Network and Integer Optimization - Problem Set 2

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1 Matchings – Quiz

(i)
- M_1 is not a maximum cardinality matching as the path described by the vertex sequence $v_1, v_4, v_{11}, v_{14}, v_{13}, v_{12}, v_6, v_8$ is augmenting.
- M_2 is a maximum cardinality matching. The only two exposed vertices are v_6 and v_8 and one can check that there is no augmenting path connecting them.

(ii)
- This statement is true. Assume for a contradiction that there are $u, v \in V \setminus C$ with $u \neq v$ and $\{u, v\} \in E$. Then, there is no vertex in C touching the edge $\{u, v\}$, contradicting that C is a vertex cover.
- This statement is false. Consider the triangle graph K_3. Then any two vertices constitute a minimum vertex cover, but the edge connecting these vertices is incident to both of them.
- This statement is **false**.

2 Night shift planning

(a) We construct the directed graph $G = (V, A)$, where:

$$\begin{aligned}
V &= \{s, t\} \cup P \cup (P \times [k]) \cup N \\
A &= (\{s\} \times P) \\
&\quad \cup \{(p, (p, i)) \mid p \in P, i \in [k]\} \\
&\quad \cup \{((p, i), n) \mid p \in P, i \in [k], n \in N_i\} \\
&\quad \cup (N \times \{t\})
\end{aligned}$$

along with a capacity function $u : A \to \mathbb{Z}_{\geq 0}$ defined as:

$$\forall j \in [q] \cdot u(s, p_j) = \alpha_j$$
$$\forall p \in P, \forall i \in [k] \cdot u(p, (p, i)) = 2$$
$$\forall j \in [q], \forall i \in [k], \forall n \in N_i \cdot u((p_j, i), n) = \begin{cases} 1 & \text{if } n \in W_j \\ 0 & \text{otherwise} \end{cases}$$
$$\forall n \in N \cdot u(n, t) = 2$$

To solve the problem, we compute an integral maximum $s - t$ flow f in G with capacities u. A feasible assignment exists iff $\nu(f) = 2n$. To retrieve the actual assignments, we use f as follows.

A physician p_j is assigned to a night $n \in N_i$ iff $f((p_j, i), n) = 1$.

We now show correctness. The first kind of arc capacities ensure that the limits on the number of night shifts physicians can be assigned to are respected. The second kind of arc capacities ensure that no physician may be assigned to more than 2 night shifts per block. The third kind of capacities ensure that physicians may only be assigned to nights on which they are available. Finally, the fourth class of constraints ensure that to each night, at most 2 physicians are assigned. In a maximum flow, these arcs have to be saturated, so a maximum flow will correspond to a feasible assignment.

The graph clearly has size polynomial in the input parameters, and one can use an algorithm such as Edmonds-Karp to compute the maximum flow in polynomial time.

(b) Let $k \in \mathbb{Z}_{\geq 0}$. Replace the numbers α_j by $\alpha'_j = \min(k, \alpha_j)$, and run the above algorithm. We use binary search on the range $[n]$ to determine the smallest possible k for which a feasible assignment exists with these modified parameters. Clearly, this increases the running time by a logarithmic factor, so the procedure is still efficient.

(c) See accompanying notebook.

3 A linear time 2-approximation for vertex cover

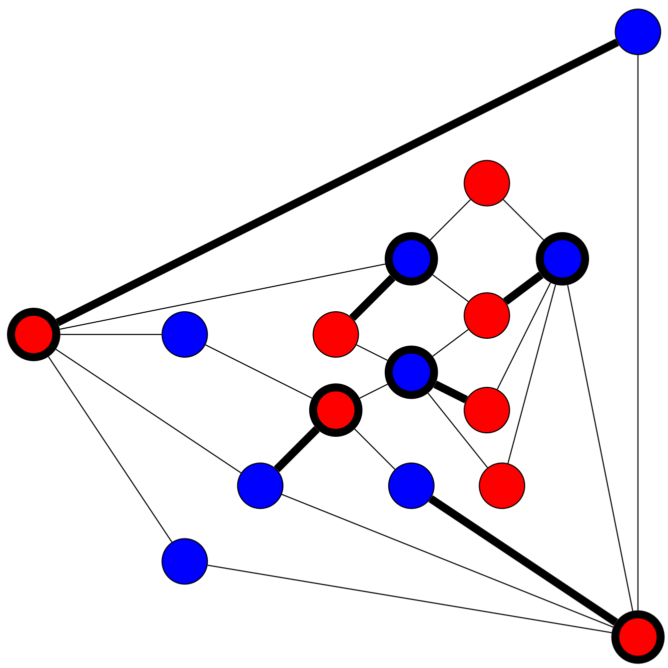
We first initialize an empty matching $M \leftarrow \emptyset$. We then iterate through all edges $e \in E$ and at each iteration, we add e to M if and only if both of its endpoints are exposed. This greedy procedure returns a maximal matching $M \subseteq E$. Then, we set $C = \cup_{e \in M} e$, i.e. the vertex cover we build consists of all non-exposed vertices in the matching M. Clearly, both phases of this algorithm can be implemented in $\mathcal{O}(|E|)$ time, so we have a linear-time algorithm. We now prove correctness and approximation quality.

Assume for a contradiction that C is not a vertex cover, i.e. there are $u, v \in V \setminus C$ with $\{u, v\} \in E$. This means, both u and v are exposed in M, so that $M \cup \{\{u, v\}\}$ is also a matching, contradicting maximality of M. Thus, C must be a vertex cover.

Now, we can bound the size of C as follows:

$$|C| = 2|M| \leq 2\nu(G) \leq 2\tau(G)$$

4 Maximum cardinality matching and minimum cardinality vertex cover in bipartite graph



We highlight the bipartition via the above color coding. Thick edges describe the maximum cardinality matching and thick vertices make up the minimum cardinality vertex cover.