Infinite Sequences Practice

Sequence Convergence

1. Use the tools we looked at in class to determine the convergence or divergence of each of the following sequences. If a sequence converges, find its limit.

(a) $a_n = \frac{4+\left(\frac{2}{3}\right)^n}{2-e^n}$

$$\lim_{n\to\infty} \frac{4+\left(\frac{2}{3}\right)^n}{2-e^n} = \frac{4+0}{2-\infty} = \frac{4}{-\infty} = 0.$$

$$\left[\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0 \text{ since } \frac{2}{3} < 1\right]$$

(b) $a_n = \frac{\cos^2 n}{3^n}$ Squeeze Thm.

$$\frac{0}{3^n} \le \frac{\cos^2 n}{3^n} \le \frac{1}{3^n}$$

$$\lim_{n\to\infty}\frac{1}{3^n} = \frac{1}{\infty} = 0. \qquad \lim_{n\to\infty}\frac{0}{3^n} = 0.$$

So $\lim_{n\to\infty}\frac{\cos^2 n}{3^n} = 0$ by Squeeze Thm.

(c) $a_n = \frac{n}{\ln n}$

$$\lim_{n\to\infty}\frac{n}{\ln n} = \lim_{x\to\infty}\frac{x}{\ln x} \overset{L'H}{=} \lim_{x\to\infty}\frac{1}{1/x} = \lim_{x\to\infty} x = \infty$$

$\frac{\infty}{\infty}$

Function Fill In Thm. →

(d) $a_n = \left(1-\frac{3}{n}\right)^{2n}$ $\to 1^\infty$

$$y = \left(1-\frac{3}{x}\right)^{2x}$$

$$\ln y = 2x\ln\left(1-\frac{3}{x}\right)$$

$$\lim_{x\to\infty} 2x\ln(1-3/x) = \lim_{x\to\infty}\frac{2\ln(1-3/x)}{1/x} \overset{L'H}{=} \lim_{x\to\infty}\frac{2\cdot\frac{1}{1-3/x}\cdot 3/x^2}{-1/x^2}$$

$$= \lim_{x\to\infty}\frac{-6}{1-3/x}\cdot\frac{1}{-1} = \frac{(-6)(-1)}{1-0} = 6.$$

So $\lim_{x\to\infty} y = e^{-6}$.

Thus, $\lim_{n\to\infty} a_n = e^{-6}$

(e) $a_n = \frac{e^{3n}+3e^n}{4e^{3n}-2}$

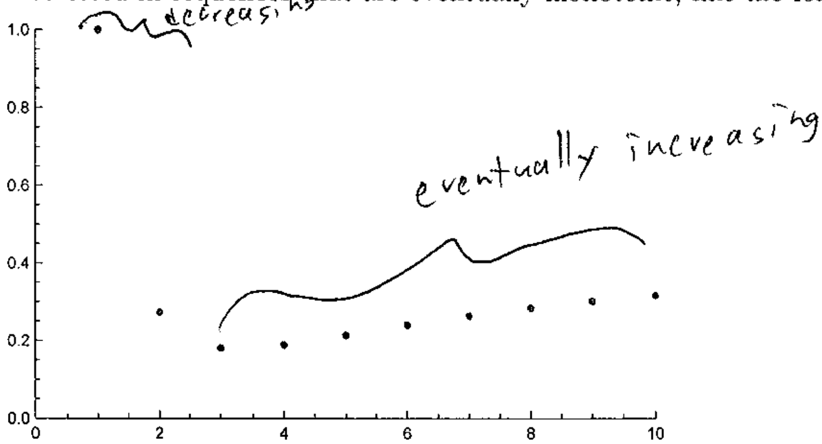
$$\lim_{n\to\infty}\frac{e^{3n}+3e^n}{4e^{3n}-2}\cdot\frac{\frac{1}{e^{3n}}}{1/e^{3n}} = \lim_{n\to\infty}\frac{1+\frac{3}{e^{2n}}}{4-2/e^{3n}} = \frac{1+0}{4-0} = \frac{1}{4}$$

(f) $a_n = (-3)^n$

diverges — $\lim_{n\to\infty} r^n$ DNE if $r \le -1$.

