Approximations to the loop space of flag manifolds

Simon Stoltze

Institut for Matematik
Aarhus University

16. January, 2017

$Y_{m,n}(\sigma)$

Definition

For a permutation $\sigma \in S_m \subset \mathrm{GL}_m(\mathbb{C})$, we define an open subset of $\mathbb{C}^{mn}$:

$$Y_{m,n}(\sigma) = \left\{ (a_1, \dots, a_n) \in (\mathbb{C}^m)^n \;\middle|\; \begin{array}{c} \text{Any } m \text{ subsequent vectors in} \\ (e_1, \dots, e_m, a_1, \dots, a_n, \sigma_1, \dots, \sigma_m) \\ \text{are linearly independent.} \end{array} \right\}.$$

Examples

Example

The space $Y_{m,1} = Y_{m,1}(\mathrm{Id})$ is homeomorphic to $(\mathbb{C}^*)^m$, since

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \lambda_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & \lambda_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \lambda_{m-1} & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & \lambda_m & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

satisfies the condition iff $\lambda_i \neq 0$ for all i.

Examples

Example

The space $Y_{2,2}$ is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a \neq 0, d \neq 0, ad - bc \neq 0 \right\},$$

which is homeomorphic to the space

$$(\mathbb{C}^*)^2 \times \left\{ (b, c) \in \mathbb{C}^2 \mid bc \neq 1 \right\}.$$

$X_{m,n}$

The torus T^n acts on the columns of $Y_{m,n}(\sigma)$,

$$(\lambda_1, \dots, \lambda_n) \cdot (a_1, \dots, a_n) = (\lambda_1 a_1, \dots, \lambda_n a_n).$$

Definition

The space $X_{m,n}(\sigma)$ is the quotient of this action,

$$X_{m,n}(\sigma) = Y_{m,n}(\sigma)/\mathrm{T}^n.$$

Identities

We have the following identities:

$$X_{m,n}(\sigma) \cong \left\{ A \in Y_{m,n}(\sigma) \;\middle|\; \begin{array}{c} \text{The last } n \text{ determinants of} \\ (e_1, \dots, e_m, a_1, \dots, a_n, \sigma_1, \dots, \sigma_m) \\ \text{are all equal to 1.} \end{array} \right\},$$

$$Y_{m,n}(\sigma) \cong X_{m,n}(\sigma) \times T^n,$$

which gives us e.g.

$$\pi_1(Y_{m,n}) \cong \pi_1(X_{m,n}) \times \mathbb{Z}^n.$$

Examples

Example

The space $X_{m,1}$ is homeomorphic to $(\mathbb{C}^*)^{m-1}$, since

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \lambda_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & \lambda_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \lambda_{m-1} & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

satisfies the condition iff $\lambda_i \neq 0$ for all i.

Examples

Example

The space $X_{2,2}$ is

$$\left\{ \begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} \;\middle|\; bc \neq 1 \right\}.$$

The limit

We are interested in what happens when n becomes large.

Definition

The limit space is

$$X_{m,\infty}(\sigma) = \left\{ (a_i) \in (\mathbb{C}^m)^{\mathbb{Z}} \;\middle|\; \begin{array}{c} \exists n : (a_1, \dots, a_n) \in X_{m,n}(\sigma), \\ a_{n+1} = \sigma_1, a_{n+2} = \sigma_2, \dots, \\ a_0 = e_m, a_{-1} = e_{m-1}, \dots \end{array} \right\}$$
$$= \varinjlim_n \left(\dots \longrightarrow X_{m,n} \xrightarrow{s} X_{m,n+1} \longrightarrow \dots \right)$$

Equivalence

Theorem

The limit space $X_{m,\infty}$ is homotopy equivalent to the loop space $\Omega = \Omega(\mathrm{SU}_m/\mathrm{T}^{m-1})$.

Defining the map

For $a_1, \dots, a_{mn} \in Y_{m,mn}$:

$$a_1 = c_1 e_1 + c_2 e_2 + \dots + c_m e_m.$$
$$\frac{1}{c_1} a_1 = e_1 + \frac{c_2}{c_1} e_2 + \dots + \frac{c_m}{c_1} e_m.$$

Get a path in $\mathrm{GL}_m(\mathbb{C})$ by

$$t \mapsto \left(e_1 + t \cdot \left(\sum_{i=2}^{m} \frac{c_i}{c_1} e_i \right), e_2, \dots, e_m \right)$$

from Id to $\left(\frac{1}{c_1} a_1, e_2, \dots, e_m\right)$.

