Problem Set 4

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Exercise 1

1. Let's calculate the following probability:

$$\mathbb{P}\{X_4 = 3\} = \mathbb{P}\{X_4 = 3|X_0 = 2\} \cdot \mathbb{P}\{X_0 = 2\} = (P^{(4)})_{23} = (P^4)_{23} = \frac{17}{128}$$

2. We will apply the chain rule based on Markov Property for calculating the probability of intersection:

$$\mathbb{P}\{X_7 = 1, X_6 = 1, X_5 = 2, X_4 = 3\} =$$

$$= \mathbb{P}\{X_7 = 1|X_6 = 1\} \cdot \mathbb{P}\{X_6 = 1|X_5 = 2\} \cdot \mathbb{P}\{X_5 = 2|X_4 = 3\} \cdot \mathbb{P}\{X_4 = 3|X_0 = 2\} \cdot \mathbb{P}\{X_0 = 2\} =$$

$$= p_{11} \cdot p_{21} \cdot p_{32} \cdot (P^4)_{23} = 0 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{17}{128} = 0$$

3. Let's calculate the invariant probabilities:

$$\bar{\pi} = \bar{\pi} P$$

We will re-write the above expression as the system of linear equations:

$$\begin{cases} \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 = \pi_0 \\ \frac{1}{2}\pi_0 + \frac{1}{2}\pi_2 = \pi_1 \\ \frac{1}{2}\pi_1 + \frac{3}{4}\pi_3 = \pi_2 \\ \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = \pi_3 \end{cases}$$

The solution for the system of equations is:

$$\pi_1 = \pi_2 = \pi_3 = \frac{3}{2}\pi_4$$

For now, let's use the normalization condition:

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

As a result, the invariant probability distribution is:

$$\bar{\pi} = \begin{pmatrix} \frac{3}{11} & \frac{3}{11} & \frac{3}{11} & \frac{2}{11} \end{pmatrix}$$

4. We have to calculate the expectation of the amount of time until the chain returns to state 2 starting from state 2:

$$T = \min\{j > 0 : X_j = 2\}$$

Using the fact that $\mathbb{E}(T)$ is inversely proportional to π_2:

$$\mathbb{E}(T) = \frac{1}{\pi_2} = \frac{1}{3/11} = \frac{11}{3}$$

5. For finding the expected amount of time until the chain reaches state 3, we will specify the Green's matrix:

$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} \qquad I - Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \end{array}$$

For now, let's calculate Green's Matrix G:

$$G = (I - Q)^{-1} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 6 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 2 \end{pmatrix} \end{array}$$

Finally, in order to calculate the expected amount of time until the chain reaches state 3 starting from state 2, we need to sum up the elements in a row 2:

$$2 - 2 + 2 = 2$$

Exercise 2

In this exercise, we have an infinite-state Markov Chain with the following probabilities for $n = 1, 2, ...$:

$$p(n, n-1) = 1 \qquad p(n, n+1) = 0 \qquad p(0, n) = e^{-1}\frac{1}{n!}$$

1. Starting from $X_0 = 0$, the number of steps until we return back is the random variable:

$$T = \min\{j > 0 : X_j = 0\}$$

After careful examination of the Markov Chain, we can see the following facts:

- The chain is irreducible, because with some non-zero probability, we can come from 0 to any state and then return to 0.
- The probability to return back to 0 from k is fully determined by the Poisson random variable: with the probability $e^{-1}\frac{1}{k!}$ we come to state k and thereafter we return back with the probability 1.
- The number of steps to return back to 0 is $k+1$: 1 step from state 0 to state k and k steps from state k to state 0.

