Network and Integer Optimization - Problem 2.3

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A linear time 2-approximation for vertex cover

We first initialize an empty matching $M \leftarrow \emptyset$. We then iterate through all edges $e \in E$ and at each iteration, we add e to M if and only if both of its endpoints are exposed. This greedy procedure returns a maximal matching $M \subseteq E$. Then, we set $C = \cup_{e \in M} e$, i.e. the vertex cover we build consists of all non-exposed vertices in the matching M. Clearly, both phases of this algorithm can be implemented in $\mathcal{O}(|E|)$ time, so we have a linear-time algorithm. We now prove correctness and approximation quality.

Assume for a contradiction that C is not a vertex cover, i.e. there are $u, v \in V \setminus C$ with $\{u, v\} \in E$. This means, both u and v are exposed in M, so that $M \cup \{\{u, v\}\}$ is also a matching, contradicting maximality of M. Thus, C must be a vertex cover.

Now, we can bound the size of C as follows:

$$|C| = 2|M| \leq 2\nu(G) \leq 2\tau(G)$$