DAYANANDA SAGAR COLLEGE OF ENGINEERING

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NOTES: MODULE 5

COURSE NAME: MIMO Communication

COURSE CODE: TE814

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CONTENTS:

Space-time coding for frequency selective fading channels: MIMO frequency-selective channels – Capacity and Information rates of MIMO FS fading channels – Space-time coding and Channel detection for MIMO FS channels – MIMO OFDM systems.

Space-Time Coding for Frequency Selective Fading Channels

In the previous module, we have concentrated mainly on flat fading channel models for wireless communications. However, if the symbol durations are decreased in order to increase the transmission rates, particularly beyond the multipath spread of the channel, flat fading channel models will no longer be appropriate. In such a case, the channel will be frequency selective and will exhibit ISI. Hence, it is of importance to consider the use of MIMO communication systems over frequency selective fading channels as well.

Our objective in this module is several fold. We review some of the information theoretical aspects of single-input single-output and MIMO communication systems. Then we consider several space-time coding approaches for frequency selective fading. These include space-time block codes, space-time trellis codes, as well as concatenated coding approaches. After considering the single carrier approach, we switch our attention to a different way of handling the ISI, namely, the use of multi-carrier communications (specifically, orthogonal frequency division multiplexing (OFDM)) for MIMO systems.

. The module is organized as follows. We first consider MIMO frequency selective channel models, study their capacity, and compute constrained information rates

MIMO Frequency Selective Channels

Let us now start with a generalization of the tapped delay line model shown in figure 1 for frequency selective Channels to the case of MIMO systems. Consider a system with *Nt* transmit and *Nr* receive antennas, and assume that the channel has *L* ISI taps.

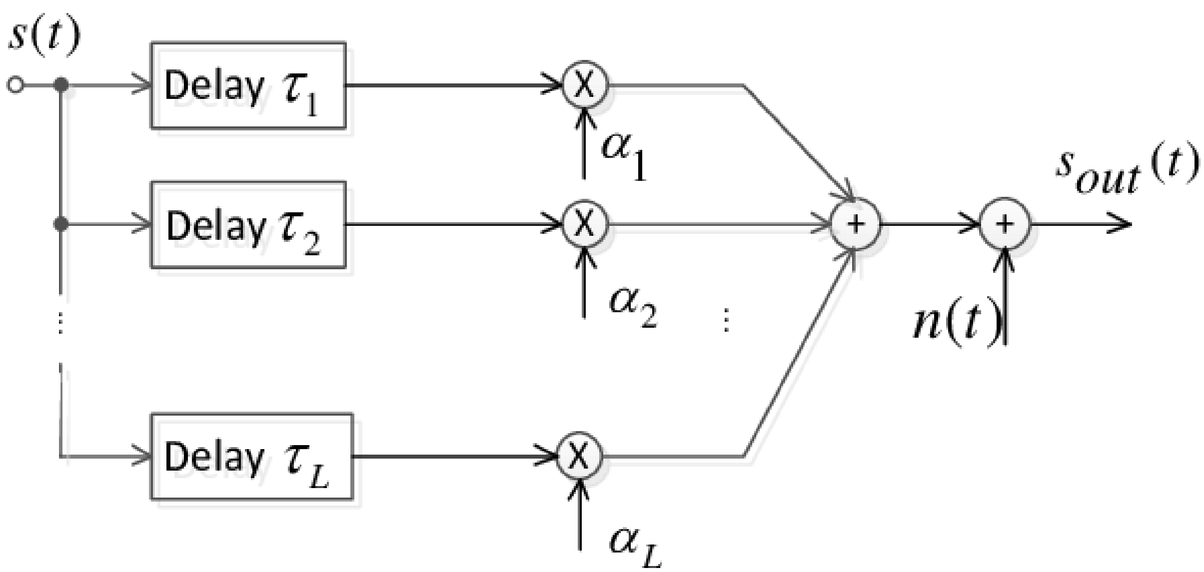

Figure 1. Tapped delay line model

The figure 2 Shows MIMO concept with two transmitter & two receiver antennas

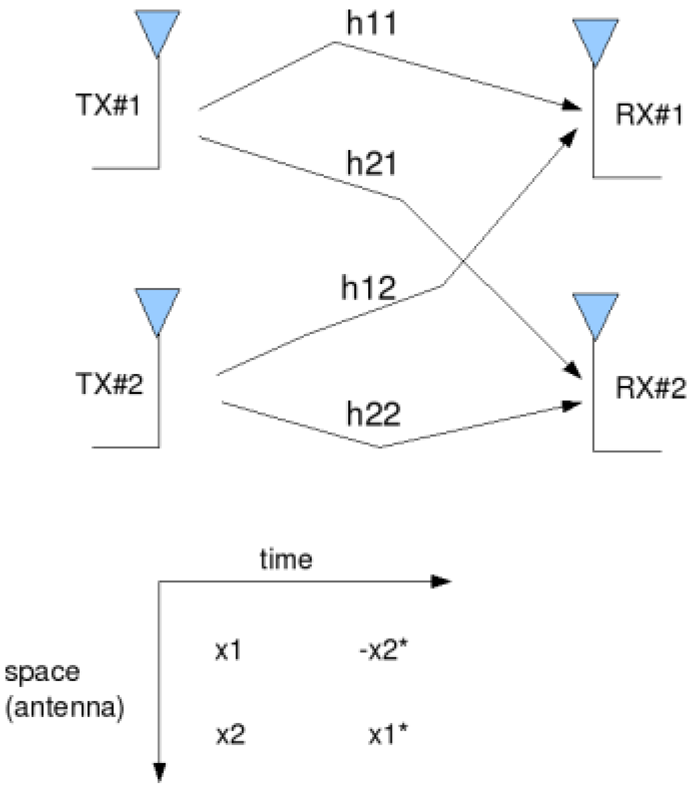

Figure 2 Showing MIMO concept with two transmitter & two receiver antennas

The received signal at the *j*th receive antenna at time *k* is given by

$$y_j(k) = \sqrt{\rho}\sum_{l=0}^{L-1}\sum_{i=1}^{N_t} h_{i,j}^{(l)}(k)x_i(k-l) + n_j(k)$$

where *xi(k)* denotes the transmitted signal from antenna *i* at time *k*, *h(l) i,j (k)* is the channel coefficient for the *l*th path from transmit antenna *i* to receive antenna *j* at time *k*, and *nj (k)* is the additive white Gaussian noise (both spatially and temporally) with a variance of 1/2 per dimension.

We need to differentiate several cases.

- For baseband MIMO ISI channels, the channel coefficients are real numbers.
- For deterministic bandpass channels (which could be useful for fixed wireless communications), the channel coefficients are in general complex, but they still can be well modeled as constants.
- For mobile communications, the channel tap coefficients are random variables.
- For instance, if the wireless channel is very slowly varying, e.g., quasi-static fading, the tap coefficients remain constant for each frame of data.
- However, for ergodic channels, they vary with time. As an example, for block fading channels, they change independently from one block of data to the next.

- For Rayleigh fading channels, the channel tap coefficients are modeled as zero mean complex Gaussian random variables.

Different channel taps are usually assumed to be independent. The average channel gains for different paths are determined from the power delay profile of the wireless channel.

For instance, for the uniform power delay profile, all the channel gains (for different paths) have equal average channel power. As another example, for the exponential power delay profile, the channel tap powers decay exponentially

Information Rates with Gaussian Inputs

For deterministic MIMO ISI channels for single input single-output systems, and uses DFT techniques. The resulting information rate is given by

$$I_{Gauss} = \int_{0}^{\infty} \log\left(\det\left(\boldsymbol{I}_{N_r} + \frac{\rho}{N_t}\tilde{\boldsymbol{H}}_f^T \tilde{\boldsymbol{H}}_f^H\right)\right) df,$$

$$\tilde{\boldsymbol{H}}_f = \sum_{l=0}^{L-1} \boldsymbol{H}_l \exp(-j2\pi f l),$$

$$\boldsymbol{H}_l = \begin{bmatrix} h_{1,1}^{(l)} & h_{1,2}^{(l)} & \cdots & h_{1,N_r}^{(l)} \\ h_{2,1}^{(l)} & h_{2,2}^{(l)} & \cdots & h_{2,N_r}^{(l)} \\ \vdots & \vdots & \ddots & \\ h_{N_t,1}^{(l)} & h_{N_t,2}^{(l)} & \cdots & h_{N_t,N_r}^{(l)} \end{bmatrix}$$

Achievable Information Rates with Practical Constellations

Consider a general MIMO FS channel model, and assume that the channel gains are known at the receiver (they can be deterministic or random). Then, the achievable information rate can be written as

$$I = \lim_{N\to\infty} \frac{1}{N} I(\boldsymbol{X}(1), \boldsymbol{X}(2), \ldots, \boldsymbol{X}(N); \boldsymbol{Y}(1), \boldsymbol{Y}(2), \ldots, \boldsymbol{Y}(N))$$

where $\boldsymbol{X}(1), \boldsymbol{X}(2), \ldots, \boldsymbol{X}(N)$ is the sequence of channel inputs, and $\boldsymbol{Y}(1), \boldsymbol{Y}(2), \ldots, \boldsymbol{Y}(N)$ is the set of channel outputs.

