Approximations to the loop space of flag manifolds

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$Y_{m,n}(\sigma)$

Definition

For a permutation $\sigma \in S_m \subset \mathrm{GL}_m(\mathbb{C})$, we define an open subset of $\mathbb{C}^{mn}$:

$$Y_{m,n}(\sigma) = \left\{ (a_1, \ldots, a_n) \in (\mathbb{C}^m)^n \;\middle|\; \begin{array}{c} \text{Any } m \text{ subsequent vectors in} \\ (e_1, \ldots, e_m, a_1, \ldots, a_n, \sigma_1, \ldots, \sigma_m) \\ \text{are linearly independent.} \end{array} \right\}.$$

Examples

Example

The space $Y_{m,1} = Y_{m,1}(\mathrm{Id})$ is homeomorphic to $(\mathbb{C}^*)^m$, since

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \lambda_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & \lambda_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \lambda_{m-1} & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & \lambda_m & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

satisfies the condition iff $\lambda_i \neq 0$ for all i.

Examples

Example

The space $Y_{2,2}$ is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a \neq 0, d \neq 0, ad - bc \neq 0 \right\},$$

which is homeomorphic to the space

$$(\mathbb{C}^*)^2 \times \left\{ (b,c) \in \mathbb{C}^2 \mid bc \neq 1 \right\}.$$

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$X_{m,n}$

The torus T^n acts on the columns of $Y_{m,n}(\sigma)$,

$$(\lambda_1, \dots, \lambda_n) \cdot (a_1, \dots, a_n) = (\lambda_1 a_1, \dots, \lambda_n a_n).$$

Definition

The space $X_{m,n}(\sigma)$ is the quotient of this action,

$$X_{m,n}(\sigma) = Y_{m,n}(\sigma)/\mathrm{T}^n.$$

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Identities

We have the following identities:

$$X_{m,n}(\sigma) \cong \left\{ A \in Y_{m,n}(\sigma) \;\middle|\; \begin{array}{c} \text{The last } n \text{ determinants of} \\ (e_1, \dots, e_m, a_1, \dots, a_{n-1}, a_n, \sigma_1, \dots, \sigma_m) \\ \text{are all equal to 1.} \end{array} \right\},$$

$$Y_{m,n}(\sigma) \cong X_{m,n}(\sigma) \times T^n,$$

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The space $X_{m,1}$ is homeomorphic to $(\mathbb{C}^*)^{m-1}$, since

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The limit

We are interested in what happens when n becomes large.

Definition

The limit space is

$$X_{m,\infty}(\sigma) = \left\{ (a_i) \in (\mathbb{C}^m)^{\mathbb{Z}} \;\middle|\; \begin{array}{c} \exists n : (a_1, \dots, a_n) \in X_{m,n}(\sigma), \\ a_{n+1} = \sigma_1, a_{n+2} = \sigma_2, \dots, \\ a_0 = e_m, a_{-1} = e_{m-1}, \dots \end{array} \right\}$$

$$= \varinjlim_n \left(\dots \longrightarrow X_{m,n} \xrightarrow{s} X_{m,n+1} \longrightarrow \dots \right)$$

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Equivalence

Theorem

The limit space $X_{m,\infty}$ is homotopy equivalent to the loop space $\Omega = \Omega(\mathrm{SU}_m/\mathrm{T}^{m-1})$.

Defining the map

For $a_1, \ldots, a_{mn} \in Y_{m,mn}$:

$$a_1 = c_1 e_1 + c_2 e_2 + \cdots + c_m e_m.$$

$$\frac{1}{c_1} a_1 = e_1 + \frac{c_2}{c_1} e_2 + \cdots + \frac{c_m}{c_1} e_m.$$

Get a path in $\mathrm{GL}_m(\mathbb{C})$ by

$$t \mapsto \left(e_1 + t \cdot \left(\sum_{i=2}^{m} \frac{c_i}{c_1} e_i \right), e_2, \ldots, e_m \right)$$

from Id to $\left(\frac{1}{c_1} a_1, e_2, \ldots, e_m \right)$.