The map, continued

Repeat for the other columns,

$$\frac{1}{d_2} a_2 = e_2 + \frac{d_1}{d_2} a_1 + \frac{d_3}{d_2} e_3 + \dots + \frac{d_m}{d_2} e_m.$$

$$t \mapsto \left(\frac{1}{c_1} a_1, e_2 + t \cdot \left(\frac{d_1}{d_2} a_1 + \sum_{i=3}^{m} \frac{d_i}{d_2} e_i \right), e_3, \dots, e_m \right).$$

Piecing all of these paths together gives a path from Id to $\mathrm{diag}(\lambda_1, \dots, \lambda_m)$. If we quotient out, this defines a map

$$f : X_{m,\infty} \to \Omega.$$

Loop space

Since $\mathrm{SU}_m/\mathrm{T}^{m-1}$ is compact, we can find $\varepsilon > 0$ such that:

- $B_\varepsilon(A)$ is geodesically convex for any $A \in \mathrm{SU}_m/\mathrm{T}^{m-1}$.
- If A and B are both in $B_\varepsilon(U)$, then any m subsequent columns in $(a_1, \dots, a_m, b_1, \dots, b_m)$ are linearly independent.

With this, we can require loops to be piecewise in some B_ε, giving

$$\Omega = \bigcup_k \Omega_k.$$

Proof

Now we can map

$$\Omega_k \to X_{m,(2^k-1)m}$$
$$\gamma \mapsto \left(\gamma\left(\frac{1}{2^k}\right), \gamma\left(\frac{2}{2^k}\right), \dots, \gamma\left(\frac{2^k - 1}{2^k}\right) \right)$$

These maps are the "inverses" of the map f, and we can now prove injectivity and surjectivity on cell complexes directly.

Corollary

The homology of $X_{m,\infty}$ is

$$T(x_1, \dots, x_{m-1}) \otimes \mathbb{Z}[y_1, \dots, y_{m-1}]/I,$$

where the degree of x_i is one, the degree of y_i is $2i$ and with relations $x_k^2 = x_p x_q + x_q x_p = 2y_1$, $p \neq q$. See [GT10, Thm 4.1].

Filtration

From now on we only consider $m = 3$. We want to do calculations:

Definition

Let $F_p \subset X_{3,n}(\sigma)$, $p \in \{0, 1, 2\}$, be the set

$$F_p = \{(a_1, \dots, a_n) \in X_{3,n}(\sigma) \mid a_n \text{ has at most } p \text{ zeroes.}\}.$$

Then $F_0 \subset F_1 \subset F_2 = X_{3,n}(\sigma)$ is an increasing sequence of open subspaces, which gives us a spectral sequence converging to the cohomology of $X_{3,n}(\sigma)$ with terms

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}).$$

Theorem

For any permutation σ, there are permutations $\sigma_\emptyset, \sigma_{\{1\}}, \sigma_{\{2\}}, \sigma_{\{1,2\}}$ such that

$$H^*(F_0) \cong H^*(X_{3,n-1}(\sigma_\emptyset) \times \mathrm{T}^2)$$
$$H^*(F_1, F_0) \cong H^*((X_{3,n-1}(\sigma_{\{1\}}) \sqcup X_{3,n-1}(\sigma_{\{2\}})) \times \mathrm{T} \times (D, D-0))$$
$$H^*(F_2, F_1) \cong H^*(X_{3,n-1}(\sigma_{\{1,2\}}) \times (D_2, D_2 - 0))$$

σ	$\sigma_\emptyset$	$\sigma_{\{1\}}$	$\sigma_{\{2\}}$	$\sigma_{\{1,2\}}$
Id	Id	(1 2)	(2 3)	(1 3 2)
(1 2)	Id	(1 2 3)	Id	(1 3)
(2 3)	Id	(1 2)	Id	(1 2)
(1 2 3)	Id	Id	Id	Id
(1 3 2)	(2 3)	(1 2 3)	(2 3)	(1 2 3)
(1 3)	(2 3)	(2 3)	(2 3)	(2 3)

