Approximations to the loop space of flag manifolds

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Contents

Preface

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This dissertation is the outcome of my Ph.D. studies at the Department of Mathematics, Aarhus University, from February 2013 to October 2016. It contains and expands on the subjects covered in my combined progress report and Master's thesis, which was finished and defended in January 2015.

I would like to thank my Ph.D. advisor, Marcel Bökstedt, for both suggesting this project and helping me with plenty of meetings and long discussions when I was stuck, as well as providing encouragement and good advice when it was needed most.

I would also like to thank my office mates and fellow Ph.D. students at the Department of Mathematics for both providing valuable feedback on ideas, and for keeping me motivated.

FiXme Fatal: Acknowledgements etc. Venner, kolleger, familie, Julie?

Notation and conventions

Throughout the dissertation, the following conventions will be used. For two topological spaces X and Y, we will write $X \cong Y$ to denote that X and Y are homeomorphic, and $X \simeq Y$ to denote that they are homotopy equivalent. Likewise, $f \simeq g$ means that the maps f and g are homotopic. For subsets, we will use $A \subset X$ when A is any subset of X, and we will generally not make a distinction between points of X and one-point subsets of X. The set difference of X and A will be written as $X - A$.

We will be working almost exclusively with topology in the complex numbers, so the disc $D \subset \mathbb{C}$ is the unit disc in the complex plane and D_p is the unit disc in $\mathbb{C}^p$. The non-zero elements in $\mathbb{C}$ will be written as $\mathbb{C}^* = \mathbb{C} - 0$. $\mathrm{GL}_m(\mathbb{C})$ will denote the group of linear automorphisms of $\mathbb{C}^m$ and will be written as matrices with respect to the standard basis $e_1, \ldots, e_m$ of $\mathbb{C}^m$, defined by

$$e_i = \begin{array}{cc} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \end{array}$$

Since we are mainly concerned with algebraic topology, a space will occasionally be replaced with a homotopy equivalent one without mention. For example, T^n will denote both the complex units $(\mathbb{C}^*)^n$, the diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$, and $(S^1)^n$ depending on context, and the punctured disc $D_p - 0$ will sometimes be identified with the sphere $S^{2p-1} \subset \mathbb{C}^p$. Likewise, we will not distinguish between the different models of the flag manifold of $\mathbb{C}^m$, given as $\mathrm{GL}_m(\mathbb{C})/\,\mathrm{B}_m \cong \mathrm{U}_m\,/\,\mathrm{T}^m \cong \mathrm{SU}_m\,/\,\mathrm{T}^{m-1}$ with $\mathrm{B}_m \subset \mathrm{GL}_m(\mathbb{C})$ the upper (or lower) triangular matrices in $\mathrm{GL}_m(\mathbb{C})$, $\mathrm{T}^m \subset \mathrm{U}_m$ the diagonal matrices in the unitary group and $\mathrm{T}^{m-1} \subset \mathrm{SU}_m$ the diagonal matrices in the special unitary group.

Homology and cohomology groups will always be singular groups with integer coefficients. If a theorem is mentioned by name without a reference, it can be found in [Hat02].

We will use permutations a great deal. The permutations of the set $\{1, \ldots, m\}$ will be denoted S_m. These will be written as products of cycles, where we multiply by considering permutations as functions and then composing. For example, the permutation $\sigma \in S_3$, given by

$$\begin{aligned} \sigma(1) &= 3, \\ \sigma(2) &= 1, \\ \sigma(3) &= 2, \end{aligned}$$

will be written as $\sigma = (1\ 3\ 2) = (2\ 3) \cdot (1\ 2)$. For our purposes, it will be useful to occasionally consider a permutation as a matrix. We will do this by working with the standard basis, $e_1, \ldots, e_m \in \mathbb{C}^m$. An element $\sigma \in S_m$ then corresponds to the matrix defined by

$$\sigma \cdot e_i = e_{\sigma(i)}.$$

So a permutation multiplied on a matrix from the left permutes the rows of the matrix.

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October, 2016

Abstract

Introduction

FiXme Fatal: Skriv det der skal være før her. Snak med Marcel. Og få indledningen til at give mening

The spaces we will be workng with were originally introduced as... blah blah, nulsnit af v.b., et eller andet, blah, matricer.

The spaces $X_{m,n}(\sigma)$ that we will be studying in the thesis were originally studied in a slightly simpler form by my advisor, Marcel Bökstedt. He looked at sequences of vectors $(v_1, \dots, v_n)$ in $\mathbb{C}^2$, with the requirement that v_i and v_{i+1} are linearly independent, that the first entry of v_1 is non-zero, and either the first or the second entry of v_n is non-zero as well. Using the notation introduced in this thesis, these are the spaces $Y_{2,n}$ or $Y_{2,n}((1\ 2))$ from Definition 8. Taking a quotient of the action

$$\begin{gathered}\mathrm{T}^n \times Y_{2,n}(\sigma) \to Y_{2,n}(\sigma)\\ (\lambda_1, \dots, \lambda_n) \cdot (v_1, \dots, v_n) = (\lambda_1 v_1, \dots, \lambda_n v_n),\end{gathered}$$

we end up with the spaces referred to as $X_{2,n}(\sigma)$ in Definition 17. He then showed that $Y_{2,n}$ was covered by the spaces $Y_{2,n-1} \times \mathrm{T}^2$ and $Y_{2,n-1}((1\ 2)) \times \mathrm{T}$, giving inclusion maps

$$s : Y_{2,n-1} \cong Y_{2,n-1} \times \{(1,1)\} \to Y_{2,n}.$$

By using the Mayer-Vietoris sequence for the cover, and the corresponding cover of $X_{2,n}$, he proved the following theorem.

Theorem 1. *The space $X_{2,n}$ has homology groups*

$$H_q(X_{2,n}) \cong \begin{cases} \mathbb{Z} & 0 \le q \le n, \\ 0 & \text{otherwise.} \end{cases}$$

The inclusion

$$s : X_{2,n-1} \to X_{2,n}$$

is an isomorphism on homology when the degree is $\le n-1$.

The homology of $X_{2,n}$ consists of a copy of the integers in all degrees up to n, so when n is large it looks like the homology groups of the loop space $\Omega(\mathrm{SU}_2 / \mathrm{T})$, which has a copy of the integers in each degree. We can define a map from $X_{2,n}$ to the loop space by sending $(v_1, \dots, v_n)$ to the path given by connecting up the matrices

$$\mathrm{Id} \rightsquigarrow [e_2, v_1] \rightsquigarrow [v_1, v_2] \rightsquigarrow \cdots \rightsquigarrow \mathrm{Id},$$

and it turns out that this map is an isomorphism on homology when the groups are non-zero.

The aim of this dissertation is to extend some of these results to $X_{m,n}$, $m > 2$, with a focus on cohomology instead of homology. The cohomology groups become much harder to calculate when $m > 2$, since the Mayer-Vietoris sequence that was used for $X_{2,n}$ does not give enough information. Instead we have to cover $X_{m,n}$ with several subspaces, which means we have to use spectral sequences to compute the cohomology groups and this leads to increased complexity. However, it is still possible to prove that $X_{m,n}$ approximates the loop space $\Omega(\mathrm{SU}_m \,/\, \mathrm{T}^{m-1})$ when n is large, which is the content of Theorem 30. This approximation allows us to compute $H^q(X_{3,n})$ when $n \geq q$, as is done in Theorem 35.

The thesis starts with a short introduction to some of the tools and subjects that will be required in the following chapters, contained in Chapter 1. Each section consists of a brief description of a subject, with references to important theorems needed for our purposes. The topics that are covered are spectral sequences, loop spaces, flags, and Coxeter groups, and since it is not possible to give a full overview of any of these in a couple of pages, every section also contain references to literature with a more extensive coverage.

In Chapter 2, we give a description of the spaces we will be working with, first introducing subspaces $Y_{m,n}(\sigma)$ of $(\mathbb{C}^m)^n$, which depend on permutations $\sigma \in S_m$, and then quotienting out with an action of the complex torus $\mathrm{T}^n = (\mathbb{C}^*)^n$, giving the spaces $X_{m,n}(\sigma)$ that we will mostly be working with. We show that these spaces are non-empty when n is large enough, and we spend some time working out how the space $Y_{m,n}(\sigma)$ can be built from the spaces $Y_{m,n-1}(\tau)$, and exactly which permutations τ are necessary to do this in Theorem 20. This allows us to take a limit, and gives us the spaces $Y_{m,\infty}(\sigma)$ and $X_{m,\infty}(\sigma)$. We end with a very brief discussion of the spaces we get when we replace the complex numbers with the reals or quaternions.

Chapter 3 starts with a description of a spectral sequence that will be used extensively. This allows us to compute the cohomology of the space $Y_{m,n}(\sigma)$ from the cohomology of the spaces $Y_{m,n-1}(\tau)$ that it is built from. We immediately use this spectral sequence to work out the first cohomology group of both $Y_{m,n}$ and $X_{m,n}$ for all m and n. The rest of the chapter is spent doing computations with the spectral sequence for the spaces $Y_{3,3}$ and $X_{3,3}$, ending with a complete description of the cohomology groups for these spaces.

The following chapter, Chapter 4, studies the limit spaces $X_{m,\infty}(\sigma)$ in detail. We start by showing that they are homotopy equivalent to the loop space $\Omega(\mathrm{SU}_m \,/\, \mathrm{T}^{m-1})$, which gives us the cohomology of the limit space. The proof does not depend on working over the complex numbers, so it easily carries over to the real and quaternionic cases. We then give a brief mention of other ways we could define the limit space, which are all equivalent from a homotopy theoretic standpoint. By working with one of these other stabilisations of $X_{3,n}(\sigma)$, we get Theorem 35 that tells us when the cohomology groups of $X_{3,n}(\sigma)$ stabilise to the cohomology groups of $X_{3,\infty}(\sigma)$.

The thesis concludes with a brief description of the open questions still remaining, contained in Chapter 5. These are questions that have been raised during the last parts of the project and thus have not been studied in detail, as well as small ideas for how the results could be applied to the free loop space instead of the space of based loops considered in the thesis.

1 Preliminaries

The purpose of this chapter is to give a brief introduction to some important topics, most importantly spectral sequences and loop spaces. Spectral sequences are an important tool in algebraic topology and can be considered a generalisation of long exact sequences, while loop spaces are a class of spaces that appear naturally when considering the spaces that will be studied in this thesis. The following pages will collect some basic info and state the results that are of interest in next chapters, while providing references for proofs and further information. The chapter also includes a very brief introduction to flags and Coxeter groups, to give enough information for the small part these two subjects will play in the main text.

1.1 Spectral sequences

A spectral sequence, as we will be using them, is a tool for computing the cohomology groups of a topological space, X, by considering an increasing sequence of subspaces

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X,$$

with the union covering all of X,

$$X = \bigcup_i X_i.$$

An example that most people should be familiar with is the long exact sequence of a pair, where we have a single subspace $A \subset X$ and try to compute the cohomology of X from the cohomology of A and the cohomology of the pair (X, A). Another simple example is the case where there are two subspaces,

$$X = A \cup B,$$

in which case the cohomology of X can often be computed from the cohomology of A, B and $A \cap B$ by using the Mayer-Vietoris sequence. This generalises to a spectral sequence to compute the cohomology of X by working with an abitrary covering. As might be expected, since spectral sequences work more generally they also carry a corresponding increase in complexity when using them for calculations. In the following section we will focus on the specific case that we will be using in this text. For a general introduction to the subject, see either [Hat04] or [McC01].

The spectral sequence of a filtration

Consider a space X with a filtration, an increasing sequence of subspaces that cover X:

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X.$$

All of the following can be modified to work if the filtration is not finite, but we will not need this here. We will augment the filtration by defining additional subspaces,

$$X_s = \begin{cases} \emptyset & s \le -1, \\ X & s \ge k+1. \end{cases}.$$

With such a sequence, we can consider the relative cohomology groups

$$E_1^{p,q} = H^{p+q}(X_p, X_{p-1}),$$

and the differentials between them coming from the long exact sequence of the triple (X_{p+1}, X_p, X_{p-1}):

$$d_1 : E_1^{p,q} = H^{p+q}(X_p, X_{p-1}) \to H^{p+q+1}(X_{p+1}, X_p) = E_1^{p+1,q}.$$

This map factors as the composition of an inclusion of X_p into the pair (X_p, X_{p-1}) and the boundary map of the pair (X_{p+1}, X_p),

$$d_1 : H^{p+q}(X_p, X_{p-1}) \xrightarrow{\jmath^*} H^{p+q}(X_p) \xrightarrow{\delta} H^{p+q+1}(X_{p+1}, X_p),$$

and in particular we get that the composition of two such maps,

$$d_1 \circ d_1 = \delta \circ \jmath^* \circ \delta \circ \jmath^*,$$

must be zero since the composition $\jmath^* \circ \delta$ is zero by the long exact seuqence of the pair (X_{p+1}, X_p). So we can compute the homology of these groups and get new groups,

$$E_2^{p,q} = \ker d_1 / \operatorname{im} d_1.$$

More surprisingly, we can also find new maps

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$$

that make these new groups into a chain complex, allowing us to form

$$E_3^{p,q} = \ker d_2 / \operatorname{im} d_2.$$

This process can be continued, giving a sequence of groups and maps

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

where the composition $d_r \circ d_r$ is zero and $E_r^{p,q}$ is formed by taking the homology of $E_{r-1}^{p,q}$ with respect to the map d_{r-1}. Since we have $E_r^{p,q} = 0$ for $p < 0$ or $p > k$, these maps will eventually always go either to or from a zero group when $r > k$, and hence the groups $E_r^{p,q}$ will not change when we increase r. This allows us to define the stable groups $E_\infty^{p,q}$ as these r-independent groups. The following theorem relates them to the cohomology groups of X that we are interested in calculating.

Theorem 2. *Given a filtration $\{X_s\}$ of a topological space X as above, there is a sequence of abelian groups $E_r^{p,q}$ and maps*

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

that satisfy the following:

- *The first groups are*
$$E_1^{p,q} = H^{p+q}(X_p, X_{p-1})$$
and the map d_1 *is the boundary map of the triples* (X_{p+1}, X_p, X_{p-1}).

- *The composition* $d_r \circ d_r$ *is zero and*
$$E_{r+1}^{p,q} = \ker d_r / \operatorname{im} d_r,$$
computed at $E_r^{p,q}$.

- *There is a filtration of the n'th cohomology group of* X,
$$0 \subset F_n^n \subset \cdots \subset F_0^n = H^n(X),$$
with the limit groups $E_\infty^{p,n-p}$ *isomorphic to the quotients*
$$E_\infty^{p,n-p} \cong F_p^n / F_{p+1}^n.$$

This theorem is a combination of Proposition 1.2 and Theorem 1.14 in [Hat04], and the proof can be found there as well. Assuming we can compute the spectral sequence, the group $H^n(X)$ can then be found by solving extension problems of the form
$$0 \to F_{p+1}^n \to F_p^n \to E_\infty^{p,n-p} \to 0,$$
starting from the short exact sequence
$$0 \to E_\infty^{n,0} \to F_{n-1}^n \to E_\infty^{n-1,1} \to 0.$$
For example, if we have that the groups $E_\infty^{p,n-p}$ are free for all p, the short exact sequences always split and $H^n(X)$ is isomorphic to the direct sum of these groups,
$$H^n(X) \cong \bigoplus_p E_\infty^{p,n-p}.$$

1.2 Loop spaces

Throughout this section, every topological space will have a distinguished basepoint. We will be looking a spaces of loops, consiting of maps from the circle to a given space X, preserving the basepoint. For a general introduction to loop spaces, consult e.g. [Mil63, Part 3] or [May99, Section 8.2]. The results in this section are taken from [Mil63] and the proofs can be found there as well.

Definition 3. If X is a topological space with basepoint $* \in X$, the loop space $\Omega(X)$ is the space of based maps from the cicle to X,
$$\Omega(X) = \left\{\gamma : (S^1, 1) \to (X, *)\right\} = \{\gamma : [0,1] \to X \mid f(0) = f(1) = *\}.$$

We equip this set with the compact-open topology, so for a compact set $K \subset S^1$ and an open set $U \subset X$, we get an open subset of the loop space consisting of functions mapping K to U,
$$V(K,U) = \{\gamma \in \Omega(X) \mid \gamma(K) \subset U\}.$$
The collection of all such sets form a subbase for the compact-open topology on $\Omega(X)$.

We will only worry about the case where X is a Riemannian manifold, so it comes equipped with a Riemannian metric g and an induced topological metric d. In this case, the compact-open topology is induced by the metric

$$d^*(\gamma, \gamma') = \max_t d(\gamma(t), \gamma'(t)),$$

so two loops are close in $\Omega(X)$ when the points $\gamma(t)$ and $\gamma'(t)$ are close for each t. By [Mil63, Theorem 17.1], this is homotopy equivalent to the space of piecewise smooth loops in X, and we will generally work with this space instead. The proof will not be repeated here, but there is a construction in the proof that deserves a mention. For an open cover $\{X_\alpha\}_{\alpha \in I}$ of X, we define subsets of the loop space by requiring that the loops are piecewise contained in a set in the cover,

$$\Omega_k(X) = \left\{ \gamma \in \Omega(X) \;\middle|\; \text{For each } j \leq 2^k, \gamma_{|[\frac{j-1}{2^k}, \frac{j}{2^k}]} \text{ is contained in } X_\alpha \text{ for some } \alpha. \right\}.$$

This gives an increasing sequence of open subsets of $\Omega(X)$ that cover the entire space. In particular, a compact subset of $\Omega(X)$ will be contained in $\Omega_k(X)$ for some k. By defining a cover of X with good properties, we will be able to work exclusively with loops where we have some control over their behaviour.

Note that instead of considering loops, we could also consider the space of paths between two points $p, q \in X$, given by

$$\Omega(X, p, q) = \{\gamma : [0, 1] \to X \mid \gamma(0) = p, \gamma(1) = q\}.$$

But by composing such a path with a fixed path α from q to p, we end up with a loop at p. If we compose a loop at p with the path from p to q given by running α in reverse, we get a path from p to q, and the composition of these two operations is homotopic to the identity. This shows that all such spaces are homotopy equivalent, so we will not make a major distinction between them.

1.3 Flags

To ease some arguments, we will need the concept of a flag in a finite dimensional vector space. In an m-dimensional vector space V, a flag is an increasing sequence of subspaces of V,

$$V_1 \subset V_2 \subset \cdots \subset V_m = V,$$

with the dimension of V_i equal to i. So the dimension of each subspace is one greater than the last. Note that if we pick a non-zero vector v_1 in V_1, another (non-zero) vector v_2 in $V_2 - V_1$, and continue picking v_i in $V_i - V_{i-1}$, we end up with m linearly independent vectors. This allows us to construct a matrix in $\mathrm{GL}(V)$ as

$$[v_1, \ldots, v_m],$$

and we can recover the flag from the matrix by taking

$$\begin{aligned} V_1 &= \operatorname{span}(v_1), \\ V_2 &= \operatorname{span}(v_1, v_2), \\ &\vdots \\ V_m &= \operatorname{span}(v_1, \ldots, v_m). \end{aligned}$$

Likewise, we could take the span of the columns starting from v_m and working backwards to get another flag.

Definition 4. Given an invertible matrix $A = [v_1, \ldots, v_m]$, the *left flag* of A is

$$\mathrm{Fl_L}(A) = \Big(\operatorname{span}(v_1) \subset \operatorname{span}(v_1, v_2) \subset \ldots \subset \operatorname{span}(v_1, \ldots, v_m) = V\Big).$$

The *right flag* of A is defined similarly as

$$\mathrm{Fl_R}(A) = \Big(\operatorname{span}(v_m) \subset \operatorname{span}(v_{m-1}, v_m) \subset \ldots \subset \operatorname{span}(v_1, \ldots, v_m) = V\Big).$$

Note that we could multiply any column of A by a non-zero scalar and we would end up with the same flag. Likewise, if we replace v_j by $v_j + \lambda v_i$ for a scalar λ and indices $j > i$, we do not change the left flag of A. The right flag is unchanged by doing the same operation with $j < i$. In particular, these opeartions correspond to multiplying with a lower-triangular matrix L for the right flag, or an upper-triangular matrix U for the left flag, giving the identities

$$\mathrm{Fl_R}(L \cdot A) = \mathrm{Fl_R}(A),$$
$$\mathrm{Fl_L}(A \cdot U) = \mathrm{Fl_L}(A).$$

The cases that will be of most importance to us is when A is a permutation matrix. Writing these out for future use, we get

$$\mathrm{Fl_R}(L) = \mathrm{Fl_R}(\mathrm{Id})$$

when A is the identity, and

$$\mathrm{Fl_L}(\sigma \cdot U) = \mathrm{Fl_L}(\sigma)$$

for a general permutation $\sigma \in S_m$.

1.4 Coxeter groups

The final subject to be introduced is Coxeter groups and the Bruhat order. For a proper introduction, consult either [BB05] or [Hil82].

Definition 5. A *Coxeter group* is a group W along with a set of generators S, where all $s \in S$ have order two. The generators are only allowed to have relations of the form

$$(ss')^{m(s,s')} = \mathrm{Id}$$

for some function $m : S \times S \to \mathbb{N} \cup \{\infty\}$, with $m(s, s) = 1$ and $m(s, s') \geq 2$ for $s \neq s'$.

An important property of Coxeter groups is that any element $w \in W$ can be written as a *reduced product*,

$$w = s_1 s_2 \ldots s_k,$$

where $k = \ell(w)$ is minimal. This number is called the *length* of w. It is a theorem, see for example [BB05, Chapter 1.5], that any two reduced products of w can be obtained from each other by using *braid moves* of the form

$$ss'ss' \ldots ss' = s'ss's \ldots s's,$$

where each side of the equality has $m(s, s')$ elements. Another important property is that any expression of an element w as a product of generators contains a reduced product for w, obtained by deleting an even number of the generators. With more work, it is possible to show that products of generators are unique up to using braid moves and inserting or deleting the identity in the product as s^2.

Our primary example of a Coxeter group will be the symmetric group S_m. The generators of S_m as a Coxeter group are the *simple transpositions*, which are the elements

$$\{(1\ 2), (2\ 3), \ldots, (m-1\ m)\}\,.$$

The length of σ is the minimal number of simple transpositions needed to write σ, which is the same as the number of *inversions* of σ,

$$\ell(\sigma) = |\{(i,j) \mid 1 \leq i < j \leq m, \sigma(i) > \sigma(j)\}|\,.$$

Coxeter groups show up in various places, but we will require two properties. The first is that the symmetric group is the Weyl group of $\mathrm{GL}_m(\mathbb{C})$, which for our purposes means that it is a Coxeter group and if we let B^U denote the upper-triangular matrices in $\mathrm{GL}_m(\mathbb{C})$, there is a *Bruhat decomposition*,

$$\mathrm{GL}_m(\mathbb{C}) = \bigcup_{\sigma \in S_m} \mathrm{B}^U \cdot \sigma \cdot \mathrm{B}^U\,.$$

Note that this is a disjoint union. For a proof of this, see [Hil82, Theorem 4.3]. We would prefer to change this slightly, which is done by considering the element σ_0 in S_m defined by

$$\sigma_0(k) = m + 1 - k.$$

This element is its own inverse, which gives us

$$\begin{aligned}\mathrm{GL}_m(\mathbb{C}) = \sigma_0 \cdot \mathrm{GL}_m(\mathbb{C}) &= \bigcup_{\sigma \in S_m} \sigma_0 \cdot \mathrm{B}^U \cdot \sigma \cdot \mathrm{B}^U \\ &= \bigcup_{\sigma \in S_m} \sigma_0 \cdot \mathrm{B}^U \cdot \sigma_0^2 \cdot \sigma \cdot \mathrm{B}^U \\ &= \bigcup_{\sigma \in S_m} \mathrm{B}^L \cdot \sigma_0 \cdot \sigma \cdot \mathrm{B}^U\,.\end{aligned}$$

This is essentially the same theorem, but we have replaced the first copy of the upper-triangular matrices B^U with lower-triangular matrices B^L, by conjugating with σ_0.

