

APPROXIMATIONS TO THE LOOP SPACE OF FLAG MANIFOLDS

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Definitions

Definition

$$X_{m,n} = \left\{ (v_1, \dots, v_n) \in (\mathbb{C}^m)^n \mid \begin{array}{l} \text{Any } m \text{ subsequent vectors in} \\ (e_1, \dots, e_m, v_1, \dots, v_n, e_1, \dots, e_m) \\ \text{are linearly independent.} \end{array} \right\}$$

$T^n = (\mathbb{C}^*)^n$ acts by multiplication on the columns of $X_{m,n}$.

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$$Y_{m,n} = X_{m,n} / T^n$$

$$X_{m,n} \cong Y_{m,n} \times (\mathbb{C}^*)^n$$

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Example

$$X_{3,3} = \left\{ \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \in \mathbb{C}^9 \mid \begin{array}{l} a \neq 0, ad - bc \neq 0, \\ adi + beh + cfg - deg - bci - afh \neq 0, \\ di - fh \neq 0, i \neq 0 \end{array} \right\}$$

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Stabilization

Definition

The map

$$\begin{aligned} s : X_{m,n} &\rightarrow X_{m,n+1} \\ (v_1, \dots, v_n) &\mapsto (Lv_1, \dots, Lv_n, Le_1) \end{aligned}$$

is injective and equivariant. The matrix is:

$$L = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \dots & \dots & 1 \end{pmatrix} \in \mathrm{GL}_m$$

The limit space

Definition

$$Y_{m,\infty} = \varinjlim_n Y_{m,n}$$

is the limit of

$$\dots \xrightarrow{s} Y_{m,n} \xrightarrow{s} Y_{m,n+1} \xrightarrow{s} \dots$$

The main theorem

Theorem

There is a collection of maps,

$$Y_{m,n} \rightarrow \Omega(\mathrm{SU}_m/T^{m-1}),$$

that fit together to form a map

$$f : Y_{m,\infty} \rightarrow \Omega(\mathrm{SU}_m/T^{m-1}).$$

This map is a (weak) homotopy equivalence.

Defining f

$$X_{m,n} \rightarrow P(\mathrm{GL}_m)$$

$$\begin{aligned}
 f(v_1, \dots, v_n) = & \left([e_1, e_2, e_3, \dots, e_m] \right. \\
 & \rightsquigarrow [d_1 v_1, e_2, e_3, \dots, e_m] \\
 & \rightsquigarrow [d_1 v_1, d_2 v_2, e_3, \dots, e_m] \\
 & \rightsquigarrow \dots \\
 & \left. \rightsquigarrow [\lambda_i e_i, \lambda_{i+1} e_{i+1}, \dots, \lambda_m e_m, \lambda_1 e_1, \dots, \lambda_{i-1} e_{i-1}] \right)
 \end{aligned}$$

Proof of theorem

$$\Omega(\mathrm{SU}_m/T^{m-1}) = \bigcup_{k=1}^{\infty} \Omega_k$$

$$\varphi_k : \Omega_k \rightarrow Y_{m, (2^k-1)m}$$

$$\gamma \mapsto \left[\gamma\left(\frac{1}{2^k}\right), \dots, \gamma\left(\frac{2^k-1}{2^k}\right) \right]$$

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Proof of surjectivity

Take a finite complex,

$$\sigma : W \rightarrow \Omega(\mathrm{SU}_m/T^{m-1})$$

$\sigma(W)$ is compact, so it is contained in Ω_k for some k .

$$\sigma \simeq f \circ \varphi_k \circ \sigma : W \rightarrow \Omega_k \rightarrow Y_{m,(2^k-1)m} \rightarrow \Omega$$

Injectivity is similar, but slightly more ugly.

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How do we get information from $Y_{m,n}$?

Definition

For $0 \leq p \leq m-1$, define

$$F_p = \{(v_1, \dots, v_n) \in X_{m,n} \mid v_n \text{ has at most } p \text{ zeroes.}\}$$

$$X_{m,n-1} \times (\mathbb{C}^*)^m \cong F_0 \subset F_1 \subset \dots \subset F_{m-1} = X_{m,n}$$

