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PARAMETER ESTIMATION ASSIGNMENT

Q1. Let $(x_1, x_2, ...)$ be a random sample of size n taken from a normal population with parameters: mean $= \theta_1$ and variance $= \theta_2$. Find the maximum likelihood estimates of these two parameters.

Ans

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ [PDF of normal distribution]

$$\mu = \theta_1 \quad , \quad \sigma^2 = \theta_2$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

Likelihood function

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$= \prod_{i=1}^{n} (\theta_2)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\prod_{i=1}^{n} \frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{\left(-\frac{1}{2\theta_2} \prod_{i=1}^{n} (x_i-\theta_1)^2\right)} \quad \text{——— } ①$$

Taking log both sides in ①

$$\ln(L(\theta_1, \theta_2)) = \ln\left[(\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2} \prod_{i=1}^{n} (x_i-\theta_1)^2}\right]$$

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$$Z = \ln(L(\theta_1, \theta_2)) = -\frac{n}{2}\ln\theta_2 - \frac{n}{2}\ln(2\pi) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i - \theta_1)^2 \quad \text{---} (2)$$

* Differentiating (2) wrt θ_1

$$\frac{\partial Z}{\partial \theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i - \theta_1)$$

as $\frac{\partial Z}{\partial \theta_1} = 0$

$$\Rightarrow \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i - \theta_1) = 0$$

$$\Rightarrow \frac{1}{\theta_2}\left(\sum_{i=1}^{n}x_i - n\theta_1\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n}x_i = n\theta_1$$

$$\Rightarrow \theta_1 = \frac{\sum_{i=1}^{n}x_i}{n} = \bar{x}_n$$

$$\Rightarrow \boxed{\theta_{1_{MLE}} = \bar{x}_n} \quad \text{---} (3)$$

* Differentiating (2) wrt θ_2

$$\frac{\partial Z}{\partial \theta_2} = \frac{-n}{2\theta_2} + \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

as $\frac{\partial Z}{\partial \theta_2} = 0$

$$\Rightarrow \frac{-n}{2\theta_2} + \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i - \theta_1)^2 = 0$$

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

From (3) $\theta_1 = \bar{x}_n$

$$\Rightarrow \boxed{\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}$$

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Q_2 let $x_1, x_2, \dots x_n$ be a random sample from $B(m, \theta)$ distribution where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known +ve integer. Compute value of θ using MLE.

Ans.

For binomial distribution $B(m, p)$

$$f(x) = {}^mC_x \, p^x (1-p)^{m-x}$$

likelihood function

$$L(p) = \prod_{i=1}^{n} {}^mC_{x_i} \, p^{x_i} (1-p)^{m-x_i}$$

Taking log on both sides.

$$\ln(L(p)) = \sum_{i=1}^{m} x_i \cdot \ln p + \ln(1-p)\left(m - \sum_{i=1}^{n} x_i\right)$$

$$= n\left(\bar{x} \ln p + (1-\bar{x}) \ln(1-p)\right)$$

Differentiating wrt p

$$\frac{\partial}{\partial p} \ln(L(p)) = n\left(\frac{\bar{x}}{p} - \frac{(1-\bar{x})}{1-p}\right) = n\left(\frac{\bar{x}-p}{p(1-p)}\right)$$

As $\frac{\partial}{\partial p} \ln(L(p)) = 0$

So $\frac{n(\bar{x}-p)}{p(1-p)} = 0$

$$p_{MLE} = \bar{x}$$

as $p = \theta$

$$\boxed{\theta_{MLE} = \bar{x}}$$

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