Problem Set 3

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Exercise 1

1. In order to find the long range fraction of time that the chain spends, we need to calculate invariant probability for the chain:

$$\bar{\pi} = \bar{\pi}P$$

The system of the equations is:

$$\begin{cases} \frac{1}{2}\pi_A + \frac{16}{36}\pi_B + \frac{4}{6}\pi_C = \pi_A \\ \frac{1}{3}\pi_A + \frac{1}{3}\pi_B + \frac{1}{3}\pi_C = \pi_B \\ \frac{1}{4}\pi_A + \frac{1}{2}\pi_B + \frac{1}{4}\pi_C = \pi_C \end{cases}$$

The solution for the system is:

$$\bar{\pi} = \begin{pmatrix}\pi_A & \pi_B & \pi_C\end{pmatrix} = \begin{pmatrix}\frac{16}{43} & \frac{15}{43} & \frac{12}{43}\end{pmatrix}$$

2. Suppose X - amount of money that the investor gains or loses at each state A, B, C. Taking into account that π is the long range fraction of time that the chain spends in each state, the long range earnings are:

$$\mathbb{E}(X) = \sum_{s\in\{A,B,C\}} x \cdot \mathbb{P}\{X = x\} = -5 \cdot \frac{16}{43} + 10 \cdot \frac{15}{43} + 5 \cdot \frac{12}{43} = \frac{130}{43}$$

3. For the first probability, let's use the time homogeneity property, that is:

$$\mathbb{P}\{X_{n+m} = k | X_n = j\} = (P^m)_{jk}$$

In our case, $m = 1$:

$$\mathbb{P}\{X_4 = B | X_3 = C\} = (P^1)_{CB} = (P)_{CB} = \frac{1}{2}$$

For the second probability, let's apply the Bayes Theorem:

$$\mathbb{P}\{X_3 = C | X_4 = B\} = \frac{\mathbb{P}\{X_4 = B | X_3 = C\} \cdot \mathbb{P}\{X_3 = C\}}{\mathbb{P}\{X_4 = B\}}$$

Using the total probability formula with $X_0 = A$:

$$\mathbb{P}\{X_4 = B\} = \sum_{x \in \{A,B,C\}} \mathbb{P}\{X_4 = B | X_0 = x\} \cdot \mathbb{P}\{X_0 = x\} = \mathbb{P}\{X_4 = B | X_0 = A\} \cdot \mathbb{P}\{X_0 = A\} =$$

$$= (P^4)_{AB} \cdot 1 = \frac{2407}{6912} \cdot 1 = \frac{2407}{6912}$$

Analogically:

$$\mathbb{P}\{X_3 = C\} = \sum_{x \in \{A,B,C\}} \mathbb{P}\{X_3 = C | X_0 = x\} \cdot \mathbb{P}\{X_0 = x\} = \mathbb{P}\{X_3 = C | X_0 = A\} \cdot \mathbb{P}\{X_0 = A\} =$$

$$= (P^3)_{AC} \cdot 1 = \frac{5}{18} \cdot 1 = \frac{5}{18}$$

Plugging in the Bayes formula above:

$$\frac{1/2 \cdot 5/18}{2407/6912} = \frac{5/36}{2407/6912} = \frac{960}{2407}$$

Exercise 2

1. Before making the stochastic matrix, we will have a look at probabilities. When we throw one dice, the probabilities are obvious:

$$\mathbb{P}\{\text{either 5 or 6}\} = \frac{2}{6} \qquad \mathbb{P}\{\text{neither 5 nor 6}\} = \frac{4}{6}$$

When we throw two dice, we should consider the three cases: two dices are bigger or equal to 5, exactly 1 dice is bigger or equal to 5, and both dices are less than 5.