Using this fact, let's calculate the expectation by definition:

$$\mathbb{E}(T) = \sum_{k=0}^{\infty}(k+1)\cdot\mathbb{P}\{X_k = k\} = \sum_{k=0}^{\infty}(k+1)e^{-1}\frac{1}{k!} = e^{-1}\left(\sum_{k=1}^{\infty}\frac{1}{(k-1)!} + \sum_{k=0}^{\infty}\frac{1}{k!}\right) =$$

Both sums are just Taylor's series for exponent with $x = 1$:

$$= e^{-1}(e+e) = 2$$

2. The invariant probabilities for the chain:

$$\pi_x = \sum_{y\in S}\pi_y p(y,x)$$

Since the Markov Chain has infinitely countable states, we can write down a few equations in order to find patterns:

$$\begin{cases} e^{-1}\frac{1}{0!}\cdot\pi_0 + \pi_1 = \pi_0 \\ e^{-1}\frac{1}{1!}\cdot\pi_0 + \pi_2 = \pi_1 \\ e^{-1}\frac{1}{2!}\cdot\pi_0 + \pi_3 = \pi_2 \\ \qquad\dots \end{cases}$$

We can organize this system as a difference equation:

$$\pi_j = \pi_{j-1} - \pi_0\cdot e^{-1}\frac{1}{(j-1)!}$$

We can check that this is positive recurrent Markov chain: if we plug π_0, we will determine all $\bar{\pi}$ in exactly one unique way. Here we apply a trick. Since we calculated $\mathbb{E}(T)$, that is he expected number of steps until we return to 0, we can use the following property, which is also suitable for infinite-state Markov Chains:

$$\pi_0 = \frac{1}{\mathbb{E}(T)} = \frac{1}{2}$$

For now, we see that we can substitute π_{j-1} to π_j (for every previous one we can restore every following one) to restore all π:

$$\pi_0 = \frac{1}{2}$$

$$\pi_1 = \pi_0 - \frac{1}{2} \cdot e^{-1} \frac{1}{0!}$$
$$\pi_2 = \pi_1 - \frac{1}{2} \cdot e^{-1} \frac{1}{1!}$$
$$\vdots$$
$$\pi_n = \pi_{n-1} - \frac{1}{2} \cdot e^{-1} \frac{1}{(n-1)!}$$

We receive the unique solution, so this infinite-state Markov Chain is positive recurrent.

Exercise 3

1. By drawing cards randomly, we can always eventually return to initial state. What happens is that we just shuffle different cards in the same deck without removing them, so each card must sooner or later return back to initial state, that's why all states communicate (many of them through other states) and the chain is irreducible.

2. Let's consider a particular ordering. For this particular ordering, we can randomly choose any of 51 cards (all except the top card) and move it to the top of the deck. Since we choose cards randomly, the probability to reach one of 51 states is $\frac{1}{51}$.

A particular state is also achievable from other 51 states with equal probabilities $\frac{1}{51}$. This is the very same logic as in the previous paragraph: a particular ordering is the result of one shuffle of 51 other orderings. For example, the 1st of those ordering is when the second card was chosen, the 2nd is when the third card chosen, ..., the 51st is when the very last card in the deck is chosen.

The first paragraph just shows the standard property of Markov Chain: the sum of each row is equal to 1, i.e. starting in state x, we will definitely go to some state in the next step and hence the sum of the probabilities equals 1:

$$\sum_{y \in S} p(x, y) = 1$$

The second paragraph shows the important observation: starting in any of 51 states x, we will come to state y in the next step with probability $\frac{1}{51}$:

$$\sum_{x \in S} p(x, y) = 1$$

The above expressions shows that sum of any column is equal to 1, that is why our transition matrix is doubly stochastic.

3. In the Problem set 3, we proved the lemma that if the matrix is doubly stochastic then the invariant probability distribution is uniform. Using the Fundamental Theorem of Markov Chains (the chain is irreducible [proven in the first part of this exercise] and finite [according to the task]), we have the unique invariant distribution. Taking into account the aforementioned lemma, this invariant distribution is uniform:

$$\bar{\pi} = \begin{pmatrix} \frac{1}{52!} & \dots & \frac{1}{52!} \end{pmatrix}$$

Let's consider the random variable T, which is the amount of time until the deck is back to the original order. Since the Markov Chain is irreducible, we can apply the following equality:

$$\mathbb{E}(T_i) = \frac{1}{\pi_i} = \frac{1}{1/52!} = 52! \approx 8.06 \cdot 10^{67}$$

On average, it takes 52! seconds (or $2.77 \cdot 10^{60}$ years) to return to the original order.