Approximations to the loop space of flag manifolds

Sarah Stoltze

Institut for Matematik
Aarhus University

30. January, 2015

Definitions

Definition

$$X_{m,n} = \left\{ (v_1, \ldots, v_n) \in (\mathbb{C}^m)^n \;\middle|\; \begin{array}{c} \text{Any } m \text{ subsequent vectors in} \\ (e_1, \ldots, e_m, v_1, \ldots, v_n, e_1, \ldots, e_m) \\ \text{are linearly independent.} \end{array} \right\}$$

$T^n = (\mathbb{C}^*)^n$ acts by multiplication on the columns of $X_{m,n}$.

Definition

$$Y_{m,n} = X_{m,n}/T^n$$

$$X_{m,n} \cong Y_{m,n} \times (\mathbb{C}^*)^n$$

Definitions

Definition

$$X_{m,n} = \left\{ (v_1, \dots, v_n) \in (\mathbb{C}^m)^n \;\middle|\; \begin{array}{c} \text{Any } m \text{ subsequent vectors in} \\ (e_1, \dots, e_m, v_1, \dots, v_n, e_1, \dots, e_m) \\ \text{are linearly independent.} \end{array} \right\}$$

$T^n = (\mathbb{C}^*)^n$ acts by multiplication on the columns of $X_{m,n}$.

Definition

$$Y_{m,n} = X_{m,n}/T^n$$

$$X_{m,n} \cong Y_{m,n} \times (\mathbb{C}^*)^n$$

Definitions

Definition

$$X_{m,n} = \left\{ (v_1, \ldots, v_n) \in (\mathbb{C}^m)^n \;\middle|\; \begin{array}{c} \text{Any } m \text{ subsequent vectors in} \\ (e_1, \ldots, e_m, v_1, \ldots, v_n, e_1, \ldots, e_m) \\ \text{are linearly independent.} \end{array} \right\}$$

$T^n = (\mathbb{C}^*)^n$ acts by multiplication on the columns of $X_{m,n}$.

Definition

$$Y_{m,n} = X_{m,n}/T^n$$

$$X_{m,n} \cong Y_{m,n} \times (\mathbb{C}^*)^n$$

Example

$$X_{3,3} = \left\{ \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \in \mathbb{C}^9 \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0, \\ adi + beh + cfg - deg - bci - afh \neq 0, \\ di - fh \neq 0, i \neq 0 \end{matrix} \right\}$$

$$Y_{3,3} \cong \left\{ \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \in \mathbb{C}^9 \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0, \\ adi + beh + cfg - deg - bci - afh = 1, \\ di - fh = 1, i = 1 \end{matrix} \right\}$$

Example

$$X_{3,3} = \left\{ \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \in \mathbb{C}^9 \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0, \\ adi + beh + cfg - deg - bci - afh \neq 0, \\ di - fh \neq 0, i \neq 0 \end{matrix} \right\}$$

$$Y_{3,3} \cong \left\{ \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \in \mathbb{C}^9 \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0, \\ adi + beh + cfg - deg - bci - afh = 1, \\ di - fh = 1, i = 1 \end{matrix} \right\}$$

Stabilization

Definition

The map

$$s : X_{m,n} \to X_{m,n+1}$$
$$(v_1, \dots, v_n) \mapsto (Lv_1, \dots, Lv_n, Le_1)$$

is injective and equivariant. The matrix is:

$$L = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \dots & \dots & 1 \end{pmatrix} \in \mathrm{GL}_m$$

The limit space

Definition

$$Y_{m,\infty} = \varinjlim_{n} Y_{m,n}$$

is the limit of

$$\dots \xrightarrow{s} Y_{m,n} \xrightarrow{s} Y_{m,n+1} \xrightarrow{s} \dots$$

The main theorem

Theorem

There is a collection of maps,

$$Y_{m,n} \to \Omega(\mathrm{SU}_m/T^{m-1}),$$

that fit together to form a map

$$f : Y_{m,\infty} \to \Omega(\mathrm{SU}_m/T^{m-1}).$$

This map is a (weak) homotopy equivalence.