Permutations

Definition

Define a function λ on S_3 by

$$\lambda(\sigma) = \begin{cases} 1 & \sigma = \mathrm{Id}, \\ 2 & \sigma \in \{(1\,2), (2\,3), (1\,2\,3)\}, \\ 3 & \sigma \in \{(1\,3\,2), (1\,3)\}. \end{cases}$$

Stabilisation

Definition

The inclusion that considers an element $(a_1, \ldots, a_n)$ in $X_{3,n}(\sigma)$ as an element in $X_{3,\infty}(\sigma)$ will be denoted

$$L_n : X_{3,n}(\sigma) \to X_{3,\infty}(\sigma).$$

Cohomological stability

Theorem

If $n \geq k + \lambda(\sigma)$, the map

$$L_n^* : H^k(X_{3,\infty}(\sigma)) \to H^k(X_{3,n}(\sigma))$$

is an isomorphism.

Proof

The proof is by induction. The induction hypothesis is:

Hypothesis (I_{k-1})

For all $r \leq k-1$ and all $\sigma \in S_3$, the following two statements hold:

- *The map $L_n^* : H^r(X_{3,\infty}(\sigma)) \to H^r(X_{3,n}(\sigma))$ is an isomorphism for $n \geq r + \lambda(\sigma)$.*
- *The map $L_n^* : H^r(X_{3,\infty}(\sigma)) \to H^r(X_{3,n}(\sigma))$ is injective for $n \geq r + \lambda(\sigma) - 1$.*

Proof

First we show the theorem for the map

$$s^* : H^k(X_{3,n+1}(\sigma)) \to H^k(X_{3,n}(\sigma_\emptyset)).$$

Since $(\mathrm{Id})_\emptyset = \mathrm{Id}$, this allows us to work out the identity case explicitly.

Proof

For $\sigma \neq \mathrm{Id}$, consider the diagram

$$\begin{tikzcd} H^k(X_{3,\infty}(\sigma)) \arrow[r, "s^*", "\cong"'] \arrow[d, "L_n^*"] & H^k(X_{3,\infty}(\sigma_\emptyset)) \arrow[d, "L_{n-1}^*"] \\ H^k(X_{3,n}(\sigma)) \arrow[r, "s^*"] & H^k(X_{3,n-1}(\sigma_\emptyset)). \end{tikzcd}$$

Use that we already know the theorem for $\sigma_\emptyset$.

Example

Example

For $n \geq k+1$, we can apply the theorem to compute

$$\begin{aligned} \bigoplus_{r=0}^{k} H^r(X_{n,3}) \otimes H^{k-r}(\mathrm{T}^3) \cong H^k(Y_{n,3}) & \\ \cong H^k(Y_{3,n}) & \\ \cong \bigoplus_{r=0}^{k} H^r(X_{3,n}) \otimes H^{k-r}(\mathrm{T}^n) & \\ \cong \bigoplus_{r=0}^{k} H^r(X_{3,\infty}) \otimes H^{k-r}(\mathrm{T}^n) & \end{aligned}$$

Example

Example ($k = 1$)

$$H^1(X_{n,3}) \oplus H^1(\mathrm{T}^3) \cong H^1(X_{3,\infty}) \oplus H^1(\mathrm{T}^n)$$
$$\implies H^1(X_{n,3}) \cong \mathbb{Z}^{n-1}.$$

Example ($k = 2$)

$$H^2(X_{n,3}) \oplus \mathbb{Z}^{3(n-1)} \oplus \mathbb{Z}^3 \cong \mathbb{Z}^2 \oplus \mathbb{Z}^{2n} \oplus \mathbb{Z}^{\binom{n}{2}}$$
$$\implies \mathrm{rank}_{\mathbb{Z}} H^2(X_{n,3}) = 1 + \binom{n-1}{2}$$

The spectral sequence

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong \bigoplus_{\{v_1,\dots,v_p\}} H^{p+q}\left(Y_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0)\right)$$

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \to E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p)$$