$$\mathbb{P}\{\text{two dices are bigger or equal to 5}\} = \frac{4}{36} \qquad \mathbb{P}\{\text{exactly one dice is bigger or equal to 5}\} = \frac{16}{36}$$

$$\mathbb{P}\{\text{two dices are less than 5}\} = \frac{16}{36}$$

When we throw three dices, we have to consider four cases likewise:

$$\mathbb{P}\{\text{three dices are bigger or equal to 5}\} = \frac{8}{216} \quad \mathbb{P}\{\text{exactly two dices are bigger or equal to 5}\} = \frac{48}{216}$$

$$\mathbb{P}\{\text{exactly one dice is less than 5}\} = \frac{96}{216} \qquad \mathbb{P}\{\text{three dices are less than 5}\} = \frac{64}{216}$$

Using this probabilities, we can write down the transition matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \frac{64}{216} & \frac{96}{216} & \frac{48}{216} & \frac{8}{216} \\ \frac{16}{36} & \frac{16}{36} & \frac{4}{36} & 0 \\ \frac{4}{6} & \frac{2}{6} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2. Let's find the invariant probability distribution for the chain:

$$\bar{\pi} = \bar{\pi} P$$

Let's re-write it with the system of equations with the normalization condition:

$$\begin{cases} \pi_0 \frac{64}{216} + \pi_1 \frac{16}{36} + \pi_2 \frac{4}{6} + \pi_3 1 = \pi_0 \\ \pi_0 \frac{96}{216} + \pi_1 \frac{16}{36} + \pi_2 \frac{2}{6} = \pi_1 \\ \pi_0 \frac{48}{216} + \pi_1 \frac{4}{36} = \pi_2 \\ \pi_0 \frac{8}{216} = \pi_3 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Then the distribution $\bar{\pi}$ is:

$$\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} \frac{27}{64} & \frac{27}{64} & \frac{9}{64} & \frac{1}{64} \end{pmatrix}$$

3. We have to calculate the following probability:

$$\mathbb{P}\{X_3 = 3 | X_0 = 0\}$$

Let's just use the fact that a 3-step stochastic matrix is just the stochastic matrix in 3rd power:

$$P^{(n)} = P^n$$

Then:

$$\mathbb{P}\{X_3 = 3 | X_0 = 0\} = (P^3)_{03} = \frac{343}{19683}$$

4. We need to calculate the expected number of steps until we return to state 0:

$$\mathbb{E}(T) = \mathbb{E}(\min\{j > 0 : X_j = 0\}) \qquad\qquad X_0 = 0$$

On the average, the expected fraction of time spent in state 0 is $\pi_0 = \frac{27}{64}$. The expected time starting at state 0 to return to state 0 is:

$$\mathbb{E}(T) = \frac{1}{\pi_0} = \frac{1}{27/64} = \frac{64}{27}$$

Exercise 3

According to the task, X_n is **finite** and **irreducible** Markov Chain. Then we can apply The Fundamental Theorem of Markov Chain (or so-called Ergodicity Theorem):

π is a **unique** invariant probability distribution

$$\bar{\pi} = \bar{\pi} P$$

Let's re-write it with the system of equations:

$$\begin{cases} \pi_1 p_{11} + \pi_2 p_{21} + ... + \pi_n p_{n1} = \pi_1 \\ \vdots \\ \pi_1 p_{n1} + \pi_2 p_{n2} + ... + \pi_n p_{nn} = \pi_n \end{cases}$$

We have to show the uniform distribution is the invariant probability distribution for the doubly stochastic chain:

$$\pi_1 = \pi_2 = ... = \pi_n = \pi$$

Then the system of equation is:

$$\begin{cases} \pi(p_{11} + p_{21} + ... + p_{n1}) = \pi \\ \vdots \\ \pi(p_{n1} + p_{n2} + ... + p_{nn}) = \pi \end{cases}$$

Using the property of doubly stochastic matrices, i.e. $\sum_{j=1}^{n} p_{jk} = 1$ for each column k:

$$\begin{cases} \pi = \pi \\ \vdots \\ \pi = \pi \end{cases}$$

We obtained n identities that are $\pi = \pi$. Since $\bar{\pi}$ is a probability distribution, it must follow the normalization condition:

$$\pi + ... + \pi = 1$$

That is why:

$$n \cdot \pi = 1 \qquad \pi = \frac{1}{n}$$

We checked that the uniform distribution is a solution and, according to the Fundamental Theorem, it is the unique solution. □