Defining f

$$
\begin{aligned}
X_{m,n} &\to P(\mathrm{GL}_m) \\
f(v_1, \dots, v_n) &= \Big([e_1, e_2, e_3, \dots, e_m] \\
&\qquad \rightsquigarrow [d_1 v_1, e_2, e_3, \dots, e_m] \\
&\qquad \rightsquigarrow [d_1 v_1, d_2 v_2, e_3, \dots, e_m] \\
&\qquad \rightsquigarrow \dots \\
&\qquad \rightsquigarrow [\lambda_i e_i, \lambda_{i+1} e_{i+1}, \dots, \lambda_m e_m, \lambda_1 e_1, \dots, \lambda_{i-1} e_{i-1}] \Big)
\end{aligned}
$$

Proof of theorem

$$\Omega(\mathrm{SU}_m/T^{m-1}) = \bigcup_{k=1}^{\infty} \Omega_k$$

$$\varphi_k : \Omega_k \to Y_{m,(2^k-1)m}$$

$$\gamma \mapsto \left[\gamma\left(\frac{1}{2^k}\right), \dots, \gamma\left(\frac{2^k-1}{2^k}\right)\right]$$

Proof of theorem

$$\Omega(\mathrm{SU}_m/T^{m-1}) = \bigcup_{k=1}^{\infty} \Omega_k$$

$$\varphi_k : \Omega_k \to Y_{m,(2^k-1)m}$$

$$\gamma \mapsto \left[\gamma\left(\frac{1}{2^k}\right), \ldots, \gamma\left(\frac{2^k-1}{2^k}\right)\right]$$

Proof of surjectivity

Take a finite complex,

$$\sigma : W \to \Omega(\mathrm{SU}_m/T^{m-1})$$

$\sigma(W)$ is compact, so it is contained in Ω_k for some k.

$$\sigma \simeq f \circ \varphi_k \circ \sigma : W \to \Omega_k \to Y_{m,(2^k-1)m} \to \Omega$$

Injectivity is similar, but slightly more ugly.

Proof of surjectivity

Take a finite complex,

$$\sigma : W \to \Omega(\mathrm{SU}_m/T^{m-1})$$

$\sigma(W)$ is compact, so it is contained in Ω_k for some k.

$$\sigma \simeq f \circ \varphi_k \circ \sigma : W \to \Omega_k \to Y_{m,(2^k-1)m} \to \Omega$$

Injectivity is similar, but slightly more ugly.

Proof of surjectivity

Take a finite complex,

$$\sigma : W \to \Omega(\mathrm{SU}_m/T^{m-1})$$

$\sigma(W)$ is compact, so it is contained in Ω_k for some k.

$$\sigma \simeq f \circ \varphi_k \circ \sigma : W \to \Omega_k \to Y_{m,(2^k-1)m} \to \Omega$$

Injectivity is similar, but slightly more ugly.

Proof of surjectivity

Take a finite complex,

$$\sigma : W \to \Omega(\mathrm{SU}_m/T^{m-1})$$

$\sigma(W)$ is compact, so it is contained in Ω_k for some k.

$$\sigma \simeq f \circ \varphi_k \circ \sigma : W \to \Omega_k \to Y_{m,(2^k-1)m} \to \Omega$$

Injectivity is similar, but slightly more ugly.

How do we get information from $Y_{m,n}$?

Definition

For $0 \leq p \leq m-1$, define

$$F_p = \{(v_1, \dots, v_n) \in X_{m,n} \mid v_n \text{ has at most } p \text{ zeroes.}\}$$

$$X_{m,n-1} \times (\mathbb{C}^*)^m \cong F_0 \subset F_1 \subset \cdots \subset F_{m-1} = X_{m,n}$$

How do we get information from $Y_{m,n}$?

Definition

For $0 \leq p \leq m-1$, define

$$F_p = \{(v_1, \ldots, v_n) \in X_{m,n} \mid v_n \text{ has at most } p \text{ zeroes.}\}$$

$$X_{m,n-1} \times (\mathbb{C}^*)^m \cong F_0 \subset F_1 \subset \cdots \subset F_{m-1} = X_{m,n}$$