The second property is that Coxeter groups come equipped with a partial order, called the Bruhat order.

Definition 6. The *Bruhat order* on the Coxeter group W with generators S is the transitive closure of the relation

$$u \to v \iff \ell(u) < \ell(v) \text{ and } u^{-1}v \in \bigcup_{w \in W} w \cdot S \cdot w^{-1}.$$

So we say that $w \leq w'$ if there is a sequence of elements in W with

$$w \to w_1 \to w_2 \to \cdots \to w_k \to w'.$$

This ordering will show up in Theorem 21. The condition on $u^{-1}v$ is usually the one to check when we want to find out if two elements are comparable, since if this is satisfied we only need to check the lengths of u and v to find out if $u \to v$ or

$v \to u$ holds. We will show a quick example to see how the order works. Take the two permutations $(1\ 2\ 3)$ and $(2\ 3)$ in S_3. These are comparable, since

$$(2\ 3)^{-1} \cdot (1\ 2\ 3) = (1\ 3) = (2\ 3)(1\ 2)(2\ 3).$$

The minimum number of simple transpositions needed to write $(1\ 2\ 3)$ as a product is two, namely

$$(1\ 2\ 3) = (1\ 2) \cdot (2\ 3),$$

while $(2\ 3)$ is already written as a product with a single factor. This shows

$$\ell((2\ 3)) = 1 < 2 = \ell((1\ 2\ 3)),$$

and we get the relation $(2\ 3) \to (1\ 2\ 3)$, showing $(2\ 3) \leq (1\ 2\ 3)$ in the Bruhat order. On the other hand, $(1\ 2)$ and $(2\ 3)$ are not comparable since they are both of length one.

To illustrate the ordering, we can draw a diagram of S_3 where the arrows denotes the relation $\sigma \to \tau$, and $\sigma \leq \tau$ exactly when we can follow a path from σ to τ in the diagram:

$$\begin{tikzcd}
 & (1\ 3) & \\
(1\ 2\ 3) \arrow[ur] & & (1\ 3\ 2) \arrow[ul] \\
(1\ 2) \arrow[u] \arrow[urr] & & (2\ 3) \arrow[u] \arrow[ull] \\
 & \mathrm{Id} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

2 Spaces of linearly independent vectors

In this chapter we will introduce the spaces that will be studied in the rest of the document, and collect various useful results.

2.1 Spaces

Before we can define the spaces that we will be working with, it will be helpful to define an approximation to them.

Definition 7. For natural numbers m and n, the space $Y_{m,n} \subset (\mathbb{C}^m)^n$ is

$$Y_{m,n} = \left\{ (a_1, \ldots, a_n) \in (\mathbb{C}^m)^n \;\middle|\; \begin{array}{c} \text{Any } m \text{ subsequent vectors in} \\ (e_1, \ldots, e_m, a_1, \ldots, a_n, e_1, \ldots, e_m) \\ \text{are linearly independent.} \end{array} \right\},$$

where $e_1, \ldots, e_m \in \mathbb{C}^m$ denotes the standard basis vectors.

We will switch between considering $(a_1, \ldots, a_n) \in Y_{m,n}$ as an ordered sequences of linearly independent vectors and an $m \times n$ matrix $[a_1, \ldots, a_n]$ as needed and without mention. We can replace the standard basis that appears in the definition of $Y_{m,n}$ with any collection of linearly independent vectors and get similarly defined spaces:

Definition 8. For two invertible matrices X and Y in $\mathrm{GL}_m(\mathbb{C})$, the space $Y_{m,n}(X, Y) \subset (\mathbb{C}^m)^n$ is

$$Y_{m,n}(X, Y) = \left\{ (a_1, \ldots, a_n) \in (\mathbb{C}^m)^n \;\middle|\; \begin{array}{c} \text{Any } m \text{ subsequent vectors in} \\ (x_1, \ldots, x_m, a_1, \ldots, a_n, y_1, \ldots, y_m) \\ \text{are linearly independent.} \end{array} \right\},$$

where $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are the column vectors of X and Y respectively.

There are two special cases that will be important. If both X and Y are the identity-matrix, we get the previously defined spaces,

$$Y_{m,n} = Y_{m,n}(\mathrm{Id}, \mathrm{Id}).$$

When X is the identity-matrix, we will usually omit it from the notation and instead write

$$Y_{m,n}(Y) = Y_{m,n}(\mathrm{Id}, Y).$$

We could also define $Y_{m,n}(X, Y)$ as the pre-image of the open set $(\mathbb{C}^*)^{m+n-1}$ in $\mathbb{C}^{m+n-1}$ under the continuous map $\mathrm{Det}_{X,Y}$ that calculates determinants of subsequent vectors:

$$\begin{aligned}
\mathrm{Det}_{X,Y} : (\mathbb{C}^m)^n &\to \mathbb{C}^{m+n-1} \\
\mathrm{Det}_{X,Y}(a_1, \ldots, a_n) = \ & (\det(x_2, \ldots, x_m, a_1), \\
& \det(x_3, \ldots, x_m, a_1, a_2), \\
& \quad \vdots \\
& \det(a_n, y_1, \ldots, y_{m-1})).
\end{aligned}$$

This shows that $Y_{m,n}(X,Y)$ is an open subset of $(\mathbb{C}^m)^n$. Another important feature of $Y_{m,n}(X,Y)$ is that it does not really depend on the particular matrices $X = [x_1, \ldots, x_m]$ and $Y = [y_1, \ldots, y_m]$, but only on the two flags in $\mathbb{C}^m$ defined from the columns:

$$\begin{aligned}
\mathrm{Fl}_{\mathrm{R}}(X) &= \Big(\mathrm{span}(x_m) \subset \mathrm{span}(x_{m-1}, x_m) \subset \ldots \subset \mathbb{C}^m \Big), \\
\mathrm{Fl}_{\mathrm{L}}(Y) &= \Big(\mathrm{span}(y_1) \ \subset \mathrm{span}(y_1, y_2) \qquad \subset \ldots \subset \mathbb{C}^m \Big).
\end{aligned}$$

This is because we are considering linear independence of the vectors

$$(x_1, \ldots, x_m, a_1, \ldots, a_n, y_1, \ldots, y_n),$$

and if $(a_1, \ldots, a_k)$ is a collection of linearly independent vectors of $\mathbb{C}^m$, then the vectors $(x_{k+1}, \ldots, x_m, a_1, \ldots, a_k)$ are linearly independent if and only if a_1 is not in the subspace spanned by $(x_{k+1}, \ldots, x_m)$, a_1 and a_2 are not in the subspace spanned by $(x_{k+2}, \ldots, x_m)$ and so forth. The same argument can be applied to the columns of Y.

At first glance it looks like we are considering a huge collection of spaces, but most of them are indistinguishable topologically, as the following lemma shows.

Lemma 9. *For $X, Y \in \mathrm{GL}_m(\mathbb{C})$, there exists a unique permutation $\sigma \in S_m$ such that the space $Y_{m,n}(X,Y)$ is homeomorphic to $Y_{m,n}(\sigma)$.*

Proof. Since X is invertible it preserves linear independence of vectors, and we get a homeomorphism

$$\begin{aligned}
Y_{m,n}(X,Y) &\to Y_{m,n}(\mathrm{Id}, X^{-1}Y) \\
(a_1, \ldots, a_m) &\mapsto (X^{-1}a_1, \ldots, X^{-1}a_m).
\end{aligned}$$

The matrix $X^{-1}Y$ is invertible, and by using the Bruhat-decomposition of $\mathrm{GL}_m(\mathbb{C})$, as in Section 1.4, we can write it uniquely as a product

$$X^{-1}Y = L\sigma U,$$

where L is lower-triangular, U is upper-triangular and σ is a permutation. Hence we get a homeomorphism,

$$Y_{m,n}(X,Y) \cong Y_{m,n}(L^{-1}, \sigma U).$$

But by the remark above, the space $Y_{m,n}(L^{-1}, \sigma U)$ only depends on the flags

$$\begin{aligned}
\mathrm{Fl}_{\mathrm{R}}(L^{-1}) &= \mathrm{Fl}_{\mathrm{R}}(\mathrm{Id}), \\
\mathrm{Fl}_{\mathrm{L}}(\sigma U) &= \mathrm{Fl}_{\mathrm{L}}(\sigma).
\end{aligned}$$

So $Y_{m,n}(L^{-1}, \sigma U)$ is the same as the space $Y_{m,n}(\sigma)$. $\square$

Due to this, we will almost exclusively consider the spaces $Y_{m,n}(\sigma)$ throughout the text. To get a feel for the spaces, we will now work with some simple examples and compute their cohomology.

Example 10. The space $Y_{m,1}$ is homeomorphic to $(\mathbb{C}^*)^m$, since

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_1 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \lambda_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_m & 0 & 0 & \dots & 1 \end{pmatrix}$$

has linearly independent columns if and only if all the scalars $\lambda_1, \dots, \lambda_m$ are invertible. The cohomology is

$$H^q(Y_{m,1}) \cong H^q((\mathbb{C}^*)^m) = \mathbb{Z}^{\binom{m}{q}}.$$

Example 11. The space $Y_{2,2}$ is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a \neq 0, d \neq 0, ad - bc \neq 0 \right\} \cong (\mathbb{C}^*)^2 \times \left\{ (b,c) \in \mathbb{C}^2 \mid bc \neq 1 \right\},$$

with the homeomorphism given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(a, d, \frac{b}{d}, \frac{c}{a} \right).$$

The product of the two fractions is

$$\frac{bc}{ad} = \frac{bc - ad}{ad} + 1 = 1 - \frac{ad - bc}{ad} \neq 1,$$

since $ad - bc \neq 0$.

The space $\left\{ (b,c) \in \mathbb{C}^2 \mid bc \neq 1 \right\}$ will be denoted Y. We would like to compute the cohomology of this space, since we can then use the Künneth formula to find the cohomology of $Y_{2,2}$. Consider Y as a subspace of the one-point compactification $S^4 = \mathbb{C}^2 \cup \{\infty\}$. Then the complement is

$$S^4 - Y = \left\{ (b,c) \in \mathbb{C}^2 \mid bc = 1 \right\} \cup \{\infty\}.$$

This can be identified with the wedge sum $S^1 \vee S^2$ by considering the subspace

$$\mathbb{C}^* \cong \left\{ \left(b, \frac{1}{b} \right) \,\middle|\, b \in \mathbb{C}^* \right\} = (S^4 - Y) - \{\infty\}.$$

The point at infinity is glued to this space when either b or c is large, which for our copy of $\mathbb{C}^*$ corresponds to b large or b near 0. Hence what we get is $S^2 = \mathbb{C} \cup \{\infty\}$ with the identification $0 \sim \infty$. This is homotopy equivalent to $S^1 \vee S^2$.

Now we can apply Alexander duality to this space to compute

$$\widetilde{H}_q(Y) \cong \widetilde{H}^{3-q}(S^4 - Y) = \begin{cases} \mathbb{Z} & q \in \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

and hence

$$H^q(Y) \cong \begin{cases} \mathbb{Z} & q \in \{0, 1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

This shows

$$H^q(Y_{2,2}) \cong H^q((\mathbb{C}^*)^2 \times Y) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^3 & q = 1, \\ \mathbb{Z}^4 & q = 2, \\ \mathbb{Z}^3 & q = 3, \\ \mathbb{Z} & q = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Example 12. The space $Y_{3,2}$ is

$$\left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \middle| \, a \neq 0, f \neq 0, ad - bc \neq 0, cf - ed \neq 0 \right\}.$$

This is homeomorphic to $(\mathbb{C}^*)^4 \times \{(x, y, z) \in \mathbb{C}^3 \mid y - z - xyz = 1\}$, with the map given by

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \mapsto \left(a, f, D = \frac{ad - bc}{af}, C = \frac{cf - de}{af}, x = \frac{bC}{fD}, y = \frac{c}{aC}, z = \frac{eD}{aC} \right).$$

Some of the coordinates have been named to shorten the writing slightly. By direct calculation, the last three coordinates satisfy the relation

$$\begin{aligned} y - z - xyz &= \frac{c}{aC} - \frac{eD}{aC} - \frac{bce}{a^2 fC} \\ &= \frac{acf - aefD - bce}{a^2 fC} \\ &= \frac{acf - e(ad - bc) - bce}{a(cf - de)} \\ &= \frac{acf - ade}{acf - ade} \\ &= 1. \end{aligned}$$

The inverse map is

$$(a, f, D, C, x, y, z) \mapsto \begin{pmatrix} a & \frac{fDx}{C} \\ aCy & fD(1 + xy) \\ \frac{aCz}{D} & f \end{pmatrix}.$$

We would like to compute the cohomology of the space

$$Z = \left\{ (x, y, z) \in \mathbb{C}^3 \mid y - z - xyz = 1 \right\}.$$

This is the same as the cohomology of S^2, but this is hard to see directly. Instead consider the space

$$\widetilde{Z} = \left\{ (b, c, d, e) \in \mathbb{C}^4 \mid d \neq bc, c \neq de \right\}.$$

This space is homeomorphic to $Z \times (\mathbb{C}^*)^2$:

$$(b, c, d, e) \mapsto \left(\frac{b(c - de)}{d - bc}, \frac{c}{c - de}, \frac{e(d - bc)}{c - de}, d - bc, c - de \right).$$

We will use Alexander duality to compute the cohomology. Consider the complement in S^8,

$$S^8 - \widetilde{Z} = \{(b,c,d,e) \mid d = bc\} \cup \{(b,c,d,e) \mid c = de\} \cup \{\infty\},$$

and denote the two sets as W_1 and W_2. Then

$$W_1 \cap W_2 = \{(b,c,d,e) \mid d = bc, c = de\} \cup \{\infty\}.$$

We glue the point at infinity to this space when c is large, but this is the same as either d or e being large. Hence there is a homeomorphism,

$$\begin{aligned} W_1 \cap W_2 &\cong \{(b,d,e) \mid d = bde\} \cup \{\infty\} \\ &= \{(b,d,e) \mid 0 = (be-1)d\} \cup \{\infty\} \\ &= \{(b,d,e) \mid d = 0\} \cup \{(b,d,e) \mid be = 1\} \cup \{\infty\}. \end{aligned}$$

These new spaces have computable homology, since they can be identified with spaces built from spheres. The first is relatively easy,

$$Z_1 = \{(b,d,e) \mid d = 0\} \cup \{\infty\} = \{(b,0,0,d)\} \cup \{\infty\} \cong S^4.$$

The second is slightly harder.

$$Z_2 = \{(b,d,e) \mid be = 1\} \cup \{\infty\} = \left\{\left(b, \frac{d}{b}, d, \frac{1}{b}\right)\right\} \cup \{\infty\}.$$

This set is homeomorphic to the one-point compactification of $\{(b,d) \in \mathbb{C}^* \times \mathbb{C}\}$. By [Dup68, Chapter 2.3], this is given by the smash product of the one-point compactifications,

$$Z_2 \simeq (\mathbb{C}^* \times \mathbb{C})^+ \cong (\mathbb{C}^*)^+ \wedge \mathbb{C}^+ \cong (S^1 \vee S^2) \wedge S^2 = S^3 \vee S^4.$$

The intersection of these spaces is

$$Z_1 \cap Z_2 = \left\{\left(b, 0, 0, \frac{1}{b}\right)\right\} \cup \{\infty\} \simeq S^1 \vee S^2.$$

From all of this, we get a commutative diagram of inclusions,

$$\begin{tikzcd} S^1 \vee S^2 \arrow[r] \arrow[d] & S^3 \vee S^4 \arrow[d] \\ S^4 \arrow[r] & W_1 \cap W_2. \end{tikzcd}$$

The two maps from $S^1 \vee S^2$ are zero in cohomology (when the degree is greater than zero) since in each degree either the domain or the codomain is zero. By applying Mayer-Vietoris, we get

$$H^q(W_1 \cap W_2) = \begin{cases} \mathbb{Z} & q = 0, \\ 0 & q = 1, \\ \mathbb{Z} & q = 2, \\ \mathbb{Z}^2 & q = 3, \\ \mathbb{Z}^2 & q = 4, \\ 0 & \text{otherwise.} \end{cases}$$

where the case $q = 0$ follows from $W_1 \cap W_2$ being connected and $q = 1$ follows from writing out the relevant part of the long exact sequence,

$$H^0(S^3 \vee S^4) \oplus H^0(S^4) \twoheadrightarrow H^0(S^1 \vee S^4) \longrightarrow H^1(W_1 \cap W_2) \longrightarrow 0.$$

W_1 and W_2 are both a copy of S^6 since we can get one of the coordinates from two of the others. Hence we get another diagram,

$$\begin{tikzcd} W_1 \cap W_2 \arrow[r] \arrow[d] & S^6 \arrow[d] \\ S^6 \arrow[r] & S^8 - \widetilde{Z}, \end{tikzcd}$$

and again the map from the intersection is zero. Repeating the above, we can compute:

$$H^q(S^8 - \widetilde{Z}) = \begin{cases} \mathbb{Z} & q = 0, \\ 0 & q = 1, \\ 0 & q = 2, \\ \mathbb{Z} & q = 3, \\ \mathbb{Z}^2 & q = 4, \\ \mathbb{Z}^2 & q = 5, \\ \mathbb{Z}^2 & q = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Alexander duality then gives

$$H^q(\widetilde{Z}) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^2 & q = 1, \\ \mathbb{Z}^2 & q = 2, \\ \mathbb{Z}^2 & q = 3, \\ \mathbb{Z} & q = 4, \\ 0 & \text{otherwise.} \end{cases}$$

The shows that the cohomology of Z is the same as the cohomology of S^2, if we factor out the torus. A more thorough description of the cohomology of Z will be given in Section 3.3.

By using this, we can compute the cohomology of $Y_{3,2}$ as

$$H^q(Y_{3,2}) \cong H^q((\mathbb{C}^*)^4 \times Z) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^4 & q = 1, \\ \mathbb{Z}^7 & q = 2, \\ \mathbb{Z}^8 & q = 3, \\ \mathbb{Z}^7 & q = 4, \\ \mathbb{Z}^4 & q = 5, \\ \mathbb{Z} & q = 6, \\ 0 & \text{otherwise.} \end{cases}$$

To work with the above spaces, it will help to know that they are not empty. For example, by considering the permutation $\big(1\ 3\big)$, we see that it is impossible to satisfy the conditions for being in $Y_{3,1}\big(\big(1\ 3\big)\big)$. To see this, take any vector a_1 in $\mathbb{C}^3$ and look at $(e_1, e_2, e_3, a_1, e_3, e_2, e_1)$. If this is to be in the space, it must have any three subsequent vectors linearly independent, but (e_3, a_1, e_3) is not invertible no matter what a_1 is. This shows that the choice of X and Y in $Y_{m,n}(X, Y)$ plays a role in determining when the spaces are empty. But as we shall now see, the spaces are non-empty as long as we are not in an obviously empty space. The exact meaning of "obviously" is given in the following lemma.

Theorem 13. *The space $Y_{m,n}(\sigma)$ is empty if and only if there exists an $i \in \{1, \ldots, m\}$ such that*

$$1 + n + i \leq \sigma(i).$$

Proof. The exact condition is slightly strangely worded, but if there is such an i, then the first and last columns of the matrix

$$(e_{\sigma(i)}, \ldots, e_m, a_1, \ldots, a_n, \sigma_1, \ldots, \sigma_i)$$

are linearly dependent, since

$$\sigma_i = e_{\sigma(i)}$$

by definition. By counting the three different groups of vectors, we see that the matrix has

$$m - \sigma(i) + 1 + n + i$$

columns. The condition on i shows that

$$m - \sigma(i) + 1 + n + i \leq m - \sigma(i) + \sigma(i) = m,$$

so the conditions for being in $Y_{m,n}(\sigma)$ can never be satisfied, no matter the choice of vectors $a_1, \ldots, a_n$. This shows one direction.

The other direction will proceed by a purely dimensional argument. Consider the two flags of interest,

$$\begin{aligned} \mathrm{Fl}_{\mathrm{R}}(\mathrm{Id}) &= (V_1 \subset V_2 \subset \ldots \subset V_m), \\ \mathrm{Fl}_{\mathrm{L}}(\sigma) &= (W_1 \subset W_2 \subset \ldots \subset W_m). \end{aligned}$$

Assume that the condition on σ and n is satisfied, so for all $i \leq m$:

$$1 + n + i > \sigma(i).$$

In the flag formulation, this means that the subspaces V_{m-n-k} and W_k intersect trivially for all k.

We must now find the vectors $a_1, \ldots, a_n$ and will do so from left to right. By looking at the matrix

$$(e_1, \ldots, e_m, a_1, \ldots, a_n, \sigma_1, \ldots, \sigma_m),$$

we see that we must choose a_1 so it is not in any of the following subspaces of $\mathbb{C}^m$:

$$\begin{aligned} &a_1 \notin V_{m-1}, \\ &a_1 \notin V_{m-n-1} \oplus W_1, \\ &\quad\vdots \\ &a_1 \notin V_1 \oplus W_{m-n-1}, \\ &a_1 \notin W_{m-n}. \end{aligned}$$

But that means choosing a_1 in $\mathbb{C}^m$ after removing a finite number of proper subspaces, since the dimensions of the spaces on the right are $m-1$ for V_{m-1} and $m-n$ for all the rest. This space is non-empty since $n \geq 1$, so this can always be done. We can continue this process inductively and get an element $(a_1, \ldots, a_n)$ of $Y_{m,n}(\sigma)$, showing that the space is non-empty and concluding the proof. □

Remark 14. Note that if $n \geq m-1$, then for any $i \geq 1$ we have

$$1 + n + i \geq m - 1 + 1 + i \geq m + i > m \geq \sigma(i),$$

and hence the space $Y_{m,n}(\sigma)$ is non-empty. So the spaces become non-empty as soon as there are "enough" columns.

It will occasionally be nice to have an explicit element of $Y_{m,n}(\sigma)$, but to do this we will need a slight amount of setup. Hence it will be delayed until Theorem 22 in Section 2.3.

2.2 Symmetries

The space $Y_{m,n}(\sigma)$ has quite a few symmetries that could be of interest. We will start with a particular one that will not be used much in the dissertation, but which can occasionally be used to ease computations.

Lemma 15. *If $(a_1, \ldots, a_n) \in Y_{m,n}$, then the transposed matrix $(a_1, \ldots, a_n)^T$ is in $Y_{n,m}$.*

Proof. This follows directly from defining $Y_{m,n}$ as $\mathrm{Det}^{-1}_{\mathrm{Id},\mathrm{Id}}(\mathrm{T}^{m+n-1})$. When we compute the determinants of the matrix, we see that the important entries of $(a_1, \ldots, a_n)$ lie in a square submatrix which is unchanged under transposition. So we get

$$\begin{aligned}\mathrm{Det}_{\mathrm{Id},\mathrm{Id}}(a_1, \ldots, a_n) &= (d_1, \ldots, d_{m+n-1}) \\ &= \mathrm{Det}_{\mathrm{Id},\mathrm{Id}}((a_1, \ldots, a_n)^T).\end{aligned}$$

□

As an example, we computed the cohomology of $Y_{3,2}$ in Example 12, and this theorem then tells us that

$$Y_{2,3} \cong Y_{3,2} \cong (\mathbb{C}^*)^4 \times Z$$

has the same cohomology groups.

Since we are interested in linear independence of vectors in $\mathbb{C}^m$, the group of complex units, $\mathrm{T} = (\mathbb{C})^*$, acts on the space in several ways. The most important for our purposes is the following:

Definition 16. For any $i \in \{1, \ldots, n\}$, the group T acts on $Y_{m,n}(\sigma)$ by scaling the i'th column vector, using the formula

$$\lambda \cdot (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) = (a_1, \ldots, a_{i-1}, \lambda a_i, a_{i+1}, \ldots, a_n).$$

Since the determinant of a matrix is linear in each column, the linear independence of the vectors is preserved.

