Solution 4:

The Hamilton operator is the integral over the Hamiltonian density operator, i.e. $\hat{H} = \int d^3x \, \hat{\mathcal{H}}$, where

$$\hat{\mathcal{H}} = \hat{\Pi}^\dagger \hat{\Pi} + (\vec{\nabla}\hat{\Phi}^\dagger) \cdot (\vec{\nabla}\hat{\Phi}) + m^2 \hat{\Phi}^\dagger \hat{\Phi} \,.$$

(a) Using

$$\hat{\phi}_j(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{a}_{j,\vec{p}} + \hat{a}^\dagger_{j,-\vec{p}} \right) \qquad (j = 1, 2)\,,$$

one obtains

$$\hat{\Phi}(\vec{x}) \equiv \frac{1}{\sqrt{2}} \left(\hat{\phi}_1(\vec{x}) + i\hat{\phi}_2(\vec{x}) \right) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}} + \hat{b}^\dagger_{-\vec{p}} \right)$$

via the identification

$$\hat{a}_{\vec{p}} = \frac{1}{\sqrt{2}} \left(\hat{a}_{1,\vec{p}} + i\hat{a}_{2,\vec{p}} \right) \qquad \text{and} \qquad \hat{b}^\dagger_{-\vec{p}} = \frac{1}{\sqrt{2}} \left(\hat{a}^\dagger_{1,-\vec{p}} + i\hat{a}^\dagger_{2,-\vec{p}} \right) .$$

The corresponding commutation relations are given by

$$[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}] = \frac{1}{2}[\hat{a}_{1,\vec{p}} + i\hat{a}_{2,\vec{p}}, \, \hat{a}^\dagger_{1,\vec{q}} - i\hat{a}^\dagger_{2,\vec{q}}] = \frac{1}{2}[\hat{a}_{1,\vec{p}}, \hat{a}^\dagger_{1,\vec{q}}] + \frac{1}{2}[\hat{a}_{2,\vec{p}}, \hat{a}^\dagger_{2,\vec{q}}] = (2\pi)^3 \delta(\vec{p} - \vec{q}) \, \hat{1}$$

and analogously

$$[\hat{b}_{\vec{p}}, \hat{b}^\dagger_{\vec{q}}] = (2\pi)^3 \delta(\vec{p} - \vec{q}) \, \hat{1} \,,$$

with all other commutators vanishing trivially.

For the momentum field, note that $\dfrac{\partial \mathcal{L}}{\partial(\partial_0 i\phi_2)} = -i \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi_2)}$ and consequently $i\hat{\phi}_2(\vec{x}) \to -i\,\hat{\pi}_2(\vec{x})$ such that $\hat{\Pi}(\vec{x}) = \frac{1}{\sqrt{2}}\left(\hat{\pi}_1(\vec{x}) - i\hat{\pi}_2(\vec{x}) \right)$. From

$$\hat{\pi}_j(\vec{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \, e^{i\vec{p}\cdot\vec{x}} \left(\hat{a}_{j,\vec{p}} - \hat{a}^\dagger_{j,-\vec{p}} \right) \qquad (j = 1, 2)$$

and the above-given identification for $\hat{a}_{\vec{p}}$ and $\hat{b}_{\vec{p}}$, the decomposition for $\hat{\Pi}(\vec{x})$ follows:

$$\hat{\Pi}(\vec{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \, e^{i\vec{p}\cdot\vec{x}} \left(\hat{b}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}} \right) \underset{\text{for use below}}{=\!=\!=\!=\!=\!=} -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \, e^{-i\vec{p}\cdot\vec{x}} \left(\hat{b}_{-\vec{p}} - \hat{a}^\dagger_{\vec{p}} \right) .$$

(b) One uses $\int d^3x \, e^{i(\vec{q}-\vec{p})\cdot\vec{x}} = (2\pi)^3 \delta(\vec{p}-\vec{q})$ to write the Hamilton operator in terms of number operators:

$$\hat{H} = \int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \, e^{i(\vec{q}-\vec{p})\cdot\vec{x}} \left(\frac{1}{2}\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}} \, (\hat{b}^\dagger_{-\vec{q}} - \hat{a}_{\vec{q}})(\hat{b}_{-\vec{p}} - \hat{a}^\dagger_{\vec{p}}) + \frac{\vec{p}\cdot\vec{q} + m^2}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} (\hat{a}^\dagger_{\vec{p}} + \hat{b}_{-\vec{p}})(\hat{a}_{\vec{q}} + \hat{b}^\dagger_{-\vec{q}}) \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2}\omega_{\vec{p}} \left(\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}} + \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^\dagger_{-\vec{p}}\hat{b}_{-\vec{p}} + \hat{b}_{-\vec{p}}\hat{b}^\dagger_{-\vec{p}} \right) = \int \frac{d^3p}{(2\pi)^3} \, \omega_{\vec{p}} \left(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}} + (2\pi)^3\delta(\vec{0})\hat{1} \right) ,$$

with the term proportional to the unit operator usually referred to as the "zero-point energy".

