Problem Set 8

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Exercise 1

1. Using the $X_3 \sim Pois(\lambda = 3)$

$$\mathbb{P}\{X_3 = 7\} = e^{-3\cdot 3}\frac{(3\cdot 3)^7}{7!} = e^{-9}\frac{9^7}{7!}$$

2. Using the independence of X_2 and $X_5 - X_2$:

$$\mathbb{P}\{X_2(X_5 - X_2) = 0\} = \mathbb{P}\{X_2 = 0\} + \mathbb{P}\{X_5 - X_2 = 0\} - \mathbb{P}\{X_2 = 0, X_5 - X_2 = 0\} =$$

$$= e^{-3\cdot 2}\frac{(3\cdot 2)^0}{0!} + e^{-3\cdot 3}\frac{(3\cdot 3)^0}{0!} - e^{-3\cdot 2}\frac{(3\cdot 2)^0}{0!}\cdot e^{-3\cdot 3}\frac{(3\cdot 3)^0}{0!} = e^{-6} + e^{-9} - e^{-15}$$

3. Expected amount of time $X_t = 6$. We will use the fact that $T_i \sim exp(\lambda = 3)$:

$$T = T_1 + T_2 + T_3 + T_4 + T_5 + T_6$$

Using the fact that $\mathbb{E}(T_i) = \frac{1}{\lambda} = \frac{1}{3}$ for an exponential random variable, we have:

$$\mathbb{E}(T) = \mathbb{E}(T_1 + ... + T_6) = 6\cdot\mathbb{E}(T_1) = \frac{6}{3} = 2$$

4. Let's use the Bayes formula:

$$\mathbb{P}\{X_1 = 2|X_3 = 7\} = \frac{\mathbb{P}\{X_3 = 7|X_1 = 2\}\mathbb{P}\{X_1 = 2\}}{\mathbb{P}\{X_3 = 7\}} =$$

Using the Memoryless property:

$$= \frac{\mathbb{P}\{X_3 - X_1 = 7 - 2\}\mathbb{P}\{X_1 = 2\}}{\mathbb{P}\{X_3 = 7\}} = \frac{e^{-3\cdot 2}\frac{(3\cdot 2)^5}{5!}e^{-3\cdot 1}\frac{(3\cdot 1)^2}{2!}}{e^{-3\cdot 3}\frac{(3\cdot 3)^7}{7!}} \approx 0.307$$

Exercise 2

a. Let's calculate the following probability:

$$\mathbb{P}\{X_1 \leq 2\} = \mathbb{P}\{X_1 = 2\} + \mathbb{P}\{X_1 = 1\} + \mathbb{P}\{X_1 = 0\} = e^{-2\cdot 1}\frac{(2\cdot 1)^2}{2!} + e^{-2\cdot 1}\frac{(2\cdot 1)^1}{1!} + e^{-2\cdot 1}\frac{(2\cdot 1)^0}{0!} =$$

$$= 2e^{-2} + 2e^{-2} + e^{-2} = 5e^{-2}$$

b. Let's apply the Memoryless property:

$$\mathbb{P}\{X_2 = 3 | X_1 = 2\} = \mathbb{P}\{\underbrace{X_2 - X_1}_{\sim Pois(\lambda} = 1\} = e^{-2\cdot 1}\frac{(2\cdot 1)^1}{1!} = 2e^{-2}$$

c. Let's apply the Bayes Theorem:

$$\mathbb{P}\{X_1 = 2 | X_2 = 4\} = \frac{\mathbb{P}\{X_2 = 4 | X_1 = 2\}\mathbb{P}\{X_1 = 2\}}{\mathbb{P}\{X_2 = 4\}} =$$

Using the Memoryless property:

$$= \frac{\mathbb{P}\{X_2 - X_1 = 2\}\mathbb{P}\{X_1 = 2\}}{\mathbb{P}\{X_2 = 4\}} = \frac{e^{-2\cdot 1}\frac{(2\cdot 1)^2}{2!}e^{-2\cdot 1}\frac{(2\cdot 1)^2}{2!}}{e^{-2\cdot 2}\frac{(2\cdot 2)^4}{4!}} = \frac{6}{16}$$

d. We know that time T_i has an exponential distribution and we need to calculate an expected value of T:

$$T = T_1 + T_2 + ... + T_{10}$$

Using the properties of an expected value:

$$\mathbb{E}(T) = \mathbb{E}(T_1 + ... + T_{10}) = 10\mathbb{E}(T_1) = 10 \cdot \frac{1}{2} = 5$$

e. Since $N \sim Pois(2\lambda)$:

$$\mathbb{E}(N) = \mathbb{V}\mathrm{ar}(N) = 2\lambda$$

Then using the standard formula for variance:

$$\mathbb{E}(N^2) = \mathbb{V}\mathrm{ar}(N) + (\mathbb{E}(N))^2 = 2\lambda + 4\lambda^2 = 4 + 4 \cdot 4 = 20$$

Exercise 3

1. Let's write the generator A:

$$A = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 4 & 0 & -4 \end{pmatrix} \end{array}$$

