Problem Set 1

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Exercise 1

1. The distributions for discrete random variables X and Y are:

X	1	2	3
$\mathbb{P}\{X = x\}$	0.4	0.2	0.4

Y	1	2	3	4
$\mathbb{P}\{Y = y\}$	0.3	0.25	0.15	0.3

2. Using the distributions found previously, let's calculate expected values for X and Y:

$$\mathbb{E}(X) = \sum_x x \cdot \mathbb{P}\{X = x\} = 1 \cdot 0.4 + 2 \cdot 0.2 + 3 \cdot 0.4 = 2$$

$$\mathbb{E}(Y) = \sum_y y \cdot \mathbb{P}\{Y = y\} = 1 \cdot 0.3 + 2 \cdot 0.25 + 3 \cdot 0.15 + 4 \cdot 0.3 = 2.45$$

Let's calculate the conditional expected values $\mathbb{E}(X|Y)$ and $\mathbb{E}(Y|X)$:

$$\mathbb{E}(X|Y = 1) = \sum_x x \cdot \mathbb{P}\{X = x|Y = 1\} = 1 \cdot \frac{1/10}{3/10} + 2 \cdot \frac{1/10}{3/10} + 3 \cdot \frac{1/10}{3/10} = 2$$

$$\mathbb{E}(X|Y = 2) = \sum_x x \cdot \mathbb{P}\{X = x|Y = 2\} = 1 \cdot \frac{1/10}{1/4} + 2 \cdot \frac{1/20}{1/4} + 3 \cdot \frac{1/10}{1/4} = 2$$

$$\mathbb{E}(X|Y = 3) = \sum_x x \cdot \mathbb{P}\{X = x|Y = 3\} = 1 \cdot 0 + 2 \cdot \frac{1/20}{15/100} + 3 \cdot \frac{1/10}{15/100} = \frac{8}{3}$$

$$\mathbb{E}(X|Y = 4) = \sum_x x \cdot \mathbb{P}\{X = x|Y = 4\} = 1 \cdot \frac{2/10}{3/10} + 2 \cdot 0 + 3 \cdot \frac{1/10}{3/10} = \frac{5}{3}$$

As a result, the conditional expectation of X given Y:

$$\mathbb{E}(X|Y) = \begin{cases} 2, & Y = 1 \\ 2, & Y = 2 \\ \frac{8}{3}, & Y = 3 \\ \frac{5}{3}, & Y = 4 \end{cases}$$

$$\mathbb{E}(Y|X=1) = \sum_y y \cdot \mathbb{P}\{Y=y|X=1\} = 1 \cdot \frac{1/10}{4/10} + 2 \cdot \frac{1/10}{4/10} + 3 \cdot 0 + 4 \cdot \frac{2/10}{4/10} = \frac{11}{4}$$

$$\mathbb{E}(Y|X=2) = \sum_y y \cdot \mathbb{P}\{Y=y|X=2\} = 1 \cdot \frac{1/10}{2/10} + 2 \cdot \frac{1/20}{2/10} + 3 \cdot \frac{1/20}{2/10} + 4 \cdot 0 = \frac{7}{4}$$

$$\mathbb{E}(Y|X=3) = \sum_y y \cdot \mathbb{P}\{Y=y|X=3\} = 1 \cdot \frac{1/10}{4/10} + 2 \cdot \frac{1/10}{4/10} + 3 \cdot \frac{1/10}{4/10} + 4 \cdot \frac{1/10}{4/10} = \frac{5}{2}$$

As a result, the conditional expectation of Y given X:

$$\mathbb{E}(Y|X) = \begin{cases} \frac{11}{4}, & X=1 \\ \frac{7}{4}, & X=2 \\ \frac{5}{2}, & X=3 \end{cases}$$