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The map, continued

Repeat for the other columns,

$$\frac{1}{d_2}a_2 = e_2 + \frac{d_1}{d_2}a_1 + \frac{d_3}{d_2}e_3 + \cdots + \frac{d_m}{d_2}e_m.$$

$$t \mapsto \left(\frac{1}{c_1}a_1, e_2 + t \cdot \left(\frac{d_1}{d_2}a_1 + \sum_{i=3}^{m} \frac{d_i}{d_2}e_i\right), e_3, \ldots, e_m\right).$$

Piecing all of these paths together gives a path from Id to $\operatorname{diag}(\lambda_1, \ldots, \lambda_m)$. If we quotient out, this defines a map

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Loop space

Since $\mathrm{SU}_m/\mathrm{T}^{m-1}$ is compact, we can find $\varepsilon > 0$ such that:

- $B_\varepsilon(A)$ is geodesically convex for any $A \in \mathrm{SU}_m/\mathrm{T}^{m-1}$.
- If A and B are both in $B_\varepsilon(U)$, then any m subsequent columns in $(a_1, \ldots, a_m, b_1, \ldots, b_m)$ are linearly independent.

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$$\Omega = \bigcup_k \Omega_k.$$

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Proof

Now we can map

$$
\begin{aligned}
\Omega_k &\to X_{m,(2^k-1)m} \\
\gamma &\mapsto \left(\gamma\left(\frac{1}{2^k}\right), \gamma\left(\frac{2}{2^k}\right), \dots, \gamma\left(\frac{2^k-1}{2^k}\right)\right)
\end{aligned}
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These maps are the "inverses" of the map f, and we can now prove injectivity and surjectivity on cell complexes directly.

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Corollary

The homology of $X_{m,\infty}$ is

$$T(x_1, \ldots, x_{m-1}) \otimes \mathbb{Z}[y_1, \ldots, y_{m-1}]/I,$$

where the degree of x_i is one, the degree of y_i is $2i$ and with relations $x_k^2 = x_p x_q + x_q x_p = 2y_1$, $p \neq q$. See [GT10, Thm 4.1].

Filtration

From now on we only consider $m = 3$. We want to do calculations:

Definition

Let $F_p \subset X_{3,n}(\sigma)$, $p \in \{0, 1, 2\}$, be the set

$$F_p = \{(a_1, \dots, a_n) \in X_{3,n}(\sigma) \mid a_n \text{ has at most } p \text{ zeroes.}\}.$$

Then $F_0 \subset F_1 \subset F_2 = X_{3,n}(\sigma)$ is an increasing sequence of open subspaces, which gives us a spectral sequence converging to the cohomology of $X_{3,n}(\sigma)$ with terms

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}).$$

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Theorem

For any permutation σ, there are permutations $\sigma_\emptyset, \sigma_{\{1\}}, \sigma_{\{2\}}, \sigma_{\{1,2\}}$ such that

$$
\begin{aligned}
H^*(F_0) &\cong H^*(X_{3,n-1}(\sigma_\emptyset) \times \mathrm{T}^2) \\
H^*(F_1, F_0) &\cong H^*((X_{3,n-1}(\sigma_{\{1\}}) \sqcup X_{3,n-1}(\sigma_{\{2\}})) \times \mathrm{T} \times (D, D-0)) \\
H^*(F_2, F_1) &\cong H^*(X_{3,n-1}(\sigma_{\{1,2\}}) \times (D_2, D_2 - 0))
\end{aligned}
$$

σ	$\sigma_\emptyset$	$\sigma_{\{1\}}$	$\sigma_{\{2\}}$	$\sigma_{\{1,2\}}$
Id	Id	(1 2)	(2 3)	(1 3 2)
(1 2)	Id	(1 2 3)	Id	(1 3)
(2 3)	Id	(1 2)	Id	(1 2)
(1 2 3)	Id	Id	Id	Id
(1 3 2)	(2 3)	(1 2 3)	(2 3)	(1 2 3)
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(1 3 2)	(2 3)	(1 2 3)	(2 3)	(1 2 3)
(1 3)	(2 3)	(2 3)	(2 3)	(2 3)