All of these actions commute with each other, so we get an action of T^n on $Y_{m,n}(\sigma)$:

$$(\lambda_1, \ldots, \lambda_n) \cdot (a_1, \ldots, a_n) = (\lambda_1 a_1, \ldots, \lambda_n a_n).$$

While all of these symmetries are nice to work with, the spaces as defined above have some issues. In particular, as we shall see in Theorem 24, the fundamental group and the cohomology ring of $Y_{m,n}(\sigma)$ both become larger as n grows. This complicates some of our arguments, but luckily it can be helped by taking the quotient with the above group action. This gives us the spaces we are actually interested in.

Definition 17. For natural numbers n and m and invertible matrices X and Y, we define the space $X_{m,n}(X,Y)$ as the quotient of $Y_{m,n}(X,Y)$ using the group action above,

$$X_{m,n}(X,Y) = Y_{m,n}(X,Y)/\,\mathrm{T}^n\,.$$

Note that since matrix multiplication is linear, it commutes with the action of T^n on $Y_{m,n}(X,Y)$ and we get directly from Lemma 9 that for each choice of $X, Y \in \mathrm{GL}_m(\mathbb{C})$ there is a permutation $\sigma \in S_m$ such that

$$X_{m,n}(X,Y) \cong X_{m,n}(\sigma).$$

Hence most of our attention will be focused on these particular spaces. The quotient spaces are naturally closely related to the spaces $Y_{m,n}(X,Y)$. In particular, we have the following:

Lemma 18. *The spaces $Y_{m,n}(X,Y)$ and $X_{m,n}(X,Y) \times \mathrm{T}^n$ are homeomorphic.*

Proof. Define a map

$$X_{m,n}(X,Y) \to Y_{m,n}(X,Y)$$

by mapping $A \cdot \mathrm{T}^n$ to the unique representative $\widetilde{A} \in Y_{m,n}(X,Y)$ with the last n determinants all equal to one,

$$\mathrm{Det}_{X,Y}(\widetilde{A}) \in (\mathbb{C}^*)^{m-1} \times \{1\}^n.$$

This map is continuous, as the composition with the quotient map,

$$Y_{m,n}(X,Y) \xrightarrow{\ q\ } X_{m,n}(X,Y) \longrightarrow Y_{m,n}(X,Y)\ ,$$

is the map that uses the group action of T^n to change the last n coordinates of $\mathrm{Det}_{X,Y}(A)$ to 1 and this is a continuous map. More precisely, it is given by the formula

$$(a_1, \ldots, a_n) \mapsto \left(a_1, \ldots, a_{m-1}, \frac{1}{d_m} a_m, \frac{d_m}{d_{m+1}} a_{m+1}, \ldots, \hat{d} a_n\right),$$

where $\mathrm{Det}_{X,Y}(a_1, \ldots, a_n) = (d_1, \ldots, d_{m+n-1})$ and the number $\hat{d}$ is an expression consisting of products and quotients of the last n of these determinants, the exact terms of which depends on n and m. If $\mathrm{Det}_{X,Y}(A) = (d_1, \ldots, d_{m+n-1})$ denotes the determinants as above, the homeomorphism is the map

$$\begin{aligned} Y_{m,n}(X,Y) &\to X_{m,n}(X,Y) \times \mathrm{T}^n \\ A &\mapsto (A \cdot \mathrm{T}^n, (d_m, \ldots, d_{m+n-1})), \end{aligned}$$

with inverse

$$(A \cdot \mathrm{T}^n, (d_m, \ldots, d_{m+n-1})) \mapsto \left(\overbrace{1, \ldots, 1}^{m-1}, d_m, \frac{d_{m+1}}{d_m}, \ldots, \frac{1}{\hat{d}}\right) \cdot f(A).$$

$\square$

The above lemma is useful, since it oftens allows us to work with the unquotiented spaces and still gain information about the quotients. This lemma also immediately proves the claim that the fundamental group and cohomology ring of $Y_{m,n}(X, Y)$ grows with n, by the formulas

$$\pi_1(Y_{m,n}(X,Y)) \cong \pi_1(X_{m,n}(X,Y)) \times \mathbb{Z}^n,$$
$$H^*(Y_{m,n}(X,Y)) \cong H^*(X_{m,n}(X,Y)) \otimes H^*(\mathrm{T}^n).$$

But more importantly, the last formula allows us to calculate the cohomology of $X_{m,n}(X, Y)$ from the cohomology of $Y_{m,n}(X, Y)$, by quotienting out the free groups coming from the cohomology ring of the torus $H^*(\mathrm{T}^n)$. For example, we will later compute

$$H^1(X_{m,n}) \cong \mathbb{Z}^{m-1}$$

by instead computing

$$H^1(X_{m,n}) \oplus \mathbb{Z}^n \cong H^1(Y_{m,n}) \cong \mathbb{Z}^{m+n-1}.$$

2.3 Stabilisation

We would like to relate the spaces $Y_{m,n}(\sigma)$ that we obtain for different values of m and n. Since an invertible, lower-triangular matix L preserves the flag $\mathrm{Fl}_R(\mathrm{Id})$, we will try to use this to define a map

$$s : Y_{m,n} \to Y_{m,n+1}$$
$$s(a_1, \dots, a_n) = (La_1, \dots, La_n, Le_1).$$

At first, consider the spaces $Y_{m,n}$ and $Y_{m,n+1}$. To ensure the right hand side actually is an element of $Y_{m,n+1}$, we will start by considering the inverse of s, mapping a subset of $Y_{m,n+1}$ to $Y_{m,n}$. Consider an element $A = (a_1, \dots, a_{n+1})$ in $Y_{m,n+1}$ where the entries of a_{n+1} are all equal to one:

$$A = \begin{pmatrix} | & | & & | & 1 \\ a_1 & a_2 & \dots & a_n & \vdots \\ | & | & & | & 1 \end{pmatrix}.$$

Then we can multiply through with the lower-triangular, invertible matrix L^{-1} with ones on the diagonal, negative ones just below the diagonal and zeroes everywhere else,

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

This reduces A to

$$L^{-1}A = \begin{pmatrix} | & | & & | & 1 \\ | & | & & | & 0 \\ L^{-1}a_1 & L^{-1}a_2 & \dots & L^{-1}a_n & \vdots \\ | & | & & | & 0 \\ | & | & & | & 0 \end{pmatrix},$$

and going a bit further we can see that $(L^{-1}a_1, \dots, L^{-1}a_n) \in Y_{m,n}$ by computing

$$L^{-1} \cdot (\mathrm{Id}, a_1, \dots, a_{n+1}, \mathrm{Id}) = \left(L^{-1}, L^{-1}a_1, \dots, L^{-1}a_n, \begin{matrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{matrix} \right).$$

This shows that $L^{-1} \cdot (a_1, \dots, a_n)$ is in $Y_{m,n}(L^{-1}, U)$, where U is the upper-triangular matrix given above. But this space is the same as $Y_{m,n}$ since the right flag of a lower-triangular matrix and the left flag of an upper-triangular matrix is the same as the corresponding flag of the identity matrix, as we saw in Section 1.3.

Since the above is done by multiplying with a matrix, it commutes with the action of the torus T^n on $Y_{m,n}$, and everything can be repeated with $Y_{m,n}$ and $Y_{m,n+1}$ replaced by the spaces $X_{m,n}$ and $X_{m,n+1}$.

The above considerations allow us define a map that relates the spaces $X_{m,n}$ and $X_{m,n+1}$, by working with the inverse matrix L.

Definition 19. The *stabilisation map* $s : X_{m,n} \to X_{m,n+1}$ is given by

$$s\,(a_1, \dots, a_n) = (La_1, \dots, La_n, Le_1)\,,$$

where L is the lower-triangular matrix with every entry below the diagonal equal to one,

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

In the above, we only considered the spaces $X_{m,n}$ and $X_{m,n+1}$ defined by the identity permutation $\mathrm{Id} \in S_m$, but there is a similar stabilisation for the other permutations. We will again start by considering the unreduced spaces $Y_{m,n}(\sigma)$. For any permutation σ, the space $Y_{m,n+1}(\sigma)$ is a union of spaces homeomorphic to a torus times $Y_{m,n}(\tau)$, for various permutations $\tau \in S_m$. The exact relation is given by the following theorem.

Theorem 20. *Any choice of indices $I = (i_1 < \dots < i_k) \subset \{1, \dots, m\}$ defines a subspace of $Y_{m,n+1}(\sigma)$:*

$$Y^I_{m,n+1}(\sigma) = \left\{ (a_1, \dots, a_{n+1}) \in Y_{m,n+1}(\sigma) \;\middle|\; \begin{matrix} (a_{n+1})_{i_j} \neq 0 \;\forall i_j \in I, \\ (a_{n+1})_j = 0 \;\forall j \notin I \end{matrix} \right\}.$$

Here $(a_{n+1})_j$ denotes the j'th entry of the vector a_{n+1}. This space is empty if I does not contain the number $\sigma(m)$, so we will assume that $\sigma(m) \in I$ but otherwise leave it out of the notation.

These subspaces are disjoint and cover $Y_{m,n+1}(\sigma)$. There is a map φ, defined for an indexing set I and a permutation σ, which gives a new permutation such that

$$Y^I_{m,n+1}(\sigma) \cong Y_{m,n}(\varphi(I, \sigma)) \times (\mathbb{C}^*)^{|I|}.$$

If we define the permutation $\widehat{\sigma} = \sigma \cdot \begin{pmatrix} m & m-1 & \dots & 1 \end{pmatrix}$ *and the indexing set is* $I = (i_1 < \cdots < i_k)$*, then* φ *is given by*

$$\varphi(I, \sigma) = \begin{cases} \widehat{\sigma} & k = 0, \\ \begin{pmatrix} i_k & \sigma(m) \end{pmatrix} \cdot \varphi\left((i_1 < \cdots < i_{k-1}), \sigma\right) & \begin{array}{l} i_k < \sigma(m) \text{ and} \\ \widehat{\sigma}^{-1}(i_k) > \widehat{\sigma}^{-1}(i_r) \, \forall r < k, \end{array} \\ \varphi\left((i_1 < \cdots < i_{k-1}), \sigma\right) & \text{otherwise.} \end{cases}$$

Note that the index $\sigma(m)$ *has once again been removed from the notation, so the* $k = 0$ *case is really* $I = (\sigma(m))$ *and* i_k *is the last index not equal to* $\sigma(m)$*.*

Proof. Fix the permutation $\sigma \in S_m$. The observation that $Y^I_{m,n+1}(\sigma)$ is empty if $\sigma(m)$ is not in I follows from considering the matrix

$$(a_{n+1}, \sigma_1, \dots, \sigma_{m-1}) = (a_{n+1}, e_{\sigma(1)}, \dots, e_{\sigma(m-1)}).$$

For this matrix to be invertible we must have $(a_{n+1})_{\sigma(m)} \neq 0$. If $\sigma(m)$ is not in I, then the conditions on $Y^I_{m,n+1}(\sigma)$ can never be satisfied and the space must be empty. We will from now on assume that I contains $\sigma(m)$.

To see that the spaces are disjoint and cover all of $Y_{m,n+1}(\sigma)$, we can see that an element $(a_1, \dots, a_{n+1}) \in Y_{m,n+1}(\sigma)$ is in exactly one subspace, namely the one corresponding to $I = \{j \in \{1, \dots, m\} \mid (a_{n+1})_j \neq 0\}$.

To define the function φ and the homeomorphism

$$Y^I_{m,n+1}(\sigma) \cong Y_{m,n}(\varphi(I, \sigma)) \times (\mathbb{C}^*)^{|I|},$$

we will proceed as in the previous case. The map to the torus $(\mathbb{C}^*)^{|I|}$ is given by remembering the non-zero entries of a_{n+1}, so we can immediately reduce to the case where all entries of a_{n+1} are either zero or one.

An element of $Y^I_{m,n+1}(\sigma)$ consists of vectors $(a_1, \dots, a_{n+1})$ with the matrix

$$M = (a_{n+1}, \sigma_1, \dots, \sigma_{m-1})$$

invertible. It is this matrix that we want to factor as $L \cdot \varphi(I, \sigma) \cdot U$, where L is an invertible, lower-triangular matrix and U is an invertible, upper-triangular matrix. To do this, we want to make row operations on the matrix corresponding to multiplying with L^{-1} until we end up with an upper-triangular matrix with permuted rows, corresponding to $\varphi(I, \sigma) \cdot U$. Multiplying with L^{-1} corresponds to making row operations where each row is only allowed to affect the rows below it. We want to remove all non-zero entries in the first column except for one, and we want this single entry to have the lowest possible index. We also want to make the matrix as close as possible to upper-triangular, which is done by making sure that each operation does not introduce a non-zero entry to the left of an already existing non-zero entry.

If $k = 0$, an element $Y^I_{m,n+1}(\sigma)$ looks like

$$(a_1, \dots, a_n, \sigma_m).$$

Since we are in $Y_{m,n+1}(\sigma)$, the linear independence condition gives us that

$$(a_1, \dots, a_n) \in Y_{m,n}(\tau),$$

where τ is the permutation

$$\tau(k) = \begin{cases} \sigma(k-1) & k > 1, \\ \sigma(m) & k = 1. \end{cases}$$

We recognize this as the permutation $\widehat{\sigma}$ and get the formula

$$\varphi((\sigma(m)), \sigma) = \widehat{\sigma}.$$

The homeomorphism is in this case given by

$$\begin{aligned} Y_{m,n+1}^{(\sigma(m))}(\sigma) &\to Y_{m,n}(\widehat{\sigma}) \times \mathbb{C}^* \\ (a_1, \ldots, a_n, \lambda \cdot \sigma_m) &\mapsto ((a_1, \ldots, a_n), \lambda) \,. \end{aligned}$$

Now consider the case $k > 0$, so $I = (i_1 < \cdots < i_k)$. If $i_k > \sigma(m)$, we can write $(a_{n+1}, \sigma_1, \ldots, \sigma_{m-1})$ in block form as

$$\begin{pmatrix} I_1 & A \\ 1 & 0 \\ I_2 & B \\ 1 & C \\ 0 & D \end{pmatrix}.$$

I_1 is a vector with ones on the entries corresponding to $\{i_j \mid i_j < \sigma(m)\}$ and zeroes elsewhere, while I_2 is similar except we consider indices greater than $\sigma(m)$. The first 1 is on the $\sigma(m)$'th row and the next is on the i_k'th row. All entries below are zero. A, B, C and D are the matrices consisting of the corresponding rows from $(\sigma_1, \ldots, \sigma_{m-1})$. But by multiplying with a lower-triangular matrix, we can reduce this matrix to

$$\begin{pmatrix} I_1 & A \\ 1 & 0 \\ I_2 & B \\ 0 & C \\ 0 & D \end{pmatrix}.$$

But this shows that $\varphi(I, \sigma) = \varphi\big((i_1 < \cdots < i_{k-1}), \sigma\big)$ when $i_k > \sigma(m)$.

By possibly using the above multiple times, we reduce to the case where $i_k < \sigma(m)$. Now we would like to reduce the length of I by one, and consider how this affects the resulting permutation. Consider $\widehat{\sigma}^{-1}(i_k)$. If there is a $j < k$ with $\widehat{\sigma}^{-1}(i_k) < \widehat{\sigma}^{-1}(i_j)$, then we can write $(a_{n+1}, \sigma_1, \ldots, \sigma_{m-1})$ as

$$\begin{pmatrix} \vdots & & & & & \\ 1 & \ldots & \ldots & \ldots & 1 & \ldots \\ \vdots & & & & & \\ 1 & \ldots & 1 & \ldots & \ldots & \ldots \\ \vdots & & & & & \\ 1 & \ldots & \ldots & \ldots & \ldots & \ldots \\ \vdots & & & & & \end{pmatrix}.$$

In the above, only the i_j'th, i_k'th and $\sigma(m)$'th rows have been written out, and only the non-zero entries. But this matrix can be reduced to

$$\begin{pmatrix} \vdots \\ 1 & \dots & \dots & \dots & 1 & \dots \\ \vdots \\ 0 & \dots & 1 & \dots & -1 & \dots \\ \vdots \\ 1 & \dots & \dots & \dots & \dots & \dots \\ \vdots \end{pmatrix}.$$

We are now unable to change the middle row, row i_k, since there are no non-zero entries above the left-most one. The additional -1 that was introduced can be removed by multiplying on the right with an upper-triangular matrix that subtracts column $\widehat{\sigma}^{-1}(i_k)$ from column $\widehat{\sigma}^{-1}(i_j)$. So we are able to remove the one in the i_k'th row without changing the resulting permutation, showing that in this case,

$$\varphi\big(I,\sigma\big) = \varphi\big((i_1 < \dots < i_{k-1}), \sigma\big).$$

Now assume that there is no such j. Then the row i_k is our best candidate for removing the bottom-most one in row $\sigma(m)$. If we do this row operation, by multiplying with $\widehat{L}$, and then use the Bruhat decomposition, we get

$$\varphi(I,\sigma)\cdot U = \widetilde{L}^{-1}\widehat{L}\cdot M.$$

But when we perform row operations on $\widehat{L}\cdot M$, we are only changing rows with index less than or equal to i_{k-1}, since this is the last non-zero entry in the first column. Hence $\widetilde{L}^{-1}$ has the form

$$\widetilde{L}^{-1} = \begin{pmatrix} \vdots \\ * & \dots & * & 1 & \dots & \dots & \dots \\ \vdots \\ 0 & \dots & \dots & \dots & \dots & 1 & \dots \\ \vdots \end{pmatrix},$$

where we once again write out row i_k and $\sigma(m)$, and use the stars to indicate entries that may be non-zero but that are not important. If we switch row i_k and row $\sigma(m)$ along with column i_k and column $\sigma(m)$, we get another lower-triangular matrix:

$$(i_k\ \sigma(m))\widetilde{L}^{-1}(i_k\ \sigma(m)) = \begin{pmatrix} \vdots \\ 0 & \dots & \dots & 1 & \dots & \dots & \dots \\ \vdots \\ * & \dots & * & \dots & \dots & 1 & \dots \\ \vdots \end{pmatrix}.$$

Inserting in our original equation, we see

$$(i_k\ \sigma(m))\varphi(I,\sigma)\cdot U = \widetilde{L}^{-1}\cdot(i_k\ \sigma(m))\cdot\widehat{L}M.$$

But $(i_k\ \sigma(m)) \cdot \widehat{L}M$ is just M with the one in the first column, row i_k, replaced by zero. This gives the formula

$$(i_k\ \sigma(m)) \cdot \varphi(I, \sigma) = \varphi\big((i_1 < \cdots < i_{k-1}), \sigma\big),$$

and our desired formula for φ follows by multiplying over. □

By looking at the formula for φ, we can deduce the following theorem that shows how the different possible permutations that can show up are related in the Bruhat order, described in Section 1.4.

Theorem 21. *If $I = (i_1 < \cdots < i_k)$ and J is a subset of I, then the permutation $\varphi(I, \sigma)$ is less than or equal to the permutation $\varphi(J, \sigma)$ in the Bruhat order.*

A different way of phrasing this is that as the indexing set becomes larger, we get closer to $\varphi\left((1 < 2 < \cdots < m), \sigma\right)$ and further away from $\varphi\left((\sigma(m)), \sigma\right)$.

Proof. The proof is mostly checking cases. If $\varphi(J, \sigma) = \varphi(I, \sigma)$ we are done. We will start by considering the simplest remaining case, where

$$\varphi(I, \sigma) = (i_k\ \sigma(m)) \cdot \varphi(J, \sigma).$$

First we note that the two permutations are related in the Bruhat order, since

$$\tau^{-1} \cdot \phi = \varphi(J, \sigma)^{-1} \cdot (i_k\ \sigma(m)) \cdot \varphi(J, \sigma),$$

which is exactly the condition described in Definition 6. So we only need to count the number of inversions of each of the permutations, and see that τ has fewer inversions than ϕ. To simplify further, assume that all the indices i_r in I and J satisfy the conditions $i_r < \sigma(m)$ and $\widehat{\sigma}^{-1}(i_r) > \widehat{\sigma}^{-1}(i_s)$ for all $s < r$. This corresponds to removing the entries of I that have no effect on $\varphi(I, \sigma)$ by the previous theorem. Then we can compute the two permutations we are interested in by using the formula for φ,

$$\begin{aligned} \tau &= \varphi(I, \sigma) = (i_1\ \ldots\ i_{k-1}\ i_k\ \sigma(m)) \cdot \widehat{\sigma}, \\ \phi &= \varphi(J, \sigma) = (i_1\ \ldots\ i_{k-1}\ \sigma(m)) \cdot \widehat{\sigma}. \end{aligned}$$

From the definition of τ and ϕ, we can see that if a is not in the set

$$\left\{\widehat{\sigma}^{-1}(i_{k-1}), \widehat{\sigma}^{-1}(i_k)\right\},$$

then $\tau(a) = \phi(a)$. We would like to find pairs (a, b) with $a < b$ that are inversions for eiher τ or ϕ, but not both. So we should only consider pairs (a, b) where at least one is either $\widehat{\sigma}^{-1}(i_k)$ or $\widehat{\sigma}^{-1}(i_{k-1})$.

Note that by our previous assumption on the indices i_r, we have $\widehat{\sigma}^{-1}(i_{k-1}) < \widehat{\sigma}^{-1}(i_k)$. So we start by checking this pair.

$$\begin{aligned} \tau(\widehat{\sigma}^{-1}(i_{k-1})) &= i_k < \sigma(m) = \tau(\widehat{\sigma}^{-1}(i_k)), \\ \phi(\widehat{\sigma}^{-1}(i_{k-1})) &= \sigma(m) > i_k = \phi(\widehat{\sigma}^{-1}(i_k)). \end{aligned}$$

This is an inversion for ϕ, but not for τ.