Space-Time Coding for MIMO FS Channels

In this section we consider several space-time coding approaches for requency selective fading. These include space-time block codes, space-time trellis codes, as well as concatenated coding approaches. As in most of our coding coverage, we consider quasi-static fading channels. That is, the fading coefficients remain constant for the entire codeword. We first consider an interpretation of MIMO FS channels, and then derive the code design criteria accordingly. Finally, we review several specific coding approaches.

Assume that the space-time codeword length is N, and additional $L - 1$ zeros are appended at the end of each codeword to clear the channel. Then, the input–output relationship of the MIMO FS channel can be written as

$$\boldsymbol{Y} = \sqrt{\rho}\boldsymbol{X}_{eq}\boldsymbol{H}_{eq} + \boldsymbol{N}$$

where the equivalent space-time codeword transmitted is an $(N + L - 1) \times Nt \cdot L$ matrix given by

$$\boldsymbol{X}_{eq} = \begin{bmatrix} x_1(1) & 0 & \cdots & 0 & \cdots & x_{N_t}(1) & 0 & \cdots & 0 \\ x_1(2) & x_1(1) & \cdots & 0 & \cdots & x_{N_t}(2) & x_{N_t}(1) & \cdots & 0 \\ & \vdots & & & \vdots & & & \vdots & \\ x_1(N) & x_1(N-1) & \cdots & 0 & \cdots & x_{N_t}(N) & x_{N_t}(N-1) & \cdots & 0 \\ 0 & x_1(N) & \cdots & 0 & \cdots & 0 & x_{N_t}(N) & \cdots & 0 \\ & \vdots & & & \cdots & & & \vdots & \\ 0 & 0 & \cdots & x_1(1) & \cdots & 0 & 0 & \cdots & x_{N_t}(1) \end{bmatrix}$$

the equivalent channel coefficient matrix is

$$\boldsymbol{H}_{eq} = \begin{bmatrix} h_{1,1}^{(0)} & h_{1,2}^{(0)} & \cdots & h_{1,N_r}^{(0)} \\ h_{1,1}^{(1)} & h_{1,2}^{(1)} & \cdots & h_{1,N_r}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1,1}^{(L-1)} & h_{1,2}^{(L-1)} & \cdots & h_{1,N_r}^{(L-1)} \\ h_{2,1}^{(0)} & h_{2,2}^{(0)} & \cdots & h_{2,N_r}^{(0)} \\ h_{2,1}^{(1)} & h_{2,2}^{(1)} & \cdots & h_{2,N_r}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{2,1}^{(L-1)} & h_{2,2}^{(L-1)} & \cdots & h_{2,N_r}^{(L-1)} \\ \vdots & \vdots & \vdots & \vdots \\ h_{N_t,1}^{(0)} & h_{N_t,2}^{(0)} & \cdots & h_{N_t,N_r}^{(0)} \\ h_{N_t,1}^{(1)} & h_{N_t,2}^{(1)} & \cdots & h_{N_t,N_r}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_t,1}^{(L-1)} & h_{N_t,2}^{(L-1)} & \cdots & h_{N_t,N_r}^{(L-1)} \end{bmatrix},$$

and the matrices $\boldsymbol{Y}$ and $\boldsymbol{N}$ are $(N + L - 1) \times Nr$ matrices of the received signals and noise terms whose (k, j)th element denotes the corresponding quantity for the kth time instant and jth antenna.

By examining the equivalent space-time code matrix, $\boldsymbol{X}eq$, we see that it is as if we have $L - 1$ additional "virtual" antennas corresponding to each actual transmit antenna that emit delayed versions of transmitted sequence. This is a useful observation in designing codes for MIMO FS channels. We illustrate this interpretation and the resulting transmission scheme in Figure 3
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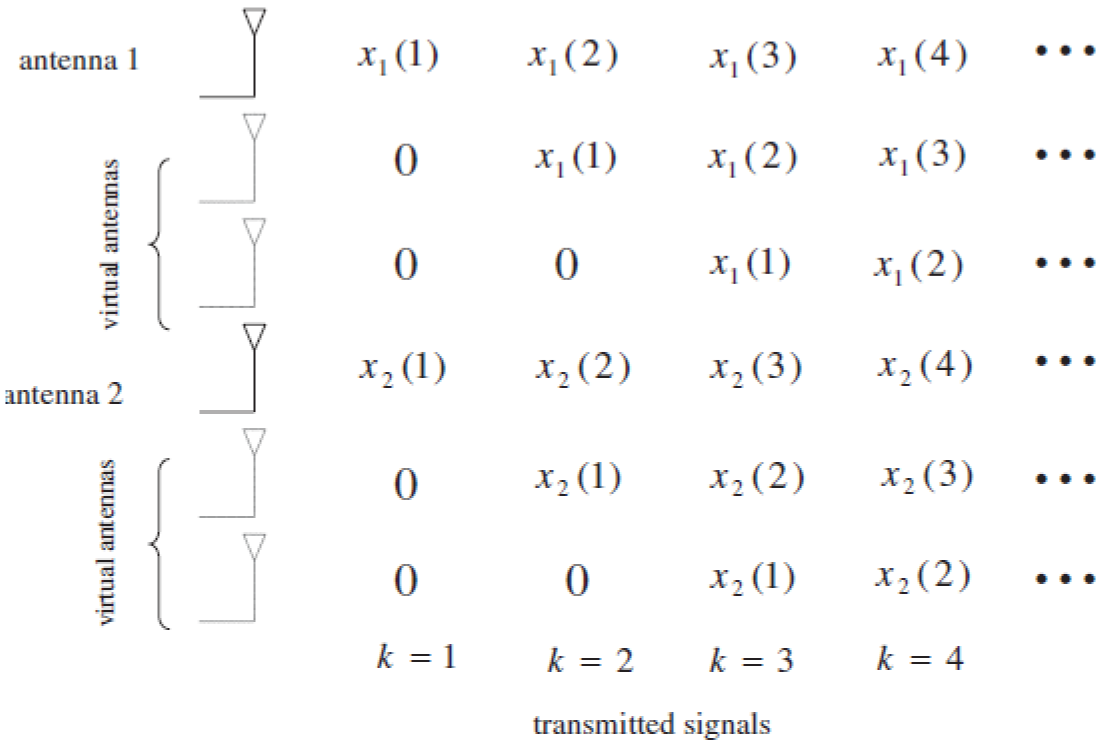

Figure 3 Virtual antenna interpretation of transmission over MIMO FS channels (with $Nt = 2$ and $L = 3$).

Although the available diversity order may be high, care must be taken in code design to exploit it. We now describe the rank criterion for achieving full diversity (over quasi-static) MIMO FS channels by extending the corresponding criterion derived in Chapter 5 in the context of MIMO flat fading channels.

Criterion for Achieving Full Diversity

A space-time code achieves full diversity over a MIMO FS channel if for any pair of codewords (of the form ***Xeq*** whose form is given above), the corresponding codeword difference matrices are of full column rank.

A Simple Full Diversity Code for MIMO FS Channels

Before we describe a simple code that achieves full diversity for MIMO FS channels, let us consider the idea of delay diversity for the flat fading scenario. For an *Nt* transmit antenna system, a delay diversity code is obtained by simply transmitting delayed versions of the sequence on the first transmit antenna from the other antennas That is, the *i*th transmit antenna emits a delayed version of the sequence (by $i - 1$ symbols) on the first transmit antenna. By this method, it is easy to see that for flat fading, space-time codewords are of the form

$$\begin{bmatrix} x(1) & 0 & 0 & \cdots & 0 \\ x(2) & x(1) & 0 & \cdots & 0 \\ x(3) & x(2) & x(1) & \cdots & 0 \\ & \vdots & & \vdots & \\ x(N_t) & x(N_t-1) & x(N_t-2) & \cdots & x(1) \\ x(N_t+1) & x(N_t) & x(N_t-1) & \cdots & x(2) \\ & \vdots & & \vdots & \\ x(N) & x(N-1) & x(N-2) & \cdots & x(N-N_t+1) \end{bmatrix}.$$

Therefore, the codeword difference matrices will clearly be full column rank if a distinct pair of codewords are considered. Hence, the delay diversity code will achieve a full rank of ***NtNr*** over flat fading.

To extend this idea to the case of MIMO FS channels, we need a slight modification. It is easy to see that delaying by a single symbol for each transmit antenna will not work, because some columns of the equivalent space-time codeword matrix (with the use of virtual antenna interpretation given in the previous section), will be identical, hence the code will not be of full rank. However, it is straightforward to see that, if we delay the transmitted signal of the first antenna by the number of ISI taps L instead of by a single symbol, we obtain the equivalent space-time codeword (for the MIMO FS channel) given as

$$\begin{bmatrix} x(1) & 0 & 0 & \cdots & 0 \\ x(2) & x(1) & 0 & \cdots & 0 \\ x(3) & x(2) & x(1) & \cdots & 0 \\ & \vdots & & \vdots & \\ x(N_tL) & x(N_tL-1) & x(N_tL-2) & \cdots & x(1) \\ x(N_tL+1) & x(N_tL) & x(N_tL-1) & \cdots & x(2) \\ & \vdots & & \vdots & \\ x(N) & x(N-1) & x(N-2) & \cdots & x(N-N_tL+1) \end{bmatrix}$$

We can then easily verify that the idea of delaying by L satisfying the full rank criterion given in the previous section, hence it provides a diversity of order *NtNrL*.
We do not go into the details here, but this delay diversity code can be described using a trellis diagram, hence decoding can be performed using the Viterbi algorithm.

