A Higher Dimensional Generalization of Subspace Theorem for Closed Subschemes by Beta Constants

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1 Introduction

1.1 Known Results

In this section, we will give some well-known results in the Diophantine approximation, and we will also show there b-Cartierv divisor's version. Main theorems will be given in [Voj23].

First, let's begin with the basic subspace theorem:

Theorem 1.1 (Schmidt's subspace theorem)**.** *Let k be a number field, let S be a finite set of places of k containing all archimedean places, let n be a positiveintefer, let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n_k$, let $\epsilon > 0$, and let $c \in \mathbb{R}$. Then there is a finite union Z of proper linear subspace of $\mathbb{P}^n_k$, depending only on $k, S, n, H_1, \dots, H_q, \epsilon$, and c, such that the inequality*

$$\sum_{v\in S} \max_J \sum_{j\in J} \lambda_{H_j,v}(x) \le (n+1+\epsilon)h(x) + c$$

holds for all $x \in (\mathbb{P}^n_{\mathbb{Q}} \backslash Z)(k)$. Here the set J ranges over all subsets of $\{1, \dots, q\}$ such that the hyperplanes $(H_j)_{j\in J}$ lie in general position.

The subspace theorem has been generalized to many cases, in Evertse and Ferretti [EF08], they generalize the theorem to the arbitrary projective varieties and to divisors which possess a common linear equivalent multiple.

Theorem 1.2 (Evertse-Ferretti,[EF08])**.** *Let X be a projective variety of dimension n defined over a number field k. Let S be a finite set of places of k. For each $v \in S$, let $D_{0,v}, \dots, D_{n,v}$ be effective Cartier divisors on X, defined over k, in general position. Suppose that there exists an ample Cartier divisor A on X and positive integers $d_{i,v}$ such that $D_{i,v} \sim d_{i,v}A$ for all i and for all $v \in S$. Let $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subset X$ such that for all points $P \in X(k) \setminus Z$,*

$$\sum_{v\in S} \sum_{i=0}^{n} \frac{\lambda_{D_{i,v},v}(P)}{d_{i,v}} < (n+1+\epsilon)h_A(P).$$

Here, $\lambda_{D_{i,v},v}$ is a local height function associated to the divisor $D_{i,v}$ and place v in S, and h_A is a global (absolute) height associated to A.

Gordon Heier and Aaron Levin further generalized this theorem in their work referenced in [HL21], extending the coefficients from the original work in [EF08] to a concept called the Seshadri constant(Definition 2.15). The specifics are as follows:

Theorem 1.3 (Heier-Levin, [HL21])**.** *Let X be a projective variety of dimension n defined over a number field k. Let S be a finite set of places of k. For each $v \in S$, let $Y_{0,v}, \dots, Y_{n,v}$ be closed subschemes of X, defined over k, and in general position. Let A be an ample Cartier divisor on X, and $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subset X$ such that for all points $P \in X(k) \backslash Z$,*

$$\sum_{v\in S} \sum_{i=0}^{n} \epsilon_{Y_i,v}(A)\lambda_{Y_i,v,v}(P) < (n+1+\epsilon)h_A(P).$$

Further, these two authors have extended this theorem in the subgeneral position(Definition 2.14), obtaining the following conclusion:

Theorem 1.4 (Heier-Levin, [HL23])**.** *Let X be a projective variety of dimension n defined over a number field k, and let S be a finite set of places of k. For each $v \in S$, let $Y_{1,v}, \ldots, Y_{q,v}$ be closed subschemes of X, defined over k (not necessarily in general position), and let $c_{1,v}, \ldots, c_{q,v}$ be nonnegative real numbers. For a closed subset $W \subset X$ and $v \in S$, let*

$$\alpha_v(W) = \sum_{\substack{i \\ W \subseteq \operatorname{Supp} Y_{i,v}}} c_{i,v}.$$

Let A be an ample Cartier divisor on X, and $\epsilon > 0$. Then there exists a proper Zariski closed subset Z of X such that

$$\sum_{v \in S} \sum_{i=1}^{q} c_{i,v} \epsilon_{Y_{i,v}}(A) \lambda_{Y_{i,v},v}(P) < \left((n+1) \max_{\substack{v \in S \\ \emptyset \subsetneq W \subsetneq X}} \left(\frac{\alpha_v(W)}{\operatorname{codim} W} \right) + \epsilon \right) h_A(P)$$

for all points $P \in X(k) \setminus Z$.

We also have the following theorem:

Theorem 1.5 ([Voj23])**.** *Let X be a complete variety over a number field k, let $\mathscr{L}$ be a line sheaf on X with $h^0(X, \mathscr{L}^n) > 1$ for some $N > 0$, and let $\mathbf{D}_1, \ldots, \mathbf{D}_p (p > 0)$ be effective $\mathbb{R}$-Cartier b-divisors on X. Take S be a finite set of places of k. Then, for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zarisiki-closed subset Z of X such that the inequality*

$$m_S(\mathbf{D}_1, \ldots, \mathbf{D}_p, x) \leq (\operatorname{Nev}_{\mathrm{bir}}(\mathscr{L}, \mathbf{D}_1, \ldots, \mathbf{D}_p) + \epsilon) h_{\mathscr{L}}(x) + C$$

holds for all points $x \in X(k) \backslash Z$.