2. This Markov Chain is irreducible because we can reach any state from any other state:

$$0 \to 1 \to 2 \to \begin{cases} \to 3 \to 1 \\ \to 0 \end{cases}$$

3. For an irreducible chain there is a unique invariant probability vector π:

$$\pi(x) = -\alpha_x \pi(x) + \sum_{y \neq x} \pi(y)\alpha(y,x)$$

Or equivalently:

$$\pi A = 0$$

$$\begin{cases} -2 \cdot \pi_0 + 1 \cdot \pi_2 = 0 \\ 2 \cdot \pi_0 - 3 \cdot \pi_1 + 4 \cdot \pi_3 = 0 \\ 3 \cdot \pi_1 - 2 \cdot \pi_2 = 0 \\ 1 \cdot \pi_2 - 4 \cdot \pi_3 = 0 \end{cases}$$

And the normalization condition:

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

As a result:

$$\pi = \begin{pmatrix} \frac{6}{29} & \frac{8}{29} & \frac{12}{29} & \frac{3}{29} \end{pmatrix}$$

4. Let's find the expected amount of time until reaching 3 from $X_0 = 0$:

$$T_{0 \to 3} = \min\{t : X_t = 3 \,|X_0 = 0\}$$

The generator A without the row 3 and the column 3:

$$\widetilde{A} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} -2 & 2 & 0 \\ 0 & -3 & 3 \\ 1 & 0 & -2 \end{pmatrix} \end{array} \qquad -\widetilde{A}^{-1} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & \frac{2}{3} & 1 \\ \frac{1}{2} & \frac{2}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \end{array}$$

$$-\widetilde{A}^{-1}1 = \begin{array}{c c} \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 8/3 \\ 13/16 \\ 11/6 \end{pmatrix} \end{array}$$

As a result:

$$\mathbb{E}(T_{0\to 3}) = \frac{8}{3}$$

5. The expected amount of time until the chain leaves state 0 for the first time:

$$\mathbb{E}(T_0) = \frac{1}{\alpha_0} = \frac{1}{2}$$

6. Suppose we have the following time:

$$T_0 \text{ - time required to leave the state 0}$$

$$T_{0\to 0} \text{ - time required to leave the state 0 and to return to the state 0}$$

Then we have the formula:

$$\frac{\mathbb{E}(T_0)}{\mathbb{E}(T_{0\to 0})} = \pi_0$$

So:

$$\mathbb{E}(T_{0\to 0}) = \frac{\mathbb{E}(T_0)}{\pi_0} = \frac{1}{\alpha_0 \pi_0} = \frac{1}{2 \cdot 6/29} = \frac{29}{12}$$

Exercise 4

1. Let's write the generator A:

$$A = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

2. This Markov Chain is irreducible because we can reach any state from any other state:

$$0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$$

3. For an irreducible chain there is a unique invariant probability vector π:

$$\pi(x) = -\alpha_x \pi(x) + \sum_{y \neq x} \pi(y)\alpha(y, x)$$

Or equivalently:

$$\pi A = 0$$

$$\begin{cases} -1 \cdot \pi_0 + 1 \cdot \pi_1 = 0 \\ 1 \cdot \pi_0 - 2 \cdot \pi_1 + 1 \cdot \pi_2 = 0 \\ 1 \cdot \pi_1 - 2 \cdot \pi_2 + 1 \cdot \pi_3 = 0 \\ 1 \cdot \pi_2 - 2 \cdot \pi_3 + 1 \cdot \pi_4 = 0 \\ 1 \cdot \pi_3 - 1 \cdot \pi_4 = 0 \end{cases}$$

And the normalization condition:

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

As a result:

$$\pi = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

4. Let's find the expected amount of time until reaching 3 from $X_0 = 0$:

$$T_{0 \to 3} = \min\{t : X_t = 3 \,|X_0 = 0\}$$

The generator A without the row 3 and the column 3:

$$\widetilde{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 4 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \qquad -\widetilde{A}^{-1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 4 \end{array} \\ \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$-\widetilde{A}^{-1}1 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 4 \end{array} \begin{pmatrix} 6 \\ 5 \\ 3 \\ 1 \end{pmatrix}$$

As a result:

$$\mathbb{E}(T_{0\to 3}) = 6$$

5. The expected amount of time until the chain leaves state 0 for the first time:

$$\mathbb{E}(T_0) = \frac{1}{\alpha_0} = \frac{1}{1} = 1$$

6. Suppose we have the following time:

$$T_0 \text{ - time required to leave the state 0}$$

$$T_{0\to 0} \text{ - time required to leave the state 0 and to return to the state 0}$$

Then we have the formula:

$$\frac{\mathbb{E}(T_0)}{\mathbb{E}(T_{0\to 0})} = \pi_0$$

So:

$$\mathbb{E}(T_{0\to 0}) = \frac{\mathbb{E}(T_0)}{\pi_0} = \frac{1}{\alpha_0\pi_0} = \frac{1}{1\cdot 1/5} = 5$$