Now we consider pairs (a, b) that are inversions for τ but not for ϕ, and in which exactly one of the two numbers are in the set $\left\{\widehat{\sigma}^{-1}(i_{k-1}), \widehat{\sigma}^{-1}(i_k)\right\}$. These satisfy the inequalities

$$\begin{gathered} a < b, \\ \tau(a) > \tau(b), \\ \phi(a) < \phi(b). \end{gathered}$$

If $a = \widehat{\sigma}^{-1}(i_{k-1})$, then the inequalities reduce to

$$i_k = \tau(a) > \tau(b) = \phi(b) > \phi(a) = \sigma(m),$$

which is impossible since $i_k < \sigma(m)$ by assumption. If $b = \widehat{\sigma}^{-1}(i_k)$ we get

$$\sigma(m) = \tau(b) < \tau(a) = \phi(a) < \phi(b) = i_k,$$

which is again impossible. So neither of these cases give rise to inversions for τ. If $a = \widehat{\sigma}^{-1}(i_k)$, the inequalities become

$$i_k < \phi(b) = \tau(b) < \sigma(m).$$

This gives us a set of inversions of the form

$$\left\{(\widehat{\sigma}^{-1}(i_k), s) \mid s > \widehat{\sigma}^{-1}(i_k), i_k < \phi(s) < \sigma(m)\right\}.$$

If $b = \widehat{\sigma}^{-1}(i_{k-1})$, we get

$$i_k < \tau(a) = \phi(a) < \sigma(m),$$

and a set of inversions of the form

$$\left\{(s, \widehat{\sigma}^{-1}(i_{k-1})) \mid s < \widehat{\sigma}^{-1}(i_{k-1}), i_k < \phi(s) < \sigma(m)\right\}.$$

Now we consider inversions for ϕ that are not inversions for τ. These satisfy

$$\begin{gathered} a < b, \\ \tau(a) < \tau(b), \\ \phi(a) > \phi(b). \end{gathered}$$

If $a = \widehat{\sigma}^{-1}(i_k)$, we get

$$\sigma(m) < \tau(b) = \phi(b) < i_k,$$

which is impossible. Likewise, $b = \widehat{\sigma}^{-1}(i_{k-1})$ gives

$$\sigma(m) < \phi(a) = \tau(a) < i_k.$$

So these cases contribute nothing. If a is $\widehat{\sigma}^{-1}(i_{k-1})$, then

$$i_k < \tau(b) = \phi(b) < \sigma(m)$$

and we get the set

$$\left\{(\widehat{\sigma}^{-1}(i_{k-1}), s) \mid s > \widehat{\sigma}^{-1}(i_{k-1}), s \neq \widehat{\sigma}^{-1}(i_k), i_k < \phi(s) < \sigma(m)\right\}.$$

The last remaining case is $b = \widehat{\sigma}^{-1}(i_k)$, which gives

$$i_k < \phi(a) = \tau(a) < \sigma(m),$$

with inversions

$$\left\{(s, \widehat{\sigma}^{-1}(i_k)) \mid s < \widehat{\sigma}^{-1}(i_k), s \neq \widehat{\sigma}^{-1}(i_{k-1}), i_k < \phi(s) < \sigma(m)\right\}.$$

But if we consider the sets of inversions, we can see that all the new inversions for τ have a corresponding inversion for ϕ. So since we certainly removed the inversion $(\widehat{\sigma}^{-1}(i_{k-1}), \widehat{\sigma}^{-1}(i_k))$ when we passed from ϕ to τ, we have not added new ones without also removing old inversions. All in all, this shows that $\varphi(I, \sigma) = \tau$ has fewer inversions than $\varphi(J, \sigma) = \phi$, giving the desired inequality in the Bruhat order,

$$\varphi(I, \sigma) \leq \varphi(J, \sigma).$$

To get the general case, note that the Bruhat order is transitive by definition, so we simply apply the above argument several times. $\square$

A summary of the above theorem is that the map $\varphi(\cdot, \sigma)$ is order-reversing when we order the indexing sets by inclusion.

Now that we have a stabilisation map, we can give explicit elements of $Y_{m,n}(\sigma)$, as was hinted at earlier.

Theorem 22. *If the space $Y_{m,n}(\sigma)$ is non-empty, then the sequence of vectors*

$$s^n(e_1) = (L^n e_1, \dots, Le_1)$$

belongs to $Y_{m,n}(\sigma)$.

Proof. For a discussion of when $Y_{m,n}(\sigma)$ is non-empty, consult Theorem 13.

We already know that the sequence

$$(e_1, \dots, e_m, L^n e_1, \dots, Le_1)$$

has all subsequent sequences of m vectors linearly independent, by the construction of the stabilisation map. So the thing to check is whether the matrices

$$(e_{k+n+1}, \dots, e_m, L^n e_1, \dots, Le_1, \sigma_1, \dots, \sigma_k)$$

are invertible, for $k \in \{1, \dots, m-1\}$. If $n + k$ is greater than or equal to m, we of course mean the matrices

$$\left(L^{m-k} e_1, \dots, Le_1, \sigma_1, \dots, \sigma_k\right).$$

Calculating the determinant of a matrix of this form by column expansion, we see that expanding along the column σ_i removes row $\sigma(i)$ from the matrix and introduces a sign. So this matrix being invertible for all σ is equivalent to the matrix

$$R = \begin{pmatrix} — r_{i_1} — \\ — r_{i_2} — \\ \vdots \\ — r_{i_s} — \end{pmatrix}$$

being invertible for any choice of $s = m - k$ numbers, $i_1, \dots, i_s \in \{1, \dots, m\}$, with $i_j < i_{j+1}$, where r_l denotes the l'th row of the matrix

$$P = (L^{m-k} e_1, \dots, Le_1).$$

Calculating P by induction, we see that it is related to Pascal's triangle,

$$P = \big(p_{i,j}\big) = \begin{pmatrix} \binom{m-k-1}{0} & \binom{m-k-2}{0} & \dots & \binom{1}{0} & \binom{0}{0} \\ \binom{m-k}{1} & \binom{m-k-1}{1} & \dots & \binom{2}{1} & \binom{1}{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{2(m-k-1)}{m-k-1} & \binom{2(m-k-1)-1}{m-k-1} & \dots & \binom{m-k}{m-k-1} & \binom{m-k-1}{m-k-1} \end{pmatrix},$$

with the entry in the i'th row and j'th column given by the formula

$$p_{i,j} = \binom{m-k-j+i-1}{i-1}.$$

We start with column $j = m-k$. This column is always Le_1, which we know consists of all ones, so the formula holds here since

$$p_{i,m-k} = \binom{m-k-(m-k)+i-1}{i-1} = \binom{i-1}{i-1}.$$

Now assume that the formula holds for column j, given as $L^{m-k-j+1}e_1$. Moving a column to the left is the same as multiplying with L, which corresponds to adding all of the entries above a given entry together. So we get

$$p_{i,j-1} = \sum_{r=1}^{i} p_{r,j} = \sum_{r=1}^{i} \binom{m-k-j+r-1}{r-1}.$$

But the entry $p_{1,j}$ is always one, so we can replace $p_{1,j}$ with $p_{1,j-1}$ in the sum without changing the result. The conclusion then follows from the standard formulas for binomial coefficients,

$$\begin{aligned} p_{2,j-1} = p_{1,j} + p_{2,j} = p_{1,j-1} + p_{2,j} &= \binom{m-k-j+1}{0} + \binom{m-k-j+1}{1} \\ &= \binom{m-k-j+2}{1} \\ &= \binom{m-k-(j-1)+2-1}{2-1}, \end{aligned}$$

and we can continue like this to get the formula for $p_{i,j}$ given above.

In particular, by the standard formulas for binomial coefficients,

$$p_{i,j} = p_{i-1,j} + p_{i,j+1},$$

so the row r_j can be calculated by taking the row above, r_{j-1}, and adding the row $\widehat{r}_j$ given by shifting every entry of r_j one to the left and adding a zero,

$$r_j = r_{j-1} + \widehat{r}_j.$$

To illustrate, if $m-k$ is three, then P is the matrix

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 6 & 3 & 1 \end{pmatrix}$$

and the third row is given by

$$\begin{pmatrix}6 & 3 & 1\end{pmatrix} = r_3 = r_2 + \widehat{r}_3 = \begin{pmatrix}3 & 2 & 1\end{pmatrix} + \begin{pmatrix}3 & 1 & 0\end{pmatrix}$$

We will prove that the matrix

$$R = \begin{pmatrix} — r_{i_1} — \\ — r_{i_2} — \\ \vdots \\ — r_{i_s} — \end{pmatrix}$$

is invertible by induction on s. The exact statement we intend to prove is that the determinant of R is strictly positive if two divides s an even number of times and is strictly negative if two divides s an odd number of times. For $s = 1$, R is always the matrix $R = (1)$ and we are done since $\det R = 1$ is strictly positive and two divides s zero times. Assume the statement is correct for $s - 1$. Since the determinant is linear in each row, we can use the equation for row r_j given above and get

$$\begin{aligned}
\det R &= \det \begin{pmatrix} — & r_{i_1} & — \\ — & r_{i_2} & — \\ & \vdots & \\ — & r_{i_{(s-1)}} & — \\ — & r_{i_s} & — \end{pmatrix} \\
&= \det \begin{pmatrix} — & r_{i_1} & — \\ — & r_{i_2} & — \\ & \vdots & \\ — & r_{i_{(s-1)}} & — \\ — & r_{i_s - 1} & — \end{pmatrix} + \det \begin{pmatrix} — & r_{i_1} & — \\ — & r_{i_2} & — \\ & \vdots & \\ — & r_{i_{(s-1)}} & — \\ — & \widehat{r}_{i_s} & — \end{pmatrix} \\
&= \det \begin{pmatrix} — & r_{i_1} & — \\ — & r_{i_2} & — \\ & \vdots & \\ — & r_{i_{(s-1)}} & — \\ — & r_{i_{(s-1)}} & — \end{pmatrix} + \sum_{j=0}^{i_s - i_{(s-1)} - 1} \det \begin{pmatrix} — & r_{i_1} & — \\ — & r_{i_2} & — \\ & \vdots & \\ — & r_{i_{(s-1)}} & — \\ — & \widehat{r}_{i_s - j} & — \end{pmatrix}.
\end{aligned}$$

The first determinant is zero, since the matrix has two equal rows, so we are left with the sum. Doing the same thing for all the other rows except for the first one gives us

$$\det R = \sum_{j_1=0}^{i_2 - i_1 - 1} \sum_{j_2=0}^{i_3 - i_2 - 1} \cdots \sum_{j_{(s-1)}=0}^{i_s - i_{(s-1)} - 1} \det \begin{pmatrix} — & r_{i_1} & — \\ — & \widehat{r}_{i_2 - j_1} & — \\ & \vdots & \\ — & \widehat{r}_{i_{(s-1)} - j_{(s-2)}} & — \\ — & \widehat{r}_{i_s - j_{(s-1)}} & — \end{pmatrix}.$$

By considering the last column of one of these matrices, we see that it has the form

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

since we have shifted every row below the first one space to the left. By expanding the determinant using this column, we remove the first row and the last column, possibly introduce a sign, and end up with

$$\det R = \sum_{j_1=0}^{i_2-i_1-1} \sum_{j_2=0}^{i_3-i_2-1} \cdots \sum_{j_{(s-1)}=0}^{i_s-i_{(s-1)}-1} (-1)^{s-1} \det \begin{pmatrix} — & \widetilde{r}_{i_2-j_1} & — \\ & \vdots & \\ — & \widetilde{r}_{i_{(s-1)}-j_{(s-2)}} & — \\ — & \widetilde{r}_{i_s-j_{(s-1)}} & — \end{pmatrix}.$$

Here we use $\widetilde{r}_i$ to denote the i'th row of P where we have removed the leftmost entry. But we can now apply the induction hypothesis. We see that all the determinants have the same sign and are non-zero, so $\det R$ is non-zero with opposite sign if $s-1$ is odd and with the same sign if $s-1$ is even. This exactly means that $\det R$ changes sign when s passes a multiple of two, as stated above, and we are done. □

2.4 The limit space

Using the stabilisation defined previously, we get a directed system of topological spaces,

$$X_{m,1} \xrightarrow{s} X_{m,2} \xrightarrow{s} \ldots \xrightarrow{s} X_{m,n} \xrightarrow{s} X_{m,n+1} \xrightarrow{s} \ldots$$

When considering such a system, it can sometimes be helpful to take the limit as n approaches infinity.

Definition 23. The space $X_{m,\infty}$ is defined as the direct limit of the directed system of spaces,

$$X_{m,\infty} = \varinjlim_n X_{m,n}.$$

As a set, $X_{m,\infty}$ is the disjoint union of all the spaces, where we identify $A \in X_{m,k}$ with $B \in X_{m,k'}$ if they eventually map to the same thing under s:

$$A \sim B \iff \exists\, i, j : s^i(A) = s^j(B).$$

We give $X_{m,\infty}$ the finest topology such that all the maps

$$\imath_n : X_{m,n} \to X_{m,\infty}$$

are continuous. So a subset $U \subset X_{m,\infty}$ is open if and only if $\imath_n^{-1}(U)$ is open in $X_{m,n}$ for all n.

It is this limit space that we will focus on in the following chapters, especially in Chapter 4. Before we do this, we will however need to set up some tools, which will be done in the next chapter.

2.5 Quaternionic and real cases

When the spaces $Y_{m,n}(X, Y)$ were introduced in Definition 8, we could have used other entries for our vectors than the complex numbers. Two possible alternatives would be the real numbers, $\mathbb{R}$, or the quaternions, $\mathbb{H}$, giving us the spaces $Y^{\mathbb{R}}_{m,n}(X, Y)$ and $Y^{\mathbb{H}}_{m,n}(X, Y)$. These behave much like the spaces studied above, with the notable exception that since the quaternions are not commutative one needs to choose whether to act with scalars from the left or from the right. Since the stabilisation map,

$$s : Y^{\mathbb{H}}_{m,n} \to Y^{Hq}_{m,n+1},$$

acts on the left with a matrix and this map should be linear to give an induced map on the quotient $X^{\mathbb{H}}_{m,n}$, we will choose to act with scalars from the right. Once this is done, everything above carries out exactly as for the complex numbers, and we get stabilisation maps and limit spaces exactly as for the complex case. These will not be important in the next chapter, but will show up again in Chapter 4.

3 Cohomology calculations

This chapter will be focused on the cohomology of $Y_{m,n}$ and $X_{m,n}$. We will introduce the spectral sequence that computes the cohomology of $Y_{m,n}$, and use it to compute the first cohomology groups of all the spaces. Afterwards, we focus on $Y_{3,3}$ and give a full description of the cohomology, which also leads to the cohomology of the quotient, $X_{3,3}$.

3.1 A filtration on $Y_{m,n}$

Before we start the calculation, we need some general setup. To work with the space $Y_{m,n}$ and compute the cohomology, it will be beneficial to build it up from simpler subspaces. This is done by considering a filtration.

For $0 \leq p \leq m-1$, let F_p be the subset

$$F_p = \{(a_1, \ldots, a_n) \in Y_{m,n} \mid a_n \text{ has at most } p \text{ zeroes.}\}.$$

F_p is open in $Y_{m,n}$ since we stay inside the subspace when we vary the coordinates of a point, as long as we do not introduce new zeroes in the last column. The subspaces form an increasing sequence:

$$F_0 \subset F_1 \subset \cdots \subset F_{m-1} = Y_{m,n}.$$

The general method of computing the homology and cohomology of $Y_{m,n}$ is by using the spectral sequence of this filtration, as described in Theorem 2. Here we will focus on cohomology, but similar arguments could of course be applied to homology.

The spectral sequence has E_1-page given by

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}),$$

with differentials given by the boundary map in the long exact sequence of the triple (F_{p+1}, F_p, F_{p-1}):

$$d_1 : H^{p+q}(F_p, F_{p-1}) = E_1^{p,q} \to E_1^{p+1,q} = H^{p+q+1}(F_{p+1}, F_p).$$

To use the spectral sequence, we need to compute the cohomology of the pair (F_p, F_{p-1}). To calculate this we will use excision, so we would like to find an open subset of F_p that contains the complement of F_{p-1}. For any choice of exactly p different indices $\{v_1, \ldots, v_p\} \subset \{1, \ldots, m-1\}$, define $Y_{\{v_1,\ldots,v_p\}}$ to be the points $(a_1, \ldots, a_n) \in F_p$ which has $(a_n)_{v_i} = 0$ for all i. Note that if we define an indexing set $I = \{1, \ldots, m\} - \{v_1, \ldots, v_p\}$, these spaces correspond to the spaces in Theorem 20 by the formula

$$Y_{\{v_1,\ldots,v_p\}} = Y^I_{m,n},$$

Since $Y_{m,n}$ is open, we can get an open set in F_p by inserting a small disc on these zero-entries. More precisely, the set we get is

$$Y_{\{v_1,\dots,v_p\}} \times D_p \cong \left\{ A \in F_p \;\middle|\; \begin{array}{c} \left|((a_n)_{v_1},\dots,(a_n)_{v_p})\right| < \min\limits_{i\notin\{v_1,\dots,v_p\}} |(a_n)_i| \,, \\ (a_1,\dots,a_{n-1},a_n - t\sum_i (a_n)_{v_i} e_{v_i}) \in F_p \forall t \in [0,1] \end{array} \right\}.$$

This is an open set in F_p, with $Y_\emptyset = F_0$, and it contains all points $(a_1,\dots,a_n) \in F_p$ where the entries of a_n indicated by $\{v_1,\dots,v_p\}$ are zero. For a different choice of indices, the condition that the radius of the disc is smaller than the norm of the non-zero entries in a_n ensures disjointness, so the intersection is empty:

$$\big(Y_{\{v_1,\dots,v_p\}} \times D_p\big) \cap \big(Y_{\{u_1,\dots,u_p\}} \times D_p\big) = \emptyset \quad \text{if } \{v_1,\dots,v_p\} \neq \{u_1,\dots,u_p\}\,.$$

The intersection of $Y_{m,n} \times D_p$ and F_{p-1} corresponds to removing 0 from the disc, since one of the entries inserted on the last column has to be non-zero,

$$\big(Y_{\{v_1,\dots,v_p\}} \times D_p\big) \cap F_{p-1} = Y_{\{v_1,\dots,v_p\}} \times (D_p - 0)\,.$$

This means that the disjoint union of all of these form an open set in F_p with

$$F_p = F_{p-1} \cup \left(\coprod_{\{v_1,\dots,v_p\}} Y_{\{v_1,\dots,v_p\}} \times D_p \right).$$

By excision, the inclusion of pairs

$$\coprod_{\{v_1,\dots,v_p\}} Y_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0) \hookrightarrow (F_p, F_{p-1})$$

induces an isomorphism of cohomology groups,

$$\begin{aligned} E_1^{p,q} &= H^{p+q}(F_p, F_{p-1}) \\ &\cong H^{p+q}\left(\coprod_{\{v_1,\dots,v_p\}} Y_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0) \right) \\ &\cong \bigoplus_{\{v_1,\dots,v_p\}} H^{p+q}(Y_{\{v_1,\dots,v_p\}} \times (D_p, D_p - 0)) \\ &\cong \bigoplus_{\{v_1,\dots,v_p\}} H^{q-p}(Y_{\{v_1,\dots,v_p\}}) \otimes H^{2p}(D_p, D_p - 0). \end{aligned}$$

The last isomorphism is given by the Künneth formula, and the fact that the pair $(D_p, D_p - 0)$ only has non-zero cohomology in degree $2p$:

$$H^q(D_p, D_p - 0) \cong \widetilde{H}^{q-1}(D_p - 0) = \widetilde{H}^{q-1}(S^{2p-1}) = \begin{cases} \mathbb{Z} & q = 2p, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it follows that

$$E_1^{p,q} = 0$$

for $q < p$.

3.2 The first cohomology of $X_{m,n}$

We would like to use the spectral sequence to compute the first cohomology group of $X_{m,n}$ for any m and n. As discussed at the end of Section 2.1, we will do this by computing $H^1(Y_{m,n})$ and then using Lemma 18 to calculate the cohomology of $X_{m,n}$ via factoring out the extra free group from the torus in the isomorphism

$$H^1(Y_{m,n}) \cong H^1(X_{m,n}) \oplus H^1(\mathrm{T}^n).$$

From Theorem 2, we get that there is a subgroup

$$0 \subset H \subset H^1(Y_{m,n})$$

and isomorphisms

$$\begin{aligned} E_\infty^{0,1} &\cong H^1(Y_{m,n})/H, \\ E_\infty^{1,0} &\cong H. \end{aligned}$$

So to calculate $H^1(Y_{m,n})$, we have to calculate the two groups $E_\infty^{1,0}$ and $E_\infty^{0,1}$ and see what groups fit into the short exact sequence

$$0 \to E_\infty^{1,0} \to H^1(Y_{m,n}) \to E_\infty^{0,1} \to 0.$$

Drawing the relevant part of the first page of the spectral sequence, we are considering the diagram illustrated in Figure 3.1. But by writing out our specific case,

$$\begin{array}{c|ccccc} 1 & H^1(F_0) & \longrightarrow & H^2(F_1,F_0) & \longrightarrow & H^3(F_2,F_1) \\ 0 & H^0(F_0) & \longrightarrow & H^1(F_1,F_0) & \longrightarrow & H^2(F_2,F_1) \\ \hline q/p & 0 & & 1 & & 2 \end{array}$$

Figure 3.1: The relevant part of E_1 for computing $H^1(Y_{m,n})$.

we can see that the the groups $H^1(F_1,F_0), H^2(F_2,F_1)$ and $H^3(F_2,F_1)$ are all zero, since $q < p$, so the diagram simplifies to Figure 3.2. From the figure, we can see

$$\begin{array}{c|cccc} 1 & H^1(Y_{m,n-1} \times \mathrm{T}^m) & \longrightarrow & H^2(F_1,F_0) & 0 \\ 0 & H^0(Y_{m,n-1} \times \mathrm{T}^m) & & 0 & 0 \\ \hline q/p & 0 & & 1 & 2 \end{array}$$

Figure 3.2: The simplified E_1.

that the group $E_\infty^{1,0}$ must be zero, so the exact sequence we are considering gives an isomorphism

$$0 \longrightarrow H^1(Y_{m,n}) \xrightarrow{\cong} E_\infty^{0,1} \longrightarrow 0\ .$$

By doing explicit calculations for small m and n, as in Example 10, 11, and 12, we are led to the following theorem.

Theorem 24. *The first cohomology group of $Y_{m,n}$ is isomorphic to $\mathbb{Z}^{m+n-1}$.*

Proof. We will proceed by induction on n. By looking at Example 10, we see that

$$H^1(Y_{m,1}) \cong H^1(\mathrm{T}^m) \cong \mathbb{Z}^m = \mathbb{Z}^{m+1-1}$$

as desired.

Now assume the theorem is correct for

$$H^1(Y_{m,n-1}) \cong \mathbb{Z}^{m+n-2}.$$

By applying the formula for φ given in Theorem 20 to an index $i < m$,

$$Y_{\{i\}} = Y_{m,n}^{\{1,\ldots,m\}-i} \cong Y_{m,n-1}\big((i\ i+1)\big) \times \mathrm{T}^{m-1},$$

and we see that excision tells us that the group $H^2(F_1, F_0)$ is given by

$$H^2(F_1, F_0) \cong \bigoplus_{i=1}^{m-1} H^0(Y_{m,n-1}\big((i\ i+1)\big) \times \mathrm{T}^{m-1}) \otimes H^2(D, D-0) \cong \mathbb{Z}^{m-1}.$$

This simplifies Figure 3.2 to Figure 3.3. Since the groups involved will never change

$$\begin{array}{c|ccc}
1 & \mathbb{Z}^{m+n-2} \oplus \mathbb{Z}^m & \longrightarrow \mathbb{Z}^{m-1} & 0 \\
0 & \mathbb{Z} & 0 & 0 \\
\hline
{}^{q}\!/_{p} & 0 & 1 & 2
\end{array}$$

Figure 3.3: E_1, assuming the induction hypothesis.

after turning the page to E_2, we see that we only have to compute the group

$$E_\infty^{0,1} = E_2^{0,1} = \ker\left(d_1 : E_1^{0,1} \to E_1^{1,1}\right).$$

The differential d_1 is given by the differential in the long exact sequence for the pair (F_1, F_0), which is the composition

$$d_1 : \ H^1\left(Y_{m,n-1} \times \mathrm{T}^m\right) \xrightarrow{\oplus\imath^*} \bigoplus_{i=1}^{m-1} H^1\left(Y_{m,n-1}\big((i\ i+1)\big) \times \mathrm{T}^{m-1} \times (D-0)\right)$$

$$\xrightarrow{\oplus\delta} \bigoplus_{i=1}^{m-1} H^2\left(Y_{m,n}\big((i\ i+1)\big) \times \mathrm{T}^{m-1} \times (D, D-0)\right).$$

The first map is induced by the inclusions

$$\imath : Y_{m,n-1}\big((i\ i+1)\big) \times \mathrm{T}^{m-1} \times (D-0) \to Y^{m,n-1} \times \mathrm{T}^m,$$

given by first using the inverse of the homeomorphism from Theorem 20, given by the indices $I = (1 < 2 < \cdots < i-1 < i+1 < \ldots m)$,

$$Y_{m,n}^I \to Y_{m,n+1}\big((i\ i+1)\big) \times \mathrm{T}^{m-1},$$

and then inserting the coordinate from $(D-0)$ on the i'th coordinate of the last column. The second map consists of the differentials from the long exact sequences of the pairs

$$Y_{m,n-1}\big((i\; i+1)\big) \times \mathrm{T}^{m-1} \times (D, D-0).$$

We want to see that d_1 is surjective, so we will compute the image of the element

$$\alpha_j = 1 \otimes \cdots \otimes 1 \otimes [T] \otimes 1 \otimes \cdots \otimes 1 \in H^1(Y_{m,n-1} \times \mathrm{T}^m),$$

where the class $[T] \in H^1(\mathrm{T})$ is in the j'th tensor coordinate. We will consider each inclusion separately. If $i \neq j$, the inclusion will not touch the j'th coordinate of T^m, so on cohomology classes it will map α_j to an element that is of the form

$$\imath^*(\alpha_j) = \widetilde{\alpha_j} \otimes 1 \in H^*(Y_{m,n-1}\big((i\; i+1)\big) \times \mathrm{T}^{m-1}) \otimes H^*(D-0)$$

And since δ is given by

$$\delta(\beta) = \begin{cases} \widetilde{\beta} \otimes [(D, D-0)] & \text{if } \beta = \widetilde{\beta} \otimes [D-0], \\ 0 & \text{otherwise.} \end{cases},$$

we see that $\delta(\imath^*(\alpha_j)) = 0$ in this case. If $i = j$, then the same considerations give us that

$$\delta \imath^*(\alpha_j) = 1 \otimes [(D, D-0)] \in H^2(Y_{m,n-1}\big((i\; i+1)\big) \times \mathrm{T}^{m-1} \times (D, D-0)).$$

The group $H^2(F_1, F_0)$ is generated by exactly these elements, so the map d_1 is surjective. Since the groups involved are all free, we know the kernel is also free and we can calculate the rank as the rank of the domain minus the rank of the codomain, giving us

$$H^1(Y_{m,n}) \cong E_\infty^{0,1} = \ker d_1 \cong \mathbb{Z}^{m+n-2+m-(m-1)} = \mathbb{Z}^{m+n-1}.$$

□

An immediate consequence is the following result.