Space-Time Trellis Codes for MIMO FS Channels

We would first like to point out that it is possible to use space-time block codes over MIMO FS channels. However, much of the benefits of space-time block coding, particularly in the decoding process, are lost. This is because, with the presence of ISI, it not trivial to perform simple linear processing to decouple the decisions of different symbols in ML decoding. Therefore, we do not describe them here, instead, we directly talk about the extension of space-time trellis codes to MIMO FS channels.

Let us now focus on the use of space-time trellis codes over MIMO FS Channels. In Figure 4, we show the system block diagram. The only difference in principle is the presence of the ISI channel as opposed to the flat fading channel considered earlier.

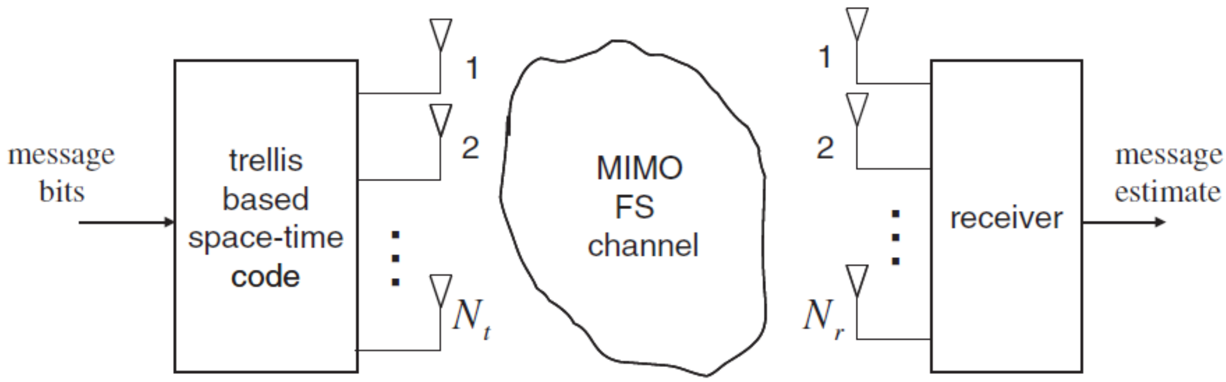

Figure 4 Block diagram of space-time trellis coding over a MIMO FS channel.

Concatenated Coding for MIMO FS Channels

In module we have talked about concatenated coding and iterative decoding techniques for MIMO channels for frequency flat fading. Similar ideas can be extended to the case of frequency selective fading as well. One basic difference is that the frequency selective fading channel can itself be used as an "inner code" in code concatenation. The basic block diagram of a concatenated coding approach for MIMO FS channels is shown in Figure 5.

An outer code which can be a space-time trellis code or a convolutional code is concatenated with an *M*-ary modulator through an interleaver with a certain length, and then the modulated symbols are transmitted through the ISI channel. This is basically the same approach we have used over MIMO flat fading channels.

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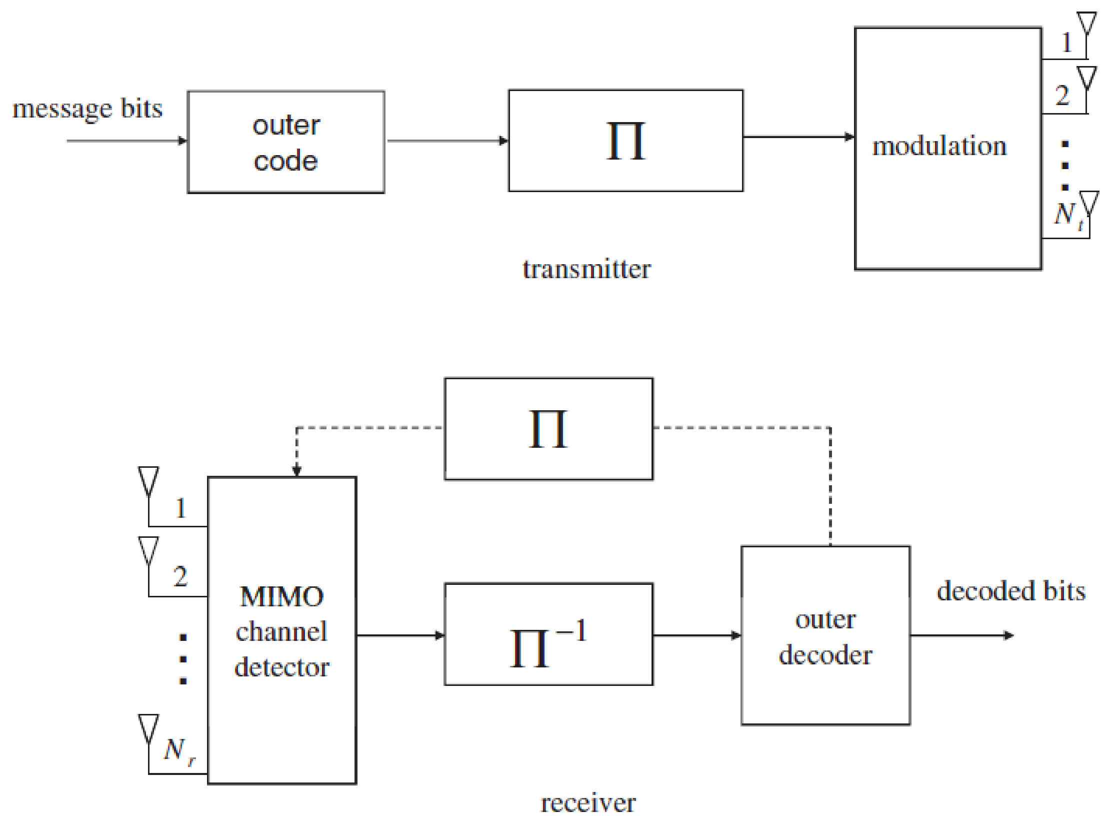

Figure 5 Block diagram of the concatenated coding approach.

For demodulation/decoding, the MIMO channel is equalized using a soft-output equalizer, for instance, implemented via the MAP algorithm. The soft-output equalizer produces likelihoods of the M-ary symbols transmitted. The symbol level likelihood information is then converted to bit level likelihoods, and these are used in the decoding of the information bits using the code constraints. The only difference in this scheme, compared with the frequency flat fading case, is the presence of the soft-output equalizer. The demodulation/ decoding is shown in Figure 5 as well.

It is also possible to use the idea of iterative "turbo" equalization at the receiver to improve the error rate performance of the system. Basically, the outer channel decoder can be implemented using a soft-output algorithm, and the extrinsic (new) information that it produces can be passed back to the MIMO channel equalizer. This information can be used to update the symbol likelihoods, hence the bit likelihoods, to be used in the next iteration of the channel decoder. This process is illustrated using the dashed lines in Figure 6

Spatial Multiplexing for MIMO FS Channels

If we are interested in increasing the transmission rates (with possibly a penalty in the diversity order provided by the system), we can employ spatial multiplexing schemes for MIMO FS channels as well. There is basically no difference in the encoding approaches. Whatever can be used for the case of flat fading will also be easily extended to the case of frequency selective fading channels. On the other hand, there are differences in the receiver design. In the case of flat fading, the receiver algorithms basically spatially equalize the received signal using different algorithms. For instance, zero forcing or MMSE receivers are employed. In the case of MIMO FS channels, the receiver works in the same way, however, in addition to the spatial equalization, it performs temporal equalization.

Channel Detection for MIMO FS Channels

MIMO FS channels can be described by a trellis with *MNt (L*−1*)* states where *M* is the constellation size. If uncoded data streams are being transmitted simultaneously from different antennas, there are *Nt* symbols for each period. Hence the number of branches emanating from each state is *MNt* . Once the trellis diagram is obtained using standard methods, the optimal detection problem is easy. If our objective is to minimize the sequence error probability, we can simply use the Viterbi algorithm. If we would like to minimize the symbol error probability, we can resort to the MAP algorithm

One problem with the full complexity algorithms, e.g., the Viterbi or MAP algorithm, is the associated complexity. The number of states in the trellis grows exponentially with the product of the number of transmit antennas and the length of the ISI channel. This quickly becomes prohibitively complex even with relatively short ISI levels. For instance, with BPSK modulation, if the number of antennas is four, with an ISI length of $L = 5$, the number of states needed for optimal equalization is 216 = 65,536 which is too large. Therefore, we need other simplified approaches for channel detection.

We note that the need for reduced complexity receivers for MIMO FS channels should not come as a surprise. Even in the case of frequency flat fading, when the number of antennas is increased, we need to resort to a BLAST type of transmission scheme (i.e., spatial multiplexing), and apply reduced complexity detection algorithms. The problem with frequency selectivity becomes even more complicated.

As reduced complexity MIMO FS channel equalizers, we consider linear equalizers and decision feedback qualizers.

Linear Equalization for MIMO FS Channels

Consider an uncoded VBLAST transmission over a MIMO FS channel, and assume that the channel coefficients for all the antenna pairs and for all the channel taps are known. A simple way of estimating the transmitted bits based on the observations at the receiver is to employ a spatially and temporally linear equalizer. The estimate of the *k*th symbol transmitted from the *i*th antenna is given by

$$\hat{x}_i(k) = \sum_{j=1}^{N_r} \sum_{n=-N_1}^{N_2} a_{j,n} y_j(k-n),$$

where *aj,n* are the set of filter coefficients and *N*1 + *N*2 + 1 is the length of the filter used. Given the fading channel coefficients, the equalizer coefficients can be found using different constraints, such as zero forcing, or MMSE.

