Quantum Field Theory Exercises week 10

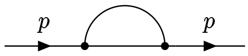
Exercise 13: UV aspects of ϕ^3-theory

Consider the scalar ϕ^3-theory, which has the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3$. The momentum-space Feynman rules for this theory are basically the same as the ones that we have derived for the scalar ϕ^4-theory, except for the fact that the interaction vertex now involves three ϕ-particles instead of four!

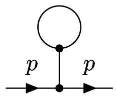
(a) Which amputated diagram contributes to the one-loop corrections to the ϕ^3 interaction?

(b) Argue that this contribution is finite. Why was this to be expected based on the discussion on page 28 of the lecture notes?

(c) Give the analytical expressions for the one-loop diagrams



and



,

with the solid lines indicating the scalar bosons.

Note: you are not supposed to perform the full calculation here!

Remark: the second diagram is usually called a "tadpole diagram". It is not 1-particle irreducible and, consequently, it is not part of the 1-particle irreducible self-energy of the scalar particle. The fact that this tadpole diagram is non-zero implies that the field $\hat{\phi}$ actually has a non-vanishing vev at one-loop order, i.e. $\langle\Omega|\hat{\phi}(x)|\Omega\rangle = v \neq 0$. For a correct particle interpretation the theory should be reformulated in terms of the field $\hat{\phi}'(x) = \hat{\phi}(x) - v$, which has a vanishing vev. This reformulation procedure that removes the tadpole diagram from the scalar self-energy is referred to in the literature as "tadpole renormalization".

(d) Suppose we regularize the loop integrals by means of a UV cutoff Λ. Discuss how the one-loop diagrams in part (c) will depend on this cutoff.

(e) Can the Lagrangian parameter m be finite?

(f) Repeat the power-counting analysis on page 87 of the lecture notes for the ϕ^3 theory. Determine the dimensionalities of spacetime d for which the theory is renormalizable, superrenormalizable and nonrenormalizable.

(g) Compare this to the discussion on page 28 of the lecture notes.

Hint: for d spacetime dimensions the action is given by $S = \int \mathrm{d}^d x\, \mathcal{L}$. Use this to determine the mass dimension of the coupling constant λ for d spacetime dimensions.

(h) Take spacetime to be four-dimensional and draw all superficially divergent 1-particle irreducible one-loop diagrams, indicating their expected dependence on the UV cutoff Λ.

Exercise 14: generators of the Lorentz group

In order to prepare for the discussion on higher-spin theories, please do parts (a) and (b) of this exercise.

(a) Consider an infinitesimal rotation by an angle $\delta\alpha$ about the $\vec{e}_n$-axis. Under this rotation the spatial components of the position four-vector x^μ transform according to

$$\vec{x} \;\to\; \vec{x}\,' \;\approx\; \vec{x} + \delta\alpha\,\vec{e}_n \times \vec{x} \;\equiv\; \vec{x} + \delta\vec{\alpha} \times \vec{x}\ .$$

Rewrite this infinitesimal transformation in the form $x'^\rho = \Lambda^\rho{}_\sigma\, x^\sigma \approx (g^\rho{}_\sigma + \omega^\rho{}_\sigma)x^\sigma$. We now want to determine $\omega_{\rho\sigma}$, with both indices down. Show that the nonzero $\omega_{\rho\sigma}$ components are given by $\omega_{12} = -\,\omega_{21} = (\delta\alpha)^3$, $\omega_{23} = -\,\omega_{32} = (\delta\alpha)^1$ and $\omega_{31} = -\,\omega_{13} = (\delta\alpha)^2$.

(b) Consider an infinitesimal boost with velocity $\delta\vec{v}$. Under this boost the position four-vector transforms according to:

$$x^0 \;\to\; x^{0'} \;\approx\; x^0 + \delta\vec{v}\cdot\vec{x} \qquad \text{and} \qquad \vec{x} \;\to\; \vec{x}\,' \;\approx\; \vec{x} + x^0\delta\vec{v}\ .$$

Rewrite this infinitesimal transformation in the form $x'^\rho = \Lambda^\rho{}_\sigma\, x^\sigma \approx (g^\rho{}_\sigma + \omega^\rho{}_\sigma)x^\sigma$. Show that the nonzero $\omega_{\rho\sigma}$ components are given by $\omega_{0k} = -\,\omega_{k0} = (\delta v)^k$ for $k = 1,2,3$.

(c) On page 11 of the lecture notes the following has been derived for infinitesimal Lorentz transformations:

$$\phi(x) \;\to\; \phi'(x) \;=\; \phi(\Lambda^{-1}x) \;\approx\; \Big(1 - \frac{i}{2}\,\omega_{\mu\nu}J^{\mu\nu}\Big)\phi(x)\ ,$$

where $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ are the six generators of the infinitesimal Lorentz transformations. Show that these generators satisfy the fundamental commutation relations

$$\big[J^{\mu\nu}, J^{\rho\sigma}\big] \;=\; i\big(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\big)\ .$$

(d) Introduce four $n\times n$ matrices γ^μ that satisfy $\big\{\gamma^\mu,\gamma^\nu\big\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_n$, with I_n the $n\times n$ unit matrix. Prove that also $S^{\mu\nu} \equiv \frac{i}{4}\big[\gamma^\mu,\gamma^\nu\big]$ satisfy the fundamental commutation relations for generators of the infinitesimal Lorentz transformations.

(e) The infinitesimal Lorentz transformations of four-vectors in parts (a) and (b) can be written as:

$$V^\alpha \;\to\; V'^\alpha \;\approx\; (g^\alpha{}_\beta + \omega^\alpha{}_\beta)V^\beta \;=\; \Big(g^\alpha{}_\beta - \frac{i}{2}\,\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta\Big)V^\beta\ ,$$

with $(J^{\mu\nu})^\alpha{}_\beta = i(g^{\mu\alpha}g^\nu{}_\beta - g^\mu{}_\beta g^{\nu\alpha})$. Guess what ... show that also these generators $J^{\mu\nu}$ satisfy the fundamental commutation relations for generators of the infinitesimal Lorentz transformations.