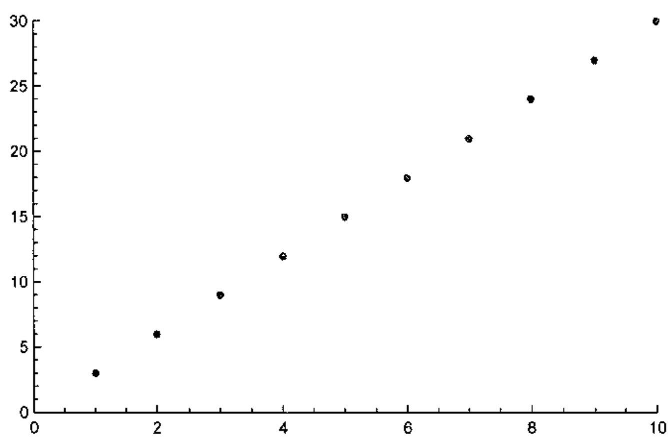
Here, $r = -3$.

2. Recall that a sequence is monotonic if it is always increasing or always decreasing. Since the limit of a sequence is determined by the end of the sequence, it really doesn't matter what happens at the beginning of a sequence. For this reason, we are also interested in sequences that are eventually monotonic, like the following.

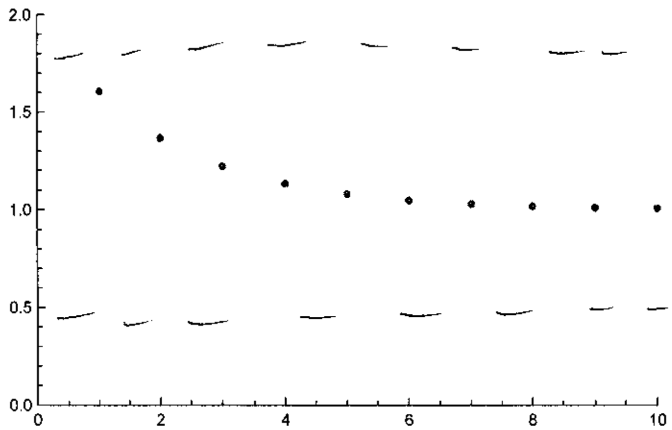


This sequence begins by decreasing, but then appears to increase after that always. This sequence is eventually monotonic. It also appears to be bounded, so the sequence must converge.

Determine whether each of the following sequences is eventually monotonic. Also determine whether each of the following sequences is bounded.

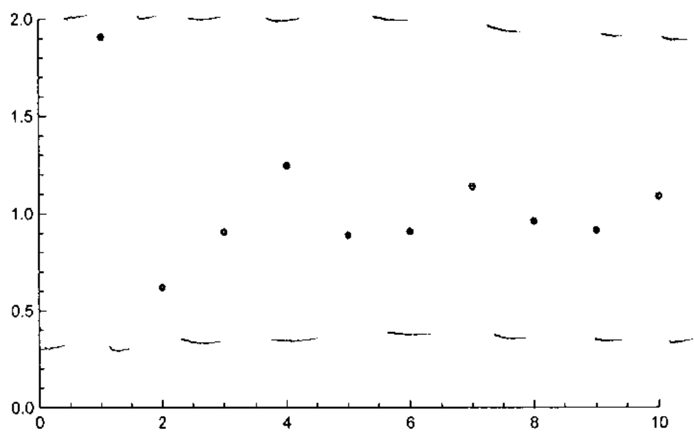
(a) 

monotonic ~~no~~
not bounded



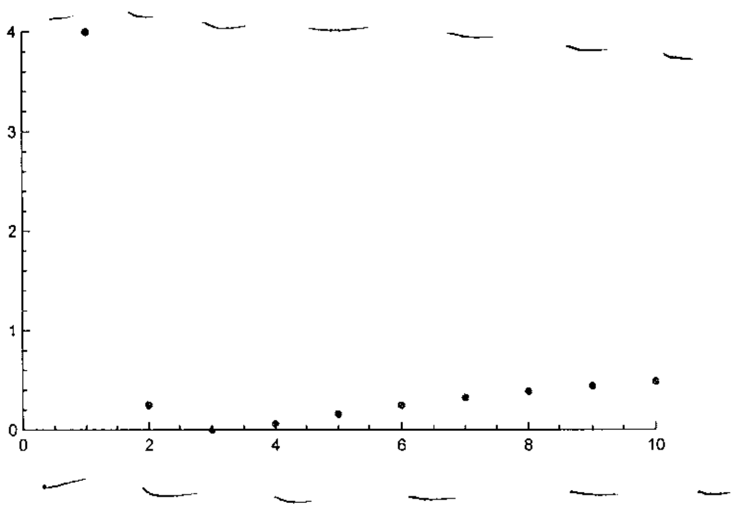
(b)

monotonic,
bounded



(c)

~~mon~~
not monotonic,
bounded



(d)

eventually monotonic,
bounded

increasing here.