Decision Feedback Equalization for MIMO FS Channels

As an alternative to linear equalization, decision feedback equalization (see Proakis (2001)) can also be extended to the case of MIMO communications. The approach in this case is to use two sets of filters (feedforward and feedback), where the feedback filter operates on already detected symbols with the objective of canceling out their contribution from the observation vector. Due to the non-linearity in the decision process (on the prior symbols), the overall equalizer is no longer linear. A block diagram of the MIMO decision feedback equalizer (DFE) is shown in Figure 6

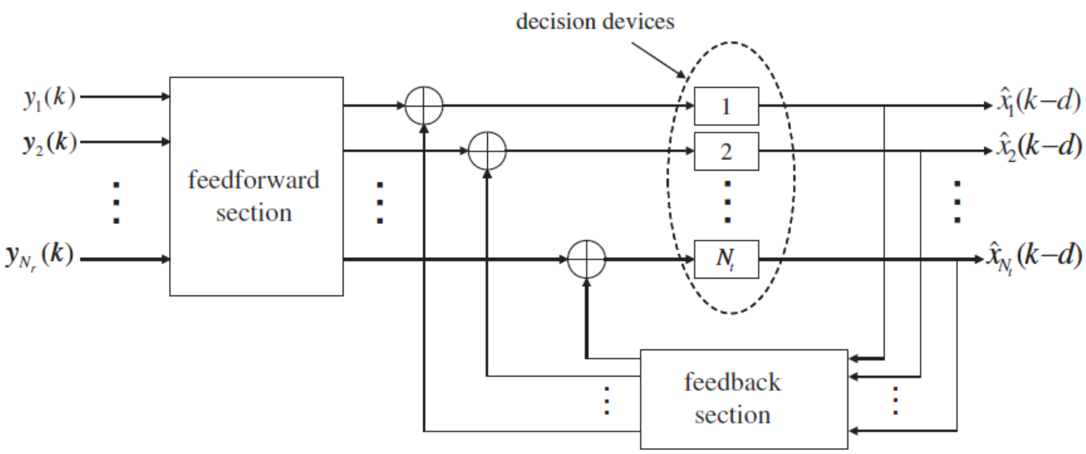

Figure 6 Block diagram of a MIMO-DFE

MIMO OFDM Systems

Until now, we have focused on single carrier transmission over frequency selective fading channels, which requires sophisticated equalization techniques to overcome the intersymbol interference problem. Another way to deal with such channels and to avoid relatively complex signal processing algorithms is to employ multi-carrier transmissions used in the same frequency band. For example, assume that the overall available bandwidth is W. The idea is to split the overall band available into N frequency sub-bands so that each sub-band (of bandwidth W/N) experiences flat fading as illustrated in Figure 7

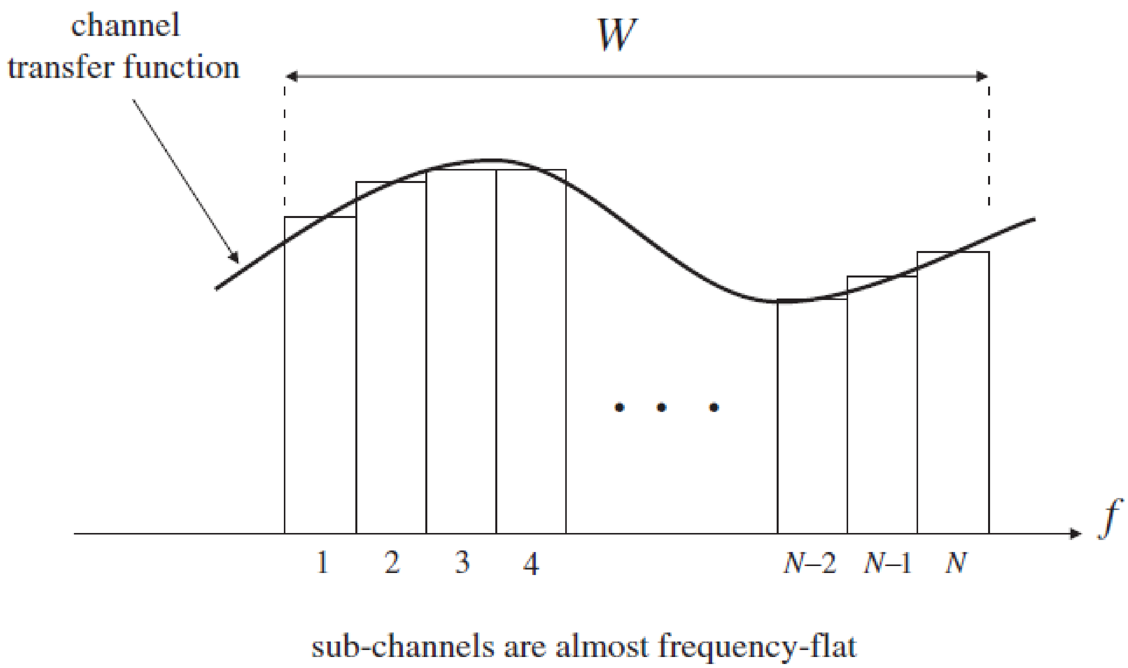

Figure 7 Illustration of multi-carrier communications in the frequency domain

The symbol duration on each multi-carrier component is N/W which can be made much larger than the multipath spread of the channel, hence the subchannels do not experience ISI (or, the effects of ISI are reduced to a desired level). The overall transmission rate is still $N \cdot W/N = W$ symbols per second, i.e., the same as the original single-carrier system

A simple and efficient way of achieving multi-carrier modulation is to employ orthogonal sub-carriers (that are separated by $1/Ts$ Hz where Ts is the symbol duration), and use the DFT pair (more precisely, fast Fourier transform (FFT) and its inverse) for implementation. The idea is to realize that with $1/Ts$ frequency separation, samples of the overall signal to be transmitted over N sub-carriers (at a rate $1/Ts$) can be obtained using the N-point inverse DFT (IDFT) or inverse FFT (IFFT) of the data sequence of the N sub-carriers. At the receiver the DFT (or FFT) is applied, and the standard receiver algorithms are used. The resulting scheme is called orthogonal frequency division multiplexing

The idea of multi-carrier modulation, or OFDM, can be applied to MIMO FS channels as well, as shown in Figure 8. For each transmit antenna element, the inverse DFT of N (uncoded or coded) M-ary symbols, denoted by

$X_i(0), X_i(1), X_i(2), \ldots, X_i(N-1)$

is first computed. Then a cyclic prefix is appended to each of these sequences, and the resulting signal is transmitted, The objective in adding the cyclic prefix is to remove possible interference between two consecutive OFDM symbols, and to make sure that the equivalent channel (after DFT at the receiver) has a simple form which allows for easier processing. For each of the received signals (at the Nr antenna elements), the DFT of the aggregate received signal (superposition of all the OFDM words from each of the transmitted antennas) is calculated, and the cyclic prefix is removed. The resulting set of signals is then used for demodulation/decoding.

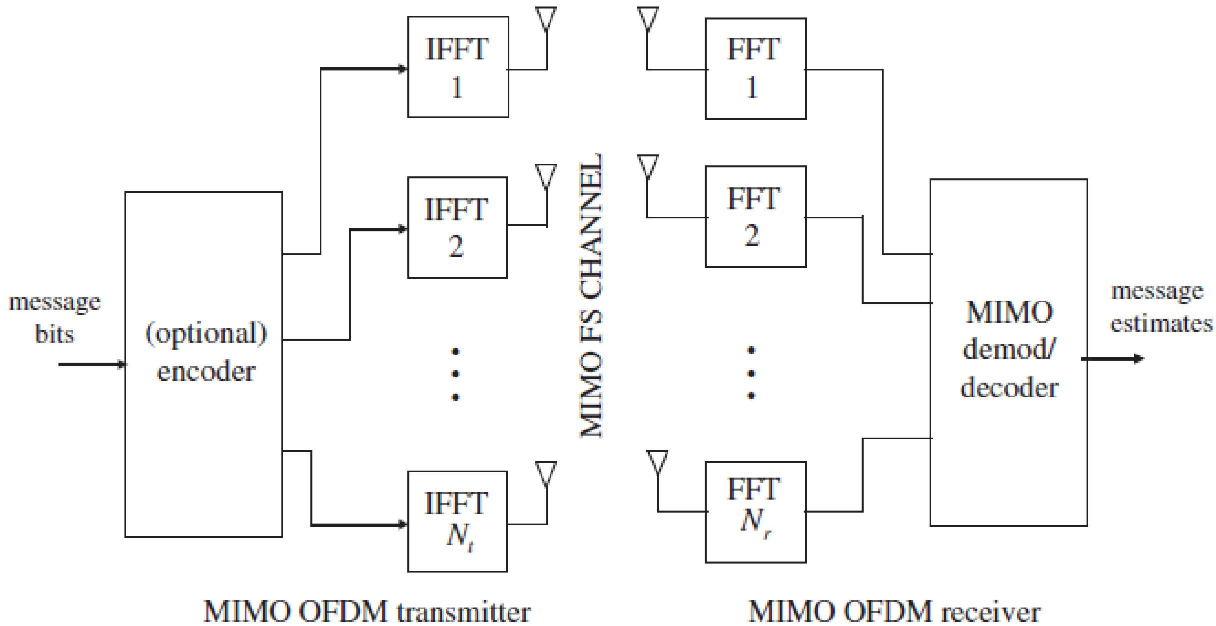

Figure 8 Block diagram of a MIMO-OFDM system.

