A Theory of Higher-Order Subtyping with Type Intervals

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CHALMERS UNIVERSITY OF TECHNOLOGY

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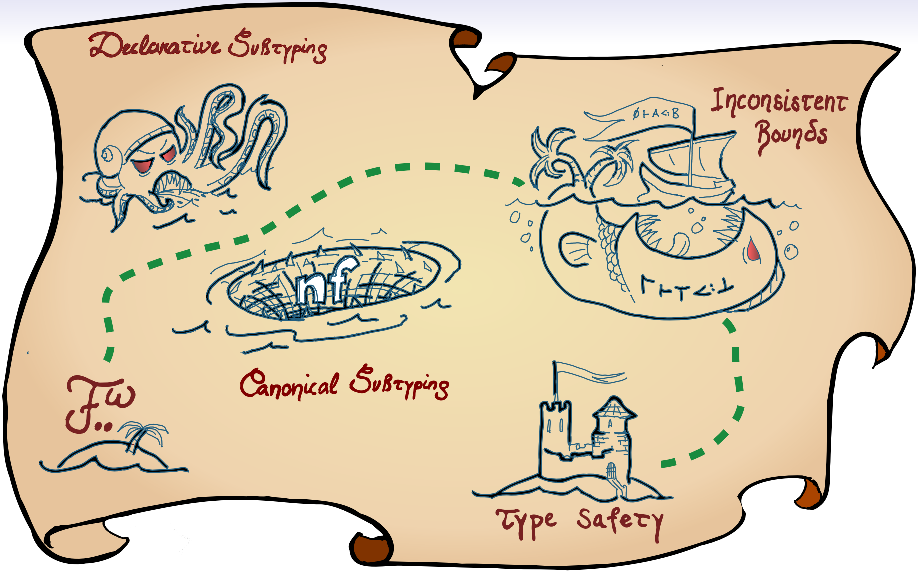
Declarative Subtyping
Inconsistent Bounds
∅ ⊢ A <: B
Γ ⊢ ⊤ <: ⊥
nf
Canonical Subtyping
Fω
Type Safety

DOT and Dotty

DOT

WadlerFest, April 2016

The Essence of Dependent Object Types

Nada Amin[1], Samuel Grütter[1], Martin Odersky[1]( ), Tiark Rompf[2], and Sandro Stucki[1]

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Abstract. Focusing on path-dependent types, the paper develops foundations for Scala from first principles. Starting from a simple calculus $D_{<:}$ of dependent functions, it adds records, intersections and recursion to arrive at DOT, a calculus for dependent object types. The paper shows an encoding of System F with subtyping in $D_{<:}$ and demonstrates the expressiveness of DOT by modeling a range of Scala constructs in it.

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- a minimal core calculus for Scala

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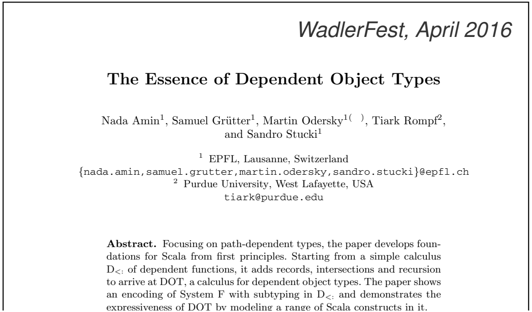
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Implementing Higher-Kinded Types in Dotty

Martin Odersky, Guillaume Martres, Dmitry Petrashko
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Abstract

dotty is a new, experimental Scala compiler based on DOT, the calculus of Dependent Object Types. Higher-kinded types are a natural extension of first-order lambda calculus, and have been a core construct of Haskell and Scala. As long as such types are just partial applications of generic classes, they can be given a meaning in DOT relatively straightforwardly. But general lambdas on the type level require extensions of the DOT calculus to be expressible. This paper is an experience report where we describe and discuss four implementation strategies that we have tried out in the last three years. Each strategy was fully implemented in the *dotty* compiler. We discuss the usability and expressive power of

proved to be challenging, so much so that we evaluated four different strategies before settling on the current direct representation encoding. The strategies are summarized as follows:

- A *simple encoding* in the DOT-inspired [9] core type structures that can express partial applications and not much more
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- a Scala compiler based on DOT
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HK Types – An Example

```
type Ordering[A] = (A, A) => Boolean

abstract class SortedView[A, B >: A](xs: List[A], ord: Ordering[B]) {
  def foldLeft[C](z: C, op: (C, A) => C): C
  def concat[C >: A <: B](ys: List[C]): SortedView[C, B]
  // declarations of further operations such as 'map', 'flatMap', etc.
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- Type parameters of operators can also have bounds!
- Type definitions can be used to introduce aliases.

The Anatomy of a Type Interval

