Solution 7:

(a) A Lagrangian typically consists of kinetic and mass terms (containing two fields) as well as interaction terms (containing three or more fields). Thus, in the scalar Yukawa theory the interaction term is given by

$$\mathcal{L}_{\text{int}} = -g\psi^\star\phi\psi = -\mathcal{H}_{\text{int}}\,.$$

(b) First we expand the exponential:

$$e^{-i\int d^4x\,\hat{\mathcal{H}}_{\text{int}}(x)} \;=\; \hat{1} - i\int d^4x\,\hat{\mathcal{H}}_{\text{int}}(x) + \mathcal{O}(g^2) \;=\; \hat{1} - ig\int d^4x\,\hat{\psi}^\dagger_{\text{I}}(x)\hat{\phi}_{\text{I}}(x)\hat{\psi}_{\text{I}}(x) + \mathcal{O}(g^2)\,.$$

At order $\mathcal{O}(g^0)$ one obtains

$$\langle 0|\,T\big(\hat{\psi}^\dagger_{\text{I}}(x_1)\hat{\phi}_{\text{I}}(x_2)\hat{\psi}_{\text{I}}(x_3)\hat{1}\big)|0\rangle = 0\,,$$

since according to Wick's theorem only a fully contracted set of fields can give rise to a nonzero vacuum expectation value. Three fields can simply not be fully contracted as each Wick contraction involves a pair of fields.

At order $\mathcal{O}(g)$ one has:

$$-ig\int d^4x\,\langle 0|\,T\big(\hat{\psi}^\dagger_{\text{I}}(x_1)\hat{\phi}_{\text{I}}(x_2)\hat{\psi}_{\text{I}}(x_3)\psi^\dagger_{\text{I}}(x)\hat{\phi}_{\text{I}}(x)\hat{\psi}_{\text{I}}(x)\big)|0\rangle\,.$$

Applying Wick's theorem one obtains only then a non-vanishing result if $\hat{\phi}_{\text{I}}(x_2)$ is Wick-contracted with $\hat{\phi}_{\text{I}}(x)$ and according to exercise 5(b)

1. $\hat{\psi}^\dagger_{\text{I}}(x_1)$ is Wick-contracted with $\hat{\psi}_{\text{I}}(x)$ and $\hat{\psi}_{\text{I}}(x_3)$ with $\hat{\psi}^\dagger_{\text{I}}(x)$,

$$-\,ig\int d^4x\langle 0|\hat{\psi}^\dagger_{\text{I}}(x_1)\hat{\phi}_{\text{I}}(x_2)\hat{\psi}_{\text{I}}(x_3)\psi^\dagger_{\text{I}}(x)\hat{\phi}_{\text{I}}(x)\hat{\psi}_{\text{I}}(x)|0\rangle$$

OR

2. $\hat{\psi}^\dagger_{\text{I}}(x_1)$ is Wick-contracted with $\hat{\psi}_{\text{I}}(x_3)$ and $\hat{\psi}^\dagger_{\text{I}}(x)$ with $\hat{\psi}_{\text{I}}(x)$,

$$-\,ig\int d^4x\langle 0|\hat{\psi}^\dagger_{\text{I}}(x_1)\hat{\phi}_{\text{I}}(x_2)\hat{\psi}_{\text{I}}(x_3)\psi^\dagger_{\text{I}}(x)\hat{\phi}_{\text{I}}(x)\hat{\psi}_{\text{I}}(x)|0\rangle\,.$$

Every Wick contraction gives a Feynman propagator, and therefore one obtains at order $\mathcal{O}(g)$:

$$-ig\int d^4x\,\Big[D_F(x-x_1;M^2)D_F(x_2-x;m^2)D_F(x_3-x;M^2)$$
$$+\;D_F(x_3-x_1;M^2)D_F(x_2-x;m^2)D_F(x-x;M^2)\Big]\,,$$

using the propagator conventions

$$\langle 0|\, T\big(\hat{\psi}_{\rm I}(x)\hat{\psi}_{\rm I}^{\dagger}(y)\big)|0\rangle \;=\; \langle 0|\, T\big(\hat{\psi}_{\rm I}^{\dagger}(y)\hat{\psi}_{\rm I}(x)\big)|0\rangle \;\equiv\; D_F(x-y;M^2)$$

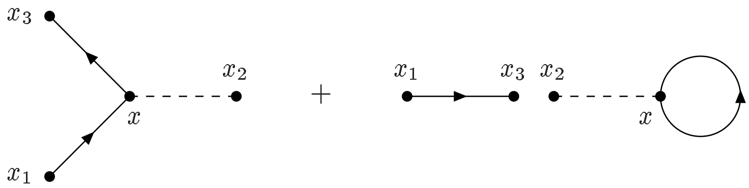
and

$$\langle 0|\, T\big(\hat{\phi}_{\rm I}(x)\hat{\phi}_{\rm I}(y)\big)|0\rangle \;\equiv\; D_F(x-y;m^2)\,.$$

