Name (same as in Gradescope): Solutions

MATH 520 MIDTERM I
SPRING 2017
BROWN UNIVERSITY
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This is a pencil-and-paper-only exam.

If you need additional space, you may use the pages at the end of the packet. Indicate where your additional work may be found by clearly writing "continued on page n" for an appropriate value of n. Please remove your staple immediately before handing in your exam (if you bend it in the middle first, it comes out quite easily). If you don't want to handle the staple, please request that your proctor remove it for you.

You only need to justify your answers when it is explicitly asked for. "Justify" means "provide an explanation for your answer which is clear and complete". You may assume facts we have learned in this course unless you are directly being asked to explain such a fact.

$\boxed{1}$ Suppose A, B, and C are real numbers such that B is the average of A and C and A and B sum to 11.

(a) (5 points) Find a system of equations whose solution set consists of all possible values for the triple (A, B, C). Write your answer in the box.

$$\boxed{\begin{array}{c} B = \frac{1}{2}A + \frac{1}{2}C \\ A + B = 11 \end{array}}$$

(b) (4 points) Write the system of equations from (a) in augmented matrix form. Write your answer in the box.

$$\boxed{\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 11 \end{array}\right]}$$

(c) (5 points) Solve the system (note: there may be more than one solution; in that case express your answer using a free variable). Write your final answer in the box.

$$\boxed{\left\{ \begin{pmatrix} \frac{22}{3} - \frac{C}{3} \\ \frac{11}{3} + C/3 \\ C \end{pmatrix} : C \in \mathbb{R} \right\}}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 11 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 3 & -1 & 11 \end{array}\right]$$

$\longrightarrow$ C free,

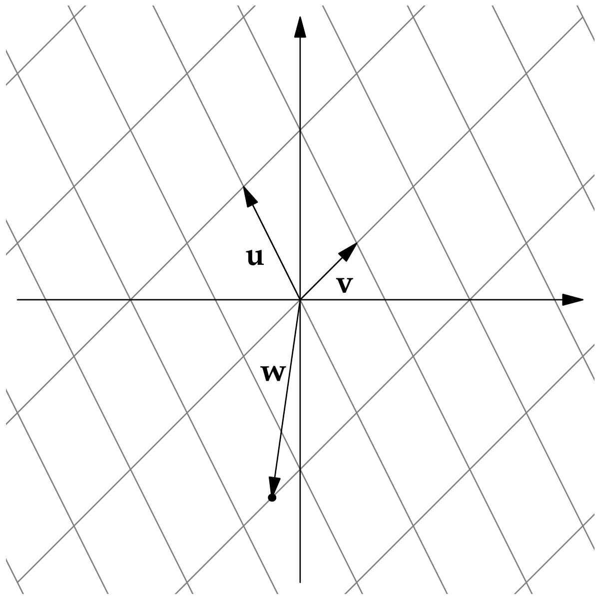
$$3B - C = 11 \Rightarrow B = \frac{11 + C}{3}$$

$$A - 2B + C = 0 \Rightarrow A = 2B - C$$

$$= 2\left(\frac{11+C}{3}\right) - C$$

$$= \frac{22}{3} - \frac{C}{3}.$$

$\boxed{2}$ Consider the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in the figure below.



(a) (3 points) Is the list $\{\mathbf{u}, \mathbf{v}\}$ linearly dependent? Write Y or N in the box (for 'yes' or 'no') and explain your reasoning.

N

$\vec{u}$ is not parallel to $\vec{v}$

(b) (3 points) Is the list $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly dependent? Write yes or no in the box and explain your reasoning.

Y

$\vec{w}$ is a lin. comb. of $\vec{u}$ & $\vec{v}$.

(c) (3 points) Find scalars a and b such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. You may assume that both a and b are integer multiples of 0.5. Express your answer as an ordered pair (a, b), written in the box.

$(-1, -1.5)$

(d) (3 points) Find scalars a and b such that $\mathbf{w} - \mathbf{u} + \mathbf{v} = a\mathbf{u} + b\mathbf{v}$. You may assume that both a and b are integer multiples of 0.5. Express your answer as an ordered pair (a, b), written in the box.

$(-2, 0.5)$

$\boxed{3}$ For each of the following statements, determine whether it is true or false. If it is false, give a counterexample demonstrating that it is false and write a big capital F in the box. If it is true, just write a big capital T in the box.

(a) (4 points) If the columns of a matrix A are linearly independent, then the system of equations $A\mathbf{x} = \mathbf{0}$ has no solutions.

$\boxed{F}$

$\vec{x} = \vec{0}$ is always a solution

(b) (4 points) If T is a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^2$, then T is injective.

$\boxed{F}$

$T(\vec{x}) = \vec{0}$ is not injective

(c) (4 points) Matrix multiplication is commutative; in other words, $AB = BA$ for all matrices A and B.