```
X >: A <: B
```

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Intuition: X has bounds $A <: X <: B$.

The Anatomy of a Type Interval

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X >: A <: B
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Intuition: X is an element of the set of types $\{\, A <: \cdots <: B \,\}$

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$$X : A \,..\, B$$

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Special cases

Upper bound	`X <: B`	$X : \bot \mathinner{..} B$

- $\bot = \texttt{Nothing} =$ minimal/bottom type;

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Alias	`X = A`	$X : A \mathrel{..} A$

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- $\bot \mathrel{..} \top = * =$ kind of all types.
- $A \mathrel{..} A =$ singleton containing only A.

The Anatomy of a Type Interval (cont.)

```
F[X >: A <: B] >: G <: H
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We can also represent bounded operators

The Anatomy of a Type Interval (cont.)

`F[X >: A <: B] >: G <: H` $\qquad F : (X{:}A\,..\,B) \to G\,..\,H$

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Examples

Alias `F1[X] = List[X]` $\qquad F_1 : (X{:}*) \to \mathsf{List}\,X\,..\,\mathsf{List}\,X$

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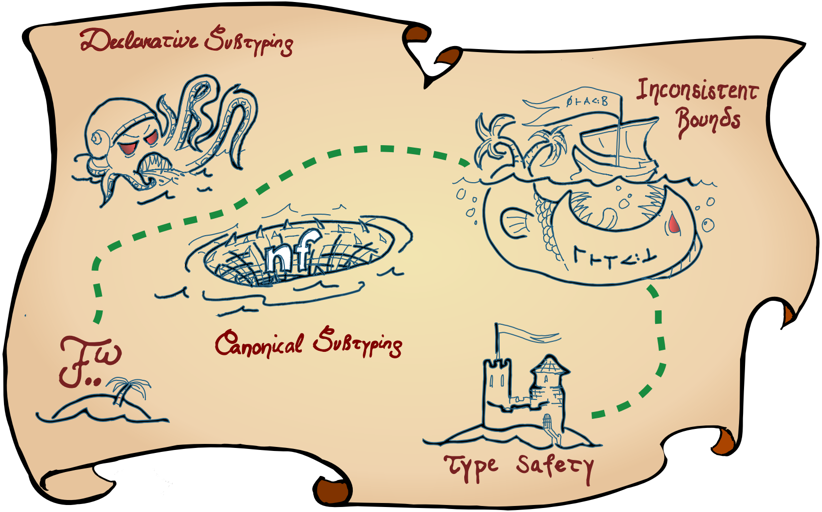
NB. The operators $F_1 - F_3$ all have dependent kinds.

Proving Type Safety of $F^{\omega}_{\cdot\cdot}$



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Problem: $\beta\eta$-conversions get in the way of inversion.

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Solution: normalize types and kinds – no redexes, no conversions!

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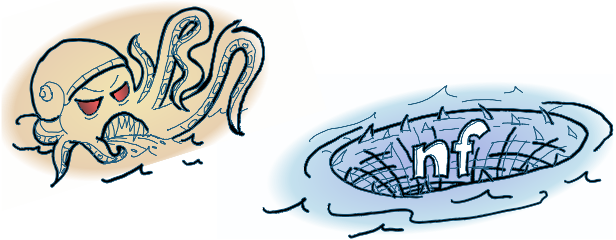

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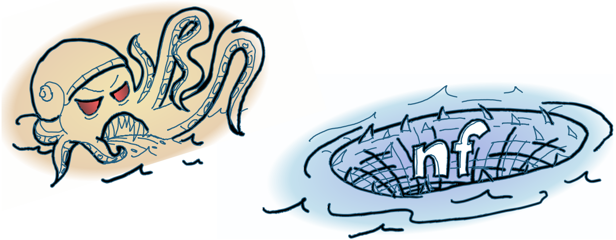

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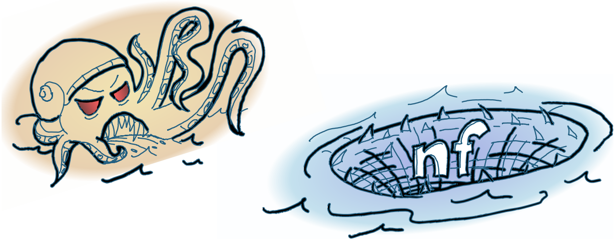
$$\frac{\Gamma \vdash X : (Y{:}J) \to K \qquad \Gamma \vdash V : J}{\Gamma \vdash X\,U : K[V/X^{|J|}]}$$