Corollary 25. *The first cohomology group of $X_{m,n}$ is $\mathbb{Z}^{m-1}$.*

Proof. By using the Künneth formula and the previous theorem, we get an isomorphism

$$H^1(X_{m,n}) \oplus H^1(\mathrm{T}^n) \cong H^1(Y_{m,n}) \cong \mathbb{Z}^{m+n-1}.$$

Since any torsion in $H^1(X_{m,n})$ would show up in $H^1(Y_{m,n})$, it must be free. We can calculate the rank from the above and get

$$H^1(X_{m,n}) \cong \mathbb{Z}^{m-1}.$$

□

To make this abstract isomorphism slightly more concrete, we will realise it as a map between spaces. Consider the image of an element $v \in Y_{m,1}$ in $Y_{m,n}$ under the stabilisation map. This has the form

$$s^{n-1}(v) = s^{n-1}\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} | & | & & | \\ L^n v & L^{n-1} e_1 & \dots & L e_1 \\ | & | & & | \end{pmatrix},$$

where L is the lower triangular matrix that defines the stabilisation map, see Definition 19. Computing the determinants gives

$$\mathrm{Det}(s^{n-1}(v)) = (\pm v_1, \dots, \pm v_m, \pm 1, \dots, \pm 1),$$

with the signs dependent on n and m. Using the action of T^n to change the last $n-1$ determinants as in the proof of Lemma 18, we can get an element of $Y_{m,n}$ with any prescribed determinant. Altogether this defines a lift, f, of the determinant map,

$$\begin{tikzcd} \mathrm{T}^{m+n-1} \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{Id}"'] & Y_{m,n} \arrow[r, "\mathrm{Det}"] & \mathrm{T}^{m+n-1} \end{tikzcd} .$$

By taking first cohomology, we get maps

$$\begin{tikzcd} H^1(Y_{m,n}) \arrow[r, shift left, "f^*"] & H^1(\mathrm{T}^{m+n-1}) \arrow[l, shift left, "\mathrm{Det}^*"] \end{tikzcd} ,$$

with

$$f^* \circ \mathrm{Det}^* = \mathrm{Id}\,.$$

This shows that Det^* is injective and f^* is surjective. But by the previous theorem, f^* is a map between free abelian groups of the same rank, hence it must also be injective and so an isomorphism with inverse Det^*. This shows that the determinant map is an isomorphism on the first cohomology groups and immediately extends to the quotiented spaces, with

$$\mathrm{Det}^* : H^1(\mathrm{T}^{m+n-1}/\,\mathrm{T}^n) \to H^1(X_{m,n})$$

an isomorphism. Note that the action of T^n on T^{m+n-1} is not the obvious one, but is given by how the determinants change when T^n acts on $Y_{m,n}$. The quotient is still isomorphic to T^{m-1}. Furthermore the stabilisation maps on degree one cohomology are surjective, since the stabilisation maps

$$s : Y_{m,n} \to Y_{m,n+1}$$

change the signs of some determinants and add an extra determinant that is always plus or minus one. So if the class $d_i \in H^1(Y_{m,n+1})$ represents the image of moving once around zero in the i'th coordinate of T^{n+1}, then

$$s^*(d_i) = \begin{cases} d_i & 1 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

3.3 The spectral sequence of $Y_{3,3}$

In this section we will calculate the cohomology of the space $Y_{3,3}$, which is the set:

$$Y_{3,3} = \left\{ \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \in \mathbb{C}^9 \;\middle|\; \begin{matrix} a \neq 0,\, ad - bc \neq 0, \\ adi + beh + cfg - deg - bci - afh \neq 0, \\ di - fh \neq 0,\, i \neq 0 \end{matrix} \right\}.$$

The way we will calculate this is by using the spectral sequence defined in Section 3.1. Some of the calculations have been done in Sage and the relevant source code can be found in Appendix A.

The groups we need to compute for the first page in the spectral sequence are:

$$E_1^{p,q} \cong \begin{cases} H^q(Y_\emptyset) & p=0, \\ H^{q+1}(Y_{\{1\}} \times (D, D-0)) \oplus H^{q+1}(Y_{\{2\}} \times (D, D-0)) & p=1, \\ H^{q+2}(Y_{\{1,2\}} \times (D_2, D_2 - 0)) & p=2 \\ 0 & \text{otherwise.} \end{cases}$$

The four spaces that appear will be handled separately.

The cohomology of $Y_{\{1,2\}}$

Consider the space $Y_{\{1,2\}}$. This consists of the elements of $Y_{3,3}$ that have two zeroes in the last column:

$$Y_{\{1,2\}} = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & i \end{pmatrix} \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0 \\ adi - bci \neq 0 \\ di \neq 0, i \neq 0 \end{matrix} \right\}.$$

The space is homeomorphic to $(\mathbb{C}^*)^3 \times \mathbb{C}^2 \times \left\{(b,c) \in \mathbb{C}^2 \mid bc \neq 1\right\}$, with the homeomorphism given by

$$\begin{aligned} Y_{\{1,2\}} &\to (\mathbb{C}^*)^3 \times \mathbb{C}^2 \times \left\{(b,c) \in \mathbb{C}^2 \mid bc \neq 1\right\} \\ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & i \end{pmatrix} &\mapsto \left(a, d, i, \frac{e}{i}, \frac{f}{i}, \frac{b}{a}, \frac{c}{d}\right). \end{aligned}$$

We already computed the cohomology of $Y = \{(b,c) \mid bc \neq 1\}$ in Example 11, so we can calculate the cohomology of $Y_{\{1,2\}}$:

$$H^q(Y_{\{1,2\}}) = \begin{cases} \mathbb{Z} & q=0, \\ \mathbb{Z}^4 & q=1, \\ \mathbb{Z}^7 & q=2, \\ \mathbb{Z}^7 & q=3, \\ \mathbb{Z}^4 & q=4, \\ \mathbb{Z} & q=5, \\ 0 & \text{otherwise.} \end{cases}$$

Since we know the product structure from the Künneth formula, we can name the generators and get the ring

$$H^*(Y_{\{1,2\}}) = \bigwedge\nolimits^* [a_{00}, d_{00}, i_{00}] \otimes H^*(Y).$$

The two generators of Y will be denoted $y_{00,1}$ for the degree one generator and $y_{00,2}$ for the degree two generator. All other generators have degree one. The ring structure is the usual one on a tensor product, with the added relation $y_{00,1} \cdot y_{00,2} = 0$. The element $y_{00,1} \in H^1(Y)$ is given by the map

$$Y \ni (b,c) \mapsto 1 - bc \in \mathbb{C}^*$$

in cohomology, since there is another map

$$\mathbb{C}^* \ni \lambda \mapsto (1, 1-\lambda) \in Y$$

and the composition is the identity on $\mathbb{C}^*$. Applying H^1 shows that the two maps give isomorphisms in degree 1.

The cohomology of $Y_{\{1\}}$

In $Y_{\{1\}}$ we have the first entry of the last column equal to zero, which is the space

$$Y_{\{1\}} = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & h \\ e & f & i \end{pmatrix} \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0 \\ adi + beh - bci - afh \neq 0. \\ di - fh \neq 0, h \neq 0, i \neq 0 \end{matrix} \right\}$$

This space is homeomorphic to $(\mathbb{C}^*)^5 \times \mathbb{C} \times Y$:

$$\begin{pmatrix} a & b & 0 \\ c & d & h \\ e & f & i \end{pmatrix} \mapsto \left(a, i, h, \frac{d}{h} - \frac{f}{i}, \frac{d}{h} - \frac{bc}{ah}, \frac{e}{i}, \frac{bhi}{adi - afh}, \frac{c}{h} - \frac{e}{i} \right).$$

Since the cohomology of Y is already known, applying the Künneth formula gives

$$H^q(Y_{\{1\}}) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^6 & q = 1, \\ \mathbb{Z}^{16} & q = 2, \\ \mathbb{Z}^{25} & q = 3, \\ \mathbb{Z}^{25} & q = 4, \\ \mathbb{Z}^{16} & q = 5, \\ \mathbb{Z}^6 & q = 6, \\ \mathbb{Z} & q = 7, \\ 0 & \text{otherwise.} \end{cases}$$

As before, the ring structure is given by

$$H^*(Y_{\{1\}}) = \bigwedge\nolimits^* [a_{01}, d_{01}, f_{01}, h_{01}, i_{01}] \otimes H^*(Y).$$

The cohomology of $Y_{\{2\}}$

The space is

$$Y_{\{2\}} = \left\{ \begin{pmatrix} a & b & g \\ c & d & 0 \\ e & f & i \end{pmatrix} \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0 \\ adi + cfg - bci - deg \neq 0 \\ di \neq 0, g \neq 0, i \neq 0 \end{matrix} \right\},$$

which can also be identified with $(\mathbb{C}^*)^5 \times \mathbb{C} \times Y$ using the homeomorphism

$$\begin{pmatrix} a & b & g \\ c & d & 0 \\ e & f & i \end{pmatrix} \mapsto \left(\frac{a}{g}, d, i, g, 1 + \frac{cfg}{adi} - \frac{eg}{ai} - \frac{bc}{ad}, \frac{f}{i}, \frac{b}{a}, \frac{c}{d} \right).$$

Since we already know the cohomology of this space, we know

$$H^q(Y_{\{2\}}) = \left(\bigwedge^*[a_{10}, d_{10}, f_{10}, g_{10}, i_{10}] \otimes H^*(Y)\right)^q = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^6 & q = 1, \\ \mathbb{Z}^{16} & q = 2, \\ \mathbb{Z}^{25} & q = 3, \\ \mathbb{Z}^{25} & q = 4, \\ \mathbb{Z}^{16} & q = 5, \\ \mathbb{Z}^6 & q = 6, \\ \mathbb{Z} & q = 7, \\ 0 & \text{otherwise.} \end{cases}$$

The cohomology of $Y_\emptyset$

This space is slightly more complicated than the others. The space consists of the elements in $Y_{3,3}$ where all entries of the last vector are non-zero.

$$Y_\emptyset = \left\{ \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \;\middle|\; \begin{matrix} a \neq 0, ad - bc \neq 0, di - fh \neq 0, \\ adi + beh + cfg - deg - bci - afh \neq 0, \\ g \neq 0, h \neq 0, i \neq 0 \end{matrix} \right\}.$$

The space is homeomorphic to $(\mathbb{C}^*)^7 \times \left\{(x, y, z) \in \mathbb{C}^3 \mid y - z - xyz = 1\right\}$, with the homeomorphism given by

$$\begin{aligned} \begin{pmatrix} a & b & g \\ c & d & h \\ e & f & i \end{pmatrix} \mapsto \Bigg(& g, h, i, \frac{a}{g}, \frac{f}{i} - \frac{d}{h}, \\ & D = \frac{ad - bc}{ah}, \\ & C = \frac{c}{h} - \frac{a}{g} + \frac{cdi - deh}{fh^2 - dhi} + \frac{beh - bci}{fgh - dgi}, \\ & \frac{bC}{aD}, \frac{c}{hC} - \frac{a}{gC}, \frac{D}{C}\frac{eh - ci}{fh - di} \Bigg). \end{aligned}$$

To get the cohomology, we refer back to Example 12, where we computed the cohomology of

$$Z = \left\{(x, y, z) \in \mathbb{C}^3 \mid y - z - xyz = 1\right\},$$

and found it to be the same as the cohomology of S^2. The generator of $H^2(Z)$ will be called z_{11} with $z_{11}^2 = 0$. If we apply this, we get

$$H^q(Y_\emptyset) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^7 & q = 1, \\ \mathbb{Z}^{22} & q = 2, \\ \mathbb{Z}^{42} & q = 3, \\ \mathbb{Z}^{56} & q = 4, \\ \mathbb{Z}^{56} & q = 5, \\ \mathbb{Z}^{42} & q = 6, \\ \mathbb{Z}^{22} & q = 7, \\ \mathbb{Z}^7 & q = 8, \\ \mathbb{Z} & q = 9, \\ 0 & \text{otherwise.} \end{cases}$$

and the generators are

$$H^*(Y_\emptyset) = \bigwedge{}^*[a_{11}, d_{11}, g_{11}, h_{11}, i_{11}, C_{11}, D_{11}] \otimes H^*(Z).$$

A note on the cohomology of Z

The space Z defined above is a manifold, which we will prove by applying the implicit function theorem. Consider the formula defining Z,

$$(x, y, z) \mapsto y - z - xyz - 1.$$

Taking partial derivatives gives the three functions

$$\begin{aligned} (x, y, z) &\mapsto yz, \\ (x, y, z) &\mapsto 1 - xz, \\ (x, y, z) &\mapsto -1 - xy. \end{aligned}$$

If the first function is zero, then at least one of the two others must be non-zero. By the implicit function theorem, the coordinate that has non-zero derivative can be written as a smooth function of the other coordinates. This shows that Z is locally the graph of a smooth function, so it is a smooth manifold.

Later on it will be necessary to have a description of $H^2(Z)$. This is done by considering the subspace

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x > 0, y < 0, xy < -1, z = \frac{y - 1}{1 + xy} > 0 \right\} \subset Z.$$

This space is closed in Z since any sequence in R that converges in Z must also converge in R. Consider a subbundle $NR \cong R \times D$ of the normal bundle of R in the tangent bundle TZ, such that the exponential map is a diffeomorphism on NR. This gives us a diagram

$$H^2(NR, NR_0) \xleftarrow{\cong} H^2(Z, Z - R) \xrightarrow{j^*} H^2(Z),$$

where NR_0 is NR without the zero section, $NR_0 \cong R \times (D - 0)$. The isomorphism is given by excision, since $Z = NR \cup (Z - R)$. The space R is homeomorphic to $\mathbb{R}^2$, so we have

$$H^2(Z, Z - R) \cong H^2(NR, NR_0) \cong H^2(R \times (D, D - 0)) \cong \mathbb{Z}.$$

From the diagram, $j^*(1)$ gives us an element of $H^2(Z)$. To see that this is a generator, consider the torus

$$\widehat{\mathrm{T}} = \{(x, y, z) \in Z \mid |x| = |y| = 2\} \subset Z.$$

This intersects R in exactly one point, namely $p = (2, -2, 1)$. By taking the intersection with $\widehat{T}$ we can extend the previous diagram,

$$\begin{tikzcd}
H^2(NR, NR_0) \arrow[d, "i^*"] & H^2(Z, Z - R) \arrow[l, "\cong"'] \arrow[r, "j^*"] \arrow[d, "i^*"] & H^2(Z) \arrow[d, "i^*"] \\
H^2(p \times (D, D - 0)) & H^2(\widehat{\mathrm{T}}, \widehat{\mathrm{T}} - p) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H^2(\widehat{\mathrm{T}}).
\end{tikzcd}$$

The left i^* is an isomorphism since the cohomology in degree two is given by the pair $(D, D - 0)$, which does not change under the inclusion. This shows that the inclusion $\widehat{\mathrm{T}} \to Z$ is an isomorphism in cohomology, so $j^*(1)$ is a generator of $H^2(Z)$. Taking the same diagram with cohomology replaced with homology reverses all arrows but is otherwise exactly the same. This shows that to calculate the homology class of a cycle σ in $H_2(Z)$, we can count the number of points of intersection between σ and R with orientation, since each of these will contribute either 1 or -1, depending on orientation, to the homology class in $(Z, Z - R)$.

There is a smiliar argument with the space Y from before and the subspace $R_Y = \{(b, c) \in \mathbb{R}^2 \mid b > 0, c > 0, bc > 1\}$. In particular, the generator of $H_2(Y)$ can be taken as $i_*\left([\widehat{T}]\right)$, where $[\widehat{T}]$ is a generator of $H_2(\widehat{\mathrm{T}})$ for the torus

$$\widehat{\mathrm{T}} = \{(b, c) \mid |b| = |c| = 2\} \subset R_Y.$$

The first page

With all of the above, we can draw the E_1-page of the spectral sequence. Remember that this is

$$E_1^{p,q} \cong \begin{cases} H^q(Y_\emptyset) & p = 0, \\ H^{q+1}(Y_{\{1\}} \times (D, D - 0)) \oplus H^{q+1}(Y_{\{2\}} \times (D, D - 0)) & p = 1, \\ H^{q+2}(Y_{\{1,2\}} \times (D_2, D_2 - 0)) & p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

By inserting the groups that were calculated above into this, we get Figure 3.4 with the non-zero differentials moving directly to the right. To get the E_2-page of the spectral sequence, we need to compute the homology of the differentials.

3.4 Differentials on E_1

By definition, the differentials are the boundary maps of the long exact sequence of the triple (F_{p+1}, F_p, F_{p-1}),

$$d_1 : E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \to H^{p+q+1}(F_{p+1}, F_p) = E_1^{p+1,q}.$$

$$
\begin{array}{c|ccccc}
9 & \mathbb{Z} & & 0 & & 0 \\
8 & \mathbb{Z}^7 & \longrightarrow & \mathbb{Z}^2 & & 0 \\
7 & \mathbb{Z}^{22} & \longrightarrow & \mathbb{Z}^{12} & \longrightarrow & \mathbb{Z} \\
6 & \mathbb{Z}^{42} & \longrightarrow & \mathbb{Z}^{32} & \longrightarrow & \mathbb{Z}^4 \\
5 & \mathbb{Z}^{56} & \longrightarrow & \mathbb{Z}^{50} & \longrightarrow & \mathbb{Z}^7 \\
4 & \mathbb{Z}^{56} & \longrightarrow & \mathbb{Z}^{50} & \longrightarrow & \mathbb{Z}^7 \\
3 & \mathbb{Z}^{42} & \longrightarrow & \mathbb{Z}^{32} & \longrightarrow & \mathbb{Z}^4 \\
2 & \mathbb{Z}^{22} & \longrightarrow & \mathbb{Z}^{12} & \longrightarrow & \mathbb{Z} \\
1 & \mathbb{Z}^7 & \longrightarrow & \mathbb{Z}^2 & & 0 \\
0 & \mathbb{Z} & & 0 & & 0 \\
\hline
{}^{q}\!/_{p} & 0 & & 1 & & 2
\end{array}
$$

Figure 3.4: The E_1-page with differentials.

We will consider the two cases $p = 0$ and $p = 1$ separately.

First column

For $p = 0$, we are looking at the boundary map of the pair (F_1, F_0). Consider the inclusion of pairs,

$$i : Y_{\{1\}} \times (D, D - 0) \sqcup Y_{\{2\}} \times (D, D - 0) \to (F_1, F_0).$$

This is an isomorphism in cohomology and gives a map between the long exact sequences that makes the following diagram commute:

$$
\begin{tikzcd}
H^{p+q}(F_0) \arrow[r, "d_1"] \arrow[d, "i^*"] & H^{p+q+1}(F_1, F_0) \arrow[d, "i^*", "\cong"'] \\
H^{p+q}((Y_{\{1\}} \sqcup Y_{\{2\}}) \times (D - 0)) \arrow[r, "\delta"] & H^{p+q+1}((Y_{\{1\}} \sqcup Y_{\{2\}}) \times (D, D - 0)).
\end{tikzcd}
$$

The bottom map is the boundary map from the long exact sequence of the pair $(D, D - 0)$. This map is well-known and will be treated at the end of this section, so the only missing part is to calculate the inclusion i^*. Since we are working with a disjoint union, we only need to calculate the inclusion on each factor.

The inclusion $Y_{\{1\}} \times (D-0) \hookrightarrow Y_\emptyset$

We can write the inclusion out in coordinates as

$$\begin{aligned}(\mathbb{C}^*)^5 \times Y \times (D-0) &\longrightarrow (\mathbb{C}^*)^7 \times Z\\ (a,d,f,h,i,b,c,g) &\mapsto \Big(g, h, i, \frac{a}{g}, -f, d,\\ &\qquad\qquad C = c - \frac{a}{g} - \frac{cd}{f} - bc^2 + \frac{abc}{g},\\ &\qquad\qquad \frac{bfC}{d}, \frac{c}{C} - \frac{a}{gC}, \frac{cd}{fC}\Big).\end{aligned}$$

On the generators, this is given by

$$\begin{aligned} i^* : H^*(Y_\emptyset) &\to H^*(Y_{\{1\}} \times (D-0))\\ a_{11} &\mapsto a_{01} - g_{01},\\ d_{11} &\mapsto f_{01},\\ g_{11} &\mapsto g_{01},\\ h_{11} &\mapsto h_{01},\\ i_{11} &\mapsto i_{01},\\ D_{11} &\mapsto d_{01},\\ C_{11} &\mapsto a_{01} - g_{01} + y_{01,1},\\ z_{11} &\mapsto y_{01,2} + (a_0 - d_{01} + f_{01} - g_{01})y_{01,1},\end{aligned}$$

where g_{01} is the generator of $H^1(D-0)$.