Using the above expressions, let's check directly $\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X)$ and $\mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(Y)$:

$$\mathbb{E}(\mathbb{E}(X|Y)) = \sum_y \mathbb{E}(X|Y=y) \cdot \mathbb{P}\{Y=y\} = 2 \cdot \frac{3}{10} + 2 \cdot \frac{1}{4} + \frac{8}{3} \cdot \frac{3}{20} + \frac{5}{3} \cdot \frac{3}{10} = 2 = \mathbb{E}(X)$$

$$\mathbb{E}(\mathbb{E}(Y|X)) = \sum_x \mathbb{E}(Y|X=x) \cdot \mathbb{P}\{X=x\} = \frac{11}{4} \cdot \frac{4}{10} + \frac{7}{4} \cdot \frac{2}{10} + \frac{5}{2} \cdot \frac{4}{10} = 2.45 = \mathbb{E}(Y)$$

As expected, the law of total expectation works!

3. Let's consider the random variable $1_A = 1_{\{Y\ is\ odd\}}$. Let's check the following facts:

$$\mathbb{E}(\mathbb{E}(X|Y)1_A) \stackrel{?}{=} \mathbb{E}(X1_A) \qquad \mathbb{E}(\mathbb{E}(Y|X)1_A) \stackrel{?}{=} \mathbb{E}(Y1_A)$$

In order to check it out, I will use the following formula (one of equivalent definitions of conditional expected value):

$$\mathbb{E}(X|A) = \frac{\mathbb{E}(X1_A)}{\mathbb{P}\{A\}} \qquad \mathbb{P}\{A\} > 0$$

We can rewrite this formula:

$$\mathbb{E}(X1_A) = \mathbb{E}(X|A) \cdot \mathbb{P}\{A\}$$

We start with calculating $\mathbb{P}\{A\}$:

$$\mathbb{P}\{A\} = \mathbb{P}\{\text{Y is odd}\} = \mathbb{P}\{Y=1 \cup Y=3\} = \mathbb{P}\{Y=1\} + \mathbb{P}\{Y=3\} = \frac{3}{10} + \frac{3}{20} = \frac{9}{20}$$

Let's consider the random variable $\hat{X} = \mathbb{E}(X|Y)$. Then use the above formula for left-hand side:

$$\mathbb{E}(\mathbb{E}(X|Y)1_A) = \mathbb{E}(\hat{X}1_A) = \mathbb{E}(\hat{X}|A) \cdot \mathbb{P}\{A\} = \left(2 \cdot \frac{3/10}{9/20} + \frac{8}{3} \cdot \frac{3/20}{9/20}\right) \cdot \frac{9}{20} = 1$$

Using the aforementioned formula for left-hand side:

$$\mathbb{E}(X1_A) = \mathbb{E}(X|A) \cdot \mathbb{P}\{A\} = \left(1 \cdot \frac{1/10}{9/20} + 2 \cdot \frac{3/20}{9/20} + 3 \cdot \frac{2/10}{9/20}\right) \cdot \frac{9}{20} = \frac{20}{9} \cdot \frac{9}{20} = 1$$

$\mathbb{E}(\mathbb{E}(X|Y)1_A) = \mathbb{E}(X1_A)$, so at the first case the fact holds.

For now, let's consider the random variable $\hat{Y} = \mathbb{E}(Y|X)$. Checking out left-hand side:

$$\mathbb{E}(\mathbb{E}(Y|X)1_A) = \mathbb{E}(\hat{Y}1_A) = \mathbb{E}(\hat{Y}|A) \cdot \mathbb{P}\{A\} = \left(\frac{11}{4} \cdot \frac{1/10}{9/20} + \frac{7}{4} \cdot \frac{3/20}{9/20} + \frac{10}{4} \cdot \frac{4/20}{9/20}\right) \cdot \frac{9}{20} = \frac{83}{80}$$

And the right-hand side:

$$\mathbb{E}(Y1_A) = \mathbb{E}(Y|A) \cdot \mathbb{P}\{A\} = \left(1 \cdot \frac{3/10}{9/20} + 3 \cdot \frac{6/20}{9/20}\right) \cdot \frac{9}{20} = \frac{15}{9} \cdot \frac{9}{20} = \frac{3}{4}$$

As a result, $\mathbb{E}(\mathbb{E}(Y|X)1_A) \neq \mathbb{E}(Y1_A)$.

