Problem Set 7

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Exercise 1

1. Let's show that M_n is a martingale by definition:

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(X_1 \cdot X_2 \cdot ... \cdot X_n \cdot X_{n+1}|\mathcal{F}_n) = X_1 \cdot X_2 ... \cdot X_n \cdot \mathbb{E}(X_{n+1}) =$$

We will calculate an expected value:

$$\mathbb{E}(X_{n+1}) = 2 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = 1$$

So:

$$= X_1 \cdot X_2 \cdot ... \cdot X_n = M_n$$

That is why M_n is a martingale.
2. Using the fact that $M_n > 0$:

$$\mathbb{E}(|M_n|) = \mathbb{E}(M_n) = \mathbb{E}(\mathbb{E}(M_n|\mathcal{F}_0)) = \mathbb{E}(M_0) = \mathbb{E}(1) = 1$$

3. Since the condition of the Martingale Convergence theorem is satisfied, we know that the limit exists and finite:

$$M_\infty = \lim_{n\to\infty} M_n$$

Let's consider the continuous function $f(x) = \log_2(x)$. Then:

$$\log_2 M_n = \log_2 X_1 + \log_2 X_2 + ... + \log_2 X_n$$

For each X_i:

$$\mathbb{E}(\log_2 X_1) = \log_2(2)\frac{1}{3} + \log_2(0.5)\frac{2}{3} = -\frac{1}{3}$$

Let's consider the following limit. We will apply the Law of Large Numbers:

$$M_\infty = \lim_{n\to\infty} 2^{\log_2 M_n} = \lim_{n\to\infty} 2^{(\log(X_1 \cdot ... \cdot X_n))} = \lim_{n\to\infty} 2^{\left(n \cdot \frac{\log_2 X_1 + ... + \log_2 X_n}{n}\right)} =$$

Since $\log_2(x)$ is a continuous function, we will apply the limit composition property:

$$= 2^{\lim\limits_{n\to\infty} n\cdot\mathbb{E}(\log_2 X_1)} = 2^{\lim\limits_{n\to\infty}(-1/3)n} = 0$$

4. Let's consider a stopping time:

$$T = \min\{j : M_j = 64\}$$

For this stopping time, we have a stopped martingale:

$$M_{n\wedge T} = \begin{cases} M_n & n < T \\ M_T & n \geq T \end{cases}$$

Since it is an unbounded stopping time, we have $\mathbb{P}\{T < \infty\} \neq 1$. But we still can apply the Optional Sampling Theorem (I):

$$\mathbb{E}(M_{n\wedge T}) = \mathbb{E}(M_0)$$

On the one hand:

$$\mathbb{E}(M_0) = 1$$

On the other hand:

$$\mathbb{E}(M_{n\wedge T}) = \mathbb{P}\{T = \infty\} \cdot M_\infty + \mathbb{P}\{T < \infty\}M_T$$

According to the theorem:

$$\mathbb{P}\{T = \infty\} \cdot 0 + \mathbb{P}\{T < \infty\}64 = 1$$

That is:

$$\mathbb{P}\{T < \infty\} = \frac{1}{64}$$

5. Let's check whether the expected value for M_n^2 is a finite or not:

$$\mathbb{E}(M_n^2) = \mathbb{E}(X_1^2 \cdot X_2^2 \cdot ... \cdot X_n^2) = \mathbb{E}(X_1^2) \cdot \mathbb{E}(X_2^2) \cdot ... \cdot \mathbb{E}(X_n^2) =$$

Let's calculate the expected value of X_i^2:

$$\mathbb{E}(X_1^2) = 4 \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} = \frac{3}{2}$$

Then:

$$= \left(\frac{3}{2}\right)^n \xrightarrow[n\to\infty]{} \infty$$

So $\nexists C < \infty$ such that:

$$\mathbb{E}(M_n^2) \leq C \qquad \forall n$$

Exercise 2

From the lecture notes, we know that winnings in the Martingale Betting Strategy is a martingale.

1. We need to check the condition of square integrable martingale:

$$\mathbb{E}(W_n^2) = 1^2 \cdot \left(1 - \left(\frac{1}{2}\right)^2\right) + (1-2^2)^2 \cdot \left(\frac{1}{2}\right)^2 = 2^n - 1 < \infty \text{ for each } n$$

Since $\mathbb{E}(W_n^2)$ is less than infinity for each n, W_n is a square integrable martingale.

