PROB 14, Given X_1, X_2 & λ parameter

$Z = X - Y$ & $W = X + Y$ (LET $Y = X_2$)

Regular Convolution integral: $f_W(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx$

For $W = X + Y$

But we want: $Z = X - Y$ which is the same as $Z = X + (-Y)$

$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_{-Y}(z-x)\,dx$

because $f_{-Y}(z-x) = f_Y(x-y)$

now we can write:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(x-z)\,dx, \quad \text{But } f_X(x) = \begin{cases} 0, & x<0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(x-z)}\,dx$$

$$= \lambda^2 e^{\lambda z} \int_{-\infty}^{\infty} e^{-2\lambda x}\,dx$$

let $u = -2\lambda x$, $\frac{du}{-2\lambda} = dx$

$$= -\frac{\lambda^2 e^{\lambda z}}{2\lambda} \int_{-\infty}^{\infty} e^{u}\,du$$

$$= \frac{\lambda e^{\lambda z}}{2}\left[e^{-2\lambda x}\right]_0^{\infty} = \frac{-\lambda e^{\lambda z}}{2}[0-1] = \frac{\lambda e^{\lambda z}}{2}$$

Now we know X, Y IID $\Rightarrow Z = X - Y$ & $-Z = Y - X$

AND By symmetry $\Rightarrow Z \stackrel{d}{=} -Z$,

$$f_Z(z)\begin{cases} \frac{\lambda}{2} e^{\lambda z}, & z<0 \\ \frac{\lambda}{2} e^{-\lambda z}, & z \geq 0 \end{cases} \Bigg\} \quad \boxed{f_Z(z) = f_Z(-z) = \frac{\lambda}{2} e^{-\lambda |z|}}$$

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