The 4/δ Bound

Designing Predictable LLM-Verifier Systems for Formal Method Guarantee
https://arxiv.org/abs/2512.02080

Probabilistic AI & Formal Verification: Bridging the Gap

We've released a technical report introducing a mathematical foundation with provable guarantees for LLM-based formal verification workflows.

The Problem: LLMs show great promise for automating tasks like:

- invariant generation
- proof synthesis
- counterexample-guided repair

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But today’s approaches lack theoretical guarantees. Without convergence bounds, LLM-driven loops may:

- run indefinitely
- waste compute
- require ad-hoc timeouts
- behave unpredictably in CI/CD pipelines

For safety-critical verification, this is unacceptable.

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Our Contribution

We present the **LLM-Verifier Convergence Theorem**. To our knowledge, the first formal convergence guarantee for iterative LLM-verification pipelines.

For sequential loops of the form: **propose → verify → repair → retry**

We prove that the system:

- terminates with high probability for any $\delta > 0$
- converges in expected time $E[n] \leq 4/\delta$
- satisfies exponential tail bounds

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Here, δ represents the probability that each LLM step makes useful progress, directly linking model quality to runtime guarantees.

Empirical Validation

- Across 90,000+ trials, observed behavior closely matches theory, showing the bounds are not just asymptotic, they’re practical.

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Why This Matters for Industry

This turns LLM-assisted verification from guesswork into engineering:

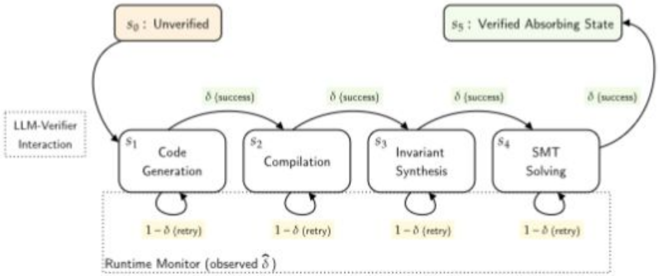
- predictable resource planning for CI/CD
- principled timeout configuration
- performance budgeting based on model capability (δ)
- clear deployment regimes: marginal, practical, and high-performance

Theoretical Background (Absorbing Markov Chain)

The **Theorem** is based on the mathematics of **Absorbing Markov Chains** and the **Geometric Distribution**.

The LLM-Verifier system is a five-state absorbing Markov Chain:

- Transient States (T): Four verification pipeline stages: s1 (CodeGen), s2 (Compilation), s3 (InvariantSynth), and s4 (SMTSolving).
- Absorbing State (A): One final, irreversible state, s5 (Verified).



Theoretical Background (Absorbing Markov Chain)

The Markovian Property holds because the probability of generating a correct verification artifact (the transition probability) depends only on the current state and is independent of the history of previous attempts (memoryless).

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Requirements for a Geometric Distribution

1. **Binary Outcomes:** Each trial has only two outcomes: success or failure.
2. **Independence:** Each trial is independent of the others.
3. **Constant Probability:** The probability of success (p) remains the same for every trial.
4. **Goal:** The experiment stops after the *first* success is achieved.

Theoretical Background (Absorbing Markov Chain)

Convergence Time Derivation (The 4/δ Bound)

The time spent in each transient state, M_j (the residence time), is modeled by a Geometric random variable.

The **Expected Single-State Time** (time to exit state j) is the expected value of the Geometric distribution: $E[M_j]=1/\delta$.

The Total **Expected Convergence Time** is the sum of the independent residence times spent in the four sequential transient states:

$E[n]=\sum_j(1^4)E[M_j]=1/\delta+1/\delta+1/\delta+1/\delta = 4/\delta$

Theoretical Background (Absorbing Markov Chain)

This theoretical foundation leads to the main result, providing formal guarantees for the system :

Almost Sure Convergence: The system reaches the Verified state ($\tau<\infty$) with probability 1, provided $\delta>0$.

Expected Iteration Bound: The mean time to convergence is $E[\tau]=\delta/4$.

Tail Bound: The probability of excessively long verification runs decays exponentially: $P(\tau>k)\leq\alpha(1-\delta)$.

Threats to Validity

1. **Assumption of Parameter Stationarity (internal)**

The most critical limitation is the assumption that the LLM's capability (δ) remains constant throughout the entire refinement process.

2. **State Space Limitation (internal)**

The model uses a simplified 5-state structure to capture the engineering pipeline.

