Test 3 Practice

Theory

1. Give the contrapositive form of the following implication: If I study a lot for a test, then I will get an A on it.

If I do not get an A on a test, I did not study a lot for it.

2. Determine whether each of the following statement is true or false. Explain your answer.

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

true - contrapositive of Test for Divergence.

(b) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

False. $\sum \frac{1}{n}$ diverges (harmonic), but $\lim_{n\to\infty} \frac{1}{n} = 0$.

(c) The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| \leq 1$ and diverges if $|r| > 1$.

False. converges when $|r| < 1$ & diverges if $|r| \geq 1$.

3. List the convergence tests that we have looked at thus far.

Test for Divergence,
telescoping
geo
p-series
harmonic

Integral Test
DCT
LCT
AST

4. Give the mathematical definition for series convergence.

5. Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where a_n is always positive and decreasing and $\lim_{n\to\infty} a_n = 0$. Explain why $s_2 < s_4 < s_6 < \ldots$. Also explain why the even partial sums must converge.

Show that $\lim_{N\to\infty} |s_{N+1} - s_N| = 0$.

Explain how this shows that the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

6. Use the tools we looked at in class to determine the convergence or divergence of each of the following sequences. If a sequence converges, find its limit.

(a) $a_n = (-3)^n$

$\lim_{n\to\infty} (-3)^n$ DNE

($\lim_{n\to\infty} r^n$ DNE when $|r| > 1$.)

(b) $a_n = \frac{4+\left(\frac{2}{3}\right)^n}{2-e^n}$

$\left(\frac{2}{3}\right)^n \to 0 \rightarrow \lim_{n\to\infty} r^n = 0$ when $|r|<1$.

$\lim_{n\to\infty} \frac{4+\cancel{\left(\frac{2}{3}\right)^n}}{2-e^n} = \lim_{n\to\infty} \frac{4}{2-e^n} = 0.$

$\frac{4}{-\infty}$

(c) $a_n = \frac{\cos^2 n}{3^n}$

$0 \le \frac{\cos^2 n}{3^n} \le \frac{1}{3^n}$

$\lim_{n\to\infty} \frac{1}{3^n} = 0$. So $\lim_{n\to\infty} a_n = 0$ by Squeeze Thm.

$1/\infty$

(d) $a_n = \left(1 - \frac{3}{n}\right)^{2n}$

l'H

$y = \left(1-\frac{3}{x}\right)^{2x}$ 1^∞

$\ln y = 2x \ln(1-3/x)$

$\infty \cdot 0$

$\lim_{x\to\infty} 2x \ln(1-3/x) = \lim_{x\to\infty} \frac{2\ln(1-3/x)}{1/x}$ $\frac{0}{0}$

$\overset{l'H}{=} \lim_{x\to\infty} \frac{\frac{2}{1-3/x}\cdot -3\cancel{x^{-2}}}{-1\cancel{x^{-2}}} = \lim_{x\to\infty} \frac{6}{1-\frac{3}{x}} = \frac{6}{1-0} = 6.$

So $\lim_{n\to\infty} \left(1-3/n\right)^{2n} = e^6$

(e) $a_n = \frac{n}{\ln n}$

~~lim~~ $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{l'H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$

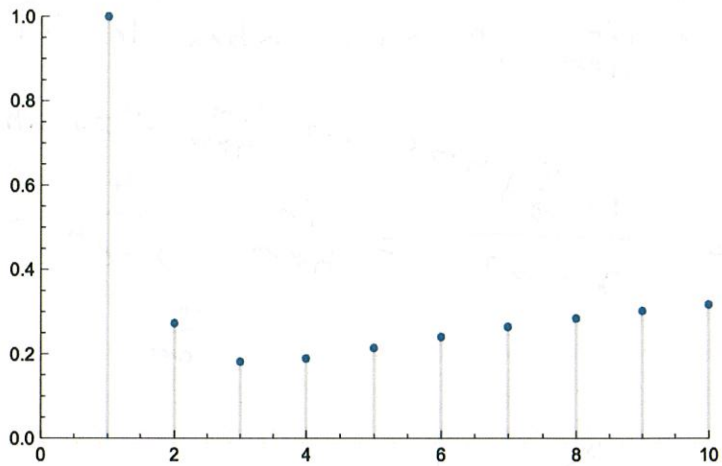
$\frac{\infty}{\infty}$

(f) $a_n = \frac{e^{3n}+3e^n}{4e^{3n}-2}$

~~$\lim_{n\to\infty} \frac{e^{3n}+3e^n}{4e^{3n}-2}$~~

$\lim_{n\to\infty} \frac{e^{3n}+3e^n}{4e^{3n}-2} = \lim_{n\to\infty} \frac{e^{3n}}{4e^{3n}} = 1/4$