The spectral sequence

$$\begin{array}{c|lll}
9 & H^9(F_0) \longrightarrow & H^{10}(F_1,F_0) \longrightarrow & H^{11}(F_2,F_1) \\
8 & H^8(F_0) \longrightarrow & H^{9}(F_1,F_0) \longrightarrow & H^{10}(F_2,F_1) \\
7 & H^7(F_0) \longrightarrow & H^{8}(F_1,F_0) \longrightarrow & H^{9}(F_2,F_1) \\
6 & H^6(F_0) \longrightarrow & H^{7}(F_1,F_0) \longrightarrow & H^{8}(F_2,F_1) \\
5 & H^5(F_0) \longrightarrow & H^{6}(F_1,F_0) \longrightarrow & H^{7}(F_2,F_1) \\
4 & H^4(F_0) \longrightarrow & H^{5}(F_1,F_0) \longrightarrow & H^{6}(F_2,F_1) \\
3 & H^3(F_0) \longrightarrow & H^{4}(F_1,F_0) \longrightarrow & H^{5}(F_2,F_1) \\
2 & H^2(F_0) \longrightarrow & H^{3}(F_1,F_0) \longrightarrow & H^{4}(F_2,F_1) \\
1 & H^1(F_0) \longrightarrow & H^{2}(F_1,F_0) \longrightarrow & H^{3}(F_2,F_1) \\
0 & H^0(F_0) \longrightarrow & H^{1}(F_1,F_0) \longrightarrow & H^{2}(F_2,F_1) \\
\hline
{}^q\!/_p & 0 & 1 & 2
\end{array}$$

$E_1 E_2 E_2 = E_\infty$

$$\begin{array}{c|ccc}
9 & \mathbb{Z} & 0 & 0 \\
8 & \mathbb{Z}^7\mathbb{Z}^5 & \mathbb{Z}^2 0 & 0 \\
7 & \mathbb{Z}^{22}\mathbb{Z}^{11} & \mathbb{Z}^{12} 0 & \mathbb{Z}0 \\
6 & \mathbb{Z}^{42}\mathbb{Z}^{15} & \mathbb{Z}^{32}\mathbb{Z} & \mathbb{Z}^4 0 \\
5 & \mathbb{Z}^{56}\mathbb{Z}^{16} & \mathbb{Z}^{50}\mathbb{Z}^3 & \mathbb{Z}^7 0 \\
4 & \mathbb{Z}^{56}\mathbb{Z}^{16} & \mathbb{Z}^{50}\mathbb{Z}^3 & \mathbb{Z}^7 0 \\
3 & \mathbb{Z}^{42}\mathbb{Z}^{15} & \mathbb{Z}^{32}\mathbb{Z} & \mathbb{Z}^4 0 \\
2 & \mathbb{Z}^{22}\mathbb{Z}^{11} & \mathbb{Z}^{12} 0 & \mathbb{Z}0 \\
1 & \mathbb{Z}^7\mathbb{Z}^5 & \mathbb{Z}^2 0 & 00 \\
0 & \mathbb{Z} & 0 & 0 \\
\hline
{}^q\!/_p & 0 & 1 & 2
\end{array}$$

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) E_2^{p,q} = \ker d_1 / \operatorname{im} d_1 E_r^{p,q} = \ker d_{r-1} / \operatorname{im}$$

$$d_1 : E_1^{p,q} \to E_1^{p+1,q} d_2 : E_2^{p,q} \to E_2^{p+2,q-1} d_r : E_r^{p,q} \to E_2^{p+r,q-r}$$

$H^*(X_{3,3})$

$$H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases} \qquad H^q(Y_{3,3}) = \begin{cases} \mathbb{Z} & q = 9 \\ \mathbb{Z}^5 & q = 8 \\ \mathbb{Z}^{12} & q = 7 \\ \mathbb{Z}^{18} & q = 6 \\ \mathbb{Z}^{19} & q = 5 \\ \mathbb{Z}^{17} & q = 4 \\ \mathbb{Z}^{15} & q = 3 \\ \mathbb{Z}^{11} & q = 2 \\ \mathbb{Z}^5 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Compare

$$H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases} \qquad H^q(\Omega(\mathrm{SU}_3/T^2)) = \begin{cases} \vdots & q > 5 \\ \mathbb{Z}^4 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases}$$