And in [RV20], they give a ‘general theorem’, and in [Voj23], Vojta generalize the general theorem to the b-divisor case:

Theorem 1.6 ([Voj23])**.** *Let k be either a number fiel, let X be a complete varitey over k, let $\mathscr{L}$ be a big line sheaf on X, let $p > 0$, and for each $i = 1, \ldots, p$ let $Y_{i,1}, \ldots, Y_{i,q_i}$ be proper closed subschemes of X that have the Autissier property. Let S be a finite set of places of k, and for all i and j and all $v \in S$ let $\lambda_{i,j,v}$ be a Weil function for $Y_{i,j}$ at v. Then, for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequlity*

$$\frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} \max_i \sum_j \beta(\mathscr{L}, Y_{i,j}) \lambda_{Y_{i,j,v}}(x) \leq (1+\epsilon) h_{\mathscr{L}}(x) + C$$

holds for all points $x \in X(k) \backslash Z$.

We omit the analytic part here. Using the approach of subspace theorems, we have proved the following theorem.

Theorem 1.7 ([CZ02])**.** *Let k be a number field with $\mathcal{O}$ ring of integers, $\tilde{C}$ a projective, absolutely irreducible curve over k, C an affine open subset of $\tilde{C}$, embedded into $\mathbb{A}^m$. Let S be a finite set of places of k. If $\tilde{C}$ has a infinitely many points in $\mathbb{A}^m(\mathcal{O}(S))$, then $\tilde{C}$ has genus 0 and moreover $\#(\tilde{C}$ $C) \leqslant 2$.*

And we have the following improvements.

Theorem 1.8 ([Lev09], Theorem 10.4A.)**.** *Let X be a nonsingular projective variety defined over a number field k. Let $q = \dim X$. Let $D = \sum_{i=1}^{r} D_i$ be a divisor on X defined over k such that the D_i are effective divisors with no irreducible components in common and such that the intersection of any $m+1$ distinct D_i is empty. Suppose also that every irreducible component of D is nonsingular. If D_i is nef and big for each i and $r > 2[(m+1)/2]q$, then $X \backslash D$ is quasi-Mordellic(See [Lev09], Definition 3.4A.).*

Definition 1.9. *Let X be a projective variety over a number field k, $X \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supseteq Y_r$ be a chain of closed subschemes of X. We say this chain satisfies condition $(*)$, if:*

1. *We blow up Y_i in order, therefore get a chain of birational morphisms:*

$$X_r \xrightarrow{Bl_{Y'_r}} X_{r-1} \xrightarrow{Bl_{Y'_{r-1}}} \dots \xrightarrow{Bl_{Y'_2}} X_1 \xrightarrow{Bl_{Y'_1}} X_0 = X$$

Let D_i(on X_r) be the strict transform of the exceptional divisor(on X_i) of Y_i. Then the divisor corresponds to Y_i on X_r is $\sum_{j=i}^{r} D_j$. The condition is $D_1, \dots, D_r$ intersect properly.

The following theorem is our main theorem:

Theorem 1.10 (Main theorem)**.** *Let X be a projective complete variety of dimension n. Let $\mathscr{L}$ be a big line sheaf on X. Let S be a finte set of places. Given a sequence of closed subschemes: $Y_1 \supset Y_2 \supset \cdots \supset Y_q$ and assume that this is a regular chain. Assume some special conditions and take $\lambda_{Y_i,v}$ to be the correspondance Weil functions, where $v \in S$. Then for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{i=1}^{q} \sum_{v \in s} (\beta(\mathscr{L}, Y_i) - \beta(\mathscr{L}, Y_{i-1}))\lambda_{Y_i}(x) \leq (1+\epsilon)h_{\mathscr{L}} + C$$

holds for all points $x \in X(k) \backslash Z$

Use the main theorem, we can give the following theorems:

Theorem 1.11. *Let X be a projective variety of dimension n defined over a number field k. Let $D_1, \dots, D_{n+1}$ be effective Cartier divisors on X. $a_1, \dots, a_{n+1}$ are positive integers such that $a_i D_i$ are all numerically equivalent to an ample Cartier divisor D. Let*

$$B = \{\cap_{i \in I \subseteq \{1,\dots,n+1\}} a_i D_i\}$$

Suppose that $\forall Y_k \in B$, $Y_k \cap a_s D_s = Y_l$, we have

$$\beta(A, Y_l) - \beta(A, Y_k) - \frac{1}{n+1} > 0 \tag{1.1}$$

Let $S \subseteq M_k$ be a finite set of places. $\epsilon > 0$ be any positive number. Then $\exists$ a Zariski closed subset $Z \subseteq X$ such that $\forall$ subset of $(\sum D_i, S)$ integral points $R \subseteq X(k)$, $\forall P \in R$
Z,

$$h_{D_i \cap D_j}(P) \leqslant \epsilon h_D(P)$$

2 Notations and Preliminaries

If X is a variety over a number field k, we use $X(M_k)$ denote the disjoint union $\coprod_{v\in M_k} X(\bar{k}_v$. Where $\bar{k}_v$ means any fixed algebraic closure of k_v

2.1 Weil functions

In this section, we will introduce some basic concepts in the Diophantine approximation.

Definition 2.1. *Let $n \geq 1$, the* multiplicative height *of a rational point $P \in \mathbb{P}^n(\mathbb{Q})$ is defined by*

$$H(P) = \max\{|x_j| : 0 \leq j \leq n\}$$

where we write $P = [x_0 : \cdots : x_n]$ with $(x_0, \cdots, x_n \in \mathbb{Z}^{n+1})$ a primitive integer vector.

