Infinite Series Practice

Definitions and Theory

1. What does it mean for a series $\sum_{n=1}^{\infty} a_n$ to converge?

$\sum_{n=1}^{\infty} a_n$ converges with sum L if $\lim_{N\to\infty} S_N = L$, where $S_N = \sum_{n=1}^{N} a_n$.

2. Give an example of an infinite series that converges to 7.

$\boxed{\sum_{n=1}^{\infty} 3.5\left(\frac{1}{2}\right)^{n-1}}$

choose a & r to have $\frac{a}{1-r} = 7$.

3. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that diverges and $\lim_{n\to\infty} a_n = 0$. If it's not possible, explain why.

$\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic)

4. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that converges and $\lim_{n\to\infty} a_n = 0$. If it's not possible, explain why.

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ (converges, geo, $r = 1/2$, $|r|<1$).

5. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that diverges and $\lim_{n\to\infty} a_n = 0$. If it's not possible, explain why.

oops.

6. If possible, give an example of a series $\sum_{n=1}^{\infty} a_n$ that converges and $\lim_{n\to\infty} a_n \neq 0$. If it's not possible, explain why.

Not possible by Test for Divergence. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

7. Give the argument for why $\sum_{n=1}^{\infty} ar^{n-1}$ converges when $|r| < 1$.

$$S_N = a + ar + ar^2 + \dots + ar^{N-1}$$
$$-\left(rS_N = ar + ar^2 + \dots + ar^{N-1} + ar^N\right)$$

$$S_N - rS_N = a - ar^N$$
$$S_N(1-r) = a - ar^N$$
$$S_N = \frac{a - ar^N}{1-r}$$

$$\lim_{N \to \infty} \frac{a - ar^N}{1-r} = \begin{cases} \frac{a}{1-r} & \text{if } |r|<1. \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

Series Convergence

8. Use the tools we looked at in class to determine the convergence or divergence of each of the following series. If a series converges, find its sum.

(a) $\sum_{n=1}^{\infty} 2\left(-\frac{1}{3}\right)^{n-1}$ converges (geo, $r = -\frac{1}{3}$, $|r|<1$).

$$\text{Sum} = \frac{a}{1-r} = \frac{2}{1-(-\frac{1}{3})} = \frac{2}{4/3} = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

(b) $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{1^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$ converges (geo, $r = \frac{1}{3}$, $|r|<1$).

$$\text{sum} = \frac{a}{1-r} = \frac{1}{1-1/3} = \frac{1}{2/3} = \frac{3}{2}$$

(c) $\sum_{n=1}^{\infty} 3(-2)^{n-1}$ diverges (geo, $r = -2$, $|r| > 1$).

(d) $\sum_{n=2}^{\infty} 7\left(-\frac{1}{2}\right)^{n+3}$ converges (geo, $r=-\frac{1}{2}$, $|r|<1$).

shifted geo series

$$\text{sum} = \frac{\text{first term}}{1-r} = \frac{7\left(-\frac{1}{2}\right)^{2+3}}{1-\left(-\frac{1}{2}\right)} = -\frac{7}{48}$$

(e) $\sum_{n=1}^{\infty} \frac{2}{5^n} = \sum_{n=1}^{\infty} 2\left(\frac{1^n}{5^n}\right) = \sum_{n=1}^{\infty} 2\left(\frac{1}{5}\right)^n$ converges (geo, $r=1/5$, $|r|<1$).

$$\text{sum} = \frac{\text{first term}}{1-r} = \frac{2\left(\frac{1}{5}\right)}{1-1/5} = 1/2$$

(f) $\sum_{n=1}^{\infty} \frac{3^{2n}}{9^{n-1}} = \sum_{n=1}^{\infty} \frac{(3^2)^n}{9^{n-1}} = \sum_{n=1}^{\infty} \frac{9^n}{9^{n-1}} = \sum_{n=1}^{\infty} 9^{n-(n-1)}$

$$= \sum_{n=1}^{\infty} 9$$

$$= 9+9+9+\ldots = \infty$$

diverges.

Also $\lim_{n\to\infty} 9 = 9 \neq 0$, so diverges by Test for Divergence.