Acknowledgment: This material is based on the text book authored by Tolga M. Duman and Ali Ghrayeb, "Coding for MIMO Communication systems", John Wiley & Sons, West Sussex, England, 2007. Some additional material are taken and/or inspired by material from various paper and / or electronic resources.

Objective Questions.

a) --------- is where the fading process is approximately constant for a number of symbol intervals.
i) **Block fading** ii) Flat fading iii) FS fading iv) Rayleigh Fading

b)A channel can be ------ block-fading' when it is block fading in both the time and frequency domains.
i) octople ii) single iii) quadruple iv) **double**

c) Channel Tap is certain delay on delay line on -------.
i) Time Axis ii) Frequency axis iii) Fourier Axis iv) Complex axis

d) -------- is the time duration over which the channel impulse response is considered to be not varying
i) Channel Time ii) **Coherence time** iii) Equalization Time iv) Interference Time

e) The ------ algorithm is an algorithm for maximum a posteriori decoding of error correcting codes defined on trellises
i) **BCJR** ii) Viterbi iii) MAP iv) Priori

f) -------is process of adjusting the spatial attribute of a sound in order to perceive desired 3D sound sensation
i) **Spatial Equalization** ii) Temporal Equalization iii) ISI iv) ISI-Tap

g) -------is transceiver architecture for offering spatial multiplexing over multiple-antenna wireless communication systems
i) D Blast ii) **BLAST** iii) V Blast iv) K-Blast

h) --------- is a detection algorithm to the receipt of multi-antenna MIMO systems.
i) V-Blast ii) D-Blast iii) K-Blast iv) Blast

i) ---------is a wireless set up that used a multi element antenna array a both the transmitter and receiver, as well as diagonally layered coding sequence.
i) BLAST ii) V-Blast iii) K-Blast iv) **D-BLAST**

j) A ------- is a filter that uses feedback of detected symbols to produce an estimate of the channel output.
i) FIR Filter ii) Matched Filter iii) ISI Filter iv) **Decision feedback equalizer (DFE)**

k) The DFE is fed with detected symbols and produces an output which typically is subtracted from the output of the ---------- equalizer.
i) FIR ii) Matched iii) **Linear** iv) Decision feedback equalizer (DFE)

Q1.Explain about MIMO Frequency Selective Fading Channels.
Consider a system with Nt transmit and Nr receive antennas, and assume that the channel has L ISI taps. The received signal at the jth receive antenna at time k is given by

$$y_j(k) = \sqrt{\rho}\sum_{l=0}^{L-1}\sum_{i=1}^{N_t} h_{i,j}^{(l)}(k)x_i(k-l) + n_j(k),$$

where xi(k) denotes the transmitted signal from antenna i at time k, $h^{(l)}_{i,j}(k)$ is the channel coefficient for the lth path from transmit antenna i to receive antenna j at time k, and $n_j(k)$ is the additive white Gaussian noise (both spatially and temporally) with a variance of 1/2 per dimension. The total signal energy for each use of the channel is normalized to unity. The constant ρ can be used as the effective signal-to-noise ratio at the receiver after proper normalization of the channel gains.
We need to differentiate several cases. For baseband MIMO ISI channels, the channel coefficients are real numbers. For deterministic bandpass channels (which could be useful for fixed wireless communications), the channel coefficients are in general complex, but they still can be well modeled as constants. For mobile communications, the channel tap coefficients are random variables. For instance, if the wireless channel is very slowly varying, e.g., quasi-static fading, the tap coefficients remain constant for each frame of data. However, for ergodic channels, they vary with time. As an example, for block fading channels, they change independently from one block of data to the next.
For Rayleigh fading channels, the channel tap coefficients are modeled as zero mean complex Gaussian random variables. Different channel taps are usually assumed to be independent. The average channel gains for different paths are determined from the power delay profile of the wireless channel. For instance, for the uniform power delay profile, all the channel gains (for different paths) have equal average channel power. As another example, for the exponential power delay profile, the channel tap powers decay exponentially.
Q2. Verify the FSFC Information Rates with Gaussian Inputs.
For deterministic MIMO ISI channels, the resulting information rate is given by

$$I_{Gauss} = \int_0^\infty \log\left(\det\left(\boldsymbol{I}_{N_r} + \frac{\rho}{N_t}\tilde{\boldsymbol{H}}_f^T\tilde{\boldsymbol{H}}_f^H\right)\right)df,$$

where

$$\tilde{H}_f = \sum_{l=0}^{L-1} H_l \exp(-j2\pi f l),$$

with

$$H_l = \begin{bmatrix} h_{1,1}^{(l)} & h_{1,2}^{(l)} & \cdots & h_{1,N_r}^{(l)} \\ h_{2,1}^{(l)} & h_{2,2}^{(l)} & \cdots & h_{2,N_r}^{(l)} \\ \vdots & \vdots & \ddots & \\ h_{N_t,1}^{(l)} & h_{N_t,2}^{(l)} & \cdots & h_{N_t,N_r}^{(l)} \end{bmatrix},$$

where the time dependence of the fading coefficients is dropped since they are constants.

It is easy to extend this result for quasi-static fading channels as well (assuming that the receiver knows the channel coefficients). In this case, the expression for a given channel matrix, should be interpreted as the achievable instantaneous information rate with independent Gaussian inputs, which is clearly random. Therefore, for a given transmission rate, one can compute the relevant outage probability, or for a given outage level, one can calculate the maximum achievable information rate. The result above can be interpreted as a lower bound on the actual Shannon type capacity (for the deterministic case), and the outage capacity (for the quasi- static fading scenario). This is not a complete capacity characterization, however it is clearly a useful quantity that can be used for system design. Furthermore, the above expression is not useful for ergodic capacity calculations.

Q3. Illustrate about Achievable Information Rates with Practical Constellations.

Consider a general MIMO FS channel model, and assume that the channel gains are known at the receiver (they can be deterministic or random). Then, the achievable information rate can be written as

$$I = \lim_{N\to\infty} \frac{1}{N} I(X(1), X(2), \ldots, X(N); Y(1), Y(2), \ldots, Y(N)),$$

where X(1),X(2),...,X(N) is the sequence of channel inputs, and Y(1),Y(2),...,Y(N) is the set of channel outputs. We note that we have omitted necessary conditioning on the channel gains for simplicity of notation (for the random channel case).

Let us compute the information rates under the constraint that the inputs are i.u.d. and are picked from a finite constellation such as BPSK. The only uncertainty that remains in the sequence of output vectors (given the channel coefficients) is due to the noise terms. Therefore, we can write the mutual information expression above as

$$I(X(1), \ldots, X(N); Y(1), \ldots, Y(N)) = H(Y(1), \ldots, Y(N)) - H(N(1), \ldots, N(N)), \tag{9.6}$$

where $N(1), N(2), \ldots, X(N)$ are the noise vectors. Thus, we can easily write

$$I = \lim_{N\to\infty} \left(\frac{1}{N} H(Y(1), \ldots, Y(N)) - \frac{1}{N} H(N(1), \ldots, N(N)) \right). \tag{9.7}$$

Clearly, the second term (entropy of the noise) can be easily computed as we simply have a sequence of independent complex Gaussian noise vectors. This is because, for the case of flat fading, given the channel coefficients, if the input symbols are independent from one time instant to the next, then the output symbols are independent as well, and the entropy can be estimated by considering the entropy of a single output vector at a time. However, due to the presence of ISI, we do not have such a simplification here and we need to resort to another technique.

By de nition, we have

$$\frac{1}{N} H(Y(1), \ldots, Y(N)) = -\frac{1}{N} \mathrm{E}\left[\log\left(p(Y(1), \ldots, Y(N))\right)\right],$$

that is, we simply need to estimate the expected value of the logarithm of the channel output sequence probability.

Assume that the size of the signal constellation is M. Since there are Nt transmit antennas, and there are L

ISI taps for each pair of antennas, one can describe the MIMO ISI channel using an ns = $M^{Nt(L-1)}$ state trellis diagram. For a given sequence of channel inputs, the noiseless channel outputs form a path through this trellis. Let us denote the state of the trellis at time k by Sk.

$$\alpha_k(m) = P(y(1), \ldots, y(k), S_k = m),$$

and

$$\gamma_k(m', m) = p(y(k) \mid S_k = m, S_{k-1} = m') \cdot P(S_k = m \mid S_{k-1} = m'),$$

where $0 \leq m', m \leq n_s - 1$.

$$= \sum_{m'=0}^{n_s-1} \gamma_k(m', m)\alpha_{k-1}(m').$$

We note that, for i.u.d. inputs, we have

$$p(S_k = m \mid S_{k-1} = m') = \frac{1}{M^{N_t}},$$

Furthermore, for a given branch of the trellis, the elements of

$$y(k) = [y_1(k)\ y_2(k)\ \ldots\ y_{N_r}(k)]^T$$

The only other issue that remains is the initialization of the algorithm. If initially encoding starts from the zero state, we can easily set $\alpha_0(0) = 1$ and $\alpha_0(m) = 0$ if m not equal to 0.

$$p(y(1), y(2), \ldots, y(N)) = \sum_{m=0}^{n_s-1} \alpha_N(m).$$

In order to estimate the information rates of deterministic or random (but ergodic) MIMO FS channels, we simply generate a large number of realizations, calculate the information rate corresponding to each realization, and average the results. In each realization, we use a long sequence of i.u.d. channel inputs, find the corresponding channel outputs, and use the forward recursion of the BCJR algorithm as described above to compute the joint probability of the channel outputs generated. It is also clear that for the case of deterministic channels or ergodic fading channels, we can simply resort to the Shannon–McMillan–Breiman theorem, and use a single long simulation.