Permutations

Definition

Define a function λ on S_3 by

$$\lambda(\sigma) = \begin{cases} 1 & \sigma = \mathrm{Id}, \\ 2 & \sigma \in \{(1\,2), (2\,3), (1\,2\,3)\}, \\ 3 & \sigma \in \{(1\,3\,2), (1\,3)\}. \end{cases}$$

Stabilisation

Definition

The inclusion that considers an element $(a_1, \ldots, a_n)$ in $X_{3,n}(\sigma)$ as an element in $X_{3,\infty}(\sigma)$ will be denoted

$$L_n : X_{3,n}(\sigma) \to X_{3,\infty}(\sigma).$$

Cohomological stability

Theorem

If $n \geq k + \lambda(\sigma)$, the map

$$L_n^* : H^k(X_{3,\infty}(\sigma)) \to H^k(X_{3,n}(\sigma))$$

is an isomorphism.

Proof

The proof is by induction. The induction hypothesis is:

Hypothesis (I_{k-1})

For all $r \leq k-1$ and all $\sigma \in S_3$, the following two statements hold:

- *The map $L_n^* : H^r(X_{3,\infty}(\sigma)) \to H^r(X_{3,n}(\sigma))$ is an isomorphism for $n \geq r + \lambda(\sigma)$.*
- *The map $L_n^* : H^r(X_{3,\infty}(\sigma)) \to H^r(X_{3,n}(\sigma))$ is injective for $n \geq r + \lambda(\sigma) - 1$.*

Proof

First we show the theorem for the map

$$s^* : H^k(X_{3,n+1}(\sigma)) \to H^k(X_{3,n}(\sigma_\emptyset)).$$

Since $(\mathrm{Id})_\emptyset = \mathrm{Id}$, this allows us to work out the identity case explicitly.

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Proof

For $\sigma \neq \mathrm{Id}$, consider the diagram

$$\begin{tikzcd} H^k(X_{3,\infty}(\sigma)) \arrow[r, "s^*", "\cong"'] \arrow[d, "L_n^*"] & H^k(X_{3,\infty}(\sigma_\emptyset)) \arrow[d, "L_{n-1}^*"] \\ H^k(X_{3,n}(\sigma)) \arrow[r, "s^*"] & H^k(X_{3,n-1}(\sigma_\emptyset)). \end{tikzcd}$$

Use that we already know the theorem for $\sigma_\emptyset$.

Example

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For $n \geq k+1$, we can apply the theorem to compute

$$\bigoplus_{r=0}^{k} H^r(X_{n,3}) \otimes H^{k-r}(\mathrm{T}^3) \cong H^k(Y_{n,3})$$

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Example

Example ($k = 1$)

$$H^1(X_{n,3}) \oplus H^1(\mathrm{T}^3) \cong H^1(X_{3,\infty}) \oplus H^1(\mathrm{T}^n)$$
$$\implies H^1(X_{n,3}) \cong \mathbb{Z}^{n-1}.$$

Example ($k = 2$)

$$H^2(X_{n,3}) \oplus \mathbb{Z}^{3(n-1)} \oplus \mathbb{Z}^3 \cong \mathbb{Z}^2 \oplus \mathbb{Z}^{2n} \oplus \mathbb{Z}^{\binom{n}{2}}$$
$$\implies \mathrm{rank}_{\mathbb{Z}} H^2(X_{n,3}) = 1 + \binom{n-1}{2}$$

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The spectral sequence

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong \bigoplus_{\{v_1,\dots,v_p\}} H^{p+q}\left(Y_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0)\right)$$

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \to E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p)$$