The spectral sequence

$$
\begin{array}{c|ccccc}
9 & H^9(F_0) & \longrightarrow & H^{10}(F_1, F_0) & \longrightarrow & H^{11}(F_2, F_1) \\
8 & H^8(F_0) & \longrightarrow & H^{9}(F_1, F_0) & \longrightarrow & H^{10}(F_2, F_1) \\
7 & H^7(F_0) & \longrightarrow & H^{8}(F_1, F_0) & \longrightarrow & H^{9}(F_2, F_1) \\
6 & H^6(F_0) & \longrightarrow & H^{7}(F_1, F_0) & \longrightarrow & H^{8}(F_2, F_1) \\
5 & H^5(F_0) & \longrightarrow & H^{6}(F_1, F_0) & \longrightarrow & H^{7}(F_2, F_1) \\
4 & H^4(F_0) & \longrightarrow & H^{5}(F_1, F_0) & \longrightarrow & H^{6}(F_2, F_1) \\
3 & H^3(F_0) & \longrightarrow & H^{4}(F_1, F_0) & \longrightarrow & H^{5}(F_2, F_1) \\
2 & H^2(F_0) & \longrightarrow & H^{3}(F_1, F_0) & \longrightarrow & H^{4}(F_2, F_1) \\
1 & H^1(F_0) & \longrightarrow & H^{2}(F_1, F_0) & \longrightarrow & H^{3}(F_2, F_1) \\
0 & H^0(F_0) & \longrightarrow & H^{1}(F_1, F_0) & \longrightarrow & H^{2}(F_2, F_1) \\
\hline
{}^{q}\!/_{p} & 0 & & 1 & & 2
\end{array}
$$

The spectral sequence

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong \bigoplus_{\{v_1,\dots,v_p\}} H^{p+q}\left(X_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0)\right)$$

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \to E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p)$$

The following slides show the computation for $X_{3,3}$.

The spectral sequence

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong \bigoplus_{\{v_1,\ldots,v_p\}} H^{p+q}\left(X_{\{v_1,\ldots,v_p\}} \times (D_p, D_p - 0)\right)$$

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \to E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p)$$

The following slides show the computation for $X_{3,3}$.

The spectral sequence

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong \bigoplus_{\{v_1,\dots,v_p\}} H^{p+q}\left(X_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0)\right)$$

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \to E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p)$$

The following slides show the computation for $X_{3,3}$.

The spectral sequence

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong \bigoplus_{\{v_1,\dots,v_p\}} H^{p+q}\left(X_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0)\right)$$

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \to E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p)$$

The following slides show the computation for $X_{3,3}$.

E_1

9	$\mathbb{Z}$	0	0
8	$\mathbb{Z}^7$	$\mathbb{Z}^2$	0
7	$\mathbb{Z}^{22}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$
6	$\mathbb{Z}^{42}$	$\mathbb{Z}^{32}$	$\mathbb{Z}^4$
5	$\mathbb{Z}^{56}$	$\mathbb{Z}^{50}$	$\mathbb{Z}^7$
4	$\mathbb{Z}^{56}$	$\mathbb{Z}^{50}$	$\mathbb{Z}^7$
3	$\mathbb{Z}^{42}$	$\mathbb{Z}^{32}$	$\mathbb{Z}^4$
2	$\mathbb{Z}^{22}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$
1	$\mathbb{Z}^7$	$\mathbb{Z}^2$	0
0	$\mathbb{Z}$	0	0
q/p	0	1	2

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1})$$

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

E_2

q			
9	$\mathbb{Z}$	0	0
8	$\mathbb{Z}^5$	0	0
7	$\mathbb{Z}^{11}$	0	0
6	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
5	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
4	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
3	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
2	$\mathbb{Z}^{11}$	0	0
1	$\mathbb{Z}^5$	0	0
0	$\mathbb{Z}$	0	0
q/p	0	1	2

$$E_2^{p,q} = \ker d_1 / \operatorname{im} d_1$$

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$$

$E_2 = E_\infty$

q \ p	0	1	2
9	$\mathbb{Z}$	0	0
8	$\mathbb{Z}^5$	0	0
7	$\mathbb{Z}^{11}$	0	0
6	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
5	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
4	$\mathbb{Z}^{16}$	$\mathbb{Z}^3$	0
3	$\mathbb{Z}^{15}$	$\mathbb{Z}$	0
2	$\mathbb{Z}^{11}$	0	0
1	$\mathbb{Z}^5$	0	0
0	$\mathbb{Z}$	0	0