Exercise 2

1. We will use the properties of conditional expected value. As X is measurable with respect to $\sigma(X)$:

$$\mathbb{E}((2X+Y)^2|X) = \mathbb{E}(4X^2 + 4XY + Y^2|X) = 4X^2 + 4X\mathbb{E}(Y|X) + \mathbb{E}(Y^2|X) =$$

Y and X are independent:

$$= 4X^2 + 4X\mathbb{E}(Y) + \mathbb{E}(Y^2) =$$

Let's calculate the moments of Y:

$$\mathbb{E}(Y) = \sum_y y \cdot \mathbb{P}\{Y = y\} = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

$$\mathbb{E}(Y^2) = \sum_y y^2 \cdot \mathbb{P}\{Y = y\} = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

Finally, substituting values into the expression:

$$= 4X^2 + 4X \cdot 3.5 + \frac{91}{6} = 4X^2 + 14X + \frac{91}{6}$$

2. Notice that $Y = \frac{Z}{X}$:

$$\mathbb{E}((2X+Y)^2|X,Z) = \mathbb{E}(4X^2 + 4XY + Y^2|X,Z) = 4X^2 + 4Z + \left(\frac{Z}{X}\right)^2$$

3. Let's consider the random variable W. Again, we use the aforementioned properties:

$$W = \mathbb{E}(Z|X) = \mathbb{E}(XY|X) = X\mathbb{E}(Y|X) = X\mathbb{E}(Y) = 3.5X$$

The distribution of W:

W	$3.5 \cdot 1$	$3.5 \cdot 2$	$3.5 \cdot 3$	$3.5 \cdot 4$	$3.5 \cdot 5$	$3.5 \cdot 6$
$\mathbb{P}\{W = w\}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

After multiplication:

W	3.5	7	10.5	14	17.5	21
$\mathbb{P}\{W = w\}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Exercise 3

1. Calculating the moments of X_1 by definition:

$$\mathbb{E}(X_1) = \sum_x x\mathbb{P}\{X_1 = x\} = 3 \cdot \frac{1}{4} + (-1) \cdot \frac{3}{4} = 0$$

$$\mathbb{E}(X_1^2) = \sum_x x^2\mathbb{P}\{X_1 = x\} = 9 \cdot \frac{1}{4} + (-1)^2 \cdot \frac{3}{4} = 3$$

$$\mathbb{E}(X_1^3) = \sum_x x^3\mathbb{P}\{X = x\} = 27 \cdot \frac{1}{4} + (-1)^3 \cdot \frac{3}{4} = 6$$

2. Calculating the moments of S_n. We use the property of linearity:

$$\mathbb{E}(S_n) = \mathbb{E}(X_1 + ... + X_n) = \mathbb{E}(X_1) + ... + \mathbb{E}(X_n) = n \cdot \mathbb{E}(X_1) = n \cdot 0 = 0$$

Taking into account that $\mathbb{V}\text{ar}(S_n) = \mathbb{E}(S_n^2) - (\mathbb{E}(S_n))^2$ and that $X_1, ..., X_n$ are iid, let's apply the properties of variance:

$$\mathbb{E}(S_n^2) = \mathbb{V}\text{ar}(S_n) + (\mathbb{E}(S_n))^2 = \mathbb{V}\text{ar}(S_n) = \mathbb{V}\text{ar}(X_1 + ... + X_n) = \mathbb{V}\text{ar}(X_1) + ... + \mathbb{V}\text{ar}(X_n) = n \cdot \mathbb{V}\text{ar}(X_1) =$$

$$= n \cdot (\mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2) = n \cdot (3 - 0) = 3n$$

For the cube of the sum, we use the formula:

$$\mathbb{E}(S_n^3) = \mathbb{E}((X_1 + ... + X_n)^3) = \mathbb{E}\left(\sum_{i=1}^n X_i + 3\sum_{i \neq j} X_i X_j + \sum_{i \neq j, j \neq k} X_i X_j X_k\right) =$$