This has the structure

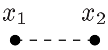
$$\text{vertex } (-ig\textstyle\int d^4x) \,*\, \psi\text{-propagator} \,*\, \phi\text{-propagator} \,*\, \psi\text{-propagator.}$$

(c) The respective Feynman diagrams are:

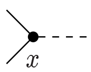


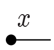
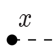
The arrows in these Feynman diagrams represent the direction of particle-number flow. The operators $\hat{\psi}_{\rm I}(x_3)$ and $\hat{\psi}_{\rm I}(x)$ correspond to antiparticle creation or particle annihilation at the spacetime points x_3 and x: hence, the arrows flow into these external/internal points of the Feynman diagrams. The operators $\hat{\psi}_{\rm I}^{\dagger}(x_1)$ and $\hat{\psi}_{\rm I}^{\dagger}(x)$ correspond to particle creation or antiparticle annihilation at the spacetime points x_1 and x: hence, the arrows flow out of these external/internal points of the Feynman diagrams.

(d) $\boxed{\text{Coordinate-space Feynman rules for the scalar Yukawa theory}}$:

1. For each ϕ-propagator  insert $D_F(x_1-x_2;m^2)$

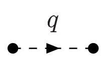
   and for each ψ-propagator  insert $D_F(x_1-x_2;M^2)\,.$

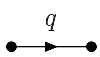
2. For each vertex  insert $(-ig)\int d^4x\,.$

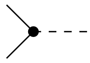
3. For each external point  or  insert $1\,.$

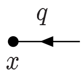
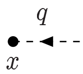
No symmetry factors are needed because all fields involved in the interaction are different.

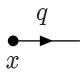
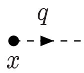
(e) $\boxed{\text{Momentum-space Feynman rules for the scalar Yukawa theory}}$:

1. For each ϕ-propagator  insert $\dfrac{i}{q^2-m^2+i\epsilon}$

   and for each ψ-propagator  insert $\dfrac{i}{q^2-M^2+i\epsilon}\,.$

2. For each vertex  insert $-ig$.

3. For each external point  or  insert $e^{-iq\cdot x}$

   and for each external point  or  insert $e^{+iq\cdot x}$.

4. Impose energy-momentum conservation at each vertex.

5. Integrate over each undetermined loop momentum p_j : $\displaystyle\int \frac{d^4p_j}{(2\pi)^4}$.

Solution 8:

Every interaction involves a pair $\hat{\psi}^\dagger(z)\hat{\psi}(z)$, and consequently for every in-flowing line there is an outflowing line. Since ψ-particles and $\bar{\psi}$-antiparticles have opposite particle number (which is a charge-like quantum number), $\hat{\psi}_\mathrm{I}(x)$ annihilates the charge associated with a ψ-particle at spacetime point x, whereas $\hat{\psi}_\mathrm{I}^\dagger(x)$ creates the same charge. As the charge of a ϕ-particle is zero, because it has no ψ-particle number, the "total charge" = "number of ψ-particles − number of $\bar{\psi}$-antiparticles" is conserved at each interaction vertex. Furthermore, a propagator

$$\langle 0|\, T\big(\hat{\psi}_\mathrm{I}(x)\hat{\psi}_\mathrm{I}^\dagger(y)\big)|0\rangle \;=\; \langle 0|\, T\big(\hat{\psi}_\mathrm{I}^\dagger(y)\hat{\psi}_\mathrm{I}(x)\big)|0\rangle$$

creates the charge associated with a ψ-particle at spacetime point y and annihilates it at spacetime point x. So, the ψ-charge flows continuously from y to x in such a propagator, as indicated by the arrow convention. This would not be the case for the propagators $\langle 0|\, T\big(\hat{\psi}_\mathrm{I}(x)\hat{\psi}_\mathrm{I}(y)\big)|0\rangle$ and $\langle 0|\, T\big(\hat{\psi}_\mathrm{I}^\dagger(x)\hat{\psi}_\mathrm{I}^\dagger(y)\big)|0\rangle$, which would lead to a clash of arrows and consequently to a violation of particle-number conservation. However, we have seen in exercise 5(b) that these propagators are zero and hence forbidden.

> *In conclusion: particle-number conservation causes the arrows in a Feynman diagram to link up and form a continuous flow.*

Solution 9:

(a) In order to calculate decay and scattering amplitudes $\langle f|\hat{S}|i\rangle = i\mathcal{M}(2\pi)^4\delta^{(4)}(p_i - p_f)$ one needs in addition to the Feynman rules derived in exercise 7 also the contractions of field operators with external lines.