$\boxed{F}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

not equal

(d) (4 points) For every function T from $\mathbb{R}^2$ to $\mathbb{R}^6$ which satisfies $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(c\mathbf{x}) = cT(\mathbf{x})$ for all scalars c and vectors $\mathbf{x}$ and $\mathbf{y}$, there exists some 6×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

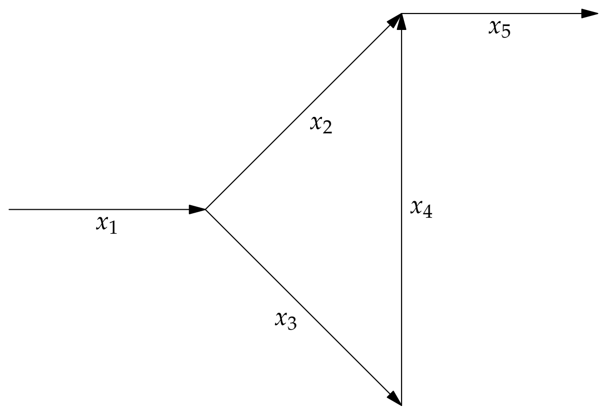
$\boxed{T}$

(e) (4 points) There are infinitely many solutions to the system of equations represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

$\boxed{T}$

$\boxed{4}$ Consider the following traffic flow diagram. Each label indicates the number of vehicles that traverse the given road over a particular time interval (which we assume begins and ends with no cars on any of the roads).



(a) (8 points) In the box, write a system of equations satisfied by the variables x_1, x_2, x_3, x_4, x_5.

$$x_1 = x_2 + x_3$$
$$x_5 = x_2 + x_4$$
$$x_3 = x_4$$

(b) (4 points) Currently, the city planners do not know any of the numbers x_1, x_2, x_3, x_4, x_5. They wish to install traffic counters, each of which directly measures exactly one of these numbers. What is the smallest number of traffic counters the city could install and still be able to determine all five numbers x_1, x_2, x_3, x_4, and x_5? Write your final answer in the box and justify it. Hint: the answer is not 5.

$\boxed{2}$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right)$$

already in row echelon form and has <u>two</u> free variables. So if we knew x_4, x_5, we'd know them all.

5 (10 points) Determine which of the columns of the matrix below can be deleted individually without changing the span of its columns. Circle an answer for each column, and briefly explain your reasoning.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Column 1: Can be deleted Cannot be deleted

top row becomes all zeros

Column 2: Can be deleted Cannot be deleted

still 3 pivots

Column 3: Can be deleted Cannot be deleted

row 3 is annihilated after one row operation

Column 4: Can be deleted Cannot be deleted

still 3 pivots

Column 5: Can be deleted Cannot be deleted

still 3 pivots

$\boxed{6}$ (a) (6 points) Suppose that A is a 3×3 matrix and $\mathbf{b}_1$ and $\mathbf{b}_2$ are vectors such that $A\mathbf{x} = \mathbf{b}_1$ has no solution and $A\mathbf{x} = \mathbf{b}_2$ has at least one solution. Explain why $A\mathbf{x} = \mathbf{b}_2$ must have infinitely many solutions.

A has at most 2 pivots; otherwise $A\vec{x} = \vec{b}_1$ would have a unique solution. So it has a non-pivot column & thus a free variable.

(b) (6 points) Suppose that the solution set of $A\mathbf{x} = \mathbf{b}_2$ is a plane in $\mathbb{R}^3$. Explain why the solution set of $A\mathbf{x} = \mathbf{0}$ is also a plane in $\mathbb{R}^3$. Note: you may explain this from first principles, or you may use a fact that we learned in class. In the latter case, be sure to give an accurate statement of that general fact.

The solution set of $A\vec{x} = \vec{b}_2$ takes the form $\{\vec{x}_0 + \vec{x}_h : A\vec{x}_h = \vec{0}\}$; in words: the nonhomogeneous solution set equals a particular solution plus the homogeneous solution set. Geometrically, the solution sets of $A\vec{x} = \vec{b}_2$ and $A\vec{x} = 0$ are related by a translation. So if one is a plane, the other is too.

$\boxed{7}$ (8 points) Using the definition of linear dependence (namely, that a list of vectors is linearly dependent if $\mathbf{0}$ can be written as a linear combination of the vectors with a list of weights which are not all zero), show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent and T is a linear transformation, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is linearly dependent.

Let c_i, not all zero, satisfy

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}.$$

Apply T:

$$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = T(\vec{0})$$

$$c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n) = \vec{0}$$

$(c_1, \ldots, c_n) \neq \vec{0}$, so $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is linearly dependent.

$\boxed{8}$ (a) (4 points) Fill in the blank: the matrix

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

represents a 30-degree counterclockwise rotation about the origin, and the matrix

$$B = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

represents a 45-degree counterclockwise rotation about the origin. Therefore, the product of these two matrices—specifically AB—represents a $\boxed{75}$-degree rotation about the origin.

(b) (4 points) Do the matrices A and B above commute? In other words, is $AB - BA$ equal to the zero matrix? Write Y or N in the box (for 'yes' or 'no'). Justify your answer, either using geometric reasoning or by direct calculation.

$\boxed{Y}$

AB & BA both represent a 75° ccw rotation, so they are equal.

(c) (4 points) Consider the matrix $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Do A and C commute? Write Y or N in the box. Justify your answer, either using geometric reasoning or by direct calculation.

$\boxed{N}$

$$AC = \begin{bmatrix} \sqrt{3}/2 & 0 \\ 1/2 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 0 & 0 \end{bmatrix}$$

not equal!

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