Q4. Describe briefly about Interpretation of MIMO FS Channels Using Virtual Antennas.

Assume that the space-time codeword length is N, and additional L−1 zeros are appended at the end of each codeword to clear the channel. Then, the input–output relationship of the MIMO FS channel can be written as

$$Y = \sqrt{\rho} X_{eq} H_{eq} + N, \tag{9.14}$$

where the equivalent space-time codeword transmitted is an $(N + L - 1) \times N_t \cdot L$ matrix given by

$$X_{eq} = \begin{bmatrix} x_1(1) & 0 & \cdots & 0 & \cdots & x_{N_t}(1) & 0 & \cdots & 0 \\ x_1(2) & x_1(1) & \cdots & 0 & \cdots & x_{N_t}(2) & x_{N_t}(1) & \cdots & 0 \\ & \vdots & & & \vdots & & & \vdots & \\ x_1(N) & x_1(N-1) & \cdots & 0 & \cdots & x_{N_t}(N) & x_{N_t}(N-1) & \cdots & 0 \\ 0 & x_1(N) & \cdots & 0 & \cdots & 0 & x_{N_t}(N) & \cdots & 0 \\ & \vdots & & & \cdots & & & \vdots & \\ 0 & 0 & \cdots & x_1(1) & \cdots & 0 & 0 & \cdots & x_{N_t}(1) \end{bmatrix},$$

the equivalent channel coefficient matrix is

$$H_{eq} = \begin{bmatrix} h_{1,1}^{(0)} & h_{1,2}^{(0)} & \cdots & h_{1,N_r}^{(0)} \\ h_{1,1}^{(1)} & h_{1,2}^{(1)} & \cdots & h_{1,N_r}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1,1}^{(L-1)} & h_{1,2}^{(L-1)} & \cdots & h_{1,N_r}^{(L-1)} \\ h_{2,1}^{(0)} & h_{2,2}^{(0)} & \cdots & h_{2,N_r}^{(0)} \\ h_{2,1}^{(1)} & h_{2,2}^{(1)} & \cdots & h_{2,N_r}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{2,1}^{(L-1)} & h_{2,2}^{(L-1)} & \cdots & h_{2,N_r}^{(L-1)} \\ \vdots & \vdots & \vdots & \vdots \\ h_{N_t,1}^{(0)} & h_{N_t,2}^{(0)} & \cdots & h_{N_t,N_r}^{(0)} \\ h_{N_t,1}^{(1)} & h_{N_t,2}^{(1)} & \cdots & h_{N_t,N_r}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_t,1}^{(L-1)} & h_{N_t,2}^{(L-1)} & \cdots & h_{N_t,N_r}^{(L-1)} \end{bmatrix},$$

and the matrices Y and N are (N +L−1)×Nr matrices of the received signals and noise terms whose (k,j)th element denotes the corresponding quantity for the kth time instant and jth antenna.
It is easy to observe the maximum diversity order achievable for a MIMO FS fading channel. With the quasi-static Rayleigh fading channel model, there are NtNrL independent fading coefficients, therefore by a proper design of the space-time code, it is possible to obtain a diversity of order NtNrL. Clearly, the presence of multipath, in fact, pro- vides us with a better performance compared with the flat fading scenario.

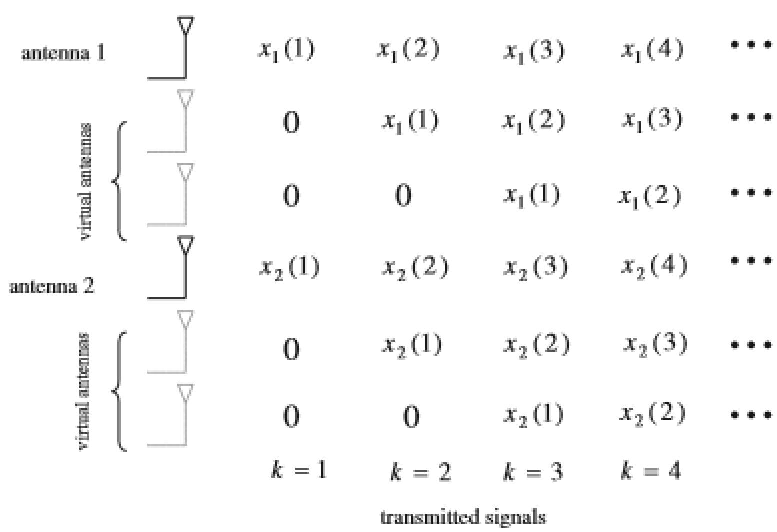

Figure 9.4 Virtual antenna interpretation of transmission over MIMO FS channels (with $N_t = 2$ and $L = 3$).

Although the available diversity order may be high, care must be taken in code design to exploit it. A space-time code achieves full diversity over a MIMO FS channel if for any pair of codewords (of the form Xeq whose form is given above), the corresponding codeword difference matrices are of full column rank.
Q5. Evaluate Full Diversity Code for MIMO FS Channels.

For an Nt transmit antenna system, a delay diversity code is obtained by simply transmitting delayed versions of the sequence on the first transmit antenna from the other antennas. That is, the ith transmit antenna emits a delayed version of the sequence (by i −1 symbols) on the first transmit antenna. By this method, it is easy to see that for flat fading, space-time codewords are of the form

$$\begin{bmatrix} x(1) & 0 & 0 & \cdots & 0 \\ x(2) & x(1) & 0 & \cdots & 0 \\ x(3) & x(2) & x(1) & \cdots & 0 \\ & \vdots & & \vdots & \\ x(N_t) & x(N_t-1) & x(N_t-2) & \cdots & x(1) \\ x(N_t+1) & x(N_t) & x(N_t-1) & \cdots & x(2) \\ & \vdots & & \vdots & \\ x(N) & x(N-1) & x(N-2) & \cdots & x(N-N_t+1) \end{bmatrix}.$$

To extend this idea to the case of MIMO FS channels, we need a slight modification. It is easy to see that delaying by a single symbol for each transmit antenna will not work, because some columns of the equivalent space-time codeword matrix (with the use of virtual antenna interpretation), will be identical, hence the code will not be of full rank. However, it is straightforward to see that, if we delay the transmitted signal of the first antenna by the number of ISI taps L instead of by a single symbol, we obtain the equivalent space-time codeword (for the MIMO FS channel) given as

$$\begin{bmatrix} x(1) & 0 & 0 & \cdots & 0 \\ x(2) & x(1) & 0 & \cdots & 0 \\ x(3) & x(2) & x(1) & \cdots & 0 \\ & \vdots & & \vdots & \\ x(N_tL) & x(N_tL-1) & x(N_tL-2) & \cdots & x(1) \\ x(N_tL+1) & x(N_tL) & x(N_tL-1) & \cdots & x(2) \\ & \vdots & & \vdots & \\ x(N) & x(N-1) & x(N-2) & \cdots & x(N-N_tL+1) \end{bmatrix}.$$

Q6. What are the disadvantages of STBC over MIMO FS Channel.

When we use space-time block codes over MIMO FS channels, much of the benefits of space-time block coding, particularly in the decoding process, are lost. This is because, with the presence of ISI, it not trivial to perform simple linear processing to decouple the decisions of different symbols in ML decoding.

Q7. Examine the features of Space-Time Trellis Codes for MIMO FS Channels.

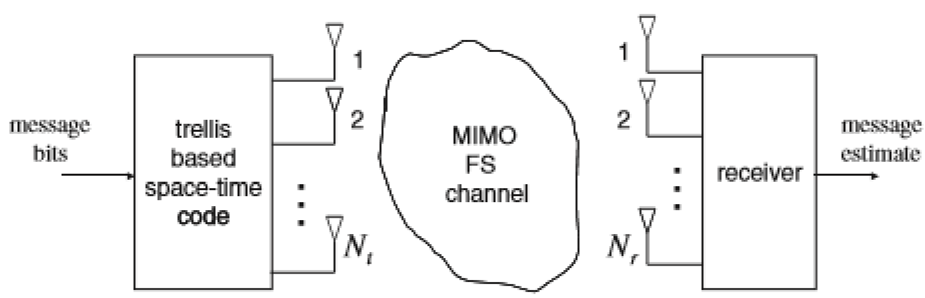

Figure 9.5 Block diagram of space-time trellis coding over a MIMO FS channel.