In degree 1, this is calculated by dualising to homology, writing out the map on generators and working out what they map to. For z_{11} it is slightly harder. To calculate this, we use the submanifold $R \subset Z$ defined previously as

$$R = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x > 0, y < 0, z = \frac{y-1}{1+xy} > 0 \right\}.$$

by choosing a generator $\alpha = [\sigma]$ of $H_2(Y_{\{1\}} \times (D-0))$ and calculating the intersection of the image of $f = \mathrm{pr}_Z \circ i \circ \sigma$ and R. Note that if the coordinates of Y are fixed at $(0,1)$, then the image of

$$f(a,d,f,h,i,0,1,g) = \left(0, \frac{1}{C} - \frac{a}{gC}, -\frac{d}{fC}\right)$$

never intersects R since $x = 0$. Likewise, the coordinates i and h do not appear in the projection to Z and can not influence the result. Hence we only have to check the duals of the following generators in degree 2:

$$a_{01}y_{01,1}, d_{01}y_{01,1}, f_{01}y_{01,1}, y_{01,1}g_{01}, y_{01,2}.$$

These will be handled individually. All non-relevant coordinates will be set to 1, except g which will be denoted ε to make it clear that this coordinate is small. If $y_{01,1}$ is involved, we choose the generator

$$\sigma(\lambda) = (1-\lambda, 1) \in Y, \quad \lambda \in S^1,$$

as this gives

$$y_{01,1} = [\lambda \mapsto 1 - (1-\lambda)\cdot 1] = [\lambda \mapsto \lambda].$$

$\mathbf{a_{01}y_{01,1}}$: The relevant map is

$$(a, \lambda) \mapsto \left((1-\lambda)C, \frac{1}{C} - \frac{a}{\varepsilon C}, \frac{1}{C}\right),$$

where $C = -1 - \frac{a\lambda}{\varepsilon}$. If this intersects R then C has to be real and positive by looking at the last coordinate, which forces λ real and negative by the first coordinate and a real and positive by the second. Hence there is at most one intersection, at $(1, -1)$:

$$f(1, -1) = \left(\frac{2-2\varepsilon}{\varepsilon}, -1, \frac{\varepsilon}{1-\varepsilon}\right).$$

By inspection, this satisfies the conditions and hence belongs to R, so we conclude that the image intersects once. This gives the following formula:

$$\langle f^*(z_{11}), a_{01}y_{01,1}\rangle = \langle z_{11}, f_*(a_{01}y_{01,1})\rangle = \pm 1.$$

To check the sign, we need to check if this map preserves the orientation of the torus in R. This is done by differentiating the map to get a map of tangent spaces

$$Tf : TT^2 \to TR$$

and checking that the determinant of this map is positive. This has been done in Sage, see Appendix A.2 for the code. The intersection preserves orientation, so the sign is plus.

$\mathbf{d_{01}y_{01,1}}$: In this case the map becomes

$$(d, \lambda) \mapsto \left(\frac{(1-\lambda)C}{d}, \frac{\varepsilon - 1}{\varepsilon C}, \frac{d}{C}\right),$$

and by a similar analysis, this intersects in exactly the point $f(1, -1)$, which is the same as before. However, this time the orientation is reversed.

$\mathbf{f_{01}y_{01,1}}$: This has the same point of intersection and preserves the orientation.

$\mathbf{y_{01,1}g_{01}}$: This also intersects and preserves the orientation.

$\mathbf{y_{01,2}}$: This is calculated by the intersection points of the fundamental class of the torus in Y, which was mentioned earlier:

$$\widehat{T} = \left\{(b, c) \in \mathbb{C}^2 \mid |b| = |c| = 2\right\} \subset Y.$$

The map becomes

$$(b, c) \mapsto \left(bC, \frac{c}{C} - \frac{1}{\varepsilon C}, \frac{c}{C}\right),$$

with $C = -bc^2 - \frac{1-bc}{\varepsilon}$. We will write x for the first coordinate, y for the second and z for the third. An analysis as before gives $bc = xz > 0$ and $c = \frac{1}{\varepsilon(z-y)} \in \mathbb{R}$. So both b and c must be real and c should be positive. Then $xy = bc - b\varepsilon^{-1} < 0$ shows that b must be positive since bc is positive. The only possible point of intersection is

$$f(2, 2) = \left(\frac{6-16\varepsilon}{\varepsilon}, \frac{2\varepsilon - 1}{3 - 8\varepsilon}, \frac{2\varepsilon}{3 - 8\varepsilon}\right).$$

Since ε is very small, this intersects R and the orientation is preserved.

The inclusion $Y_{\{2\}} \times (D-0) \hookrightarrow Y_\emptyset$

The inclusion in coordinates is

$$\begin{aligned} (\mathbb{C}^*)^5 \times Y \times (D-0) &\longrightarrow (\mathbb{C}^*)^7 \times Z \\ (a,d,f,g,i,b,c,h) &\mapsto \Bigg(g, h, i, a, -\frac{d}{h}, \\ &\qquad D = \frac{d}{h}(1-bc), \\ &\qquad C = -af - \frac{a^2bh}{d} + \frac{a^2bhf}{d} + \frac{a^2b^2ch}{d}, \\ &\qquad \frac{bC}{D}, \frac{cd}{hC} - \frac{a}{C}, \frac{cD}{C} - \frac{ahD}{dC} + \frac{afhD}{dC} + \frac{abchD}{dC} \Bigg). \end{aligned}$$

We calculate as before and get the following on generators:

$$\begin{aligned} i^* : H^*(Y_\emptyset) &\to H^*(Y_{\{2\}} \times (D-0)) \\ a_{11} &\mapsto a_{10}, \\ d_{11} &\mapsto d_{10} - h_{10}, \\ g_{11} &\mapsto g_{10}, \\ h_{11} &\mapsto h_{10}, \\ i_{11} &\mapsto i_{10}, \\ D_{11} &\mapsto d_{10} + y_{10,1} - h_{10}, \\ C_{11} &\mapsto f_{10} + a_{10}, \\ z_{11} &\mapsto y_{10,2} + (a_{10} - d_{10} + f_{10} + h_{10})y_{10,1}, \end{aligned}$$

where h_{10} is the generator of $H^1(D-0)$.

The calculations have been left out, but are very similar to the previous ones.

Second column

The differential on the second column is the boundary map

$$\delta : H^q((Y_{\{1\}} \sqcup Y_{\{2\}}) \times (D, D-0)) \cong H^q(F_1, F_0) \to H^{q+1}(F_2, F_1).$$

To calculate this map, consider the inclusion of triples

$$Y_{\{1,2\}} \times (D_2, D_2 - 0, (D-0) \times (D-0)) \hookrightarrow (F_2, F_1, F_0)$$

given by inserting a small disc, sphere or torus on the zero-entries of $Y_{\{1,2\}}$. There are homeomorphisms of pairs,

$$\begin{aligned} \big(D_2 - 0, (D-0) \times (D-0)\big) &\cong \big(D \times (D-0) \sqcup (D-0) \times D, (D-0) \times (D-0)\big) \\ \big(D_2, D_2 - 0\big) &\cong \big(D \times D, D \times (D-0) \cup (D-0) \times D\big) = (D, D-0) \times (D, D-0). \end{aligned}$$

These gives a commuting diagram,

$$\begin{tikzcd} H^q(F_1, F_0) \arrow[r] \arrow[d] & H^{q+1}(F_2, F_1) \arrow[d, "\cong"] \\ H^q(Y_{\{1,2\}} \times (D_2 - 0, (D-0) \times (D-0))) \arrow[r] & H^{q+1}(Y_{\{1,2\}} \times (D_2, D_2 - 0)). \end{tikzcd}$$

The marked isomorphism follows from excision and the left map is induced by an inclusion. If we apply the homeomorphism of pairs to the bottom of the diagram, the boundary map at the bottom becomes the sum of the two boundary maps,

$$\begin{tikzcd}[column sep=small] H^q(Y_{\{1,2\}} \times (D, D-0) \times (D-0)) \arrow[dr, "\delta"] & \\ & H^{q+1}(Y_{\{1,2\}} \times (D, D-0) \times (D, D-0)). \\ H^q(Y_{\{1,2\}} \times (D-0) \times (D, D-0)) \arrow[ur, "\delta"'] & \end{tikzcd}$$

The boundary maps are given by the long exact sequence of the pair $(D, D-0)$, so we only need to consider the inclusion on the left. By choosing the discs sufficiently small and using the identification of $H^q(F_1, F_0)$ defined in the first section, we see that the inclusion maps the space $Y_{\{1,2\}} \times (D, D-0) \times (D-0)$ to $Y_{\{1\}} \times (D, D-0)$ and $Y_{\{1,2\}} \times (D-0) \times (D, D-0)$ to $Y_{\{2\}} \times (D, D-0)$. To get the differential, we must calculate the sum of these two maps:

$$\begin{aligned} i_1^* &: H^q(Y_{\{1\}} \times (D, D-0)) \to H^q(Y_{\{1,2\}} \times (D, D-0) \times (D-0)), \\ i_2^* &: H^q(Y_{\{2\}} \times (D, D-0)) \to H^q(Y_{\{1,2\}} \times (D-0) \times (D, D-0)). \end{aligned}$$

The inclusion $Y_{\{1,2\}} \times (D, D-0) \times (D-0) \to Y_{\{1\}} \times (D, D-0)$

In coordinates, the map becomes

$$\begin{aligned} (\mathbb{C}^*)^3 \times Y \times (D, D-0) \times (D-0) &\to (\mathbb{C}^*)^5 \times Y \times (D, D-0) \\ (a, d, i, b, c, g, h) &\mapsto \left(a, i, h, \frac{d}{h}, \frac{d}{h}(1-bc), \frac{bh}{d}, \frac{dc}{h}, g\right). \end{aligned}$$

Again, we calculate the map on cohomology by dualising to homology and working with the dual generators. To check what happens to the elements of $H^q(Y)$, we count intersections with the subspace $R_Y = \{(b, c) \in \mathbb{R}^2 \mid b > 0, c > 0, bc > 1\}$ of Y, exactly as for Z above.

$$\begin{aligned} i_1^* : H^q(Y_{\{1\}} \times (D, D-0)) &\to H^q(Y_{\{1,2\}} \times (D, D-0) \times (D-0)) \\ a_{01} &\mapsto a_{00}, \\ d_{01} &\mapsto d_{00} - h_{00} + y_{00,1}, \\ f_{01} &\mapsto d_{00} - h_{00}, \\ g_{01} &\mapsto g_{00}, \\ h_{01} &\mapsto h_{00}, \\ i_{01} &\mapsto i_{00}, \\ y_{01,1} &\mapsto y_{00,1}, \\ y_{01,2} &\mapsto y_{00,2} - d_{00}y_{00,1} + h_{00}y_{00,1}. \end{aligned}$$

We denote the generator of $H^2(D, D-0)$ in $H^*(Y_{\{1\}} \times (D, D-0))$ by g_{01} and for $H^*(Y_\emptyset \times (D, D-0) \times (D-0))$ we consider $g_{00} \in H^2(D, D-0)$ and $h_{00} \in H^1(D-0)$.

The inclusion $Y_{\{1,2\}} \times (D-0) \times (D, D-0) \to Y_{\{2\}} \times (D, D-0)$

This is exactly as above for the map

$$\begin{aligned}(\mathbb{C}^*)^3 \times Y \times (D-0) \times (D, D-0) &\to (\mathbb{C}^*)^5 \times Y \times (D, D-0)\\ (a, d, i, b, c, g, h) &\mapsto \left(\frac{a}{g}, d, i, g, 1-bc, b, c, h\right).\end{aligned}$$

In cohomology, this becomes

$$\begin{aligned} i_2^* : H^q(Y_{\{2\}} \times (D, D-0)) &\to H^q(Y_{\{1,2\}} \times (D-0) \times (D, D-0))\\ a_{10} &\mapsto a_{00} - g_{00},\\ d_{10} &\mapsto d_{00},\\ f_{10} &\mapsto y_{00,1},\\ g_{10} &\mapsto g_{00},\\ h_{10} &\mapsto h_{00},\\ i_{10} &\mapsto i_{00},\\ y_{10,1} &\mapsto y_{00,1},\\ y_{10,2} &\mapsto y_{00,2},\end{aligned}$$

with $g_{01} \in H^2(D, D-0)$ inside $H^*(Y_{\{2\}} \times (D, D-0))$ and $g_{00} \in H^2(D, D-0)$, $h_{00} \in H^1(D-0)$ inside $H^*(Y_\emptyset \times (D-0) \times (D, D-0))$.

Boundary maps

We have now reduced everything to calculating some well-known boundary maps for the pair $(D, D-0)$. By writing out the long exact sequence, the boundary map

$$\delta : H^*(D-0) \to H^*(D, D-0)$$

is an isomorphism in degree 1,

$$H^1(D-0) \cong H^2(D, D-0),$$

and zero in all other degrees. The boundary map for the pair $X \times (D, D-0)$ is $\mathrm{Id} \otimes \delta$ in tensor-notation. So the boundary map in the cases above picks out the "new" coordinate added to $X \times (D-0)$ and throws away everything that does not depend on it. To be more precise, the map is given on generators by:

$$\begin{aligned} H^*(X) \otimes H^*(D-0)) &\to H^*(X) \otimes H^*(D, D-0))\\ x \otimes d &\mapsto \begin{cases} x \otimes \delta d & \text{if } d \in H^1(D-0),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

3.5 Turning the page

We can now use the above to calculate the E_2-page. We have calculated the groups appearing on the first page and the differential. To get the second page, we need to compute the homology groups of d_1. This reduces to linear algebra over the integers, and is done using Sage. The result can be found in Figure 3.5.

$$
\begin{array}{c|ccc}
9 & \mathbb{Z} & 0 & 0 \\
8 & \mathbb{Z}^5 & 0 & 0 \\
7 & \mathbb{Z}^{11} & 0 & 0 \\
6 & \mathbb{Z}^{15} & \mathbb{Z} & 0 \\
5 & \mathbb{Z}^{16} & \mathbb{Z}^3 & 0 \\
4 & \mathbb{Z}^{16} & \mathbb{Z}^3 & 0 \\
3 & \mathbb{Z}^{15} & \mathbb{Z} & 0 \\
2 & \mathbb{Z}^{11} & 0 & 0 \\
1 & \mathbb{Z}^5 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 \\
\hline
{}^{q}\!/_{p} & 0 & 1 & 2
\end{array}
$$

Figure 3.5: The E_2-page.

The differentials from E_2 onwards are given as maps:

$$d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}.$$

For $r \geq 2$, this increases p by at least 2. This means that the domain or codomain of d_r will always be zero, so taking homology does not change anything. Hence $E_2 = E_\infty$, the spectral sequence collapses at the 2$^{\text{nd}}$ term, and there are no extension problems since all the groups are free and abelian. This allows us to read off the cohomology of $Y_{3,3}$. Since we are also interested in factoring out the T^3 torus, we note this as well.

$$
H^q(Y_{3,3}) = E_2^{0,q} \oplus E_2^{1,q-1} = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^5 & q = 1, \\ \mathbb{Z}^{11} & q = 2, \\ \mathbb{Z}^{15} & q = 3, \\ \mathbb{Z}^{17} & q = 4, \\ \mathbb{Z}^{19} & q = 5, \\ \mathbb{Z}^{18} & q = 6, \\ \mathbb{Z}^{12} & q = 7, \\ \mathbb{Z}^5 & q = 8, \\ \mathbb{Z} & q = 9, \\ 0 & \text{otherwise.} \end{cases}
\qquad
H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^2 & q = 1, \\ \mathbb{Z}^2 & q = 2, \\ \mathbb{Z}^2 & q = 3, \\ \mathbb{Z}^3 & q = 4, \\ \mathbb{Z}^2 & q = 5, \\ \mathbb{Z} & q = 6, \\ 0 & \text{otherwise.} \end{cases}
$$

4 Loops and limits

We now return to the general case. We want to show that the limit space $X_{m,\infty}$ is equivalent to the space of loops in the flag manifold $\mathrm{GL}_m(\mathbb{C})/\mathrm{B}_m$.

Definition 26. The loop space $\Omega = \Omega(\mathrm{GL}_m(\mathbb{C})/\mathrm{B}_m)$ is the space

$$\Omega = \{\gamma : [0,1] \to \mathrm{GL}_m(\mathbb{C})/\mathrm{B}_m \mid \gamma(0) = \gamma(1) = [\mathrm{Id}]\}$$

with the compact-open topology. This was briefly described in Section 1.2. Since $\mathrm{GL}_m(\mathbb{C})/\mathrm{B}_m$ is a Riemannian manifold, the loop space becomes a metric space where loops are close if they are close at every point.

4.1 The map $X_{m,n} \to \Omega$

As has been done before, we will start by considering $Y_{m,n}$ and then quotient out with the torus T^n afterwards. For $A = [a_1, \ldots, a_n] \in Y_{m,n}$ we can define a path γ_A in $\mathrm{GL}_m(\mathbb{C})$ by starting with $\gamma_A(0) = \mathrm{Id}$. Since the matrix $[e_2, \ldots, e_m, a_1]$ is invertible by the definition of $Y_{m,n}$, we know that a_1 is linearly independent of $e_2, \ldots, e_m$. Hence it can be written as

$$a_1 = \sum_{i=1}^{m} c_i e_i,$$

with $c_1 \neq 0$. Rewriting the expression leads to the formula

$$\frac{1}{c_1} a_1 = e_1 + \frac{1}{c_1}\left(\sum_{i=2}^{m} c_i e_i\right).$$

This gives us a path in $\mathrm{GL}_m(\mathbb{C})$ from the identity to $\left[\frac{1}{c_1}a_1, e_2, \ldots, e_m\right]$:

$$[0,1] \ni t \mapsto \left[e_1 + t\left(\sum_{i=2}^{m} \frac{c_i}{c_1} e_i\right), e_2, \ldots, e_m\right] \in \mathrm{GL}_m(\mathbb{C}).$$

This process can be continued, since we can rewrite the similar expression for a_2 as

$$\frac{1}{d_2} a_2 = e_2 + \frac{1}{d_2}\left(\sum_{i=3}^{m} d_i e_i\right) + \frac{d_1}{d_2 c_1} a_1,$$

and get a path

$$t \mapsto \left[\frac{1}{c_1} a_1, e_2 + t\left(\sum_{i=3}^{m} \frac{d_i}{d_2} e_i + \frac{d_1}{d_2 c_1} a_1\right), e_3, \ldots, e_m\right]$$

that starts where the other path stopped. If we continue through the vectors in A, we will eventually end at a point,

$$[\lambda_i e_i, \dots, \lambda_m e_m, \lambda_1 e_1, \dots, \lambda_{i-1} e_{i-1}] \in \mathrm{GL}_m(\mathbb{C}),$$

which is diagonal, except the columns may have been permuted if n is not divisible by m. Choosing a fixed path in $\mathrm{GL}_m(\mathbb{C})$ from $[e_i, \dots, e_{i-1}]$ to Id, scaling the columns and appending it to all the other paths then gives a path γ_A with

$$\gamma_A(0) = \mathrm{Id}, \qquad \gamma_A(1) = [\lambda_1 e_1, \dots, \lambda_m e_m].$$

If we consider this as a path in $\mathrm{GL}_m(\mathbb{C})/\,\mathrm{B}_m$ we get a loop, since $\gamma_A(1) \in \mathrm{B}_m$ is a diagonal matrix. The assignment

$$f(A) = \gamma_A \in \Omega(\mathrm{GL}_m(\mathbb{C})/\,\mathrm{B}_m)$$

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defines a continuous function from $Y_{m,n}$ to the loop space. Note that this function descends to the quotient space $X_{m,n} = Y_{m,n}/\,\mathrm{T}^n$, since elements of $Y_{m,n}$ that are equivalent under the T^n-action gives paths in $\mathrm{GL}_m(\mathbb{C})$ that are equivalent under the B_m-action.

The goal of this chapter is to prove the following theorem:

Theorem 27. *The map $f : X_{m,n} \to \Omega$ induces a map*

$$f : X_{m,\infty} \to \Omega.$$

This is a homotopy equivalence.

Since the space Ω is relatively well-studied, this gives information about $X_{m,\infty}$. In particular, it allows us to calculate the homology and cohomology of this space.

We want to show that the stabilisation map

$$s : X_{m,n} \to X_{m,n+1}$$
$$A \cdot \mathrm{T}^n = (a_1, \dots, a_n) \cdot \mathrm{T}^n \mapsto \left(La_1, \dots, La_n, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right) \cdot \mathrm{T}^n = (LA, Le_1) \cdot \mathrm{T}^n$$

does not affect the map f, or more precisely that f induces a well-defined map on $X_{m,\infty}$. To do this we will work with the subsequence $X_{m,nm}$ of the sequence $X_{m,n}$, as the two limits are isomorphic. To ease the notation, we will leave out the coset specifier $\cdot\,\mathrm{T}^n$ when discussing elements of $X_{m,n}$ for the remainder of this chapter.

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We will show that the diagram

$$\begin{tikzcd} X_{m,nm} \arrow[rr, "s^m"] \arrow[dr, "f"'] & & X_{m,(n+1)m} \arrow[dl, "f"] \\ & \Omega & \end{tikzcd}$$

commutes up to homotopy. This is done by showing that the map s^m is homotopic to the map $\widetilde{s}$ that inserts the identity on the last m columns of $A \in X_{m,nm}$:

$$\widetilde{s}(a_1, \dots, a_{nm}) = (a_1, \dots, a_{nm}, e_1, \dots, e_m).$$

If we connect Id and L^{-m} by a path γ in the space of invertible lower triangular matrices, we get a homotopy

$$s_t(A) = (\gamma(t)L^m A, \gamma(t)L^m e_1, \gamma(t)L^{m-1}e_1, \ldots, \gamma(t)Le_1)$$

which goes from $s_0 = s^m$ to the map

$$s_1 : A \mapsto (A, e_1, L^{-1}e_1, \ldots, L^{-m+1}e_1).$$

The matrix $T = [e_1, L^{-1}e_1, \ldots, L^{-m+1}e_1]$ is upper triangular with entries given by the formula:

$$T_{ij} = (L^{-j+1}e_1)_i = (-1)^{i-1}\binom{j-1}{i-1} \quad \text{when } i \leq j.$$

This is proven by considering negative powers of L and showing the identity:

$$(L^{-j+1})_{ik} = (-1)^{i-k}\binom{j-1}{i-k} \quad \text{when } i \geq k.$$

This is automatically true when j is one, so assume it is true for a given j. We will then show it for $j+1$. We are considering the entry

$$(L^{-(j+1)+1})_{ik} = (L^{-j})_{ik} = (L^{-1} \cdot L^{-j+1})_{ik} = \sum_{l=1}^{m} (L^{-1})_{il} \cdot (L^{-j+1})_{lk}.$$

But if we use that L^{-1} has entries

$$(L^{-1})_{il} = \begin{cases} -1 & l = i-1, \\ 1 & l = i, \\ 0 & \text{otherwise,} \end{cases}$$

this simplifies to

$$\begin{aligned} (L^{-(j+1)+1})_{ik} &= (-1)\cdot(-1)^{i-1-k}\binom{j-1}{i-1-k} + (-1)^{i-k}\binom{j-1}{i-k} \\ &= (-1)^{i-k}\cdot\left(\binom{j-1}{i-k-1} + \binom{j-1}{i-k}\right) \\ &= (-1)^{i-k}\binom{j}{i-k} \\ &= (-1)^{i-k}\binom{(j+1)-1}{i-k} \end{aligned}$$

as desired. Since we are considering only the first column, we get

$$T_{ij} = (L^{-j+1}e_1)_i = (L^{-j+1})_{i1} = (-1)^{i-1}\binom{j-1}{i-1}.$$

As T is an upper triangular matrix, it can be homotoped to the identity without affecting the linear independence of the columns, and the maps s^m and $\widetilde{s}$ are therefore homotopic. The composition $f \circ \widetilde{s}$ is the same as f, except that the loop $f \circ \widetilde{s}$ is stationary at the identity for a while before it terminates. Putting it all together shows the following:

Lemma 28. *The diagram*

$$\begin{tikzcd}
X_{m,nm} \arrow[rr, "s^m"] \arrow[dr, "f"'] & & X_{m,(n+1)m} \arrow[dl, "f"] \\
& \Omega &
\end{tikzcd}$$

commutes up to homotopy and f induces a map:

$$f : X_{m,\infty} \to \Omega.$$

4.2 Homotopy equivalence

To show that the map $f : X_{m,\infty} \to \Omega$ gives an isomorphism of homotopy groups, we will again follow [Mil63]. For each element U of $\mathrm{SU}_m \,/\, \mathrm{T}^{m-1}$, we can find a geodesically convex open ball centered on U. Since $\mathrm{SU}_m \,/\, \mathrm{T}^{m-1}$ is compact, there is some $\varepsilon > 0$ such that any ball of radius ε or less is geodesically convex. By the same argument, we can also choose ε so small that for any two points $A = [a_1, \dots, a_m]$ and $B = [b_1, \dots, b_m]$ in $\mathrm{SU}_m \,/\, \mathrm{T}^{m-1}$, with the distance from A to B less than ε, any m subsequent vectors in the sequence $(a_1, \dots, a_m, b_1, \dots, b_m)$ are linearly independent. With this, we get an increasing sequence of open sets in Ω, defined by requiring the loops to be piecewise contained in a ball of radius ε:

$$\Omega_k = \left\{ \gamma \in \Omega \;\middle|\; \gamma_{|\left[\frac{j-1}{2^k}, \frac{j}{2^k}\right]} \text{ is contained in an } \varepsilon\text{-ball } \mathrm{B}_\varepsilon \right\}.$$