$\boxed{\text{In coordinate space}}$: the field operator $\hat{\psi}_\mathrm{I}(x)$ can be contracted with an initial ψ-particle state on the right providing a factor $e^{-ik_\psi\cdot x}$, since

$$\overline{\hat{\psi}_\mathrm{I}(x)|\vec{k}_\psi}\rangle_0 \;=\; \int \frac{d^3q}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{q}}}}\,e^{-i\omega_{\vec{q}}\,x^0 + i\vec{q}\cdot\vec{x}}\,\hat{b}_{\vec{q}}\sqrt{2\omega_{\vec{k}_\psi}}\,\hat{b}^\dagger_{\vec{k}_\psi}|0\rangle \;=\; e^{-ik_\psi\cdot x}\,|0\rangle\,.$$

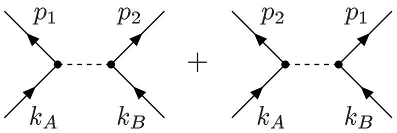
Analogously, a contraction ${}_0\langle\overline{\vec{p}_{\bar\psi}|\hat{\psi}_{\mathrm{I}}(x)}$ of the field operator $\hat{\psi}_{\mathrm{I}}(x)$ with a final $\bar\psi$-particle state on the left provides a factor $e^{ip_{\bar\psi}\cdot x}$, a contraction $\overline{\hat{\psi}^\dagger_{\mathrm{I}}(x)|\vec{k}_{\bar\psi}}\rangle_0$ of the field operator $\hat{\psi}^\dagger_{\mathrm{I}}(x)$ with an initial $\bar\psi$-particle state on the right provides a factor $e^{-ik_{\bar\psi}\cdot x}$, and a contraction ${}_0\langle\overline{\vec{p}_{\psi}|\hat{\psi}^\dagger_{\mathrm{I}}(x)}$ of the field operator $\hat{\psi}^\dagger_{\mathrm{I}}(x)$ with a final ψ-particle state on the left provides a factor $e^{ip_\psi\cdot x}$.

To mark this in Feynman diagrams an arrow will be drawn on $\psi/\bar\psi$-lines, representing the direction of particle number flow: ψ-particles flow along the arrow, $\bar\psi$-particles $=$ ψ-antiparticles against the arrow.

$\boxed{\text{In momentum space}}$: these extra Feynman rules all provide a trivial factor 1.

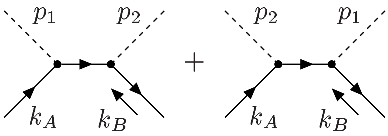
(b) The lowest-order amplitude for the decay process $\phi(k_A) \to \psi(p_1)\bar\psi(p_2)$ is given by the elementary vertex, hence $i\mathcal{M}^{LO}(\phi \to \psi\bar\psi) = -ig$.

(c) In all three $2 \to 2$ scattering cases there are two Feynman diagrams possible at lowest order. With the help of energy-momentum conservation, the lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B) \to \psi(p_1)\psi(p_2)\big)$ reads



$$= -ig^2\left(\frac{1}{(k_A-p_1)^2-m^2+i\epsilon}+\frac{1}{(k_A-p_2)^2-m^2+i\epsilon}\right).$$

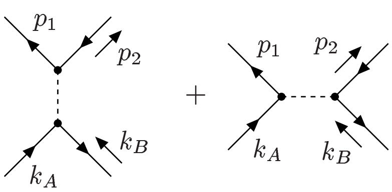
The lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\bar\psi(k_B) \to \phi(p_1)\phi(p_2)\big)$ is given by



$$= -ig^2\left(\frac{1}{(k_A-p_1)^2-M^2+i\epsilon}+\frac{1}{(k_A-p_2)^2-M^2+i\epsilon}\right).$$

Note that the incoming momentum k_B of the antiparticle is defined in the opposite direction of the arrow! This is typical for the arrow convention for antiparticles.

The lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\bar\psi(k_B) \to \psi(p_1)\bar\psi(p_2)\big)$ is given by



$$= -ig^2\left(\frac{1}{(k_A+k_B)^2-m^2+i\epsilon}+\frac{1}{(k_A-p_1)^2-m^2+i\epsilon}\right).$$

This amplitude can be obtained from the first one by means of crossing, i.e. changing the momenta according to $k_B\,, p_2 \to -p_2\,, -k_B$ in order to switch from particles to antiparticles in the opposite reaction state.

(d) The lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B) \to \phi(p_1)\phi(p_2)\big)$ vanishes, since an initial state with particle number 2 cannot scatter into a final state with particle number 0 (cf. exercise 8). In the Feynman diagrams this manifests itself by the absence of a consistent (continuous) flow for the particle number in this particular process.