Infinite Series Practice

Definitions

1. What does it mean for the series $\sum_{n=1}^{\infty} a_n$ to converge?

$\sum a_n$ converges with sum L if

$\lim_{N\to\infty} S_N = L$, where $S_N = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_N$.

2. For the following series $\sum_{n=1}^{\infty} a_n$, fill out the table for a_n and s_n. (This problem is supposed to reinforce the idea that a series has two sequences associated to it. The first sequence consists of the terms of the series; the second is the sequence of partial sums, and is found by adding up the terms of the first sequence.)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

n	a_n	s_N
1	1/1	1
2	1/4	1.25
3	1/9	1.36
4	1/16	1.42
5	1/25	1.46
6	1/36	1.49

$s_1 = 1$

$s_2 = 1 + \frac{1}{4} = 1.25$

$s_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.36$

$\vdots$

3. What is the harmonic series? Explain how you know whether it converges or diverges.

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is the harmonic series.

$$\sum_{n=1}^{\infty} \cdot \; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{1/2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{1/2} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + {}_{1} \; \frac{1}{2} + \dots$$

$$= \infty.$$

Since it is larger than something infinite, it diverges.

Series Convergence

see below

Use the tools we looked at in class to determine the convergence or divergence of each of the following series. If a series converges, find its sum.

1. $\sum_{n=1}^{\infty} \frac{3}{n^2+n} = \sum_{n=1}^{\infty} \frac{3}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{3}{n} - \frac{3}{n+1}\right)$

partial fractions: $\left[\frac{3}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}\right] n(n+1)$

$$3 = A(n+1) + Bn$$

$$0n + 3 = (A+B)n + A$$

$$3 = A$$

$$0 = A + B \longrightarrow B = -3$$

$S_1 = \frac{3}{1} - \frac{3}{2}$

$S_2 = \left(\frac{3}{1} - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \frac{3}{3}\right) = 3 - 1\frac{3}{3}$

$S_3 = \left(\frac{3}{1} - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \frac{3}{4}\right) = 3 - \frac{3}{4}$

$S_N = 3 - \frac{3}{N+1}$

$\lim_{N\to\infty} 3 - \frac{3}{N+1}$

$= 3 - 0 = 3.$

Converges with sum 3.

2. $\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right)$

$S_1 = \sin\frac{1}{1} - \sin\frac{1}{2}$

$S_2 = \left(\sin\frac{1}{1} - \cancel{\sin\frac{1}{2}}\right) + \left(\cancel{\sin\frac{1}{2}} - \sin\frac{1}{3}\right) = \sin 1 - \sin\frac{1}{3}$

$S_3 = \left(\sin\frac{1}{1} - \cancel{\sin\frac{1}{2}}\right) + \left(\cancel{\sin 1/2} - \cancel{\sin 1/3}\right) + \left(\cancel{\sin 1/3} - \sin 1/4\right)$

$= \sin 1 - \sin 1/4$

$\vdots$

$S_N = \sin 1 - \sin\frac{1}{N+1}$

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sin 1 - \sin\frac{1}{N+1} = \sin 1 - \cancel{\sin 0}$

$= \sin 1$

law of logarithms

3. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \left(\ln(n) - \ln(n+1)\right)$

$S_1 = \ln(1) - \ln(2)$

$S_2 = (\ln(1) - \cancel{\ln(2)}) + (\cancel{\ln(2)} - \ln(3)) = \ln(1) - \ln 3$

$S_3 = (\ln(1) - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + (\cancel{\ln 3} - \ln 4)$

$= \ln 1 - \ln 4$

$S_N = \ln 1 - \ln(N+1)$

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln 1 - \ln(N+1)$

$= \ln 1 - \ln \infty$

$= 0 - \infty = -\infty.$

series diverges

4. $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+3} - \frac{n+3}{n+5}\right)$

$S_1 = \left(\frac{2}{4} - \frac{4}{6}\right)$

$S_2 = \left(\frac{2}{4} - \frac{4}{6}\right) + \left(\frac{3}{5} - \frac{5}{7}\right)$

$S_3 = \left(\frac{2}{4} - \cancel{\frac{4}{6}}\right) + \left(\frac{3}{5} - \frac{5}{7}\right) + \left(\cancel{\frac{4}{6}} - \frac{6}{8}\right) = \frac{2}{4} + \frac{3}{5} - \frac{5}{7} - \frac{6}{8}$

$S_4 = \left(\frac{2}{4} - \cancel{\frac{4}{6}}\right) + \left(\frac{3}{5} - \cancel{\frac{5}{7}}\right) + \left(\cancel{\frac{4}{6}} - \frac{6}{8}\right) + \left(\cancel{\frac{5}{7}} - \frac{7}{9}\right) = \frac{2}{4} + \frac{3}{5} - \frac{6}{8} - \frac{7}{9}$

pattern: $S_N = \frac{2}{4} + \frac{3}{5} - \frac{n+2}{n+4} - \frac{n+3}{n+5}$

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{2}{4} + \frac{3}{5} - \frac{n+2}{n+4} - \frac{n+3}{n+5} = \frac{2}{4} + \frac{3}{5} - 1 - 1$

$= \frac{-9}{10}$