By the choice of ε made above, we can define a map

$$\begin{aligned}
\varphi_k : \Omega_k &\to X_{m,(2^k-1)m} \\
\gamma &\mapsto \left[\gamma\left(\frac{1}{2^k}\right), \gamma\left(\frac{2}{2^k}\right), \dots, \gamma\left(\frac{2^k-1}{2^k}\right) \right].
\end{aligned}$$

Note that $\gamma(t)$ is an element of $\mathrm{SU}_m \,/\, \mathrm{T}^{m-1}$, so the notation $[\gamma(t), \dots]$ is shorthand for $[\gamma(t)e_1, \dots, \gamma(t)e_m, \dots]$. The map is continuous since two paths being close in Ω_k means that the maximum distance between them is small. But this translates to the vectors of the image being close together. The maps φ_k and φ_{k+1} are related in the following fashion:

Lemma 29. *The following diagram commutes up to homotopy:*

$$\begin{tikzcd}
\Omega_k \arrow[r, hook] \arrow[d, "\varphi_k"] & \Omega_{k+1} \arrow[d, "\varphi_{k+1}"] \\
X_{m,(2^k-1)m} \arrow[r, "s"] & X_{m,(2^{k+1}-1)m}.
\end{tikzcd}$$

Proof. The proof is by direct computation.

$$\begin{aligned}
\varphi_{k+1} &= \left(\gamma \mapsto \left[\gamma\left(\frac{1}{2^{k+1}}\right), \gamma\left(\frac{2}{2^{k+1}}\right), \dots, \gamma\left(\frac{2^{k+1}-1}{2^{k+1}}\right)\right]\right) \\
&\simeq \left(\gamma \mapsto \left[\gamma\left(\frac{2}{2^{k+1}}\right), \gamma\left(\frac{2}{2^{k+1}}\right), \gamma\left(\frac{4}{2^{k+1}}\right), \dots, \gamma\left(\frac{2^{k+1}}{2^{k+1}}\right)\right]\right) \\
&\simeq \left(\gamma \mapsto \left[\gamma\left(\frac{1}{2^{k}}\right), \gamma\left(\frac{2}{2^{k}}\right), \dots, \gamma\left(\frac{2^{k}-1}{2^{k}}\right), \mathrm{Id}, \dots, \mathrm{Id}\right]\right) \\
&= s \circ \varphi_k.
\end{aligned}$$

The first homotopy is given by sliding along the minimal geodesic between the points and gives an allowed path by our choice of ε. The second homotopy is somewhat similar. If $(a_1, \dots, a_m)$ and $(b_1, \dots, b_m)$ are two elements of $\mathrm{SU}_m \,/\, \mathrm{T}^{m-1}$ and we have an element of $X_{m,n}$ of the form

$$[\dots, a_1, \dots, a_m, a_1, \dots, a_m, b_1, \dots, b_m, \dots],$$

then we can change the middle columns one by one, as we did in the construction of f, and get a path in $X_{m,n}$ that changes the middle from $[a_1, \dots, a_m]$ to $[b_1, \dots, b_m]$:

$$\begin{aligned}
[a_1, \dots, a_m, a_1, a_2, \dots, a_m, b_1, \dots, b_m] &\rightsquigarrow [a_1, \dots, a_m, b_1, a_2, \dots, a_m, b_1, \dots, b_m] \\
&\rightsquigarrow [a_1, \dots, a_m, b_1, b_2, \dots, a_m, b_1, \dots, b_m] \\
&\rightsquigarrow [a_1, \dots, a_m, b_1, b_2, \dots, b_m, b_1, \dots, b_m].
\end{aligned}$$

This allows us to take all the repetitions $[\dots, \gamma(t), \gamma(t), \dots]$ and move them to the end as diagonal matrices, which results in the stabilisation map s. □

The above lemma shows that the maps φ_k fit together to define a map from Ω to $X_{m,\infty}$. The idea is to use the maps φ_k as homotopy inverses of the loop space map f. We are now ready to prove the result.

Theorem 30. *The map $f : X_{m,\infty} \to \Omega$ is a homotopy equivalence, i.e. the map*

$$f_* : \pi_i(X_{m,\infty}) \to \pi_i(\Omega)$$

is an isomorphism for all i.

Proof. We will prove that f is a weak homotopy equivalence. Since the loop space is a CW complex and $X_{m,\infty}$ is a limit of CW complexes, both are CW complexes by respectively Corollary 17.2 and the appendix of [Mil63].

We will start with surjectivity. Let

$$g : W \to \Omega$$

be a finite cell complex in Ω. We want to show that we can construct a map $\widetilde{g}$ from W to $X_{m,\infty}$ such that the composition $f \circ \widetilde{g}$ is homotopic to g.

The image of g is compact and hence contained in Ω_k for some k, since $\{\Omega_k\}_{k\in\mathbb{N}}$ is an open cover of Ω. Consider the map

$$\widetilde{g} = \varphi_k \circ g : W \to X_{m,(2^k-1)m}.$$

Evaluating at a point $w \in W$ gives

$$\widetilde{g}(w) = \varphi_k(\gamma_w) = \left[\gamma_w\left(\frac{1}{2^k}\right), \dots, \gamma_w\left(\frac{2^k-1}{2^k}\right)\right].$$

Evaluating f on this gives a path connecting the points $\gamma_w\left(\frac{j}{2^k}\right)$. But by our choice of ε, any such path is homotopic to the path connecting these points with minimal geodesics, so $f \circ \widetilde{g} \simeq g$.

To show injectivity, let

$$g : W \to X_{m,\infty}$$

be a finite cell complex with $f \circ g$ null-homotopic. We want to show that g is null-homotopic in $X_{m,\infty}$. In the direct limit, a compact set is contained in the image of $X_{m,n}$ for some n (see [Hat02, Proposition A.1] or [May99, Chapter 9.4]), so g is represented by a map:

$$g : W \to X_{m,n}.$$

We will work with this representative. The map $f \circ g$ is null-homotopic in Ω and the image is compact, so it is contained in Ω_k for some k. By applying φ_k, we get a function

$$\varphi_k \circ f \circ g : W \to X_{m,(2^k-1)m}.$$

Since $f \circ g$ is null-homotopic, this function is null-homotopic. But by the same argument that was used in Lemma 29, the map

$$\varphi_k(f(g(w))) = \left[\gamma_w\left(\frac{1}{2^k}\right), \dots, \gamma_w\left(\frac{2^k-1}{2^k}\right)\right]$$

is homotopic to the stabilisation map

$$s(g(w)) = [g(w), \mathrm{Id}, \dots, \mathrm{Id}],$$

with the homotopy given by sliding along the path $\gamma_w = f(g(w))$. So the map g can be stabilised to a null-homotopic map, so it must be null-homotopic in $X_{m,\infty}$. This shows that f_* is injective on homotopy groups. □

Since f is a homotopy equivalence, it follows that it induces an isomorphism on homology and cohomology groups. The Pontrjagin homology ring of the loop space, with product given by loop concatenation, has been calculated in [GT10, Theorem 4.1] as

$$H_*(\Omega(\mathrm{SU}_m / \mathrm{T}^{m-1})) \cong T(x_1, \dots, x_{m-1}) \otimes \mathbb{Z}[y_1, \dots, y_{m-1}]/I,$$

where $T(x_1, \dots, x_{m-1})$ is the tensor algebra generated by $x_1, \dots, x_{m-1}$. The product has the relation $x_k^2 = x_p x_q + x_q x_p = 2y_1$ for $1 \le k, p, q \le m-1$ and $p \ne q$. The degree of x_i is 1 and the degree of y_i is $2i$. By the above result, this is also the homology of $X_{m,\infty}$. The groups are finitely generated and torsion free, so by the Universal Coefficient Theorem the homology and cohomology groups are isomorphic as abelian groups. Note that the first cohomology group is the same as the first cohomology of $X_{m,n}$, as computed in Corollary 25.

4.3 Other coefficients

In the above, we did not use the structure of the complex numbers in any noticeable way. In particular, we could replace them by the reals or the quaternions without any change to the proofs. This immediately gives us two theorems, very similar to Theorem 30. These require computing Borel subgroups and maximal tori for various spaces, an overview of which can be found in e.g. [MT11, Example 6.7].

Theorem 31. *The space $X^{\mathbb{R}}_{m,\infty}$ is homotopy equivalent to the loop space of the real flag manifold, $\Omega(\mathrm{GL}_m(\mathbb{R})/\mathrm{B}_m) = \Omega(\mathrm{SO}_m/\mathrm{T}^k)$, where SO_m are the $m \times m$ special orthogonal matrices and T^{k} is a maximal torus in the special orthogonal group, where k is half of m rounded down.*

By [GT10], the Pontrjagin homology ring of this space depends on whether m is odd or even. For odd m, it is

$$H_*(\Omega(2k+1)/\mathrm{T}^k) \cong T(x_1,\dots,x_k) \otimes \mathbb{Z}[y_1,\dots,y_{k-1},2y_n,\dots,2y_{2k-1}]/I,$$

where x_i is degree 1, the degree of y_i and $2y_i$ is $2i$, and I is generated by the relations

$$\begin{aligned} x_1^2 - y_1, x_i^2 - x_{i+1}^2, &\quad 1 \le i \le k-1, \\ x_a x_b + x_b x_a, &\quad a \ne b, \\ y_i^2 - 2y_{i-1}y_{i+1} + \cdots \pm 2y_{2i}, &\quad 1 \le i \le k-1. \end{aligned}$$

For even m, the homology ring is

$$H_*(\Omega(2k)/\mathrm{T}^k) \cong T(x_1,\dots,x_k) \otimes \mathbb{Z}[y_1,\dots,y_{k-2},y_{k-1}+z,y_{k-1}-z,2y_k\dots,2y_{2k-2}]/I,$$

where x_i is degree 1 and anything involving a y_i is of degree $2i$. The relations are generated by

$$\begin{aligned} x_1^2 - y_1, x_i^2 - x_{i+1}^2, &\quad 1 \le i \le k-1, \\ x_a x_b + x_b x_a, &\quad a \ne b, \\ y_i^2 - 2y_{i-1}y_{i+1} + \cdots \pm 2y_{2i}, &\quad 1 \le i \le k-2 \\ (y_{n-1}+z)(y_{n-1}-z) - 2y_{n-2}y_n + \cdots \pm 2y_{2n-2}. & \end{aligned}$$

Theorem 32. *The space $X^{\mathbb{H}}_{m,\infty}$ is homotopy equivalent to the loop space of the quaternionic flag manifold $\Omega(\mathrm{GL}_m(\mathbb{H})/\mathrm{B}_m) = \Omega(\mathrm{Sp}_m/\mathrm{T}^m)$. Here, Sp_m is the symplectic group and T^m is a maximal torus.*

The homology ring is

$$H_*(\Omega(\mathrm{Sp}_m/\mathrm{T}^m)) \cong T(x_1,\dots,x_m) \otimes \mathbb{Z}[y_2,\dots y_m]/I,$$

with x_i of degree 1 and y_i of degree $4i-2$. The relations are

$$\begin{aligned} x_i^2 &= x_j^2, \\ x_a x_b &= -x_b x_a, \quad a \ne b. \end{aligned}$$

This then also gives the homology of $X^{\mathbb{R}}_{m,\infty}$ and $X^{\mathbb{H}}_{m,\infty}$.

4.4 Other stabilisations

Now that we know the space $X_{m,\infty}$, we can consider different ways of reaching this space from the spaces $X_{m,n}(\sigma)$. The first thing to note is that the space $X_{m,\infty}$, so far described as sequences of linearly independent vectors $(v_1, v_2, \ldots, v_k)$ in $\mathbb{C}^m$ of varying length, may as well be described as infinite sequences $(v_i)_{i=1}^{\infty}$ for which there is some k such that the matrix $(v_{k+r*m+1}, v_{k+r*m+2}, \ldots, v_{k+(r+1)*m})$ is the identity matrix for all integers $r \geq 0$, with the same requirements for linear independence. We could also consider sequences that are infinite in the other direction, or doubly infinite sequences that start and end with the identity matrix and get the same space. Likewise, $X_{m,\infty}(\sigma)$ could be defined as doubly infinite series that start with the identity matrix and end with infinite copies of σ. The homeomorphisms between these spaces are the obvious ones of inclusion or cutting off the tail or start of an infinite sequence. As we saw in the proof of Lemma 28, the two inclusions of $X_{m,n}$ in $X_{m,\infty}$ by either using the stabilisation map s or by adding on the identity matrix are homotopic, so this does not change any of the above results. To differentiate the two ways of stabilising, we will refer to the inclusion by adjoining an infinite sequence of identity matrices on the left as

$$L_n : X_{m,n}(\sigma) \to X_{m,\infty}(\sigma).$$

This map is important since it respects the filtration defined in Section 3.1,

$$F_0 \subset F_1 \subset \cdots \subset X_{m,n}(\sigma),$$

and hence defines a map between the spectral sequences of $X_{m,n}(\sigma)$ and $X_{m,\infty}(\sigma)$. These spectral sequences have not played a role so far, but are defined exactly as for $X_{m,n}(\mathrm{Id})$.

We will be using L_n to prove the following theorem.

Theorem 33. *The spectral sequence for $X_{3,\infty}(\sigma)$ collapses on the second page, with all cohomology concentrated in the first column.*

At first this theorem may seem slightly strange, but we know that the spaces $X_{3,\infty}(\sigma)$ are homotopy equivalent for all σ, since they are all equivalent to the loop space $\Omega(\mathrm{SU}_3 \,/\, \mathrm{T}^2)$, so the theorem just tells us that all the cohomology of $X_{3,\infty}(\sigma)$ can be found in the $X_{3,\infty}(\sigma_\emptyset)$ part of $F_0 \cong X_{3,\infty}(\sigma_\emptyset) \times T^2$.

Proof. We will show that the spectral sequence for $X_{3,\infty}(\sigma)$ is determined entirely by the torus-part. To see this, note that since all the spaces $X_{3,\infty}(\sigma)$ are equivalent, they all have the same cohomology ring R. The first page of the spectral sequence has the form

$$R \otimes H^*(\mathrm{T}^2) \to (R \oplus R) \otimes H^*(T \times (D, D-0)) \to R \otimes H^*(D_2, D_2 - 0).$$

The boundary map d_1 is an inclusion of spaces followed by the boundary map for the pair $(D, D-0)$,

$$d_1 = \delta \circ \imath^*,$$

and we would like to show that the inclusion map, $\imath^*$ essentially does nothing. We know that for cohomology classes r and t, we have

$$\imath^*(r \otimes t) = \imath^*((r \otimes 1) \cdot (1 \otimes t)) = \imath^*(r \otimes 1)\imath^*(1 \otimes t),$$

so we need to know that happens in both these cases. Instead of reducing all the way to what would correspond to $X_{3,n-1}(\tau)$, as we have often done in the finite-dimensional case, it is in this case simpler to only use the torus action to change all of the non-zero entries in the last column to one and avoid the multiplication with the lower-triangular matrix. The inclusion can be illustrated by considering a concrete example. For the case

$$\imath : X_{3,\infty}(\mathrm{Id})_{\{1\}} \times (D-0) \to X_{3,\infty}(\mathrm{Id})_{\emptyset},$$

we get the map

$$\imath\left(\begin{pmatrix} — r_1 — & 0 \\ — r_2 — & 1 \\ — r_3 — & 1 \end{pmatrix}, t_2, \varepsilon\right) = \left(\begin{pmatrix} — \frac{1}{\varepsilon} r_1 — & 1 \\ — r_2 — & 1 \\ — r_3 — & 1 \end{pmatrix}, t_2, \varepsilon\right).$$

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When ε is a fixed, small real number, we get a homotopy commutative diagram

$$\begin{tikzcd} X_{3,\infty}(\mathrm{Id})_{\{1\}} \arrow[r, hook] \arrow[d, "\imath"] & X_{3,\infty}(\mathrm{Id}) \\ X_{3,\infty}(\mathrm{Id})_{\emptyset} \arrow[ur, hook] & \end{tikzcd},$$

where the homotopy is given by first multiplying the top row with a number moving from one to ε and then moving the last coordinate of the top row to zero, illustrated below:

$$\begin{aligned} \begin{pmatrix} — \frac{1}{\varepsilon} r_1 — & 1 \\ — r_2 — & 1 \\ — r_3 — & 1 \end{pmatrix} &\rightsquigarrow \begin{pmatrix} — r_1 — & \varepsilon \\ — r_2 — & 1 \\ — r_3 — & 1 \end{pmatrix} \\ &\rightsquigarrow \begin{pmatrix} — r_1 — & 0 \\ — r_2 — & 1 \\ — r_3 — & 1 \end{pmatrix}. \end{aligned}$$

But the inclusions into $X_{3,\infty}(\mathrm{Id})$ are homotopy equivalences, since they correspond to considering paths with a different endpoint in the flag manifold and we know these are all equivalent. Hence we must have $\imath$ a homotopy equivalence as well, and so it is an isomorphism on cohomology groups:

$$\imath^*(r \otimes 1) = r \otimes 1.$$

Likewise, we can see that if we choose our matrix to be e.g.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

the inclusion does not change the rows of A since we have quotiented out the torus action on the columns. Hence we conclude that we preserve the other part of the cohomology as well. All in all, we get

$$d_1(r \otimes t) = r \otimes \delta(t).$$

But we know the map δ, and hence we can compute the map on the first column as

$$\begin{aligned} d_1(r \otimes t_1 t_2) &= (r \otimes t_2, -r \otimes t_1) \otimes [D, D-0] \\ d_1(r \otimes t_1) &= (0, -r \otimes t_1) \otimes [D, D-0] \\ d_1(r \otimes t_2) &= (r \otimes t_2, 0) \otimes [D, D-0] \\ d_1(r \otimes 1) &= 0. \end{aligned}$$

By doing the exact same thing for the second column, only including the non-zero maps and leaving out the class $[D, D-0]$ to ease on notation, we see

$$\begin{aligned} d_1(r_1 \otimes t_2, r_2 \otimes t_1) &= (r_1 + r_2) \otimes [D_2, D_2 - 0] \\ d_1(r_1 \otimes t_2, 0) &= 0 \\ d_1(0, r_2 \otimes t_1) &= 0 \end{aligned}$$

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But by using this we can compute the second page and see that it is concentrated as $R \otimes 1$ in the first column, as desired. $\square$

4.5 Cohomological stability

Since we now know the space $X_{m,\infty}(\sigma)$ and its cohomology, it would be nice to relate this to the cohomology of $X_{m,n}(\sigma)$, for finite n. We once again specialise to the case $m = 3$ in everything that follows. First, we need an offset that depends on the permutation we are considering:

Definition 34. Define a function λ on the symmetric group of three elements by

$$\lambda(\sigma) = \begin{cases} 0, & \sigma = \mathrm{Id}, \\ 1, & \sigma \in \{(1\,2), (2\,3), (1\,2\,3)\}, \\ 2, & \sigma \in \{(1\,3\,2), (1\,3)\}. \end{cases}$$

We will need to distinguish the spectral sequences for various permutations. We will refer to the groups of the spectral sequence to compute the cohomology of $X_{3,n}(\sigma)$ as $E_r^{p,q}(n, \sigma)$. We will also use the notation $\sigma_\emptyset$ for the permutation that shows up in the column $p = 0$ when using the spectral sequence to compute cohomology, corresponding to

$$X_{3,n}(\sigma)_\emptyset \cong X_{3,n-1}(\sigma_\emptyset) \times \mathrm{T},$$

and likewise for $\sigma_{\{1\}}, \sigma_{\{2\}}$ and $\sigma_{\{1,2\}}$.

This section will prove the following theorem:

Theorem 35. *If $n \geq k + \lambda(\sigma)$, the stabilisation map*

$$L_n^* : H^k(X_{3,\infty}(\sigma)) \to H^k(X_{3,n}(\sigma))$$

is an isomorphism.

Proof. The proof will proceed by induction on k. Our induction hypothesis is divided into two parts, were we require the stabilisation to be an isomorphism when n is greater than $k + \lambda(\sigma)$, and injective when n is $k + \lambda(\sigma) - 1$. The exact statement we want to prove is the following:

Hypothesis (I_{k-1})**.** For all $r \leq k-1$ and all $\sigma \in S_3$, the following two statements hold:

- The map $L_n^* : H^r(X_{3,\infty}(\sigma)) \to H^r(X_{3,n}(\sigma))$ is an isomorphism for $n \geq r + \lambda(\sigma)$.
- The map $L_n^* : H^r(X_{3,\infty}(\sigma)) \to H^r(X_{3,n}(\sigma))$ is injective for $n \geq r + \lambda(\sigma) - 1$.

By direct calculation, the groups $H^0(X_{3,n}(\sigma))$ are all $\mathbb{Z}$ when $n \geq \lambda(\sigma)$, so the isomorphism in I_0 is automatically true.

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We now assume the hypothesis I_{k-1} and want to prove I_k. Before we start, we consider the spectral sequence for $X_{3,n+1}(\sigma)$ and observe how the different permutations that show up relate to each other. This is summarised in the following table:

σ	$\sigma_\emptyset$	$\sigma_{\{1\}}$	$\sigma_{\{2\}}$	$\sigma_{\{1,2\}}$
Id	Id	(1 2)	(2 3)	(1 3 2)
(1 2)	Id	(1 2 3)	Id	(1 3)
(2 3)	Id	(1 2)	Id	(1 2)
(1 2 3)	Id	Id	Id	Id
(1 3 2)	(2 3)	(1 2 3)	(2 3)	(1 2 3)
(1 3)	(2 3)	(2 3)	(2 3)	(2 3)

By considering the table, we can see the following inequalities for the function λ:

$$\begin{aligned}
&\lambda(\sigma) \geq \lambda(\sigma_\emptyset),\\
&\lambda(\sigma) \geq \lambda(\sigma_{\{1\}}) - 1,\\
&\lambda(\sigma) \geq \lambda(\sigma_{\{2\}}) - 1,\\
&\lambda(\sigma) \geq \lambda(\sigma_{\{1,2\}}) - 2.
\end{aligned}$$

These will be used during the proof.