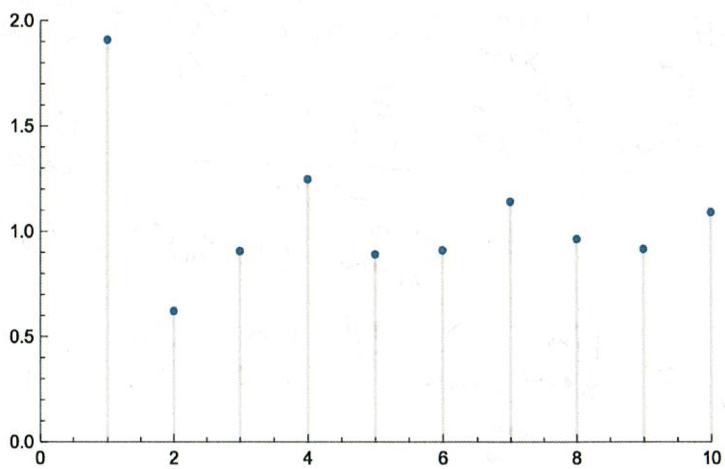
7. Recall that a sequence is monotonic if it is always increasing or always decreasing. Since the limit of a sequence is determined by the end of the sequence, it really doesn't matter what happens at the beginning of a sequence. For this reason, we are also interested in sequences that are eventually monotonic, like the following.

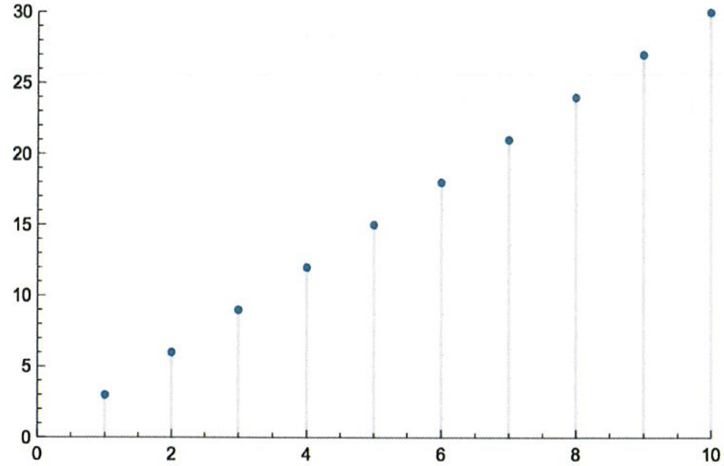


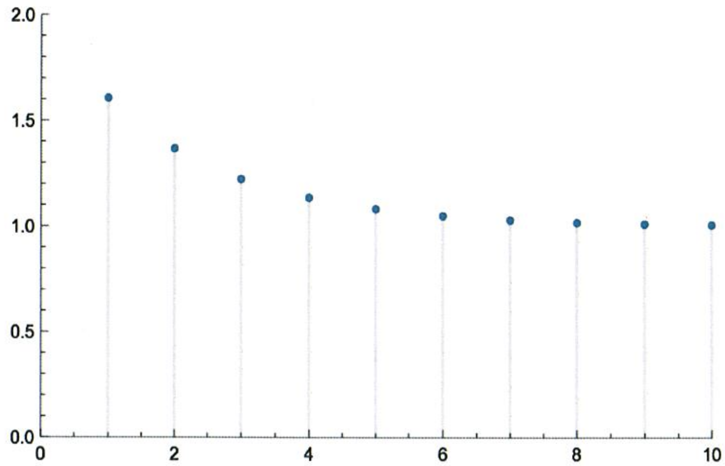
This sequence begins by decreasing, but then appears to increase after that always. This sequence is eventually monotonic. It also appears to be bounded, so the sequence must converge.

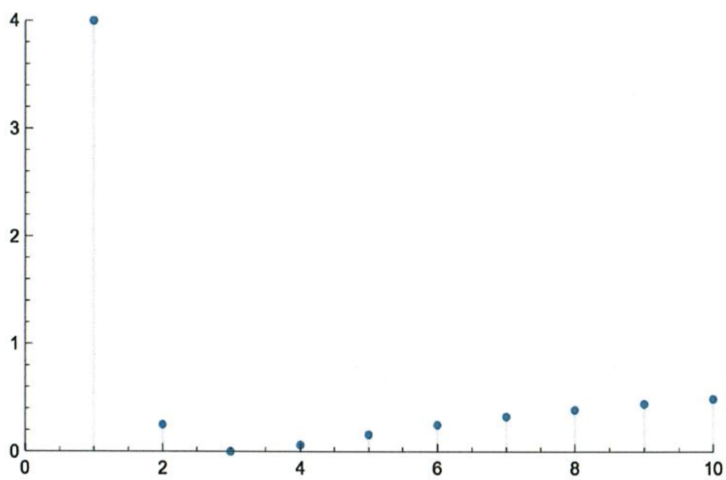
Determine whether each of the following sequences is eventually monotonic. Also determine whether each of the following sequences is bounded.

