1}\Bigg)^n \xrightarrow{n\to\infty} \infty$$

That is why $\nexists C < \infty$ such that the expression above holds.