2. Let's consider the difference:

$$\Delta_n = W_n - W_{n-1}$$

Then:

$$\mathbb{E}(\Delta_n^2) = \mathbb{E}(W_n - W_{n-1})^2 =$$

Using the fact that for the Martingale Betting strategy:

$$W_n = \sum_{j=1}^{n} B_j \Delta M_j = \sum_{j=1}^{n} B_j X_j$$

After substituting:

$$= \mathbb{E}(B_n^2 X_n^2) = \mathbb{E}(\mathbb{E}(B_n^2 X_n^2)|\mathcal{F}_{n-1}) =$$

Since a bet B_n is $\mathcal{F}_{n-1}$ measurable:

$$= \mathbb{E}(B_n^2 \cdot \mathbb{E}(X_n^2)|\mathcal{F}_{n-1}) = \mathbb{E}(B_n^2 \cdot \mathbb{E}(X_n^2)) = \mathbb{E}\left(B_n^2 \cdot \left(1^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2}\right)\right) =$$

For B_n, we have the following distribution:

$$B_n = \begin{cases} 2^{n-1} & \left(\frac{1}{2}\right)^{n-1} \\ 0 & 1 - \left(\frac{1}{2}\right)^{n-1} \end{cases}$$

As a result:

$$= 2^{2(n-1)} \left(\frac{1}{2}\right)^{n-1} + 0 \cdot \left(1 - \left(\frac{1}{2}\right)^{n-1}\right) = 2^{n-1}$$

3. We already calculated $\mathbb{E}(W_n^2)$ in the part 1; however, let's do so using the property of square integrable martingale:

$$\mathbb{E}(W_n^2) = \mathbb{E}(W_0^2) + \sum_{j=1}^{n} \mathbb{E}(W_n - W_{n-1})^2 = 0 + \frac{1}{2} \sum_{j=1}^{n} 2^j = 2^n - 1$$

4. Let's calculate the conditional expected value:

$$\mathbb{E}(B_n^2 X_n^2 | \mathcal{F}_{n-1}) =$$

Since B_n is $\mathcal{F}_{n-1}$ measurable and X_n^2 does not depend on $\mathcal{F}_{n-1}$:

$$= B_n^2 \mathbb{E}(X_n^2) = B_n^2 \cdot 1 = B_n^2 = \begin{cases} 2^{2(n-1)} & \left(\frac{1}{2}\right)^{n-1} \\ 0 & 1 - \left(\frac{1}{2}\right)^{n-1} \end{cases}$$

Exercise 3

1. Let's check the expected value of a martingale:

$$\mathbb{E}(M_n) = \mathbb{E}(\mathbb{E}(M_n|\mathcal{F}_0)) = \mathbb{E}(M_0) = \mathbb{E}(1) = 1$$

This is **true** statement.
2. Let's consider a simple symmetric random walk:

$$S_n = S_0 + X_1 + ... + X_n \qquad S_0 = 1$$

with:

$$\mathbb{P}\{X_j = 1\} = 1 - \mathbb{P}\{X_j = -1\} = 0.5$$

And a martingale:

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_n + X_{n+1}|\mathcal{F}_n) = S_n + (1 \cdot 0.5 + (-1) \cdot 0.5) = S_n = M_n$$

This example is suitable for our task, because it has an expected value:

$$\mathbb{E}(M_n) = \mathbb{E}(\mathbb{E}(M_n|\mathcal{F}_0)) = \mathbb{E}(M_0) = \mathbb{E}(S_0) = \mathbb{E}(1) = 1$$

Let's check the condition of the Martingale Convergence Theorem:

$$\mathbb{E}(|M_n|) = \mathbb{E}(|S_n|) = \mathbb{E}(|S_0 + X_1 + ... + X_n|) \xrightarrow[n\to\infty]{} \infty$$

So, the condition of the theorem does not hold. That is why we have:

$$M_\infty^{up} = \lim_{n\to\infty} \sup M_n = \lim_{n\to\infty} \sup S_n = +\infty$$

$$M_\infty^{down} = \lim_{n\to\infty} \inf M_n = \lim_{n\to\infty} \inf S_n = -\infty$$

Since $M_\infty^{up} \neq M_\infty^{down}$, $\nexists M_\infty$ such that:

$$M_\infty = \lim_{n\to\infty} M_n$$

This is **not a true** statement.

3. Let's consider a martingale from the exercise 1:

$$M_n = X_1 \cdot ... \cdot X_n$$

$$\mathbb{P}\{X_j = 2\} = \frac{1}{3} \qquad \mathbb{P}\{X_j = 0.5\} = \frac{2}{3}$$

This example is suitable for our task, because it has an expected value:

$$\mathbb{E}(M_n) = \mathbb{E}(\mathbb{E}(M_n|\mathcal{F}_0)) = \mathbb{E}(M_0) = \mathbb{E}(1) = 1$$

However, in the part 2 it was shown that the martingale satisfies the condition of the Martingale Convergence theorem, so in the part 3 it was calculated that:

$$\exists M_\infty = \lim_{n\to\infty} M_n = 0$$

That is why:

$$1 = \mathbb{E}(M_n) \neq \mathbb{E}(M_\infty) = 0$$

This is **not a true** statement.

4. Since M_n is a martingale by the task and $M_n \geq 0$ *almost surely* for any n then:

$$\mathbb{E}(|M_n|) = \mathbb{E}(M_n) = 1 < \infty \qquad \forall n$$

That is why the condition of the Martingale Convergence Theorem is satisfied, so $\exists M_\infty$ such that:

$$M_\infty = \lim_{n\to\infty} M_n$$

This is a **true** statement.

5. The case that satisfies the condition that $M_n \geq 0$ *almost surely* was already considered in the part 3:

$$1 = \mathbb{E}(M_n) \neq \mathbb{E}(M_\infty) = 0$$

This is **not a true** statement.