Usually, we say height we are concern about logarithmic height, i.e

$$h(P) \coloneqq \log H(P)$$

To generalize this concept, we have the following theorem:

Theorem 2.2 (Weil's height function)**.** *For each irreducible, smooth, projective variety X and invertible sheaf $\mathscr{L}$ on it, both defined over $\mathbb{Q}$, there is a function:*

$$h_{X,\mathscr{L}} : X(\mathbb{Q}) \to \mathbb{R}$$

uniquely defined up to adding a bounded function, such that the following properties hold:

(i) *(Normalization) For $X = \mathbb{P}^n_{\mathbb{Q}}$ and $\mathscr{L} = \mathscr{O}(1)$ er have $h_{\mathbb{P}^1_{\mathbb{Q}},\mathscr{O}(1)}(P) = h(P) + O(1)$*

(ii) *(Functoriality) Given a morphism $f : X \to Y$ of projective varieties over $\mathbb{Q}$ and a line sheaf $\mathscr{L}$ on Y, we have $h_{X,f^*\mathscr{L}}(P) = h_{Y,\mathscr{L}}(f(P)) + O(1)$*

(iii) *(Additivity) Given $\mathscr{L}, \mathscr{M}$ line sheaves on X, we have $h_{X,\mathscr{L}^\vee}(P) = -h_{X,\mathscr{L}}(P) + O(1)$ and $h_{X,\mathscr{L}\otimes\mathscr{M}}(P) = h_{X,\mathscr{L}}(P) + h_{X,\mathscr{M}}(P) + O(1)$*

(iv) *(Isomorphism) if $\mathscr{L} \simeq \mathscr{M}$ on X, then $h_{X,\mathscr{L}}(P) = h_{X,\mathscr{M}}(P) + O(1)$*

(v) *(Effective positivity) If $\mathscr{L}$ is an effective line sheaf on X, then there is $c > 0$ such that $h_{X,\mathscr{L}}(P) \geq c$ for all $P \in X(\mathbb{Q})$ outside the base locus of $\mathscr{L}$*

(vi) *(Ample finiteness) If $\mathscr{A}$ is an ample line sheaf on X, then for each $B > 0$ the set $\{P \in X(\mathbb{Q}) : h_{X,\mathscr{A}}(P) \leq B\}$ is finite*

(vii) *(Numerical equivalence) Let $\mathscr{A}$ and $\mathscr{L}$ be line sheaves on X with $\mathscr{A}$ ample and $\mathscr{L}$ numerically trivial(i.e $\mathscr{L} \equiv \mathscr{O}_X$). Let $\epsilon > 0$. Then for all but finitely many $P \in X(\mathbb{Q})$ we have $|h_{X,\mathscr{L}}(P)| < \epsilon \cdot h_{X,\mathscr{A}}(P)$*

You can view the above definitions in the global cases, now we will work on the local field. For our case, we will only treat the number field k, and we define $M_{\mathbb{Q}}$ is the set of places on k. For simplicity, one can just treat the case $k = \mathbb{Q}_v$.

Definition 2.3 (Weil function)**.** *For each smooth, projective, irreducible variety X over $k = \mathbb{Q}_v$ and any $D \in \mathrm{Div}_k(X)$, a* Weil function *for D is a function*

$$\lambda_{X,v}(D, -) : X(k) - \mathrm{supp}(D) \to \mathbb{R}$$

with the following property: For every $x \in X(k)$ and every $f \in k(X)^\times$ local equation for D near X, there is a v-adic neighborhood $W \subset V$ of x and a bounded continuous function $\alpha : W \to \mathbb{R}$ such that for all $P \in W - \mathrm{supp}(D)$ we have

$$\lambda_{X,v} = -\log|f(P)|_v + \alpha(P)$$

Finally, we give our definiton of (D,S)-integral:

Definition 2.4. *Let X be a smooth, irreducible, projective variety over $\mathbb{Q}$. Let $D \in \mathrm{Div}_{\mathbb{Q}}$ be an effective divisor. Suppose that $U = X - \mathrm{supp}(D)$. We say that a set of rational points $\sigma \subset U(\mathbb{Q})$ is (D,S)-integral if for all $P \in \sigma$ we have*

$$\sum_{v \in S} \lambda_{X,v}(D,P) = h_{X,\mathscr{O}(D)}(P) + O(1)$$

In [Sil87], Silverman generalized the Weil height machine for Cartier divisors to height functions on projective varieties with respect to closed subschemes. More precisely, let X be a projective variety over a number field K, and let $Z(X)$ denote the set of closed subschemes of X. Let M_K be the set of places of K. Note that the closed subschemes $Y \in Z(X)$ are in one-to-one correspondence with quasi-coherent ideal sheaves $\mathcal{I}_Y \subseteq \mathcal{O}_X$, and we identify a closed subscheme Y with its ideal sheaf $\mathcal{I}_Y$. Generalizing the Weil height machine for Cartier divisors, Silverman assigned to each $Y \in Z(X)$ and each place $v \in M_K$ a local height function $\lambda_{Y,v}$, and to each $Y \in Z(X)$ a global height function $h_Y = \sum_{v \in M_K} \lambda_{Y,v}$ (both uniquely determined up to a bounded function). We now summarize some of the basic properties of height functions associated to closed subschemes.

