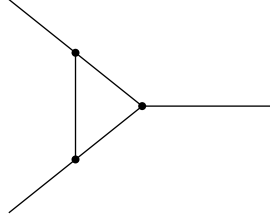


Solution 13:

- (a) Only one amputated diagram contributes to the one-loop correction to the ϕ^3 interaction:

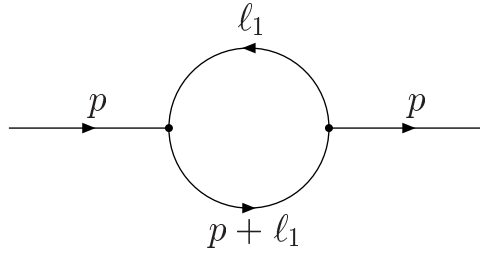


- (b) In the diagram of part (a) we have three propagators and one loop momentum. After Wick rotation and Feynman parametrization, the integral will have the form

$$\int_0^{\Lambda^2} d\ell_E^2 \frac{\ell_E^2}{(\ell_E^2 + \Delta - i\epsilon)^3} .$$

Since there are four powers of ℓ_E in the numerator and six in the denominator, the integral is finite. This can be understood from a dimensional reasoning. For a 4-dimensional theory we have $[\mathcal{L}] = 4$. Since \mathcal{L} contains a term of the form $m^2\phi^2$, that means that $[\phi] = 1$, so $[\phi^3] = 3$ and therefore $[\lambda] = 4 - 3 = 1$. According to the discussion on page 28 of the lecture notes the ϕ^3 interaction is thus weak at high energies, which explains its finiteness.

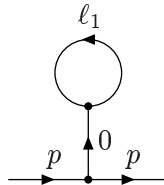
- (c+d) First we consider the self-energy diagram



Taking into account the symmetry factor, this diagram yields:

$$\begin{aligned} \frac{1}{2} \int \frac{d^4 \ell_1}{(2\pi)^4} \frac{\lambda^2}{[\ell_1^2 - m^2 + i\epsilon][(\ell_1 + p)^2 - m^2 + i\epsilon]} &= \frac{i\lambda^2}{32\pi^2} \int_0^1 d\alpha \int_0^{\Lambda^2} d\ell_E^2 \frac{\ell_E^2}{(\ell_E^2 + \Delta(\alpha, m^2, p^2) - i\epsilon)^2} \\ &= \frac{i\lambda^2}{32\pi^2} \log(\Lambda^2) + \text{finite terms for } \Lambda \rightarrow \infty . \end{aligned}$$

Note that for the leading behaviour we do not need to determine the precise form of Δ . For the calculation of the integral, see exercise 12.



The second (“tadpole”) diagram yields:

$$\begin{aligned} \frac{1}{2} \int \frac{d^4 \ell_1}{(2\pi)^4} \frac{-\lambda^2/m^2}{[\ell_1^2 - m^2 + i\epsilon]} &= \frac{i\lambda^2}{32\pi^2 m^2} \int_0^{\Lambda^2} d\ell_E^2 \frac{\ell_E^2}{\ell_E^2 + m^2 - i\epsilon} \\ &= \frac{i\lambda^2}{32\pi^2} \frac{\Lambda^2}{m^2} + \text{less divergent terms for } \Lambda \rightarrow \infty. \end{aligned}$$

(e) The full propagator in the ϕ^3 -theory reads:

$$\frac{i}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} \equiv \frac{iZ}{p^2 - m_{ph}^2 + i\epsilon} + \text{regular terms at the pole } m_{ph}^2,$$

with $\Sigma(p^2)$ the self-energy. The physical mass m_{ph} is determined by the location of the pole:

$$m_{ph}^2 - m^2 - \Sigma(m_{ph}^2) = 0.$$

We know that the physical mass m_{ph} is an observable and thus cannot depend on Λ . According to part (c+d) the self-energy is divergent for $\Lambda \rightarrow \infty$, so the same should hold for the Lagrangian parameter m .