(a)



(b) 

(c) 

(d) 

8. Suppose $f(x)$ is a positive, decreasing, continuous function, and that $a_k = f(k)$ for whole numbers k. Draw a graph to explain why $\sum_{k=2}^{\infty} a_k \leq \int_1^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k$. Explain why this shows that the Integral Test holds.

9. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ with the 5th partial sum.

   Find a bound on the error used in this approximation.

10. Give an example of an infinite series that converges to 7.

11. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that diverges and $\lim_{n\to\infty} a_n = 0$. If it's not possible, explain why.

12. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that converges and $\lim_{n\to\infty} a_n = 0$. If it's not possible, explain why.

13. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that diverges and $\lim_{n\to\infty} a_n \overset{\neq}{\cancel{=}} 0$. If it's not possible, explain why.

14. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that converges and $\lim_{n\to\infty} a_n \neq 0$. If it's not possible, explain why.

15. Give the argument for why $\sum_{n=1}^{\infty} ar^{n-1}$ converges when $|r| < 1$.

16. For the following series $\sum_{n=1}^{\infty} a_n$, fill out the table for a_n and s_n.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

n	a_n	s_N
1		
2		
3		
4		
5		
6		

17. What is the harmonic series? Explain how you know whether it converges or diverges.

18. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+3n+1}$ converges.

Estimate the sum of the series with S_5 and find a bound on the error.

19. Determine whether each of the following series converges or diverges. If a series converges and it is possible to do so, please find the sum. Be sure to justify your answer carefully.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin^2 n}{e^n}$

not Test for Divergence

DCT

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{2n^2+1}}$

not Test for Divergence

DCT doesn't work

$\boxed{\text{LCT will work}}$

$\frac{1}{\sqrt{2n^2+1}} \leq \frac{1}{\sqrt{2n^2}} = \frac{1}{\sqrt{2}n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n-1}$

DCT

or

LCT

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3+n+1}{2n^2-5}$

~~AST~~

Test for Divergence.

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$

$\lim_{n\to\infty} ne^{-n^2} = \lim_{n\to\infty} \frac{n}{e^{n^2}}$ l'H

$= 0.$

Integral Test

(f) $\sum_{n=1}^{\infty} e^{\frac{1}{n}}$

Test for Divergence

(g) $\sum_{n=1}^{\infty} \tan^{-1} n$

Test for Divergence

(h) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1-e^n}{2+3e^n}$

AST or Test for Divergence

(i) $\sum_{n=1}^{\infty} \frac{3^{2n}}{8^{n-1}} = \sum \frac{(3^2)^n}{8^{n-1}} = \sum \frac{9^n}{8^{n-1}} = \sum \frac{9^n}{8^n \cdot 8^{-1}}$

geo

$= \sum 8\left(\frac{9}{8}\right)^n$ diverges (geo, $r = \frac{9}{8}$, $|r| \geq 1$).

(j) $\sum_{n=1}^{\infty} 4n^{-1}$

harmonic

(k) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^3}$

DCT or LCT

(l) $\sum_{n=3}^{\infty} \frac{2}{3^n}$

geo

(m) $\sum_{n=2}^{\infty}\left(\frac{n}{n+1}-\frac{n+2}{n+3}\right)$

telescoping

(n) $\sum_{n=1}^{\infty}\frac{1}{2}(-2)^n$

geo

(o) $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$

telescoping

(p) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2-n+3}{n^3+4}$

AST

(q) $\sum_{n=1}^{\infty}\frac{7+6n+2n^3}{8n^6+12}$

LCT

(r) $\sum_{n=1}^{\infty}\frac{n+3}{2n^2+7}$

LCT

(s) $\sum_{n=1}^{\infty}\frac{3}{n^2+n}$

telescoping in disguise!

(t) $\sum_{n=1}^{\infty} 2\left(-\frac{1}{3}\right)^{n-1}$

geo

(u) $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$

geo

(v) $\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right)$

telescoping

(w) $\sum_{n=1}^{\infty} 3(-2)^{n-1}$

geo

(x) $\displaystyle\sum_{j=2}^{\infty} \frac{1}{\ln j + 2^j}$

DCT

(y) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2+3n-1}{(2n+1)(n-2)}$

Test for Divergence.