Theorem 2.5. *([Sil87]) Let X be a projective variety over a number field K. Let $Z(X)$ be the set of closed subschemes of X. There are maps*

$$\begin{aligned} Z(X) \times M_K &\to \{\text{functions } X(K) \to [0,+\infty]\}, \\ (Y,v) &\mapsto \lambda_{Y,v}, \\ Z(X) &\to \{\text{functions } X(K) \to [0,+\infty]\}, \\ Y &\mapsto h_Y, \end{aligned}$$

satisfying the following properties (we also write $\lambda_{X,Y,v}$ and $h_{X,Y}$ for clarity in (6)):

1. *If $D \in Z(X)$ is an effective Cartier divisor, then $\lambda_{D,v}$ and h_D agree with the classical height functions associated to D.*
2. *If $W, Y \in Z(X)$ satisfy $W \subseteq Y$, then $h_W \leq h_Y + O(1)$ and $\lambda_{W,v} \leq \lambda_{Y,v} + O(1)$ for all $v \in M_K$.*
3. *If $W, Y \in Z(X)$ satisfy $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(Y)$, then there exists a constant C such that $h_W \leq C \cdot h_Y + O(1)$ and $\lambda_{W,v} \leq C \cdot \lambda_{Y,v} + O(1)$ for all $v \in M_K$.*
4. *For all $W, Y \in Z(X)$, $\lambda_{W \cap Y,v} = \min\{\lambda_{W,v}, \lambda_{Y,v}\} + O(1)$.*
5. *For all $W, Y \in Z(X)$, we have $h_{W+Y} = h_W + h_Y + O(1)$ and $\lambda_{W+Y,v} = \lambda_{W,v} + \lambda_{Y,v} + O(1)$ for all $v \in M_K$.*
6. *Let $\phi : X' \to X$ be a morphism of projective varieties over K, and let $Y \in Z(X)$. Then*

$$\begin{aligned} h_{X',\phi^*Y} &= h_{X,Y} \circ \phi + O(1), \\ \lambda_{X',\phi^*Y,v} &= \lambda_{X,Y,v} \circ \phi + O(1), \end{aligned}$$

   for all $v \in M_K$.

7. *If D and E are numerically equivalent Cartier divisors on X and A is an ample divisor on X, then for any $\varepsilon > 0$, we have*

$$|h_D(P) - h_E(P)| < \varepsilon h_A(P) + O(1)$$

   for all $P \in X(K)$.

Here, $Y \subset Z$, $Y + Z$, and ϕ^*Y are all defined in terms of the associated ideal sheaves (see [Sil87]). For a closed subscheme Y and finite set of places of S of K, we let $m_{Y,S}(P) = \sum_{v \in S} \lambda_{Y,v}(P)$. For Cartier divisors D and E on a variety X, we will also write $D \geq E$ (or $E \leq D$) if $D - E$ is an effective divisor.

2.2 Birational Part

We will use languages of birational divisors, here are some basic definitions and properties. Here we introduce the notaitons in the [Voj23], anyone interested in this topic can find the explicit proof in it.

we follow the notion of b-divisor given by Shokurov; see [Cor07], where the 'b' stands for the *birational*.

Definition 2.6. *Let X be a complete variety overa field k.*

(a) *A* model *of X is a proper birational morphism $Y \to X$ over k, where Y is a variety over k. We often use Y to denote the model.*

(b) *The category of models of X is the category whose objects are models of X and whose morphisms are morphism over X. We say that a model Y_1 of X* dominates *a model Y_2 of X if there is a morphism $Y_1 \to Y_2$(necessarily unique) in this category.*

(c) *A* b-Cartier divisor *(resp. $\mathbb{Q}$-b-Cartier divisor) on X is an equivalence class of pairs (Y, D), where Y is a model of X and D is a Cartier divisor (resp. $\mathbb{Q}$-Cartier divisor) on Y; here equivalence classes are those for the equivalence relation generated by the relation $(Y_1, D_1) \sim (Y_2, D_2)$ if Y_1 dominates Y_2 via $\phi : Y_1 \to Y_2$ and $D_1 = \phi^* D_2$.*

(d) *A b-Cartier divisor or $\mathbb{Q}$-b-Cartier divisor $\mathbf{D}$ on X is* effective *if it is represented by a pair (Y, D) such that D is effective.*

Remark 2.7. *Here is some basic remarks on the definition above:*

1. *Here we can also given the definition more directly, i.e we can define:*

$$\mathbf{D} = (\mathbf{D}_Y)Y \in \varprojlim_{Y} \mathrm{Div}(Y)$$

   when the X is required to be normal. In this case, one can check that the two definitons conincide, at the same time, we will always work on the case that X is normal.