Consider a simple two-antenna four-state space-time code (of rate 1 bits per channel use) with BPSK described by two convolutional codes with connection polynomials 1+D2 (D refers to the delay element) for the first antenna, and 1+D+D2 for the second antenna, i.e., $(5,7)_{octal}$ convolutional code. Clearly, this describes a four-state space-time code. Let us assume that each sub-channel has two taps. For instance, if there is a single receive antenna, the received signal is given by (ignoring noise)

$$y(n) = h_1^{(1)} x_1(n) + h_1^{(2)} x_1(n-1) + h_2^{(1)} x_2(n) + h_2^{(2)} x_2(n-1),$$

Where x1 and x2 denote the transmitted signals from the two antennas. Clearly, if the sequence of bits being encoded by the space-time code are represented by s(n), we have

$x1(n) = s(n) \oplus s(n-2)$ and

$x2(n) = s(n) \oplus s(n-1) \oplus s(n-2)$.

Combining these with the expression in Equation and assuming that the channel state information is available at the receiver, it is clear that one can describe the “noiseless” received signal using an eight-state trellis diagram where the states are formed by the three previous input bits ($s(n-1)$,$s(n-2)$ and $s(n-3)$). The resulting trellis diagram is shown in Figure where we have used a vector of length four to illustrate the branch labels. These denote the signals $x1(n)$,$x1(n-1)$,$x2(n)$,$x2(n-1)$. As an example, corresponding to the label $(-1,+1,-1,-1)$, the received signal in the absence of noise is $-h_1^{(1)} + h_1^{(2)} - h_2^{(1)} - h_2^{(2)}$.

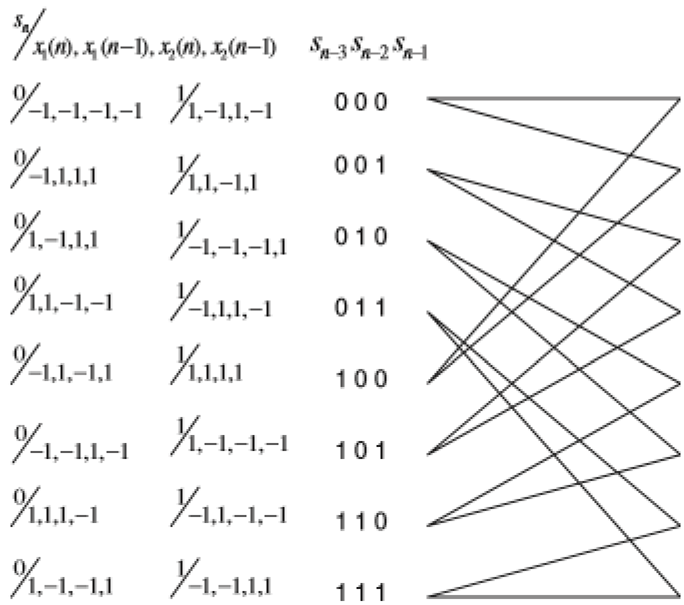

Figure 9.6 Combined code and channel trellis for $(5, 7)_{octal}$ convolutional code over a two-input two-tap FS channel.

The same representation can be used for more than one receive antenna as well in a straight- forward manner. We further note that this is by no means a unique way of representing the space-time code together with the ISI channel; it only represents one possible approach. The joint trellis representation of the space-time trellis code and the frequency selective channel is useful in the decoding of the code. Since the channel state information is assumed to be known at the receiver, and the noise terms are independent both spatially and tem- porally, minimization of the squared Euclidean distance between the noiseless sequences through the joint trellis and the received signal sequence will minimize the sequence error probability, hence it is optimal. Furthermore, this can be implemented using the well-known Viterbi algorithm.

Q8. Demonstrate Concatenated Coding for MIMO FS Channels.

The basic block diagram of a concatenated coding approach for MIMO FS channels is shown in Figure. An outer code which can be a space-time trellis code or a convolutional code is concatenated with an M-ary modulator through an interleaver with a certain length, and then the modulated symbols are transmitted through the ISI channel. This is basically the same approach we have used over MIMO flat fading channels.

For demodulation/decoding, the MIMO channel is equalized using a soft-output equalizer, for instance, implemented via the MAP algorithm. The soft-output equalizer produces likelihoods of the M-ary

symbols transmitted. The symbol level likelihood information is then converted to bit level likelihoods, and these are used in the decoding of the information bits using the code constraints. The only difference in this scheme, compared with the frequency flat fading case, is the presence of the soft-output equalizer.
The outer code may take many different forms. For instance, it can be a convolutional code, a parallel concatenated convolutional code , a serial concatenated convolutional code, a space-time trellis code, or even an LDPC code. For the case of a convolutional code or a space-time trellis code, we simply have its concatenation with an ISI channel which may be referred to as a serial concatenation scheme. We note that in such a scheme using a precoder after interleaving may improve the performance by effectively making the channel constraints "recursive" (as in serially concatenated convolutional code in the standard turbo-coding context). The outer decoder, in this case, can be implemented using the Viterbi algorithm, or a MAP type algorithm. For the case of a turbo code, as the decoder of the outer code, we would need an iterative algorithm (implemented using component MAP type decoders). For the case of an outer LDPC code, decoding can be performed using the usual message-passing algorithm.
It is also possible to use the idea of iterative "turbo" equalization at the receiver to improve the error rate performance of the system. Basically, the outer channel decoder can be implemented using a soft-output algorithm, and the extrinsic (new) information that it produces can be passed back to the MIMO channel equalizer. This information can be used to update the symbol likelihoods, hence the bit likelihoods, to be used in the next iteration of the channel decoder. This process is illustrated using the dashed lines in Figure of decoder.

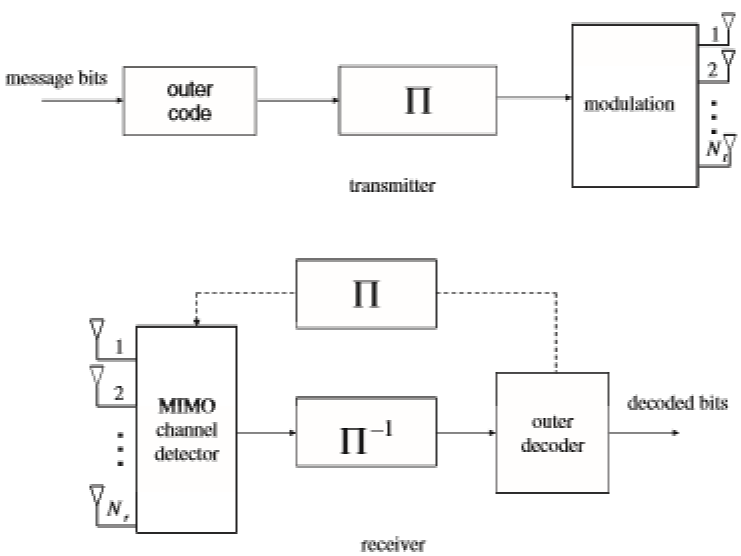

Figure 9.9 Block diagram of the concatenated coding approach.

Q9. Elaborate on Spatial Multiplexing for MIMO FS Channels.
There is basically no difference in the encoding approaches for MIMO flat fading and FS fading channels.. Whatever can be used for the case of flat fading will also be easily extended to the case of frequency selective fading channels. On the other hand, there are differences in the receiver design. In the case of flat fading, the receiver algorithms basically spatially equalize the received signal using different algorithms. For instance, zero forcing or MMSE receivers are employed. In the case of MIMO FS channels, the receiver works in the same way, however, in addition to the spatial equalization, it performs temporal equalization.

Q10. List out the features for Channel estimation for MIMO FS Channel.
In addition to channel equalizers for uncoded MIMO systems, we also consider soft-input soft-output type

MIMO equalizers to be used in conjunction with channel coding. The fundamental problem is that, we need to do equalization both temporally and spatially due to the simultaneous transmission of different data streams from different antennas.

MIMO FS channels can be described by a trellis with $M^{Nt(L-1)}$ states where M is the constellation size. If uncoded data streams are being transmitted simultaneously from different antennas, there are Nt symbols for each period. Hence the number of branches emanating from each state is M^{Nt}. Once the trellis diagram is obtained using standard methods, the optimal detection problem is easy. If our objective is to minimize the sequence error probability, we can simply use the Viterbi algorithm. If we would like to minimize the symbol error probability, we can resort to the MAP algorithm (which is the same one used in the turbo-decoding process as a component decoder for each constituent convolutional code).

One problem with the full complexity algorithms, e.g., the Viterbi or MAP algorithm, is the associated complexity. The number of states in the trellis grows exponentially with the product of the number of transmit antennas and the length of the ISI channel. This quickly becomes prohibitively complex even with relatively short ISI levels. For instance, with BPSK modulation, if the number of antennas is four, with an ISI length of L =5, the number of states needed for optimal equalization is 216 =65,536 which is too large. Therefore, we need other simplified approaches for channel detection.

Q11. Evaluate Linear Equalization for MIMO FS Channels.

Consider an uncoded VBLAST transmission over a MIMO FS channel, and assume that the channel coefficients for all the antenna pairs and for all the channel taps are known. A simple way of estimating the transmitted bits based on the observations at the receiver is to employ a spatially and temporally linear equalizer.

The estimate of the kth symbol transmitted from the ith antenna is given by

$$\hat{x}_i(k) = \sum_{j=1}^{N_r} \sum_{n=-N_1}^{N_2} a_{j,n} y_j(k-n),$$

where $a_{j,n}$ are the set of filter coefficients and N1 +N2 +1 is the length of the filter used. Given the fading channel coefficients, the equalizer coefficients can be found using different constraints, such as zero forcing, or MMSE. The solutions will be very similar to the case of single-transmit single-receive antenna systems. In practice, if the channel coefficients are not available, adaptive solutions can also be derived.