(c) Doing the same for the "charge" operator $\hat{Q}$ results in

$$\hat{Q} = \int d^3x \, \hat{j}^0(\vec{x}) = -i \int d^3x \left(\hat{\Phi}^\dagger(\vec{x})\hat{\Pi}^\dagger(\vec{x}) - \hat{\Pi}(\vec{x})\hat{\Phi}(\vec{x}) \right)$$

$$= -\frac{1}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\, e^{i(\vec{p}-\vec{q})\cdot\vec{x}}\left(\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\,(\hat{a}_{\vec{q}}^{\dagger}+\hat{b}_{-\vec{q}})(\hat{a}_{\vec{p}}-\hat{b}_{-\vec{p}}^{\dagger}) + \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}\,(\hat{a}_{\vec{q}}^{\dagger}-\hat{b}_{-\vec{q}})(\hat{a}_{\vec{p}}+\hat{b}_{-\vec{p}}^{\dagger})\right)$$

$$= -\int \frac{d^3p}{(2\pi)^3}\,(\hat{a}_{\vec{p}}^{\dagger}\hat{a}_{\vec{p}} - \hat{b}_{-\vec{p}}\hat{b}_{-\vec{p}}^{\dagger}) = \int \frac{d^3p}{(2\pi)^3}\left(-\hat{a}_{\vec{p}}^{\dagger}\hat{a}_{\vec{p}} + \hat{b}_{\vec{p}}^{\dagger}\hat{b}_{\vec{p}} + (2\pi)^3\delta(\vec{0})\hat{1}\right),$$

with the term proportional to the unit operator usually referred to as the "zero-point charge".

(d) Apparently we are dealing here with two types of particles with mass m:

- particles with energy $\omega_{\vec{p}}$ and "charge" -1, corresponding to $\hat{a}_{\vec{p}}^{\dagger}$ and $\hat{a}_{\vec{p}}$,

- anti-particles with energy $\omega_{\vec{p}}$ and "charge" $+1$, corresponding to $\hat{b}_{\vec{p}}^{\dagger}$ and $\hat{b}_{\vec{p}}$.

(e) By reversing the order of the quantum fields in the definitions of $\hat{H}$ and $\hat{Q}$, the order of creation and annihilation operators is swapped in the final expressions. This gives rise to exactly the same expression for $\hat{H}$, with the same counting operators and the same zero-point energy. The counting operators occurring in $\hat{Q}$ will also be unaffected. However, the zero-point charge will no longer be generated by the b-terms (i.e. the antiparticles) but by the a-terms (i.e. the particles) instead, resulting in an opposite sign for the zero-point charge. Hence, the zero-point energy is a robust quantity and the zero-point charge is not!

(f) As

$$i\vec{\nabla}\hat{\Phi}^{\dagger}(\vec{x}) = i\vec{\nabla}\int \frac{d^3p}{(2\pi)^3}\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}}(\hat{a}_{\vec{p}}^{\dagger}+\hat{b}_{-\vec{p}}) = \int \frac{d^3p}{(2\pi)^3}\,\vec{p}\,\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}}(\hat{a}_{\vec{p}}^{\dagger}+\hat{b}_{-\vec{p}})$$

and similarly for $-i\vec{\nabla}\hat{\Phi}(\vec{x})$, one obtains

$$\langle 0|\hat{\vec{j}}(\vec{x})|0\rangle = -\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\,(\vec{p}+\vec{q})\,\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}}\frac{e^{i\vec{q}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{q}}}}\,\langle 0|\hat{a}_{\vec{p}}^{\dagger}\hat{a}_{\vec{q}}+\hat{a}_{\vec{p}}^{\dagger}\hat{b}_{-\vec{q}}^{\dagger}+\hat{b}_{-\vec{p}}\hat{a}_{\vec{q}}+\hat{b}_{-\vec{p}}\hat{b}_{-\vec{q}}^{\dagger}|0\rangle.$$

The first three terms in the sum vanish as $\hat{a}_{\vec{q}}|0\rangle = 0$ and $\langle 0|\hat{a}_{\vec{p}}^{\dagger} = 0$, and for the last term we use

$$\langle 0|\hat{b}_{-\vec{p}}\hat{b}_{-\vec{q}}^{\dagger}|0\rangle = \langle 0|\hat{b}_{-\vec{q}}^{\dagger}\hat{b}_{-\vec{p}} + [\hat{b}_{-\vec{p}},\hat{b}_{-\vec{q}}^{\dagger}]|0\rangle = 0 + (2\pi)^3\delta(\vec{p}-\vec{q})\langle 0|0\rangle = (2\pi)^3\delta(\vec{p}-\vec{q})$$

to get

$$\langle 0|\hat{\vec{j}}(\vec{x})|0\rangle = -\int \frac{d^3p}{(2\pi)^3}\frac{e^0}{2\omega_{\vec{p}}}2\vec{p} = -\int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}}{\omega_{\vec{p}}} = \vec{0}.$$

(g) A Lorentz transformation mixes time and spatial components of a contravariant four-vector $\hat{j}^{\mu}(\vec{x})$, $\hat{j}'^{\mu}(\vec{x}) = \Lambda^{\mu}{}_{\nu}\hat{j}^{\nu}(\vec{x})$, such that under a Lorentz boost $\langle 0|\hat{j}^0(\vec{x})|0\rangle \neq 0$ will generate a non-vanishing $\langle 0|\hat{\vec{j}}'(\vec{x})|0\rangle$ even if $\langle 0|\hat{\vec{j}}(\vec{x})|0\rangle = \vec{0}$. Hence, the additional condition $\langle 0|\hat{j}^0(\vec{x})|0\rangle = 0$ should be imposed if one requires that in all Lorentz frames $\langle 0|\hat{\vec{j}}(\vec{x})|0\rangle = \vec{0}$.

(h) Normal ordering puts all creation operators in front of the annihilation operators, thereby removing all zero-point contributions from the observables. This means that the vacuum indeed would have no charge. At the same time it would imply that the vacuum has no energy, which seems to be in conflict with the experimental observation of the Casimir effect. If we would instead apply Weyl-ordering, by averaging over all possible orderings of the quantum fields in the observables, then according to part (e) the zero-point charge would vanish (as required) but the zero-point energy would be unaffected.

To phrase it provocatively: "once the dust settles over the correct interpretation of the Casimir experiment, the days of either normal ordering or Weyl-ordering could be over".