2. *The definition for the effective is well-defined, for one can work on the models which is dominate two given models on X, then one can move the effective properties along it.*

3. *For the blow up morphism is always projective, hence proper, we know that any closed subschemes can be lift to a model, in which it is a divisor. Hence any closed subschemes in X can view as one b-divisor, that is one reason why the b-divisors genrealize the usual divisors.*

We have just given the definition of the b-divisors, so it is natural to consider the b-Weil functions:

Definition 2.8. *Let X be a complete variety over a number field k. Then a* b-Weil function *on X (resp. a $\mathbb{Q}$-b-Weil function on X) is an equivalence class of pairs (U, λ), where U is a nonempty Zariski-open subset of X and $\lambda : U(M_k) \to \mathbb{R}$ is a function such that there exist a model $\phi : Y \to X$ of X and a Cartier divisor(resp. $\mathbb{Q}$-b-Cartier divisor) D on Y such that $\lambda \circ \phi$ extend to a Weil function for D(resp. such that $n\lambda \circ \phi$ extends to a Weil function for nD for some(and hence all) nonzero integers n for which nD is a Cartier divisor). Pairs (U, λ) and (U', λ') are* equivalent *if $\lambda = \lambda'$ on $(u \cap U')(M_k)$. Local b-Weil functions and local $\mathbb{Q}$-b-Weil functions on X are defined similarly.*

Definition 2.9. *Let X be a complete variety over a number field k, let λ be a b-Weil function on X, and let $\mathbf{D}$ be a b-cartier divisor on X. We say that λ is a b-Weil function for $\mathbf{D}$ if $\mathbf{D}$ is represented by a pair (Y, D) as above, such that if $\phi : Y \to X$ is the structural morphism of Y, then $\lambda \circ \phi$ extends to a Weil function for D on Y.*

The conncetion between the b-Weil funciton and b-Cartier divisor is just the same as the usual case. Here we just give a list of the properties:

Proposition 2.10. *Let X be a complete variety over a number field k. Given $\mathbf{D}_i$ and λ_i respectively/*

(a) $-\lambda_i$ *correspond to* $-\mathbf{D}_i$ *and* $\lambda_1+\lambda_2$ *correspond to* $\mathbf{D}_1+\mathbf{D}_2$

(b) λ_i *is* M_k *bounded iff* $\mathbf{D}_i$ *is effective.*

(c) *Modulo a* M_k *constant, we have the fact that the b-Weil function and the b-Cartier funciton are one-to-one corresponding.*

One could define a partial order on the set of b-Cartier divisors, more explicitly, one may define $\mathbf{D}_1 \geq \mathbf{D}_2$ iff $\mathbf{D}_1 - \mathbf{D}2$ is effective. One surprising result is that with the partial order given above, the b-Cartier divisors form a lattice, i.e. it has a least upper bound and the greatest lower bound. As for the partial order is compatible with the group action, we only need give the description of the least lower bound.

Lemma 2.11. *Let X/k as above, let $\mathbf{D}$ and $\mathbf{D}_1,\dots,\mathbf{D}_l$ be b-Cartier divisors on X. Choose a model Y of X such that $\mathbf{D}_i$ is represented D_i where D_i is a traditional Cartier divisor on Y, and $\mathbf{D}$ is represented by D. Then we have the fact that $\mathbf{D}$ is a least upper bound of $\mathbf{D}_1,\dots,\mathbf{D}_l$ if and only each $\mathbf{D}-\mathbf{D}_i$ is effective and we have:*

$$\bigcap_{i=1}^{l} \mathrm{Supp}(D-D_i)=\emptyset$$

This lemma is easy to proof, briefly, one just consider the models which blow up the $D-D_i$, and then one can find a lower upper bound. Hence one must require that $\bigcap_{i=1}^{l}(D-D_i)$ is $\emptyset$.

We also have that if we assume λ_i for the correseponding b-Weil divisor to the $\mathbf{D}_i$, then we have $max\{\lambda_1,\lambda_2\}=\lambda_{\mathbf{D}_1\vee\mathbf{D}_2}$, where we use the $\mathbf{D}_1\vee\mathbf{D}_2$ to represent the least upper bound of the $\mathbf{D}_1$ and $\mathbf{D}_2$.

We can define the global section of b-divisor.

Definition 2.12 ([Voj23], Definition 6.3.)**.** *Let $\mathscr{L}$ be a line sheaf on X and let $\mathbf{D}$ be an effective Cartier b-divisor on X. Then*

$$H^0_{bir}(X,\mathscr{L}(-\mathbf{D}))=H^0(W,\pi^*\mathscr{L}(-D))$$

, where $\pi: W\to X$ is any normal model of X on which $\mathbf{D}$ is represented by a Cartier divisor D. Also,

$$h^0_{bir}(X,\mathscr{L}(-\mathbf{D}))=\dim_k H^0_{bir}(X,\mathscr{L}(-\mathbf{D}))$$

.

Lemma 2.13 ([Voj23], Lemma 6.4.)**.** *Let $\mathscr{L}$ be a line sheaf on X, let D be a nonzero effective Cartier divisor on X, and let $d=\dim X$. Then*

$$h^0_{bir}(X,\mathscr{L}^N)=h^0(X,\mathscr{L}^N)+O(N^{d-1})$$

and

$$\sum_{m=1}^{\infty} h^0(X,\mathscr{L}^N(-mD))=\sum_{m=1}^{\infty} h^0(X,\mathscr{L}^N(-mD))+O(N^d)$$

.

2.3 Some constants

We will firmly work on this special constant, here we first give the basic definition of it:

Definition 2.14 (m-Subgeneral Position). *Let X be a projective variety of dimension n. We say that closed subschemes $Y_1, \dots, Y_q$ of X are in m-subgeneral position if for every subset $I \subset \{1, \dots, q\}$ with $|I| \leq m+1$, we have*

$$\operatorname{codim} \bigcap_{i \in I} Y_i \geq |I| + n - m,$$

where we use the convention that $\dim \emptyset = -1$. In the case $m = n$, we say that the closed subschemes are in general position. If V is a subset of X, we say that closed subschemes $Y_1, \dots, Y_q$ of X are in general position outside of V if for every subset $I \subset \{1, \dots, q\}$ with $|I| \leq n+1$ we have $\operatorname{codim}((\bigcap_{i \in I} Y_i) \setminus V) \geq |I|$.