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The spectral sequence

$$
\begin{array}{c|ccccc}
9 & H^9(F_0) & \longrightarrow & H^{10}(F_1, F_0) & \longrightarrow & H^{11}(F_2, F_1) \\
8 & H^8(F_0) & \longrightarrow & H^{9}(F_1, F_0) & \longrightarrow & H^{10}(F_2, F_1) \\
7 & H^7(F_0) & \longrightarrow & H^{8}(F_1, F_0) & \longrightarrow & H^{9}(F_2, F_1) \\
6 & H^6(F_0) & \longrightarrow & H^{7}(F_1, F_0) & \longrightarrow & H^{8}(F_2, F_1) \\
5 & H^5(F_0) & \longrightarrow & H^{6}(F_1, F_0) & \longrightarrow & H^{7}(F_2, F_1) \\
4 & H^4(F_0) & \longrightarrow & H^{5}(F_1, F_0) & \longrightarrow & H^{6}(F_2, F_1) \\
3 & H^3(F_0) & \longrightarrow & H^{4}(F_1, F_0) & \longrightarrow & H^{5}(F_2, F_1) \\
2 & H^2(F_0) & \longrightarrow & H^{3}(F_1, F_0) & \longrightarrow & H^{4}(F_2, F_1) \\
1 & H^1(F_0) & \longrightarrow & H^{2}(F_1, F_0) & \longrightarrow & H^{3}(F_2, F_1) \\
0 & H^0(F_0) & \longrightarrow & H^{1}(F_1, F_0) & \longrightarrow & H^{2}(F_2, F_1) \\
\hline
{}^{q}\!/_{p} & 0 & & 1 & & 2
\end{array}
$$

E_1

$q \diagup p$	0	1	2
9	$\mathbb{Z}$	0	0
8	$\mathbb{Z}^7$	$\mathbb{Z}^2$	0
7	$\mathbb{Z}^{22}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$
6	$\mathbb{Z}^{42}$	$\mathbb{Z}^{32}$	$\mathbb{Z}^4$
5	$\mathbb{Z}^{56}$	$\mathbb{Z}^{50}$	$\mathbb{Z}^7$
4	$\mathbb{Z}^{56}$	$\mathbb{Z}^{50}$	$\mathbb{Z}^7$
3	$\mathbb{Z}^{42}$	$\mathbb{Z}^{32}$	$\mathbb{Z}^4$
2	$\mathbb{Z}^{22}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$
1	$\mathbb{Z}^7$	$\mathbb{Z}^2$	0
0	$\mathbb{Z}$	0	0

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1})$$

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

E_2

$q \diagup p$	0	1	2
9	$\mathbb{Z}$	0	0
8	$\mathbb{Z}^5$	0	0
7	$\mathbb{Z}^{11}$	0	0
6	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
5	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
4	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
3	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
2	$\mathbb{Z}^{11}$	0	0
1	$\mathbb{Z}^5$	0	0
0	$\mathbb{Z}$	0	0

$$E_2^{p,q} = \ker d_1 / \operatorname{im} d_1$$

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$$

$E_2 = E_\infty$

$q \diagup p$	0	1	2
9	$\mathbb{Z}$	0	0
8	$\mathbb{Z}^5$	0	0
7	$\mathbb{Z}^{11}$	0	0
6	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
5	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
4	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
3	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
2	$\mathbb{Z}^{11}$	0	0
1	$\mathbb{Z}^5$	0	0
0	$\mathbb{Z}$	0	0

$$E_r^{p,q} = \ker d_{r-1} / \operatorname{im} d_{r-1}$$

$$d_r : E_r^{p,q} \to E_2^{p+r,q-r+1}$$

$H^*(X_{3,3})$

$$H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases} \qquad H^q(Y_{3,3}) = \begin{cases} \mathbb{Z} & q = 9 \\ \mathbb{Z}^5 & q = 8 \\ \mathbb{Z}^{12} & q = 7 \\ \mathbb{Z}^{18} & q = 6 \\ \mathbb{Z}^{19} & q = 5 \\ \mathbb{Z}^{17} & q = 4 \\ \mathbb{Z}^{15} & q = 3 \\ \mathbb{Z}^{11} & q = 2 \\ \mathbb{Z}^5 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Compare

$$H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases} \qquad H^q(\Omega(\mathrm{SU}_3/T^2)) = \begin{cases} \vdots & q > 5 \\ \mathbb{Z}^4 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases}$$