The property of linearity:

$$= n \cdot \mathbb{E}(X_1^3) + 3\sum_{i \neq j} \mathbb{E}(X_i X_j^2) + \sum_{i \neq j, j \neq k} \mathbb{E}(X_i X_j X_k) =$$

Because of independence of $X_1, ...X_n$, $E(X_i X_j^2) = E(X_i) \cdot \mathbb{E}(X_j^2) = 0$ if $i \neq j$ and $E(X_i X_j X_k) = E(X_i) \cdot \mathbb{E}(X_j) \cdot \mathbb{E}(X_k) = 0$ if $i \neq j, j \neq k$:

$$= 6n + 3 \sum_{i \neq j} \mathbb{E}(X_i)\mathbb{E}(X_j^2) + \sum_{i \neq j, j \neq k} \mathbb{E}(X_i)\mathbb{E}(X_j)\mathbb{E}(X_k) = 6n$$

3. $m < n$.

$$\mathbb{E}(S_n|\mathcal{F}_m) = \mathbb{E}(S_n - S_m + S_m|\mathcal{F}_m) = \mathbb{E}(S_m|\mathcal{F}_m) + \mathbb{E}(S_n - S_m|\mathcal{F}_m) = S_m + \mathbb{E}(X_{m+1} + ... + X_n|\mathcal{F}_m) =$$

$$= S_m + \mathbb{E}(X_{m+1} + ... + X_n) = S_m + \mathbb{E}(X_{m+1}) + ... + \mathbb{E}(X_n) = S_m + (n-m)\mathbb{E}(X_1) = S_m + (n-m) \cdot 0 = S_m$$

$$\mathbb{E}(S_n^2|\mathcal{F}_m) = \mathbb{E}((S_n - S_m) + S_m)^2|\mathcal{F}_m) = \mathbb{E}((S_n - S_m)^2)|\mathcal{F}_m) + \mathbb{E}(2S_m(S_n - S_m)|\mathcal{F}_m) + \mathbb{E}(S_m^2|\mathcal{F}_m) =$$

Let's calculate these conditional expectations:

$$\mathbb{E}((S_n - S_m)^2|\mathcal{F}_m) = \mathbb{E}((X_{m+1} + ... + X_n)^2|\mathcal{F}_m) = \mathbb{V}\text{ar}(X_{m+1} + ... \mathbb{V}\text{ar}(X_n)) = (n-m)\,\mathbb{V}\text{ar}(X_1) =$$

$$= (n-m)(\mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2) = (n-m)\mathbb{E}(X_1^2) = 3(n-m)$$

$$\mathbb{E}((S_n - S_m)S_m|\mathcal{F}_m) = 2S_m\mathbb{E}(X_{m+1} + ... + X_n|\mathcal{F}_m) = 2S_m(n-m)\mathbb{E}(X_1) = 0$$

$$\mathbb{E}(S_m^2|\mathcal{F}_m) = S_m^2$$

Let's substitute the above expressions to the conditional expectation:

$$= 3(n-m) + S_m^2$$

4. Since $X_1, ..., X_n$ are identically distributed:

$$\mathbb{E}(X_1|S_n) = \mathbb{E}(X_2|S_n) = ... = \mathbb{E}(X_m|S_n) = ... = \mathbb{E}(X_n|S_n)$$

Let's sum them up:

$$\mathbb{E}(X_1|S_n) + ... + \mathbb{E}(X_n|S_n) = \mathbb{E}(X_1 + ... + X_n|S_n) = S_n$$

On the other hand:

$$\mathbb{E}(X_1|S_n) + ... + \mathbb{E}(X_n|S_n) = n\mathbb{E}(X_m|S_n)$$

The left-hand sides of these expressions are equal, so:

$$n\mathbb{E}(X_m|S_n) = S_n$$

$$\mathbb{E}(X_m|S_n) = \frac{S_n}{n}$$