Surjectivity of s^*

Let σ be a permutation in S_3 and let $n \geq k + \lambda(\sigma)$. We start by showing that the map

$$s^* : H^k(X_{3,n+1}(\sigma)) \to H^k(X_{3,n}(\sigma_\emptyset))$$

is surjective in this case. To see this, we will look at the spectral sequences for $X_{3,\infty}(\sigma)$ and $X_{3,n+1}(\sigma)$, and the maps between them induced by the stabilisation map L_{n+1}^*. This is summarised in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H^k(X_{3,\infty}(\sigma)) \arrow[r, "i^*"] \arrow[d, "L_{n+1}^*"] & E_1^{0,k}(\infty,\sigma) \arrow[r, "d_1"] \arrow[d, "A_0"] & E_1^{1,k}(\infty,\sigma) \arrow[r, "d_1"] \arrow[d, "A_1"] & E_1^{2,k}(\infty,\sigma) \arrow[d, "A_2"] \\
0 \arrow[r] & H^k(X_{3,n+1}(\sigma)) \arrow[r, "i^*"] & E_1^{0,k}(n+1,\sigma) \arrow[r, "d_1"] & E_1^{1,k}(n+1,\sigma) \arrow[r, "d_1"] & E_1^{2,k}(n+1,\sigma).
\end{tikzcd}$$

The upper row is exact, since it is part of the spectral sequence for $X_{3,\infty}(\sigma)$ and this collapses at the second page by Theorem 33. The bottom row is a chain complex, by considering the spectral sequence, and is exact at $E_1^{0,k}(n+1,\sigma)$. To see this, consider the induction hypothesis applied to the spectral sequence for $X_{3,n+1}(\sigma)$, illustrated in Figure 4.1. Since we choose to stabilise from the left, as discussed in Section 4.4

above, L_n becomes a map of spectral sequences and respects the stabilisation map s. We use the notation $\boxed{E_1^{p,q}}$ to show that the map L_n^* is an isomorphism on $E_1^{p,q}$ according to I_{k-1}.

k	$\boxed{H^{k-2}(X_{3,n}(\sigma_\emptyset)) \otimes t_1t_2}$ $\boxed{H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes t_1}$ $\boxed{H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes t_2}$ $H^{k}(X_{3,n}(\sigma_\emptyset)) \otimes 1$	$\boxed{H^{k-2}(X_{3,n}(\sigma_{\{1\}})) \otimes t_2}$ $\boxed{H^{k-2}(X_{3,n}(\sigma_{\{2\}})) \otimes t_1}$ $\boxed{H^{k-1}(X_{3,n}(\sigma_{\{1\}})) \otimes 1}$ $\boxed{H^{k-1}(X_{3,n}(\sigma_{\{2\}})) \otimes 1}$	$\boxed{H^{k-2}(X_{3,n}(\sigma_{\{1,2\}}))}$
$k-1$	$\boxed{H^{k-3}(X_{3,n}(\sigma_\emptyset)) \otimes t_1t_2}$ $\boxed{H^{k-2}(X_{3,n}(\sigma_\emptyset)) \otimes t_1}$ $\boxed{H^{k-2}(X_{3,n}(\sigma_\emptyset)) \otimes t_2}$ $\boxed{H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes 1}$	$\boxed{H^{k-3}(X_{3,n}(\sigma_{\{1\}})) \otimes t_2}$ $\boxed{H^{k-3}(X_{3,n}(\sigma_{\{2\}})) \otimes t_1}$ $\boxed{H^{k-2}(X_{3,n}(\sigma_{\{1\}})) \otimes 1}$ $\boxed{H^{k-2}(X_{3,n}(\sigma_{\{2\}})) \otimes 1}$	$\boxed{H^{k-2}(X_{3,n}(\sigma_{\{1,2\}}))}$
	$\dots$	$\dots$	$\dots$
0	$\boxed{E_1^{0,0}}$	$\boxed{E_1^{1,0}}$	$\boxed{E_1^{2,0}}$

Figure 4.1: The E_1 page when we assume I_{k-1} and $n \geq k + \lambda(\sigma)$.

If we pass to the second page of the spectral sequence, we can see that at row k the groups will no longer change. This is because everything except $H^k(X_{3,n}(\sigma_\emptyset)) \otimes 1$ is isomorphic to the $X_{3,\infty}(\sigma)$-case, where the columns $p = 1$ and $p = 2$ disappears on the second page, according to Theorem 33. We then know

$$H^k(X_{3,n+1}(\sigma)) = \ker d_1.$$

The map A_1 in the diagram is an isomorphism, since it is induced from the maps

$$L_n^* : H^{k-1}(X_{3,\infty}(\sigma_{\{i\}})) \to H^{k-1}(X_{3,n}(\sigma_{\{i\}})).$$

These are isomorphisms by applying the first part of I_{k-1}, using the assumption

$$n \geq k + \lambda(\sigma) \geq (k-1) + \lambda(\sigma_{\{i\}}).$$

The map A_2 is an isomorphism by the exact same argument. This allows us to do a diagram chase: Consider an element x in $E_1^{0,k}(n+1,\sigma)$. Then $d_1(x) = A_1(y)$ for a y in $E_1^{1,k}(\infty,\sigma)$. But

$$A_2(d_1(y)) = d_1(A_1(y)) = d_1(d_1(x)) = 0,$$

so $y = d_1(z)$ for some z in $E_1^{0,k}(\infty,\sigma)$. The element $x - A_0(z)$ is in the kernel of d_1,

$$d_1(x - A_0(z)) = d_1(x) - A_1(d_1(z)) = d_1(x) - A_1(y) = 0.$$

So there is a w in $H^k(X_{3,n+1}(\sigma))$ with $i^*(w)$ equal to $x - A_0(z)$. So we have found w and z with

$$A_0(z) + i^*(w) = A_0(z) + x - A_0(z) = x,$$

showing that

$$H^k(X_{3,n+1}(\sigma)) \oplus E_1^{0,k}(\infty,\sigma) \xrightarrow{i^*+A_0} E_1^{0,k}(n+1,\sigma)$$

is surjective. The map s^* that we are considering is the composition

$$H^k(X_{3,n+1}(\sigma)) \xrightarrow{i^*} E_1^{0,k}(n+1,\sigma) = H^k(X_{3,n}(\sigma_\emptyset \times T^2)) \xrightarrow{\mathrm{pr}} H^k(X_{3,n}(\sigma_\emptyset)).$$

The projection, which is induced from the inclusion of $X_{3,n}(\sigma_\emptyset)$ into the product, is surjective, so we only need to show that the image of $E_1^{0,k}(\infty,\sigma)$ under the map $\mathrm{pr} \circ A_0$ is contained in the image of s^* to finish the proof of surjectivity of s^*. Consider the diagram

$$\begin{tikzcd} E_1^{0,k}(\infty,\sigma) \arrow[r, "A_0"] \arrow[d, "\mathrm{pr}"] & E_1^{0,k}(n+1,\sigma) \arrow[d, "\mathrm{pr}"] \\ H^k(X_{3,\infty}(\sigma_\emptyset)) \arrow[r, "L_n^*"] & H^k(X_{3,n}(\sigma_\emptyset)). \end{tikzcd}$$

This shows that $\mathrm{pr} \circ A_0$ is contained in the image of L_n^*. But the stabilisation map L_n factors up to homotopy over s, so we get that

$$\mathrm{im}(\mathrm{pr} \circ A_0) \subset \mathrm{im}\, L_n^* \subset \mathrm{im}\, s^*.$$

By applying the surjectivity of $\mathrm{pr} \circ (i^* + A_0)$ that was just shown, we see that

$$s^* : H^k(X_{3,n+1}(\sigma)) \to H^k(X_{3,n}(\sigma_\emptyset))$$

is surjective.

Injectivity

We will now prove injectivity of L_n^*, so assume that $n \geq k + \lambda(\sigma) - 1$. As above, we will prove that the stabilisation map

$$s^* : H^k(X_{3,n+1}(\sigma)) \to H^k(X_{3,n}(\sigma_\emptyset))$$

is injective. Consider the spectral sequence for computing $H^k(X_{3,n+1}(\sigma))$. Figure 4.2 is an image of the first page of the spectral sequence, with $E_1^{p,q}$ denoting that L_n^* is an isomorphism on the group and $E_1^{p,q}$ denoting that it is injective.

k	$H^{k-2}(X_{3,n}(\sigma_\emptyset)) \otimes t_1t_2$ $H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes t_1$ $H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes t_2$ $H^{k}(X_{3,n}(\sigma_\emptyset)) \otimes 1$	$H^{k-2}(X_{3,n}(\sigma_{\{1\}})) \otimes t_2$ $H^{k-2}(X_{3,n}(\sigma_{\{2\}})) \otimes t_1$ $H^{k-1}(X_{3,n}(\sigma_{\{1\}})) \otimes 1$ $H^{k-1}(X_{3,n}(\sigma_{\{2\}})) \otimes 1$	$H^{k-2}(X_{3,n}(\sigma_{\{1,2\}}))$
$k-1$	$H^{k-3}(X_{3,n}(\sigma_\emptyset)) \otimes t_1t_2$ $H^{k-2}(X_{3,n}(\sigma_\emptyset)) \otimes t_1$ $H^{k-2}(X_{3,n}(\sigma_\emptyset)) \otimes t_2$ $H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes 1$	$H^{k-3}(X_{3,n}(\sigma_{\{1\}})) \otimes t_2$ $H^{k-3}(X_{3,n}(\sigma_{\{2\}})) \otimes t_1$ $H^{k-2}(X_{3,n}(\sigma_{\{1\}})) \otimes 1$ $H^{k-2}(X_{3,n}(\sigma_{\{2\}})) \otimes 1$	$H^{k-2}(X_{3,n}(\sigma_{\{1,2\}}))$
	$\dots$	$\dots$	$\dots$
0	$E_1^{0,0}$	$E_1^{1,0}$	$E_1^{2,0}$

Figure 4.2: The E_1 page when we assume I_{k-1} and $n \geq k + \lambda(\sigma) - 1$.

If the map

$$s^* : H^k(X_{3,n+1}(\sigma)) \to H^k(X_{3,n}(\sigma_\emptyset))$$

is not injective, there is an element α of the kernel of the boundary map that is also in the group

$$H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes t_1 \oplus H^{k-1}(X_{3,n}(\sigma_\emptyset)) \otimes t_2 \oplus H^{k-2}(X_{3,n}(\sigma_\emptyset)) \otimes t_1t_2.$$

But on these last groups, the stabilisation map is an isomorphism so we get an element $\widehat{\alpha}$ in the corresponding group in the spectral sequence for $X_{3,\infty}(\sigma)$. Since the stabilisation maps are maps of spectral sequences, we have

$$0 = d_1(L_n(\widehat{\alpha})) = L_n(d_1(\widehat{\alpha}))$$

But the right hand side is the composition of L_n, which we know by the induction hypothesis to be injective, and the boundary map in the spectral sequence for the infinite spaces, which we know to be injective on the groups we are considering by Theorem 33. So $\widehat{\alpha}$ must be zero, and hence so is α. This shows that s^* is injective when $n \geq k + \lambda(\sigma) - 1$. Now we consider the different choices of σ.

$\sigma = \mathrm{Id}$

Since $\sigma_\emptyset$ is Id, and all the maps

$$s^* : H^k(X_{3,j+1}) \to H^k(X_{3,j}), \quad j \geq n$$

are surjective, we know that the cohomology of $X_{3,\infty}$ is the inverse limit of the cohomology groups of $X_{3,n}$ using the stabilisation map, see e.g. [May99, Chapter 19.4]. When considered in this way, the map L_n^* is just the projection on the corresponding factor. But if we project and get zero, we know that the only possible lift to the next group is zero, since s^* is injective. Continuing like this, we see that the only element of $H^k(X_{3,\infty})$ that can project to zero is the zero element, and hence L_n^* is injective in this case.

$\lambda(\sigma) \geq 1$

If $\lambda(\sigma) = 1$, we know $\sigma \in \{(1\ 2), (2\ 3), (1\ 2\ 3)\}$, $\sigma_\emptyset = \mathrm{Id}$ and that $n \geq k + \lambda(\sigma) - 1 = k$, giving $n - 1 \geq k - 1$. Consider the following commutative diagram:

$$\begin{tikzcd}
H^k(X_{3,\infty}(\sigma)) \arrow[r, "s^*", "\cong"'] \arrow[d, "L_n^*"] & H^k(X_{3,\infty}) \arrow[d, "L_{n-1}^*"] \\
H^k(X_{3,n}(\sigma)) \arrow[r, "s^*"] & H^k(X_{3,n-1}).
\end{tikzcd}$$

By the previous case, the map on the right is injective and so the composition $s^* \circ L_{n-1}^*$ is as well. Since the map commutes, this means that the left vertical map, L_n^* must be injective as well, as desired.

The last case has $\sigma_\emptyset = (2\ 3)$ and proceeds exactly as the case $\lambda(\sigma) = 1$, using the injectivity that has already been proved for this permutation and that $\lambda((2\ 3))$ is $\lambda(\sigma) - 1$.

Surjectivity

We finish by proving that L_n^* is surjective, so we assume $n \geq k + \lambda(\sigma)$. We have already worked with the stabilisation map s^* and we now split into cases based on σ.

$\sigma = \mathrm{Id}$

We still know that the cohomology group of $X_{3,\infty}$ is the inverse limit of the cohomology groups of $X_{3,n}$ and that the map L_n^* is just the projection. But since we know all the maps

$$s^* : H^k(X_{3,j+1}) \to H^k(X_{3,j}), \quad j \geq n$$

are surjective, we can hit anything in $H^k(X_{3,n})$ by choosing a lift in each $H^k(X_{3,j})$ for $j \geq n$ and use these to get an element of the inverse limit. Hence L_n^* is surjective in this case.

$\lambda(\sigma) \geq 1$

For the other permutations, define groups $C_n(\sigma)$ as the cokernel of the map L_n^*:

$$0 \longrightarrow H^k(X_{3,\infty}(\sigma)) \xrightarrow{L_n^*} H^k(X_{3,n}(\sigma)) \longrightarrow C_n(\sigma) \longrightarrow 0.$$

The maps

$$s^* : H^k(X_{3,n+1}(\sigma)) \to H^k(X_{3,n}(\sigma_\emptyset))$$

induces maps between these cokernels and gives a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & H^k(X_{3,\infty}(\sigma)) \arrow[r, "L_n^*"] \arrow[d, "s^*"] & H^k(X_{3,n}(\sigma)) \arrow[r] \arrow[d, "s^*"] & C_n(\sigma) \arrow[r] \arrow[d, "s^*"] & 0 \\
0 \arrow[r] & H^k(X_{3,n}(\sigma_\emptyset)) \arrow[r, "L_{n-1}^*"] & H^k(X_{3,n-1}(\sigma_\emptyset)) \arrow[r] & C_{n-1}(\sigma_\emptyset) \arrow[r] & 0.
\end{tikzcd}$$

If $\lambda(\sigma) = 1$, we know that $\sigma_\emptyset = \mathrm{Id}$, and if we combine this with the inequality

$$n \geq k + \lambda(\sigma) = k + 1 \implies n - 1 \geq k,$$

we conclude $C_{n-1}(\mathrm{Id}) = 0$ by the previous case. The maps not between cokernels are isomorphisms, since we have already proved injectivity and surjectivity of s^*, in the infinity case in Theorem 33 and earlier in this proof in the case of $n < \infty$. Hence the map

$$s^* : C_n(\sigma) \to C_{n-1}(\sigma_\emptyset) = 0$$

is an isomorphism by the 5-lemma, and so L_n^* is surjective.

In the final cases, $\lambda(\sigma) = 2$, we apply the above argument, using

$$\sigma_\emptyset = (2\ 3), \quad \lambda(\sigma_\emptyset) = 1.$$

Since

$$n \geq k + \lambda(\sigma) = k + 2 \implies n - 1 \geq k + 1 = k + \lambda((2\ 3)),$$

we conclude $C_{n-1}((2\ 3)) = 0$ by the previous case, and the same arguments as last time give us $C_n(\sigma) = 0$ and L_n^* surjective.

This shows that I_k follows from I_{k-1}, and hence finishes the proof of the theorem. □

We will end this chapter with a quick application of this theorem to $X_{3,3}$, since this is a group we have previously calculated.

Example 36. Recall that

$$H^q(X_{3,3}) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^2 & q = 1, \\ \mathbb{Z}^2 & q = 2, \\ \mathbb{Z}^2 & q = 3, \\ \mathbb{Z}^3 & q = 4, \\ \mathbb{Z}^2 & q = 5, \\ \mathbb{Z} & q = 6, \\ 0 & \text{otherwise.} \end{cases}$$

If we consider the above theorem, we have $n = 3$ and $\lambda(\mathrm{Id}) = 0$, giving

$$3 = n \geq k + \lambda(\sigma) = k.$$

So the theorem tells us the first three cohomology groups of $X_{3,3}$ are isomorphic to the corresponding ones for $\Omega(\mathrm{SU}_3 / \mathrm{T}^2)$. This fits with what is expected if we use the groups computed in [GT10],

$$H_*(\Omega(\mathrm{SU}_3 / \mathrm{T}^2)) \cong T(x_1, x_2) \otimes \mathbb{Z}[y_1, y_2]/I.$$

In degree one we get the elements x_1 and x_2, in degree two we get y_1 and x_1x_2 and in degree three we get y_1x_1 and y_1x_2 as generators in the homology of the loop space, which correspond to the right dimensions of the cohomology groups by the universal coefficient theorem. Since the theorem also works for $n > 3$, we conclude that for $q \in \{1, 2, 3\}$ and $n \geq 3$, we have

$$H^q(X_{3,n}) \cong \mathbb{Z}^2.$$

5 Conjectures and ideas

I will end with a brief discussion of possible future work that could be done to study the spaces $X_{m,n}(\sigma)$ and their relation to the loop space.

5.1 Stabilisation

An obvious first place to start would be to prove Theorem 35 for m greater than 3, giving something like the following.

Conjecture. *For each m, there is a function λ from the symmetric group S_m to the integers, such that the stabilisation map*

$$L_n^* : H^k(X_{m,\infty}(\sigma)) \to H^k(X_{m,n}(\sigma))$$

is an isomorphism for $n \geq k + \lambda(\sigma)$.

The exact definition of λ is probably one of the hard parts of this conjecture. It will need to satisfy something similar to the inequalities stated in the previous chapter, which would probably look like

$$\begin{aligned}
\lambda(\sigma) &\geq \lambda(\sigma_\emptyset), \\
\lambda(\sigma) &\geq \lambda(\sigma_{\{1\}}) - 1, \\
&\vdots \\
\lambda(\sigma) &\geq \lambda(\sigma_{\{m\}}) - 1, \\
&\vdots \\
\lambda(\sigma) &\geq \lambda(\sigma_{\{1,\dots,m\}}) - m + 1.
\end{aligned}$$

By considering the function defined on S_3 in the previous chapter, it seems like the definition should depend on $\sigma(1)$, but it may also be related to the Bruhat ordering on S_m, which means it depends on the minimum number of simple transpositions required to write σ as a product.

FiXme Fatal: Overvej dette igen. Og burde nok kun være 1,...,m-1 i indekset, men kan ikke lige få det formuleret på en ordentlig måde

The diagram obtained by considering L_n^* as a map of spectral sequences,

$$L_n^* : E_*^{*,*}(\infty, \sigma) \to E_*^{*,*}(n+1, \sigma),$$

also played a role in the proof. The conjecture above will probably also require a study of this diagram in more generality than was considered here. There is a risk that the spectral sequence for $X_{m,\infty}(\sigma)$ does not collapse as readily for high m as it does for $m = 3$, in particular because increasing m increases the number of non-zero columns on the first page of the spectral sequence, $E_1^{*,*}(\infty,\sigma)$. Likewise, the maps induced from L_n^* will also have to be well-behaved on the different columns of the spectral sequence if the proof is to carry over without too many changes. This, in particular, is where the choice of λ plays a crucial role.

Inverse stabilisation

During the study of these spaces, computer programs were written to simplify some computations. By continuing to work with these, and applying some conjectures about the structure of the spectral sequence, it seems like there is a relation between the cohomology of $X_{3,n}$ and the cohomology of the double suspension $\Sigma^2(X_{3,n-1})$, when the degree of the cohomology group is larger than n. With some work, it might be possible to define a space W_3 that encodes this relation, giving a formula along the lines of

$$H^k(W_3) \cong H^{2n-k}(X_{3,n}), \quad k < n.$$

This is the opposite situation to what was considered in Theorem 35, so a proof of this would mean that we would know almost all the cohomology groups of $X_{3,n}(\sigma)$, without doing any explicit calculations. And like above, if this could be established it would seem natural to consider $m > 3$ and try to prove a similar theorem for $X_{m,n}(\sigma)$.

Other coefficients

In a similar fashion, it seems very natural to try and prove the stabilisation theorem for the other possible coefficient-groups. Nothing in the proof seems to rely on the coefficients being complex numbers, so the proof should be almost automatic if a suitable λ could be defined.

5.2 Structure

There is also more structure to study on the spaces $X_{m,n}(\sigma)$. For example, since $X_{m,\infty}$ is a loop space, it comes equipped with a product. This product can be defined in various very similar ways, and it could be interesting to see how they relate. One possible definition would be

$$(A,B) \mapsto [A, \mathrm{Id}, B],$$

which is directly inspired by the product on the loop space being concatenation of loops. But we also have the other spaces $X_{m,n}(\sigma)$ which are also equivalent to a loop space, and for these the product is slightly harder to define. A possiblity would be to use Theorem 22 to define a map

$$\begin{aligned} X_{m,\infty}(\sigma) \times X_{m,\infty}(\tau) &\to X_{m,\infty}(\tau) \\ (A,B) &\mapsto [A, \sigma, \widehat{L}, \mathrm{Id}, B], \end{aligned}$$

where $\widehat{L}$ is the matrix $(L^m e_1, \ldots, L e_1)$. Another way of doing it would be

$$(A,B) \mapsto [A, \sigma, \sigma \cdot B] \in X_{m,\infty}(\sigma \cdot \tau).$$

All of these possibilites could be used to give a coproduct in cohomology, and it would be interesting to see how this relates to the usual coproduct on the cohomology of $\Omega(\mathrm{SU}_m \,/\, \mathrm{T}^{m-1})$.

Likewise, all of these maps have equivalents in the spaces $X_{m,n}(\sigma)$, $n < \infty$, except here we need to add on extra columns to the spaces. This gives various maps that could be used to probe the cohomology and structure of the spaces.

5.3 Generalised spaces

Another avenue of study could be to consider all the spaces $X_{m,n}(\sigma)$ at the same time:

$$\widehat{X}_{m,n} = \bigcup_{A \in \mathrm{GL}_m(\mathbb{C})} X_{m,n}(A, A).$$

Since the space $X_{m,n}$ approximates the loop group when n is large, by having a sequence of vectors map to a loop at the identity, one could expect $\widehat{X}_{m,n}$ to approximate the free loop group of all maps from the circle to $\mathrm{GL}_m(\mathbb{C})$, by mapping a sequence of vectors starting and ending at A to a loop at A. This would require a study of similarly defined stabilisation maps for these spaces and seeing what happens when we take a limit. This is summarised in the following conjecture.

Conjecture. *There are suitable stabilisations maps*

$$s : \widehat{X}_{m,n} \to \widehat{X}_{m,n+1},$$

so that the space

$$\widehat{X}_{m,\infty} = \varinjlim_{n} \widehat{X}_{m,n}$$

is homotopy equivalent to the free loop space

$$\mathrm{L}(\mathrm{GL}_m(\mathbb{C})) = \left\{ \gamma : S^1 \to \mathrm{GL}_m(\mathbb{C}) \right\}.$$

Like the spaces $X_{m,n}$ considered here, this would give another way to study a complicated space, $\mathrm{L}(\mathrm{GL}_m)$, by working with simpler versions that consist of a finite sequence of points that we connect with nice paths.

A Sage code

The Sage code that was used for computations, referenced in Chapter 3, is included below. Note that a red arrow at the beginning of a line of code denotes a linebreak that was inserted to make the code fit on the page and is not there in the original code.

A.1 The boundary map

The code below computes the kernel and image of the first boundary map in the spectral sequence for $Y_{3,3}$, as described in Section 3.5. It does so by creating a ring with the correct generators and relations and encoding the boundary map in two steps, firstly using the inclusion of the relevant spaces and then using the boundary map from the pair $(D, D - 0)$. It the splits the cohomology groups appearing in the spectral sequence into ranks and computes kernel and image for the boundary. To get a readable output, load the Sage file and call either the function pretty_print for the full E_2-page of $Y_{3,3}$ or print_mod for the cohomology of $X_{3,3}$. FiXme Fatal: Skriv kommentarer i koden

A.2 Orientation of intersections

The code below computes the orientation of the points of intersection of the inclusion maps, as described in Section 3.4. Note that the code is old, and hence uses a slightly different naming convention than is otherwise used in this thesis. In particular, the space X_{001} in the code corresponds to the space $Y_{\{1,2\}}$ in the thesis, X_{101} corresponds to $Y_{\{1\}}$, X_{011} to $Y_{\{2\}}$ and X_{111} to $Y_{\emptyset}$.

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