PSet 10

1. (a) $\sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}$, so $\sum_{n=1}^{\infty} \frac{(3/5)^n}{n!} = \boxed{e^{3/5} - 1}$

(b) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, by geometric series, so differentiating gives

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \quad \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

we throw out the $n=0$ term because it's zero

again

continued on next page...

So

$$\frac{2}{(1-x)^3} = 2\cdot 1 + 3\cdot 2\cdot x + 4\cdot 3 x^2 + 5\cdot 4\cdot x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)x^n.$$

So

$$\frac{2}{(1-\frac{1}{2})^3} = 16 = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2^n}$$

2. Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, we have

$$\sin x^4 = x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \frac{x^{28}}{7!} + \cdots,$$

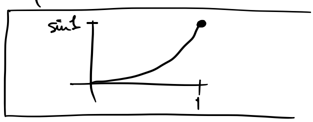
which means the RHS is the Maclaurin series for $\sin x^4$.

$$\left| \int_0^1 \sin(x^2)\,dx - \int_0^1 x^4\,dx \right| = 0.01243\ldots$$

$$\left| \int_0^1 \sin x^2\,dx - \int_0^1 \left(x^4 - \frac{x^{12}}{3!}\right) dx \right| = 0.000390\ldots$$

$$\left| \int_0^1 \sin x^2\,dx - \int_0^1 \left(x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!}\right) dx \right| = 0.0000067\ldots$$

So the minimal order is $\boxed{20}$.



The function and its degree-20 Maclaurin series are not distinguishable to the eye.

3. $f(x) = (1+x^3)^{44} = (1+x^3)(1+x^3) \cdots (1+x^3)$

$= 1 + 44x^3 +$ other terms where x is raised to a multiple of 3.

Since f is equal to a power series, its Taylor Series must be equal to this power series. Therefore, every $f^{(k)}(0)$ must be $\boxed{0}$ whenever k is not a multiple of 3.

4. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$f(0) = a_0 = 0 \Rightarrow a_0 = 0$

$f'(0) = a_1 = 1 \Rightarrow a_1 = 1.$

$f''(x) = -f(x) \Rightarrow 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + \cdots$

$= -a_0 - a_1 x - a_2 x^2 - \cdots$

So $2a_2 = -a_0 = 0$. So $a_2 = 0$. Then $a_3 = -\frac{a_1}{3 \cdot 2} = \frac{-1}{6}$. Then $a_4 = \frac{-a_2}{4 \cdot 3} = 0$ and

$a_5 = \frac{1/6}{-5 \cdot 4} = \frac{-1}{120}$. So $a_n = 0$ when n is even and $\frac{(-1)^{(n-1)/2}}{n!}$ if n is odd. So $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$= \boxed{\sin x}$

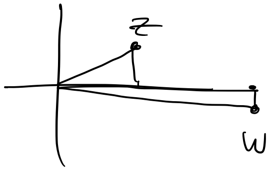
by Sage

5. $z^4 w = 114244 + 114244i = 114244\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right)$,

So the angle of $z^4 w$ is $\pi/4$. Also,

$$z = re^{i\arctan(1/5)}$$

$$w = se^{i\arctan(1/239)}$$



for some $r, s > 0$. So

$$z^4 w = rse^{i(4\arctan(1/5) - \arctan(1/239))},$$

which means the angle of $z^4 w$ is also $4\arctan(1/5) - \arctan(1/239)$. So

$$\pi = 16\arctan(1/5) - 4\arctan(1/239).$$

The Maclaurin series for arctan is

$$\int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

& subbing in these first 3 terms gives

$$\pi \approx 3.141621\ldots,$$

which is 0.00002... off. Using $4\arctan(1) \approx$

$$4\left(1-\frac{x^3}{3}+\frac{x^5}{5}\right)\Big|_{x=1} = 4\left(1-\frac{1}{3}+\frac{1}{5}\right) = \frac{8}{3}+\frac{4}{5} = \frac{52}{15}$$

$$= 3.4\overline{6},$$

which is way less good. The problem is that 1 is much farther from the center, $x=0$, than $1/5$ or $1/239$.

6. The third one is $(0.1, 4, 0.3)$, since $\cos 3x$ is the predominant wave.
The second one is $(3,-1,1)$ since that one has more $\cos 11x$ in it than the first one.
So the first one is $(3,-1,0.1)$.

7. $\int_a^{a+2\pi} g(x)\,dx = \int_a^b g(x)\,dx + \int_b^{a+2\pi} g(x)\,dx$ where b is the multiple of 2π with $a < b \le a+2\pi$.

