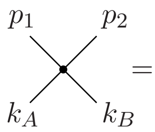
Solution 12:

(a) The only Feynman diagram at lowest order is given by



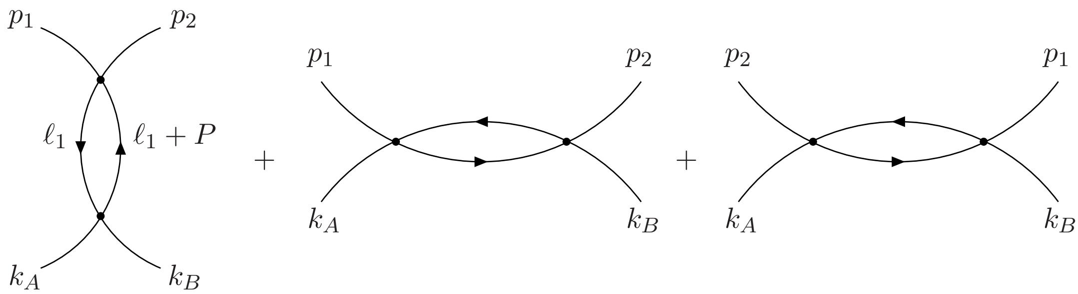
$$= -i\lambda = i\mathcal{M} .$$

So

$$\frac{1}{2}\int d\Pi_2 |\mathcal{M}|^2 = \frac{1}{2}\lambda^2 \int d\Pi_2 \stackrel{p.\,61}{=\!=} \frac{1}{2}\frac{\lambda^2 |\vec{p}\,|}{16\pi^2 E_{\mathrm{CM}}} \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \stackrel{|\vec{p}|=E_{\mathrm{CM}}/2}{=\!=\!=} \frac{\lambda^2}{16\pi} ,$$

where the factor $\frac{1}{2}$ is due to the fact that we have to restrict the integration to the inequivalent configurations of the identical particles (see page 62 in the lecture notes).

(b) The three diagrams that contribute now are:



with ℓ_1 the loop momentum and P the total initial state momentum. These diagrams are exactly the same, except that the momenta have been interchanged. Examining the combinations of momenta closely, you can see that (not surprisingly) the combinations yield the Mandelstam variables (e.g. $P^2 = s$). So at one loop we have (if you do not see it, work it out explicitly):

$$i\mathcal{M}|_{1-loop} = f(s) + f(t) + f(u) ,$$

where s, t and u are the Mandelstam variables and f is a function that we will determine from the first diagram (the s-channel diagram). For massless particles the first diagram yields

$$f(s) = \frac{1}{2}\int \frac{d^4\ell_1}{(2\pi)^4} \frac{\lambda^2}{[\ell_1^2 + i\epsilon][(\ell_1 + P)^2 + i\epsilon]} .$$

We can rewrite this integral using a Feynman parameter α and shifting the integration variable:

$$\begin{aligned} f(s) &= \frac{\lambda^2}{2(2\pi)^4}\int_0^1 d\alpha \int d^4\ell_1 \frac{1}{[\alpha(\ell_1 + P)^2 + (1-\alpha)\ell_1^2 + i\epsilon]^2} \\ &= \frac{\lambda^2}{2(2\pi)^4}\int_0^1 d\alpha \int d^4\ell \frac{1}{[\ell^2 - \Delta_s + i\epsilon]^2} , \end{aligned}$$

with

$$\Delta_s \;=\; \alpha^2 s - \alpha s \;\leq\; 0\;.$$

For the t and u channels, we have similar quantities Δ_t and Δ_u. However, these are always positive (or 0) because $t\,, u \leq 0$, whereas $s \geq 0$. A Wick-rotation of the integral yields

$$f(s) \;=\; \frac{i\lambda^2}{2(2\pi)^4}\int_0^1 d\alpha \int d^4\ell_E\,\frac{1}{[\ell_E^2+\Delta_s-i\epsilon]^2} \;=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha\int_0^\infty d\ell_E^2\,\frac{\ell_E^2}{[\ell_E^2+\Delta_s-i\epsilon]^2}\;.$$