(f) The analysis on page 87 of the lecture notes is for ϕ^4 -theory. For ϕ^3 -theory we have:

- three lines meet in each vertex, each external line is connected to one vertex and each propagator is connected to two vertices or to the same vertex twice, so $3V = N + 2P \Rightarrow P = \frac{3}{2}V - \frac{1}{2}N$.
- By the same reasoning as on page 87, we have $L = P - V + 1 = \frac{1}{2}V - \frac{1}{2}N + 1$ so we only have loops if $V \geq N$.
- The superficial degree of divergence is given by $D = dL - 2P = (\frac{1}{2}d - 3)V + (1 - d/2)N + d$.



Now as for renormalizability:

$d = 4$ In our four-dimensional world we have $D = 4 - V - N$, so D decreases at every order since the coefficient in front of V is negative. The only divergences occur at $N = 1$ and $V = 1$ or 3 (tadpole graphs) and $V = N = 2$ (the one-loop self-energy graph from part c), bearing in mind that $V = N + 2(L - 1)$. So this case is superrenormalizable.

$d = 6$ In this case we have $D = 6 - 2N$, so $D \geq 0$ for $N \leq 3$. That means that divergences occur at all orders, but there is only a finite number of divergent amplitudes. So this theory is renormalizable.

$d < 6$ For any theory with $d < 6$, a similar argumentation as in the $d = 4$ case shows that it is superrenormalizable.

$d > 6$ All amplitudes become divergent at sufficiently high loop order, since the coefficient in front of V is positive, so the theory is nonrenormalizable.

- (g) For a d dimensional theory, we have $[\mathcal{L}] = d$. Since \mathcal{L} contains a term of the form $m^2\phi^2$, that means that $[\phi] = (d-2)/2$, so $[\phi^3] = 3(d-2)/2$ and therefore $[\lambda] = d - 3(d-2)/2 = 3 - d/2$. That means that λ is dimensionless for $d = 6$, which is the renormalizable case. For $d < 6$, $[\lambda] > 0$, so the ϕ^3 interaction is weak at high energies, which explains why it is superrenormalizable. For $d > 6$, $[\lambda] < 0$, so the ϕ^3 interaction is strong at high energies, which explains why the integrals diverge and the theory is nonrenormalizable.
- (h) According to part (f), the superficially divergent 1-particle irreducible one-loop diagrams for four-dimensional spacetime are the self-energy diagram  $\propto \log(\Lambda^2)$ and the tadpole diagram  $\propto \Lambda^2$.

Solution 14:

- (a) Infinitesimal rotation in terms of rotation angle $\delta\alpha$ and rotation axis \vec{e}_n :
 $\vec{x} \rightarrow \vec{x}' \approx \vec{x} + \delta\alpha \vec{e}_n \times \vec{x} \equiv \vec{x} + \delta\vec{\alpha} \times \vec{x}$.
 Rewritten as infinitesimal Lorentz transformation $x'^\rho = \Lambda^\rho_\sigma x^\sigma \approx (g^\rho_\sigma + \omega^\rho_\sigma) x^\sigma$:
 $x'^0 = x^0 \Rightarrow \omega_{0\sigma} = 0$ for $\sigma = 0, 1, 2, 3$.
 $x'^j \approx x^j + (\delta\vec{\alpha} \times \vec{x})^j = x^j + \epsilon^{jlk}(\delta\alpha)^l x^k \Rightarrow \omega_{j0} = 0$ and $\omega_{jk} = -\epsilon^{jlk}(\delta\alpha)^l = \epsilon^{jkl}(\delta\alpha)^l$.
 The non-vanishing components are: $\omega_{12} = -\omega_{21} = (\delta\alpha)^3$ and cyclic permutations thereof.
- (b) Infinitesimal boost with boost velocity $\delta\vec{v}$: $x^0 \rightarrow x'^0 \approx x^0 + \delta\vec{v} \cdot \vec{x}$, $\vec{x} \rightarrow \vec{x}' \approx \vec{x} + x^0 \delta\vec{v}$.
 Rewritten as infinitesimal Lorentz transformation $x'^\rho = \Lambda^\rho_\sigma x^\sigma \approx (g^\rho_\sigma + \omega^\rho_\sigma) x^\sigma$:
 $\omega_{00} = 0$, $\omega_{0j} = \delta v^j$ from the temporal part and $\omega_{ij} = 0$, $\omega_{j0} = -\delta v^j$ from the spatial part.