New solution: use hereditary substitution

Challenge 1: Getting Rid of $\beta\eta$-Conversions



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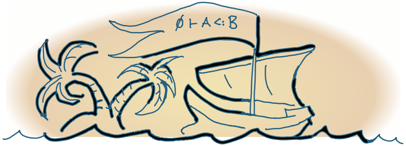
New solution: use hereditary substitution (introducing further problems. . .)

Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce arbitrary subtyping relationships.



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NB. This causes all sorts of problems:

- subject reduction (preservation) fails,
- subtyping becomes undecidable,
- …

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NB. This causes all sorts of problems:

- subject reduction (preservation) fails,
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Solution: invert $<:$ only for closed types
– no variables, no inconsistencies!

Inversion – Step by Step

declarative

$$\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B'$$

Inversion – Step by Step

declarative canonical

$$\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B' \xrightarrow{\mathsf{nf}} \varnothing \vdash_{\mathsf{c}} U \to V <: U' \to V'$$

- $U = \mathsf{nf}(A)$, $V = \mathsf{nf}(B)$, ...

Inversion – Step by Step

declarative | canonical | transitivity-free

$$\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B' \xrightarrow{\mathsf{nf}} \varnothing \vdash_{\mathsf{c}} U \to V <: U' \to V' \xrightarrow{\simeq} \vdash_{\mathsf{tf}} U \to V <: U' \to V'$$

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Inversion – Step by Step

$$
\begin{array}{ccccc}
\text{declarative} & & \text{canonical} & & \text{transitivity-free} \\
\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B' & \xrightarrow{\mathsf{nf}} & \varnothing \vdash_{\mathsf{c}} U \to V <: U' \to V' & \xrightarrow{\simeq} & \vdash_{\mathsf{tf}} U \to V <: U' \to V' \\
& & & & \Big\downarrow \text{invert} \\
& & & & \begin{array}{l} \vdash_{\mathsf{tf}} U' <: U \\ \vdash_{\mathsf{tf}} V \;\; <: V' \end{array}
\end{array}
$$

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Inversion – Step by Step

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 & & & & \Big\downarrow \text{invert} \\
 & & \begin{array}{l} \varnothing \vdash_{\mathsf{c}} U' <: U \\ \varnothing \vdash_{\mathsf{c}} V <: V' \end{array} & \xleftarrow[\simeq]{} & \begin{array}{l} \vdash_{\mathsf{tf}} U' <: U \\ \vdash_{\mathsf{tf}} V <: V' \end{array}
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Inversion – Step by Step

declarative | canonical | transitivity-free

$$
\begin{tikzcd}
\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B' \arrow[r, "\mathsf{nf}"] \arrow[d, dashed] & \varnothing \vdash_{\mathsf{c}} U \to V <: U' \to V' \arrow[r, "\simeq"] & \vdash_{\mathsf{tf}} U \to V <: U' \to V' \arrow[d, "\text{invert}"] \\
\begin{array}{l} \varnothing \vdash_{\mathsf{d}} A' = U' <: U = A \\ \varnothing \vdash_{\mathsf{d}} B = V <: V' = B' \end{array} & \begin{array}{l} \varnothing \vdash_{\mathsf{c}} U' <: U \\ \varnothing \vdash_{\mathsf{c}} V <: V' \end{array} \arrow[l, "\mathsf{nf}\ \text{sound}"'] & \begin{array}{l} \vdash_{\mathsf{tf}} U' <: U \\ \vdash_{\mathsf{tf}} V <: V' \end{array} \arrow[l, "\simeq"']
\end{tikzcd}
$$

- $U = \mathsf{nf}(A)$, $V = \mathsf{nf}(B)$, ...
- nf sound: $\Gamma \vdash A = \mathsf{nf}_\Gamma(A)$ for all Γ and A.

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...and in the artifact (`https://zenodo.org/record/5060213`).

- Mechanization of the full metatheory!

Thank you!

Coauthor

Paolo Giarrusso

Collaborators

- Guillaume Martres
- Nada Amin
- Martin Odersky
- Andreas Abel
- Jesper Cockx

Check out the Agda mechanization!

`https://github.com/sstucki/f-omega-int-agda`
`https://zenodo.org/record/5060213`