Network and Integer Optimization - Problem Set 1

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5 Characterization of bipartite graphs

Assume G is bipartite and let $v_1, v_2, \ldots, v_k, v_1$ be the vertices along some arbitrary cycle $C \subseteq E$ in G. We have $|C| = k$, so we aim to show that k is even. As $V = X \cup Y$ is a partition, we assume w.l.o.g. that $v_1 \in X$. We prove that for all $i \in [k]$ the following holds:

If i is odd, then $v_i \in X$ and otherwise $v_i \in Y$.

We proceed by induction. For $i = 1$ the statement holds by our assumption. Now assume the statement holds for some arbitrary but fixed $i \in [k-1]$. If i is odd, then by inductive hypothesis, $v_i \in X$ and we know that $\{v_i, v_{i+1}\} \in C \subseteq E$, so we must have $v_{i+1} \in Y$ as G is bipartite. The other case where i is even is analogous.

Now, as $\{v_k, v_1\} \in C \subseteq E$ and $v_1 \in X$, we must have $v_k \in Y$, meaning k is even by the above.

We now prove the other direction. Assume that G contains no odd cycles. We assume w.l.o.g. that G is connected, as otherwise, one can apply this proof onto each connected component. We provide a partitioning of V that certifies bipartiteness. Let $s \in V$ be arbitrary and define:

$$X = \{v \in V \mid d(s, v) \text{ is even}\}$$
$$Y = \{v \in V \mid d(s, v) \text{ is odd}\}$$

We proceed by contradiction. Assume that the given partition does not satisfy the requirements, so w.l.o.g. there are vertices $v, w \in X$ with $v \neq w$ s.t. $\{v, w\} \in E$. We must have that $d(s, v) = d(s, w)$, for otherwise, as both quantities are even, we would have $|d(s, v) - d(s, w)| \geq 2$, which would contradict the fact that $\{v, w\} \in E$. Let $k = d(s, v) = d(s, w)$ and consider an $s - v$ path P of length k, described via the vertex sequence:

$$v_0, v_1, \ldots, v_k$$

where $v_0 = s$ and $v_k = v$. Similarly, we describe an $s - w$ path Q of length k via the vertex sequence:

$$w_0, w_1, \ldots, w_k$$

where $w_0 = s$ and $w_k = w$. We claim that for any $i, j \in [k]$, if $v_i = w_j$, then we must have $i = j$. Indeed, if there are $i, j \in [k]$ with $i < j$ and $v_i = w_j$, then the $s - w$ walk given by the vertices:

$$v_0, \ldots, v_i, w_{j+1}, \ldots, w_k$$

has length $i + k - j = k + (i - j) < k$, a contradiction. Now, define:

$$i^* = \max\{i \in \{0, \ldots, k-1\} \mid v_i = w_i\}$$

This quantity is well-defined as $v_0 = s = w_0$. We have that the two subpaths:

$$v_{i^*}, \ldots, v_k$$
$$w_{i^*}, \ldots, w_k$$

are disjoint except for their first common vertex $v_{i^*} = w_{i^*}$. This means that the sequence:

$$v_{i^*}, \ldots, v_k, w_k, w_{k-1}, \ldots, w_{i^*}$$

is a cycle in G of length $2(k - i^*) + 1$, which is odd, a contradiction. This proves that our proposed partition of V indeed certifies bipartiteness of G.