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The spectral sequence

q/p	0	1	2
9	$H^9(F_0)$	$\longrightarrow H^{10}(F_1, F_0)$	$\longrightarrow H^{11}(F_2, F_1)$
8	$H^8(F_0)$	$\longrightarrow H^9(F_1, F_0)$	$\longrightarrow H^{10}(F_2, F_1)$
7	$H^7(F_0)$	$\longrightarrow H^8(F_1, F_0)$	$\longrightarrow H^9(F_2, F_1)$
6	$H^6(F_0)$	$\longrightarrow H^7(F_1, F_0)$	$\longrightarrow H^8(F_2, F_1)$
5	$H^5(F_0)$	$\longrightarrow H^6(F_1, F_0)$	$\longrightarrow H^7(F_2, F_1)$
4	$H^4(F_0)$	$\longrightarrow H^5(F_1, F_0)$	$\longrightarrow H^6(F_2, F_1)$
3	$H^3(F_0)$	$\longrightarrow H^4(F_1, F_0)$	$\longrightarrow H^5(F_2, F_1)$
2	$H^2(F_0)$	$\longrightarrow H^3(F_1, F_0)$	$\longrightarrow H^4(F_2, F_1)$
1	$H^1(F_0)$	$\longrightarrow H^2(F_1, F_0)$	$\longrightarrow H^3(F_2, F_1)$
0	$H^0(F_0)$	$\longrightarrow H^1(F_1, F_0)$	$\longrightarrow H^2(F_2, F_1)$

The spectral sequence

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong \bigoplus_{\{v_1, \dots, v_p\}} H^{p+q} \left(X_{\{v_1, \dots, v_p\}} \times (D_p, D_p - 0) \right)$$

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \rightarrow E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p)$$

The following slides show the computation for $X_{3,3}$.

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The following slides show the computation for $X_{3,3}$.

E_1

9	\mathbb{Z}	0	0
8	\mathbb{Z}^7	\mathbb{Z}^2	0
7	\mathbb{Z}^{22}	\mathbb{Z}^{12}	\mathbb{Z}
6	\mathbb{Z}^{42}	\mathbb{Z}^{32}	\mathbb{Z}^4
5	\mathbb{Z}^{56}	\mathbb{Z}^{50}	\mathbb{Z}^7
4	\mathbb{Z}^{56}	\mathbb{Z}^{50}	\mathbb{Z}^7
3	\mathbb{Z}^{42}	\mathbb{Z}^{32}	\mathbb{Z}^4
2	\mathbb{Z}^{22}	\mathbb{Z}^{12}	\mathbb{Z}
1	\mathbb{Z}^7	\mathbb{Z}^2	0
0	\mathbb{Z}	0	0
q/p	0	1	2

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1})$$

$$d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$$

E_2

9	\mathbb{Z}	0	0
8	\mathbb{Z}^5	0	0
7	\mathbb{Z}^{11}	0	0
6	\mathbb{Z}^{15}	\mathbb{Z}	0
5	\mathbb{Z}^{16}	\mathbb{Z}^3	0
4	\mathbb{Z}^{16}	\mathbb{Z}^3	0
3	\mathbb{Z}^{15}	\mathbb{Z}	0
2	\mathbb{Z}^{11}	0	0
1	\mathbb{Z}^5	0	0
0	\mathbb{Z}	0	0
q/p	0	1	2

$$E_2^{p,q} = \ker d_1 / \operatorname{im} d_1$$

$$d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$$

$$E_2 = E_\infty$$

9	\mathbb{Z}	0	0
8	\mathbb{Z}^5	0	0
7	\mathbb{Z}^{11}	0	0
6	\mathbb{Z}^{15}	\mathbb{Z}	0
5	\mathbb{Z}^{16}	\mathbb{Z}^3	0
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2	\mathbb{Z}^{11}	0	0
1	\mathbb{Z}^5	0	0
0	\mathbb{Z}	0	0
q/p	0	1	2

$$E_r^{p,q} = \ker d_{r-1} / \operatorname{im} d_{r-1}$$

$$d_r : E_r^{p,q} \rightarrow E_2^{p+r, q-r+1}$$

$$H^*(X_{3,3})$$

$$H^q(Y_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases} \quad H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 9 \\ \mathbb{Z}^5 & q = 8 \\ \mathbb{Z}^{12} & q = 7 \\ \mathbb{Z}^{18} & q = 6 \\ \mathbb{Z}^{19} & q = 5 \\ \mathbb{Z}^{17} & q = 4 \\ \mathbb{Z}^{15} & q = 3 \\ \mathbb{Z}^{11} & q = 2 \\ \mathbb{Z}^5 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Compare

$$H^q(Y_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases} \quad H^q(\Omega(\mathrm{SU}_3/T^2)) = \begin{cases} \vdots & q > 5 \\ \mathbb{Z}^4 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases}$$

Ideas for future work

- Show that $s : Y_{m,n} \rightarrow Y_{m,n+1}$ is:
 - Injective on homology.
 - Surjective on cohomology.
- Compute $\pi_1(Y_{m,n})$. We already know $H_1(Y_{m,n}) \cong \mathbb{Z}^{m-1}$.
- Consider the space,

$$Y'_{m,n} = \bigcup_{A \in \mathrm{GL}_m} Y_{m,n}(A, A)$$

See if this approximates the free loop space $L(\mathrm{SU}_m/T^{m-1})$.

- Replace \mathbb{C} by \mathbb{R} or \mathbb{H} .

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