Definition 2.15 (Seshadri constants). *Let Y be a closed subscheme of a projective variety X and let $\pi : \tilde{X} \to X$ be the blowing-up of X along Y. Let A be a nef Cartier divisor on X. We define the Seshadri constant $\epsilon_Y(A)$ of Y with respect to A to be the real number*

$$\epsilon_Y(A) = \sup\{\gamma \in \mathbb{Q}_{\geq 0} \mid \pi^* A - \gamma E \text{ is } \mathbb{Q}\text{-nef}\},$$

where E is an effective Cartier divisor on $\tilde{X}$ whose associated invertible sheaf is the dual of $\pi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{X}}$.

Definition 2.16 (β constant). *Let $\mathscr{L}$ be a big line sheaf on X, let Y be a nonempty proper closed subscheme of X, and let $\mathscr{I}$ be the sheaf of ideals corresponding to Y. Then*

$$\beta(\mathscr{L}, Y) = \liminf_{N \to \infty} \frac{\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N \otimes \mathscr{I}^m)}{N h^0(X, \mathscr{L}^N)}$$

Since we have talked about so much with the b-divisor, it is a natural question that can we generalize the concept of the β constant into some 'b-divisor' case. Use the definition above, we have the following defintion:

Definition 2.17.

$$\beta(\mathscr{L}, \mathbf{D}) = \liminf_{N \to \infty} \frac{\sum_{m=1}^{\infty} h^0_{bir}(X, \mathscr{L}^N(-m\mathbf{D}))}{N h^0_{bir}(X, \mathscr{L}^N)}$$

. This definition is well-defined. Acoording to the lemma above, we know when we take different representative element for the b-divisor $\mathbf{D}$, the variation of the numerator of the β constant is at most $O(N^d)$. And one reason that we define above is the following proposition:

Proposition 2.18 ([Voj23], Corollary 6.9.). *Let $\mathbf{Y}$ be the b-divisor corresponding to a proper closed subscheme of X. Let $\mathscr{L}$ be a big line sheaf on X. Then:*

$$\beta(\mathscr{L}, Y) = \beta(\mathscr{L}, \mathbf{Y})$$

We give another constant, the Nevanlinna constant:

Definition 2.19 (Nevanlinna constant). *Let X be a complete variety, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. If X is normal, then we define*

$$\operatorname{Nev}_{bir}(\mathscr{L}, D) = \inf_{N,V,\mu} \frac{\dim V}{\mu}$$

where the infimun passes over all triples (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, and $\mu \in \mathbb{Q}_{>0}$, with the following property. There exist a variety Y and a proper birational morphism $\phi : Y \to X$ such that the following condition holds. For all $Q \in Y$ there is a basis $\mathcal{B}$ of V such that

$$\phi^*(\mathcal{B}) \geq \mu N \phi^* D$$

in a Zariski-open neighborhood U of Q, relative to the cone of effective $\mathbb{Q}$-divisors on U. If there are no such triples (N, V, μ), then Nev_{bir} is defined to be $+\infty$. For a general complete variety X, $\operatorname{Nev}_{bir}(\mathscr{L}, D)$ is defined by pulling back to the normalization of X.

3 Filtration method

3.1 The closed subscheme cases

We will give our definition for the intersect properly. When our base ringis Cohen-Macauley, we know that the concept of intersect peoperly and in general position defined above are the same.

Definition 3.1. *Let $I_1, \dots, I_q$ be ideals of A, with $q \in \mathbb{N}$. Then $I_1, \dots, I_q$ intersect properly if*

(i) *for each $i = 1, \dots, q$ there is a nonempty regular sequence $\phi_{i1}, \dots, \phi_{ir_i}$ in A such that I_i is of monomial type with repect to $\phi_{i1}, \dots, \phi_{ir_i}$, i.e. $I = \mathscr{J}(N)$ with respect to $\phi_{i1}, \dots, \phi_{ir_i}$ and some saturated N.*

(ii) *the sequence $\phi_{11}, \dots, \phi_{1r_1}, \dots, \phi_{q1}, \dots, \phi_{qr_q}$ is a reugular sequence.*

Back to the scheme cases, as in the [Voj23], we have the following definition:

Definition 3.2 ([Voj23], Definition 4.1)**.** *Let $\mathscr{J}_1, \dots, \mathscr{J}_q$ be the ideal sheaves that corresponding to $Y_1, \dots, Y_q$, where Y_i are proper closed subscheme of X, a complete variety.*

(a) *We say that $Y_1, \dots, Y_q$ intersect properly at a point $P \in X$ if the subsequence of proper ideals in the sequence $(\mathscr{J}_1)_P, \dots, (\mathscr{J}_q)_P$ of ideals of the local ring $\mathscr{O}_{X,P}$ intersect properly. If $P \notin \cup Y_i$, this is naturally correct.*

(b) *We say that $Y_1, \dots, Y_q$ intersect properly if $Y_1, \dots, Y_q$ intersect properly at all points of X.*

(c) *We say that $Y_1, \dots, Y_q$ weakly intersect properly if they intersect properly at all $P \in \bigcup_{i \neq j}(Y_i \cap Y_j)$*

3.2 Filtration Method

In this section, we will know why should have the Autissier property in our works. We will introduce a poweful method in our job, called the filtration method. Its basic is the following.