Now introducing the cut-off Λ^2, we can calculate the integral:

$$\begin{aligned}
f(s) \;&=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha\int_0^{\Lambda^2} d\ell_E^2\,\frac{\ell_E^2+\Delta_s-i\epsilon-\Delta_s+i\epsilon}{[\ell_E^2+\Delta_s-i\epsilon]^2}\\
&=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha\left[\log(\ell_E^2+\Delta_s-i\epsilon)+\frac{\Delta_s-i\epsilon}{\ell_E^2+\Delta_s-i\epsilon}\right]_{\ell_E=0}^{\ell_E=\Lambda^2}\\
&=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha\left[\log(\Lambda^2)-\log(\Delta_s-i\epsilon)-1\right]\;,
\end{aligned}$$

where in the last step we used that $\Lambda^2 \gg s$. Now the sign of Δ matters. For the t and u channel, $\Delta \geq 0$ and the imaginary part in the logarithm does not matter. However, $\Delta_s \leq 0$, so the logarithm picks up a $-i\pi$ term. Taking everything together gives

$$\begin{aligned}
i\mathcal{M}|_{1-loop} \;&=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha\left[3\log(\Lambda^2)-3-\log(\Delta_s-i\epsilon)-\log(\Delta_t)-\log(\Delta_u)\right]\\
&=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha\left[3\log(\Lambda^2)-3-\log\big(\alpha[1-\alpha]s\big)+i\pi-\log\big(-\alpha[1-\alpha]t\big)-\log\big(-\alpha[1-\alpha]u\big)\right]\\
&=\; \frac{i\lambda^2}{32\pi^2}\left[3\log(\Lambda^2)+3-\log(s)-\log(-t)-\log(-u)+i\pi\right]\;.
\end{aligned}$$

(c) From (b) it follows that $2\,\mathrm{Im}(\mathcal{M}) = \frac{\lambda^2}{16\pi}$, which is exactly the answer of (a). This imaginary part is caused by the imaginary $i\epsilon$ terms in the propagators and not by the imaginary factors occurring in the vertices.

(d) In order to see the explicit link to on-shell intermediate particles, we replace $f(s)$ by

$$\frac{1}{2}\int\frac{d^4\ell_1}{(2\pi)^4}\,\lambda^2\big[-2i\pi\delta(\ell_1^2)\big]\big[-2i\pi\delta(\ell_1^2+2\ell_1\cdot P+P^2)\big] \;=\; -\frac{\lambda^2}{8\pi^2}\int d^4\ell_1\,\delta(\ell_1^2)\,\delta(\ell_1^2+2\ell_1\cdot P+P^2)\;.$$

Since the delta-functions and integration measure are Lorentz invariant, we can calculate this integral in the CM frame of the collision where $P^\mu = (E_{\rm CM}, \vec{0}\,)$ and $P^2 = s = E_{\rm CM}^2$:

$$-\frac{\lambda^2}{8\pi^2}\int d^4\ell_1\, \delta(\ell_1^2)\, \delta(\ell_1^2 + 2\ell_1\!\cdot\! P + P^2) \;=\; -\frac{\lambda^2}{8\pi^2}\int d^4\ell_1\, \delta(\ell_1^2)\, \delta(2\ell_1^0 E_{\rm CM} + E_{\rm CM}^2)$$

$$=\; -\frac{\lambda^2}{16\pi^2 E_{\rm CM}}\int d\vec{\ell}_1\, \delta(E_{\rm CM}^2/4 - \vec{\ell}_1^{\,2}) \;=\; -\frac{\lambda^2}{4\pi E_{\rm CM}}\int d\,|\vec{\ell}_1|\, |\vec{\ell}_1|^2\, \frac{\delta(E_{\rm CM}/2 - |\vec{\ell}_1|\,)}{2\,|\vec{\ell}_1|} \;=\; -\frac{\lambda^2}{16\pi}\,.$$

The t- and u-channel one-loop diagrams vanish upon putting the intermediate particles on-shell. In that case both delta-function requirements cannot be satisfied simultaneously since that would imply the existence of a CM frame of the two on-shell intermediate particles, which in turn would cause t or u to be positive rather than negative.

(e) Apart from a minus sign this coincides with the imaginary part derived in part (c).