Threats to Validity

3. Lack of Correlation Between Stages (Advanced Modeling Gap)

The model assumes independence between the transient states, allowing the total time T to be a simple sum of the times spent in each state ($\tau = \sum Mj$).

4. Generalizability to Real-World Systems (External)

The core validation relies on a simulation framework rather than real-world LLM deployments.

Threats to Validity

5. Focus on Iteration Count Over Time (Construct)

The primary metric of convergence is the iteration count (E[n]), not the actual time spent.

6. Binary Success Definition (Construct Validity Threat)

The model uses a simple binary definition for progress.

Theorem 1 (LLM-Verifier Convergence Theorem) *We model the LLM-verifier process using a discrete-time Markov Chain, denoted as* $X = \{X_n\}_{n \geq 0}$. *The state space* S *consists of a sequential engineering pipeline* ($S = T \cup A$). *First,* $T = \{s_1, s_2, s_3, s_4\}$ *represents the set of transient pipeline stages (* `CodeGen` *,* `Compilation` *,* `InvariantSynth` *,* `SMTSolving` *). Second,* $A = \{s_5\}$ *acts as the single absorbing state (* `Verified` *). Assuming a fixed success probability* $\delta \in (0, 1]$ *for passing any single stage, we construct the transition matrix* $P = (P_{i,j})$. *The individual entries* $P_{i,j} = \mathbb{P}(X_{n+1} = s_j \mid X_n = s_i)$ *are arranged as follows:*

(i) ***Transient Pipeline States***: *For the transient states* $i \in \{1, 2, 3, 4\}$, *the transition probabilities are:*

$$P_{i,j} = \begin{cases} \delta & \text{if } j = i + 1 \quad \text{(success: advance to next stage)}, \\ 1 - \delta & \text{if } j = i \quad \text{(failure: retry current stage)}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) ***Absorbing State***: *State* s_5 *is absorbing, meaning the verification is complete. Hence,* $P_{5,5} = 1$.

Let $\tau = \inf\{n \geq 0 : X_n \in A\}$ *be the iteration count until verification. The following bounds hold:*

1. ***Almost Sure Convergence***:

$$\mathbb{P}(\tau < \infty \mid X_0 \in T) = 1$$

2. ***Expected Iteration Bound***: *The mean time to convergence is given by:*

$$\mathbb{E}[\tau \mid X_0 = s_1] = \frac{4}{\delta}$$

3. ***Tail Bound***: *Consider constants* $\alpha > 0$ *and* $\lambda_Q \in (0, 1)$, *where* λ_Q *is the spectral radius of the transient submatrix* Q. *Then, for all* $k \geq 0$:

$$\mathbb{P}(\tau > k \mid X_0 = s_1) \leq \alpha \lambda_Q^k.$$

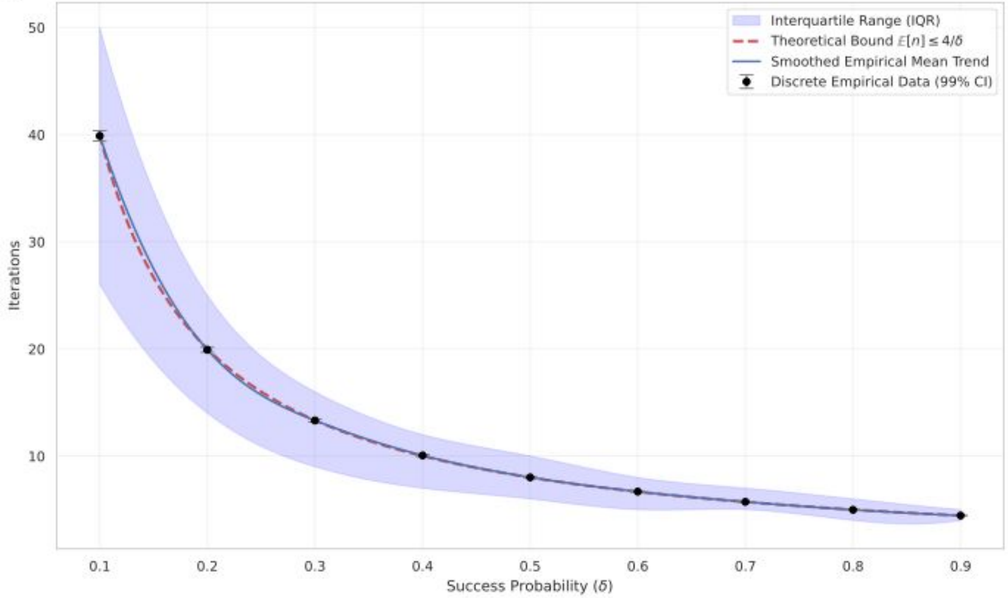
Substituting $\lambda_Q = 1 - \delta$ *results in the following exponential bound:*

$$\mathbb{P}(\tau > k \mid X_0 = s_1) \leq \alpha(1 - \delta)^k.$$

Proof We analyze the absorbing Markov Chain structure to establish the three guarantees.