We first give the most basic type of the filtration method:

Lemma 3.3 ([Lev09], Lemma 10.1.)**.** *Let V be a vector space of finite dimension d over a field k. Let $V = W_1 \supset W_2 \supset \cdots \supset W_h$ and $V = W_1^* \supset W_2^* \supset \cdots \supset W_{h^*}^*$ be two filtrations on V. There is a basis $v_1, \dots, v_d$ of V that contains a basis of each W_j and W_j^*.*

Definition 3.4. *Let $r \in \mathbb{Z}_{>0}$. A subset N of $\mathbb{N}^r$ is saturated if it is nonempty and if $N \supset \mathbf{a} + \mathbb{N}^r$ for all $\mathbf{a} \in N$.*

Definition 3.5. *Let $\phi_1, \dots, \phi_r \in A$ with $r > 0$, and let N be a saturated subset of $\mathbb{N}^r$. Then $\mathscr{J}(N)$ is the ideal of A of generated by the set $\{\phi_1^{b_1}, \dots \phi_r^{b_r} : \mathbf{b} \in N\}$.*

In the [Aut11], Autissier find the following key lemma:

Lemma 3.6. *Let $\phi_1, \dots, \phi_r (r > 0)$ be a regular sequence in A, and let N_1 and N_2 be saturated subsets of $\mathbb{N}^r$. Then*

$$\mathscr{J}(N_1 \cap N_2) = \mathscr{J}(N_1) \cap \mathscr{J}(N_2)$$

For our purpose, we will give a bit generalization of this concept into the ideal case.

Definition 3.7. *Let I be an ideal of A and let $\phi_1, \dots, \phi_r$ be a sequence of elements of A. Then I is of monomial type with respect to $\phi_1, \dots, \phi_r$ if $r > 0$ and $I == \mathscr{J}(N)$ (taken relative to $\phi_1, \dots, \phi_r$) for some saturated subset N of $\mathbb{N}^r$.*

Here, we define the n-multiple of the saturated set as below:

Definition 3.8. *When n=0, we define $0N = \mathbb{N}^r$, when $N > 0$, we define*

$$nN := \{\mathbf{b}_1 + \cdots + \mathbf{b}_n : \mathbf{b}_1, \dots, \mathbf{b}_n \in N\}$$

We now replace the divisors into closed subschemes, we have the following definiton:

Definition 3.9. *Let $q \in \mathbb{Z}_{>0}$, let $I_1, \ldots, I_q$ be ideals in A, let N be a saturated subset of N^q. Then $\mathscr{I}(N)$ is the ideal of A defined by*

$$\mathscr{I}(N) = \sum_{\mathbf{b} \in N} I_1^{b_1} \cdots I_q^{b_q}$$

Hence, we define the Autissier property as below:

Definition 3.10. *Let $I_1, \ldots, I_q$ be ideals in A. We say that they have the Autissier peoperty if*

$$\mathscr{I}(N \cap N') = \mathscr{I}(N) \cap \mathscr{I}(N')$$

Definition 3.11 ([Voj23], Definition 4.2)**.** *Let $\mathscr{I}_1, \ldots, \mathscr{I}_q$ be as in 3.2*

(a) *Let $P \in X$, and let $j_1, \ldots, j_r$ be the subsequence of $1, \ldots, q$ consisting of those j such that $P \in Y_j$. We say that $Y_1, \ldots, Y_q$ have the Autissier property at P if*

$$\mathscr{I}(N \cap N') = \mathscr{I}(N) \cap \mathscr{I}(N')$$

for all saturated subsets N and N' of $\mathbb{N}^r$, where $\mathscr{I}$ is taken with respect to the sequence $(\mathscr{I}_{j1})_p, \ldots, (\mathscr{I}_{jr})_P$ of proper ideals of $\mathscr{O}_{X,P}$.

(b) *We say that $Y_1, \ldots, Y_q$ have the Autissier property if they have the Autissier property at all $P \in X$.*

In the [Voj23], Vojta deeply study the Autissier property, and given the following useful propersition:

Proposition 3.12. *If $Y_1, \ldots, Y_q$ weakly intersect properly, then they have the Autissier properly.*

Definition 3.13. *Let W be a vector space of finite dimension. A filration of W is a family of subspaces $\mathcal{F} = (\mathcal{F}_x)_{x \in \mathbb{R}^+}$ of subspaces of W such that $\mathcal{F}_x \supset \mathcal{F}_y$ whenever $x \leq y$, and such that $\mathcal{F}_x = 0$ for x big enough. A basis $\mathcal{B}$ of W is said to be adapted to $\mathcal{F}$ if $\mathcal{B} \cap \mathcal{F}_x$ is a basis of $\mathcal{F}_x$ for every real number $x \geq 0$.*

Lemma 3.14 (Corvaja-Zannier[CZ04],Levin[Lev09],Autissier[Aut11])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations of W. Then there exists a basis of W which is adapted to both $\mathcal{F}$ and $\mathcal{G}$.*

Turning to the consequence of the Autissier property, we need the following setting.

