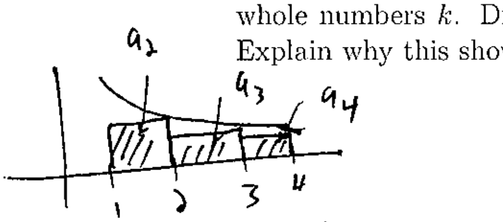
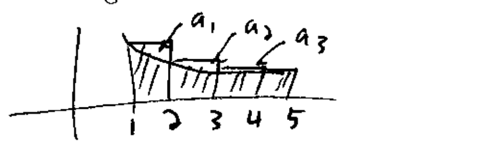
Infinite Series Practice

Definitions and Theory

1. Suppose $f(x)$ is a positive, decreasing, continuous function, and that $a_k = f(k)$ for whole numbers k. Draw a graph to explain why $\sum_{k=2}^{\infty} a_k \leq \int_1^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k$. Explain why this shows that the Integral Test holds.



$$\int_1^{\infty} f(x)dx \geq a_2 + a_3 + \cdots = \sum_{k=2}^{\infty} a_k$$



$$\int_1^{\infty} f(x)dx \leq a_1 + a_2 + \cdots = \sum_{k=2}^{\infty} a_k.$$

If $\int_1^{\infty} f(x)dx$ ~~converges~~ is finite, then $\sum_{k=2}^{\infty} a_k$ is finite since it is smaller. If $\int_1^{\infty} f(x)dx$ is infinite, then $\sum_{k=1}^{\infty} a_k$ is infinite because it is bigger.

2. Determine whether each series converges or diverges. Explain your answer.

(a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

$$\int_1^{\infty} \frac{x}{e^x}dx = \lim_{A\to\infty} \int_1^A x e^{-x}\,dx = \lim_{A\to\infty} -xe^{-x}\Big|_1^A - \int_1^A -e^{-x}dx$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= \lim_{A\to\infty} -xe^{-x}\Big|_1^A + \int_1^A e^{-x}dx = \lim_{A\to\infty} -xe^{-x} - e^{-x}\Big|_1^A$$

$e^{-\infty} = 0$

$$= \lim_{A\to\infty} -Ae^{-A} - \cancel{e^{-A}} - (-1e^{-1} - e^{-1})$$

$$= \lim_{A\to\infty} -\frac{A}{e^A} + e^{-1} + e^{-1}$$

$\frac{\infty}{\infty}$

$$\overset{l'H}{=} \lim_{A\to\infty} \cancel{\frac{-1}{e^A}} \to 0 + e^{-1} + e^{-1}$$

$$= e^{-1} + e^{-1} = 2e^{-1}.$$

Since the integral converges, so does the series by the Integral Test.

(b) $\sum_{n=1}^{\infty} \frac{3}{n^4}$

$$= 3\sum_{n=1}^{\infty} \frac{1}{n^4}$$

Converges (p-series, $p=4$, $p>1$).

Error Bound

3. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ with the 5th partial sum.

$$S_5 = \sum_{n=1}^{5} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} \approx \cancel{1.185662} \; 1.19$$

$$E_n < \int_n^{\infty} f(x)\,dx$$

Find a bound on the error used in this approximation.

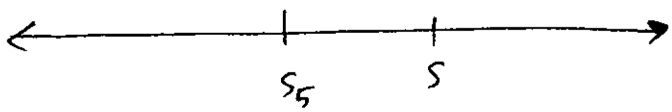
$$E_5 < \int_5^{\infty} \frac{1}{x^3}\,dx = \lim_{A\to\infty} \int_5^A \frac{1}{x^3}\,dx$$

$$= \lim_{A\to\infty} \int_5^A x^{-3}\,dx = \lim_{A\to\infty} \frac{x^{-2}}{-2}\Big|_5^A$$

$$= \lim_{A\to\infty} \frac{-1}{2x^2}\Big|_5^A = \lim_{A\to\infty} \cancel{\frac{-1}{2A^2}}^{\,0} - \left(\frac{-1}{2(5)^2}\right) = \frac{1}{50} = .02$$

$1/\infty = 0$

Use the bound to find a range of values for the sum of the series.



we know $S > S_5$, since all terms in series are positive. $E_5 = S - S_5 = .02$. Our approximation is off by at most 0.02. So at most the sum could be in

~~1.185662 + .02~~

$1.19 + .02 = 1.21$.

So $1.19 < S < 1.21$.

How many terms must be used to approximate the sum of the series with error less than 0.0005?

$$E_n < \int_n^{\infty} \frac{1}{x^3}\,dx$$

$$= \lim_{A\to\infty} \int_n^A \frac{1}{x^3}\,dx = \lim_{A\to\infty} \int_n^A x^{-3}\,dx$$

$$= \lim_{A\to\infty} -\frac{1}{2x^2}\Big|_n^A = \lim_{A\to\infty} \cancel{\frac{-1}{2A^2}}^{\,0} - \left(\frac{-1}{2n^2}\right)$$

$$= \frac{1}{2n^2}.$$

want $\frac{1}{2n^2} < 0.0005$

$$2n^2 > \frac{1}{0.0005}$$

$$n^2 > \frac{1}{2}\left(\frac{1}{0.0005}\right)$$

$$n > \sqrt{\frac{1}{2}\left(\frac{1}{0.0005}\right)} \approx 31.6$$

So take 32 terms