$$E_r^{p,q} = \ker d_{r-1} / \operatorname{im} d_{r-1}$$

$$d_r : E_r^{p,q} \to E_2^{p+r,q-r+1}$$

$H^*(X_{3,3})$

$$H^q(Y_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases} \qquad H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 9 \\ \mathbb{Z}^5 & q = 8 \\ \mathbb{Z}^{12} & q = 7 \\ \mathbb{Z}^{18} & q = 6 \\ \mathbb{Z}^{19} & q = 5 \\ \mathbb{Z}^{17} & q = 4 \\ \mathbb{Z}^{15} & q = 3 \\ \mathbb{Z}^{11} & q = 2 \\ \mathbb{Z}^5 & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Compare

$$H^q(Y_{3,3}) = \begin{cases} \mathbb{Z} & q = 6 \\ \mathbb{Z}^2 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases} \qquad H^q(\Omega(\mathrm{SU}_3/T^2)) = \begin{cases} \vdots & q > 5 \\ \mathbb{Z}^4 & q = 5 \\ \mathbb{Z}^3 & q = 4 \\ \mathbb{Z}^2 & q = 3 \\ \mathbb{Z}^2 & q = 2 \\ \mathbb{Z}^2 & q = 1 \\ \mathbb{Z} & q = 0 \end{cases}$$

Ideas for future work

- Show that $s : Y_{m,n} \to Y_{m,n+1}$ is:
  - Injective on homology.
  - Surjective on cohomology.
- Compute $\pi_1(Y_{m,n})$. We already know $H_1(Y_{m,n}) \cong \mathbb{Z}^{m-1}$.
- Consider the space,

$$Y'_{m,n} = \bigcup_{A \in \mathrm{GL}_m} Y_{m,n}(A, A)$$

  See if this approximates the free loop space $L(\mathrm{SU}_m / T^{m-1})$.
- Replace $\mathbb{C}$ by $\mathbb{R}$ or $\mathbb{H}$.

Ideas for future work

- Show that $s : Y_{m,n} \to Y_{m,n+1}$ is:
  - Injective on homology.
  - Surjective on cohomology.
- Compute $\pi_1(Y_{m,n})$. We already know $H_1(Y_{m,n}) \cong \mathbb{Z}^{m-1}$.
- Consider the space,

$$Y'_{m,n} = \bigcup_{A \in \mathrm{GL}_m} Y_{m,n}(A, A)$$

  See if this approximates the free loop space $L(\mathrm{SU}_m/T^{m-1})$.
- Replace $\mathbb{C}$ by $\mathbb{R}$ or $\mathbb{H}$.

Ideas for future work

- Show that $s : Y_{m,n} \to Y_{m,n+1}$ is:
  - Injective on homology.
  - Surjective on cohomology.
- Compute $\pi_1(Y_{m,n})$. We already know $H_1(Y_{m,n}) \cong \mathbb{Z}^{m-1}$.
- Consider the space,

$$Y'_{m,n} = \bigcup_{A \in \mathrm{GL}_m} Y_{m,n}(A, A)$$

  See if this approximates the free loop space $L(\mathrm{SU}_m / T^{m-1})$.
- Replace $\mathbb{C}$ by $\mathbb{R}$ or $\mathbb{H}$.

Ideas for future work

- Show that $s : Y_{m,n} \to Y_{m,n+1}$ is:
  - Injective on homology.
  - Surjective on cohomology.
- Compute $\pi_1(Y_{m,n})$. We already know $H_1(Y_{m,n}) \cong \mathbb{Z}^{m-1}$.
- Consider the space,

$$Y'_{m,n} = \bigcup_{A \in \mathrm{GL}_m} Y_{m,n}(A, A)$$

  See if this approximates the free loop space $L(\mathrm{SU}_m / T^{m-1})$.
- Replace $\mathbb{C}$ by $\mathbb{R}$ or $\mathbb{H}$.

Ideas for future work

- Show that $s : Y_{m,n} \to Y_{m,n+1}$ is:
  - Injective on homology.
  - Surjective on cohomology.
- Compute $\pi_1(Y_{m,n})$. We already know $H_1(Y_{m,n}) \cong \mathbb{Z}^{m-1}$.
- Consider the space,

$$Y'_{m,n} = \bigcup_{A \in \mathrm{GL}_m} Y_{m,n}(A, A)$$

  See if this approximates the free loop space $L(\mathrm{SU}_m / T^{m-1})$.
- Replace $\mathbb{C}$ by $\mathbb{R}$ or $\mathbb{H}$.