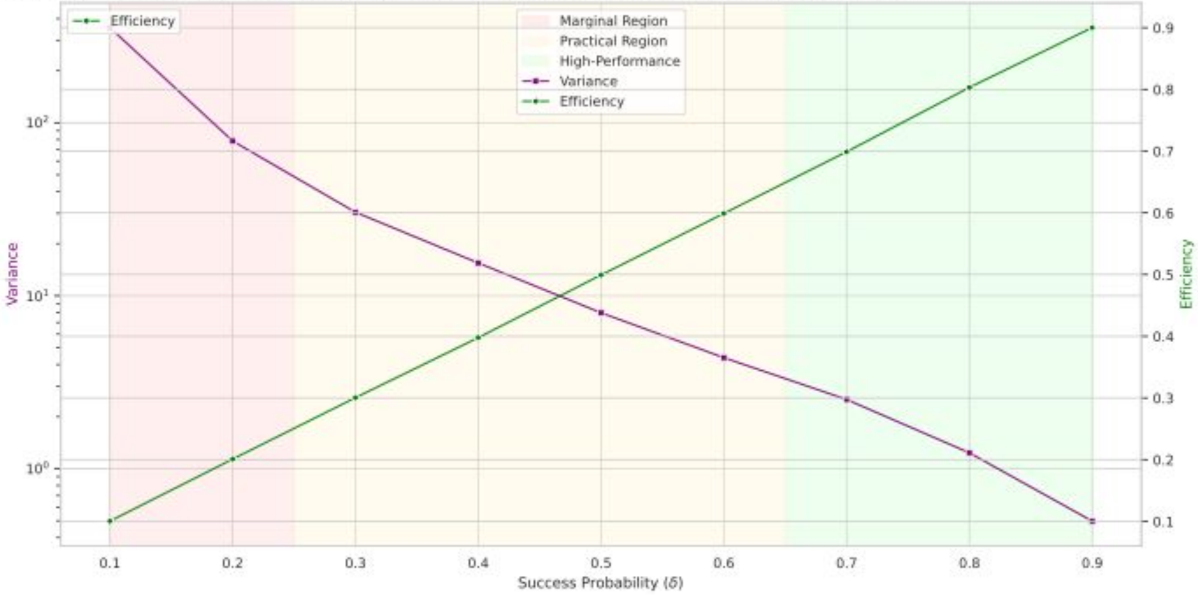
Empirical Validation

Fig. 6 Theoretical Bound Alignment and Empirical Convergence Rate. The figure compares the theoretical expected bound ($\mathbb{E}[n] \leq 4/\delta$) against the empirical mean (μ) across the δ spectrum. The extremely close tracking between the empirical curve (blue) and the theoretical curve (red dashed) demonstrates the tight alignment ($C_f \approx 1.0$) of the model prediction. The blue shaded area represents the IQR ($P_{25} - P_{75}$), which visually captures the dramatic decrease in system variance (σ^2) when transitioning from the marginal region ($\delta < 0.3$) to the high-performance region ($\delta > 0.6$)



Empirical Validation

Fig. 7 Operational Regions Map: Performance and Stability Analysis. This dual-axis visualization maps the framework's behavior across the δ spectrum. The left axis (purple, log scale) tracks the empirical variance (σ^2), demonstrating the system's predictability. The right axis (green, linear scale) tracks iteration efficiency (η). The plot clearly illustrates the sharp phase transition in stability: variance collapses rapidly upon entering the practical region ($\delta \geq 0.3$), confirming the empirical boundaries for safe and efficient deployment



Empirical Validation

Fig. 8 Tail Probability Analysis: Empirical Validation of Exponential Decay. The figure plots the Complementary Cumulative Distribution Function (CCDF, $P(n > k)$) on a log-linear scale. The approximately linear decay observed across all tested δ values confirms the fundamental assumption that the convergence time follows an exponential tail behavior ($\mathbb{P}(n > k) \propto e^{-c\delta k}$). This validation ensures that the system's worst-case convergence time is reliably bounded and predictable, even in the marginal region