Quantum Field Theory Exercises week 2

Exercise 2 (continued)
Complete exercise 2.1 from Peskin and Schroeder. Don't forget to use the additional hints given in last week's exercises.

Exercise 3: A free complex scalar doublet (example of extended gauge symmetries)

Consider the free quantum field theory for two complex-valued scalar fields $\phi_1(x)$ and $\phi_2(x)$ with the same mass. The Lagrangian of this theory is given by

$$\mathcal{L} \;=\; (\partial_\mu \phi_1^*)(\partial^\mu \phi_1) + (\partial_\mu \phi_2^*)(\partial^\mu \phi_2) - m^2(\phi_1^* \phi_1 + \phi_2^* \phi_2).$$

This free quantum field theory resembles the one for a doublet of Higgs fields. It is easiest to analyze this theory by considering $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$ as basic dynamical variables, rather than the real and imaginary parts of $\phi_{1,2}(x)$.

(a) Find the equations of motion for $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$.

(b) Determine the conjugate momenta belonging to $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$.

(c) Show that the Hamiltonian is given by

$$H \;=\; \int \mathrm{d}\vec{x} \left[\pi_1^*\pi_1 + \pi_2^*\pi_2 + (\vec{\nabla}\phi_1^*)\cdot(\vec{\nabla}\phi_1) + (\vec{\nabla}\phi_2^*)\cdot(\vec{\nabla}\phi_2) + m^2(\phi_1^*\phi_1 + \phi_2^*\phi_2) \right] .$$

(d) View the fields $\phi_{1,2}$ as being components of a vector $\vec{\phi}$ and rewrite the Lagrangian in terms of this vector. Then show that the Lagrangian is invariant (i.e., does not change) under the continuous transformation

$$\vec{\phi}(x) \;\to\; \exp(i\alpha)\,\vec{\phi}(x) \qquad\quad (\alpha \in \mathbb{R} \text{ independent of } x) .$$

This is called a global $U(1)$ transformation, where global refers to the fact that α is independent of x and $U(1)$ indicates that all fields are multiplied by a phase factor.

Determine the corresponding conserved Noether current and charge.

(e) Show that the Lagrangian is invariant under the continuous global $SU(2)$-transformations

$$\vec{\phi}(x) \;\to\; \exp(i\alpha^k\sigma^k)\,\vec{\phi}(x) \qquad\quad (\alpha^{1,2,3} \in \mathbb{R} \text{ independent of } x) ,$$

with σ^k the Pauli matrices and with summation over $k = 1,2,3$ implied.

Determine the corresponding three conserved Noether currents and charges.