Proposition 3.15 ([Aut11],[RV20])**.** *Let $q \in \mathbb{Z}_{>0}$, let*

$$\square = \mathbb{R}^q_{\geq 0} \backslash \{\mathbf{0}\}$$

and for all $\mathbf{t} in \square$ and all $x \in \mathbb{R}_{\geq 0}$ let

$$N(\mathbf{t}, x) = \{\mathbf{b} \in \mathbb{N}^q : t_1 b_1 + \cdots + t_q b_q \geq x\}.$$

Let $I_1, \ldots, I_q$ be ideals in A that have the Autissier property. Then

$$\mathscr{I}(N(\mathbf{t}, x)) \cap \mathscr{I}(N(\mathbf{u}, y)) \subset \mathscr{I}(N(\lambda \mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda) y))$$

for all $\mathbf{t}, \mathbf{u} \in \square$, all $x, y \in \mathbb{R}_{\geq 0}$, and all $\lambda \in [0, 1]$

Definition 3.16 ([Voj23])**.** *Let $\square$ and $N(\mathbf{t}, x)$ be as above, fix a complete variety X over a field k and proper closed subschemes $Y_1, \ldots, Y_q$ of X. Let $\mathscr{I}_1, \ldots, \mathscr{I}_q$ be the corresponding ideal sheaf.*

(a) *Let N be a saturated subset of $\mathbb{N}^q$. Then*

$$\mathscr{I}_X(N) = \sum_{\mathbf{b} \in N} \mathscr{I}_1^{b_1} \cdots \mathscr{I}_q^{b_q}$$

This is a coherent ideal sheaf in $\mathscr{O}_X$

(b) *For each* $\mathbf{t} \in \square$ *and all* $x \in \mathbb{R}_{\geq 0}$, *let*

$$\mathscr{J}_X(\mathbf{t}, x) = \mathscr{J}_X(N(\mathbf{t}, x)) = \sum_{b \in N(\mathbf{t}, x)} \mathscr{J}_1^{b_1} \cdots \mathscr{J}_q^{b_q}$$

(c) *Fix a line sheaf* $\mathscr{L}$ *on* X, *and let* $\mathbf{t}$ *and* x *be as above. Then we let*

$$\mathscr{F}(\mathbf{t})_x = \mathscr{F}_{\mathscr{L}}(\mathbf{t})_x = H^0(X, \mathscr{L} \otimes \mathscr{J}_X(\mathbf{t}, x))$$

Then $(\mathscr{F}(\mathbf{t})_x)_{x \in \mathbb{R}_{\geq 0}}$ *is a descending filtration of* $H^0(X, \mathscr{L})$ *that satisfies* $\mathscr{F}(\mathbf{t})_x = 0$ *for all* $x \gg 0$.

(d) *Finally, for all* $\mathbf{t} \in \square$ *we let*

$$F(\mathbf{t}) = F_{\mathscr{L}}(\mathbf{t}) = \frac{1}{h^0(X, \mathscr{L})} \int_0^\infty (\dim \mathscr{F}(\mathbf{t})_x) dx.$$

Here the function we defined $F(\mathbf{t})$ was widely used in the proof of the 1.6. And will be used in our situations.

4 Main Theorem

In this section, we will give our approach of proving the main theorem, Here we repeat the main theorem again:

Theorem 4.1 (Main theorem)**.** *Let X be a projective complete variety of dimension n. Let $\mathscr{L}$ be a big line sheaf on X. Let S be a finte set of places. Given a sequence of closed subschemes: $Y_1 \supset Y_2 \supset \cdots \subset Y_q$ and assume that this is a regular chain. Assume some special conditions and take $\lambda_{Y_i,v}$ to be the correspondance Weil functions, where $v \in S$. Then for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{i=1}^{q} \sum_{v \in s} (\beta(\mathscr{L}, Y_i) - \beta(\mathscr{L}, Y_{i-1}))\lambda_{Y_i}(x) \leq (1+\epsilon)h_{\mathscr{L}} + C$$

holds for all points $x \in X(k) \backslash Z$

We have two ways to approach the proof of the main theorem, we will represent them in order.

4.1 Filtration method

In this section, we will follow the ideal of traditional method and in some case prove our theorem. First, we call a function is pseudo concave if:

Definition 4.2. *Given a positive-valued function $F(t)$, we call $F(t)$ is pseudo concave if for any $\mathbf{t} = (t_1, \ldots, t_n)$ with $t_i \geq 0$, $(\beta_1, \ldots, \beta_n)$ with $\beta_i \geq 0$ and the property: $\sum_{i=1}^{n} \beta_i t_i = 1$, we have:*

$$F(\mathbf{t}) \geq \min\{\frac{1}{\beta_i - \beta_{i-1}}(F(\mathbf{e}_i) - F(\mathbf{e}_{i-1}))\}$$

Now we will give our first approach in proving the main theorem:

Theorem 4.3 (Main theorem: Filtration version)**.** *Let X be a projective complete variety of dimension n. Let $\mathscr{L}$ be a big line sheaf on X. Let S be a finte set of places. Given a sequence of closed subschemes: $Y_1 \supset Y_2 \supset \cdots \supset Y_q$ and assume that this is a regular chain. Assume the function $F(t)$ in 3.16 is pseudo concave. Take $\lambda_{Y_i,v}$ to be the correspondance Weil functions, where $v \in S$. Then for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{i=1}^{q} \sum_{v \in s} (\beta(\mathscr{L}, Y_i) - \beta(\mathscr{L}, Y_{i-1}))\lambda_{Y_i}(x) \leq (1+\epsilon)h_{\mathscr{L}} + C$$

holds for all points $x \in X(k) \backslash Z$

Proof. We denote by $m_s(s,)$ □

5 Corollaries

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