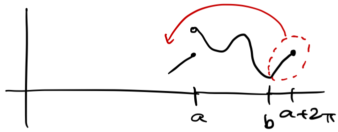
$\downarrow$ sub $u = x-2\pi$

$$= \int_a^b g(x)\,dx + \int_{b-2\pi}^{a} g(x)\,dx$$

$$= \int_{b-2\pi}^{b} g(x)\,dx$$

$u = x-b+2\pi$

$$= \int_0^{2\pi} g(x)\,dx$$



So $\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_0^{2\pi} f(x)\cos nx\,dx$, & same for sin.

8. We have: $a_0 = \frac{1}{2\pi}\int_0^{\pi}\cos x\,dx = \frac{1}{2\pi}\sin x\Big|_0^{\pi} = 0.$

$2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos x\cos nx\,dx = \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n+1)x + \cos(n-1)x\right]dx = \frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right]_0^{\pi}$$

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$$= \frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right]_0^{\pi} = 0 \quad \text{if } n \neq 1, \text{ and } a_1 = \frac{1}{2\pi}\int_0^{\pi} \cos 0\,dx$$
$$= \frac{1}{2}.$$

$$\boxed{\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha\end{aligned}}$$

And for $n \neq 1$, $b_n = \frac{1}{\pi}\int_0^{\pi} \cos x \sin nx\,dx$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin(n+1)x + \sin(n-1)x\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos((n-1)\pi)}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \begin{cases} 0 & n \text{ odd} \\ \frac{1}{2\pi}\cdot\frac{4n}{n^2-1} & n \text{ even}\end{cases} = \begin{cases} 0 & n \text{ odd} \\ \frac{2n}{\pi(n^2-1)} & n \text{ even}\end{cases}$$

Also $b_1 = \frac{1}{2\pi}\int_0^{\pi} \sin(n+1)x + \sin 0x\,dx = 0$. So the Fourier series is

$$\frac{1}{2}\cos x + \frac{4}{3\pi}\sin 2x + \frac{8}{15\pi}\sin 4x + \frac{12}{35\pi}\sin 6x \cdots$$

9. We have $a_n = 0$ for all $n \geq 0$ by oddness and for $n \geq 1$,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \overset{\left(-\frac{\cos nx}{n}\right)'}{\sin nx}\,dx$$

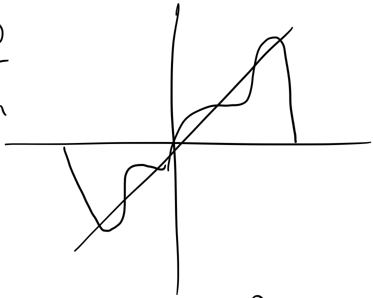
(basic wave integrates to 0 over full period)

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\Big|_{-\pi}^{\pi} - \cancel{\int_{-\pi}^{\pi} \frac{-\cos nx}{n}\,dx}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos \pi n}{n} + \frac{-\pi\cos(-\pi n)}{n}\right] = \frac{-2\cos(\pi n)}{n} = \frac{(-1)^{n+1}\cdot 2}{n}$$

So the Fourier series is $2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \cdots$

the graph of the 4^{th} order approximation is:



the F.S. converges to $f(x)$ at all numbers x which are not odd multiples of π.

10. (a) is just squaring out the integrand and distributing:

$$\int_0^{2\pi} P(x)^2 - 2P(x)f(x) + f(x)^2\,dx$$

$$= \int_0^{2\pi} P(x)^2\,dx - 2\int_0^{2\pi} P(x)f(x)\,dx + \int_0^{2\pi} f(x)^2\,dx$$

(b) Now $\int_0^{2\pi} (a_0 + a_1\cos x + a_2\cos 2x)^2\,dx$

$$= 2\pi a_0^2 + \pi a_1^2 + \pi a_2^2$$

by theorem 13.1.

(c) Like (a): $\int f(x)\,P(x) = a_0\int f(x)\,dx + a_1\int f(x)\cos x\,dx$

$+ a_2\int f(x)\sin x\,dx.$

(d) RHS of (1) is

$$2\pi a_0^2 - 2a_0\int_0^{2\pi} f(x)\,dx + \pi a_1^2 - 2a_1\int_0^{2\pi} f(x)\cos x\,dx + \pi b_1^2 - 2a_1\int_0^{2\pi} f(x)\sin x\,dx + \int f^2$$

Only the highlighted terms depend on a_0, so to make this as small as possible, we set:

(e)
$$\frac{d}{da_0}\left(2\pi a_0^2 - 2a_0\int_0^{2\pi} f(x)\,dx\right) = 0 \quad \Rightarrow$$

$$4\pi a_0 - 2\int_0^{2\pi} f(x)\,dx = 0 \quad \Rightarrow$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx.$$

(f) Similarly, $a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$, and $b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$ minimize the next two highlighted expressions.

(g) So the first order Fourier approximation $P(x)$ minimizes $\int (P(x) - f(x))^2\,dx$ over all linear combinations of $1, \sin x, \cos x$.