Q12. Explain Decision Feedback Equalization for MIMO FS Channels.

As an alternative to linear equalization, decision feedback equalization can also be extended to the case of MIMO communications. The approach in this case is to use two sets of filters (feedforward and feedback), where the feedback filter operates on already detected symbols with the objective of canceling out their contribution from the observation vector. Due to the non-linearity in the decision process (on the prior symbols), the overall equalizer is no longer linear.

A block diagram of the MIMO decision feedback equalizer (DFE) is shown in Figure. As in the case of a linear equalizer, different criteria with different restrictions on the feedforward and feedback filters can be employed, such as the use of finite-length filters with MMSE. Also, adaptive solutions can be derived using a training-based approach if the channel coefficients are not known at the receiver.

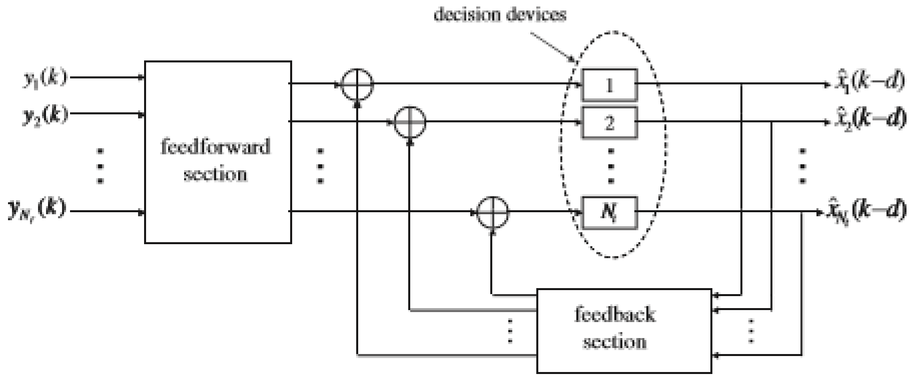

Figure 9.12 Block diagram of a MIMO-DFE.

Q13.Hypothesize Soft-Input Soft-Output Channel Detection.

Consider coded transmission over MIMO FS channels. In particular, for the case of con- catenated coding (with an interleaver) over MIMO FS channels, in order to be able to run the outer decoder we often need soft outputs about the coded bits. This is directly provided by the MAP type equalization algorithms. However, for other equalization approaches they may not be readily available. Both linear and decision feedback equalizers can be made to accept soft information about the input symbols, and produce soft outputs. Therefore, instead of the MAP type optimal (soft-output) channel equalizer, these suboptimal solutions can be used.

After the likelihood information about the symbols transmitted are obtained, we can compute the likelihoods of the coded bits, then these can be used in decoding the information bits using the outer code constraints. Therefore, reduced complexity linear and DFE based MIMO equalizers can be incorporated in equalization/decoding of coded systems (with or without turbo equalization

Q14. Demonstrate MIMO OFDM systems.

we have focused on single carrier transmission over frequency selective fading channels, which requires sophisticated equalization techniques to overcome the intersymbol interference problem. Another way to deal with such channels and to avoid relatively complex signal processing algorithms is to employ multi-carrier transmissions used in the same frequency band.

The idea is to split the overall band available into N frequency sub-bands so that each sub-band (of bandwidth W/N) experiences flat fading. The symbol duration on each multi-carrier component is ~ N/W which can be made much larger than the multipath spread of the channel, hence the subchannels do not experience ISI (or, the effects of ISI are reduced to a desired level). The overall transmission rate is still ~ N ·W/N = W symbols per second, i.e., the same as the original single-carrier system. A simple and efficient way of achieving multi-carrier modulation is to employ orthogonal sub-carriers (that are separated by 1/Ts Hz where Ts is the symbol duration), and use the DFT pair (more precisely, fast Fourier transform (FFT) and its inverse) for implementation. The idea is to realize that with 1/Ts frequency separation, samples of the overall signal to be transmitted over N sub-carriers (at a rate 1/Ts) can be obtained using the N-point inverse DFT (IDFT) or inverse FFT (IFFT) of the data sequence of the N sub-carriers. At the receiver the DFT (or FFT) is applied, and the standard receiver algorithms are used. The resulting scheme is called orthogonal frequency division multiplexing. The idea of multi-carrier modulation, or OFDM, can be applied to MIMO FS channels as well, as shown in Figure.

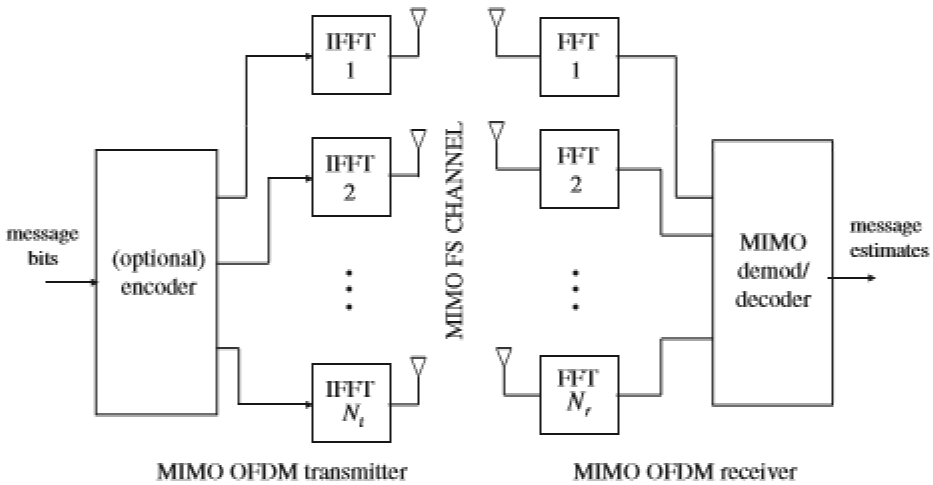

Figure 9.14 Block diagram of a MIMO-OFDM system.

For each transmit antenna element, the inverse DFT of N (uncoded or coded) M-ary symbols, denoted by

$$X_i(0), X_i(1), X_i(2), \ldots, X_i(N-1)$$

is first computed. Then a cyclic prefix is appended to each of these sequences, and the resulting signal is transmitted (after digital-to-analog conversion). The objective in adding the cyclic prefix is to remove possible interference between two consecutive OFDM symbols, and to make sure that the equivalent channel (after DFT at the receiver) has a simple form which allows for easier processing. For each of the received signals (at the Nr antenna elements), the DFT of the aggregate received signal (superposition of all the OFDM words from each of the transmitted antennas) is calculated, and the cyclic prefix is removed. The resulting set of signals is then used for demodulation/decoding.

Q15. Model the MIMO OFDM Channel.

Consider a MIMO-OFDM system with N subcarriers used over a MIMO FS channel with L ISI taps. Assume that the fading coefficients are spatially uncorrelated, and that they remain constant over one OFDM symbol. The transmitted signal over Nt transmit antennas can be represented by an N ×Nt matrix, XOFDM whose (n,i)th element is the symbol transmitted at subcarrier n on transmit antenna i, Xi(n). We assume that there is a power constraint over the MIMO-OFDM word such that

$$\sum_{n=0}^{N-1}\sum_{i=1}^{N_t} E[|X_i(n)|^2] = N.$$

The transmitter simply takes the IFFT of the columns of the codeword matrix XOFDM, appends cyclic prefix, and transmits the resulting sequence. The role of the cyclic prefix is to convert the circular convolution to linear convolution as described below. Also, it can be used for synchronization purposed. Obviously, there is some overhead involved with the inclusion the cyclic prefix, however, the length of the ISI is usually much smaller than the OFDM word, thus the overhead required is small. At the jth receive antenna, after applying the FFT and removing of the cyclic prefix, the resulting signal for the nth subcarrier is given by

$$y_j(n) = \sqrt{\rho}\sum_{i=1}^{N_t} X_i(n)H_{i,j}(n) + n_j(n), \tag{9.20}$$

where the equivalent channel coefficient from the ith to the jth antenna for the nth sub-carrier is given by

$$H_{i,j}(n) = \sum_{l=0}^{L-1} h_{i,j}^{(l)} e^{-j2\pi nl/T}. \tag{9.21}$$

Denoting the channel coefficients for the (i, j)th antenna pair by

$$\boldsymbol{H}_{i,j} = [H_{i,j}(0) \quad H_{i,j}(1) \quad \cdots \quad H_{i,j}(N-1)], \tag{9.22}$$

we can see that

$$\boldsymbol{H}_{i,j} = \boldsymbol{W}\boldsymbol{A}_{i,j}, \tag{9.23}$$

where $\boldsymbol{W}$ is the DFT (or FFT) matrix, and

$$\boldsymbol{A}_{i,j} = [h_{i,j}^{(0)} \quad h_{i,j}^{(1)} \quad \cdots \quad h_{i,j}^{(L-1)}],$$

i.e., the vector of L independent fading coefficients (for the L different paths of the ISI channel). Since we have a total of N equivalent channel coefficients derived from L in- dependent random variables (with N typically much larger than L), it is clear that in the equivalent channel model above the fading coefficients for distinct subcarriers are different but dependent. The maximum rank of the channel gain matrix (in the frequency domain) is the number of ISI taps L, hence it is a low-rank matrix in general.