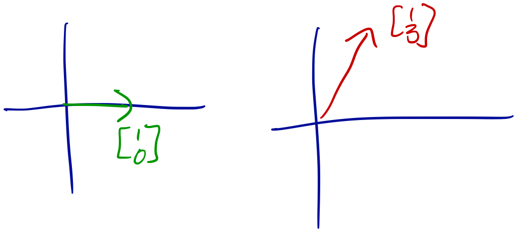
§ 5.1 Eigenvectors

6 Apr 2017

$A\vec{x}$ and $\vec{x}$ often point in different directions:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$



However, we will see that if we can find vectors whose directions are preserved, then that will give us some very useful insight into how A acts on vectors.

Def^n $\vec{v}$ is an **eigenvector** of A if $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in \mathbb{R}$. We require that $\vec{v} \neq \vec{0}$.

λ is called $\vec{v}$'s **eigenvalue**.

Example Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Solution: We want to show that $A\vec{v} = \lambda \vec{v}$

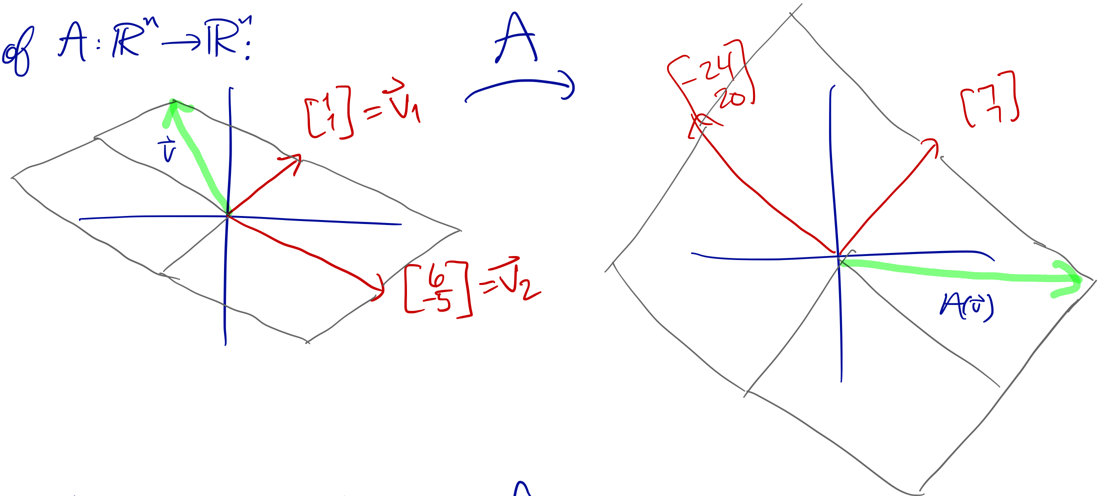
has a nontrivial solution. So we look at

$$
\begin{aligned}
A\vec{v} = \lambda\vec{v} &\Leftrightarrow A\vec{v} - \lambda\vec{v} = \vec{0} \\
&\Leftrightarrow A\vec{v} - \lambda I\vec{v} = \vec{0} \\
&\Leftrightarrow (A - \lambda I)\vec{v} = \vec{0}.
\end{aligned}
$$

We have $A - \lambda I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$.

Clearly this matrix is not injective, so it will have nonzero solutions. For example, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ∎

We see that once we know an eigen<u>value</u>, finding corresponding eigen<u>vectors</u> is a matter of finding vectors in the kernel of $A - \lambda I$. The set of all eigenvectors corresponding to a given eigenvalue is a linear subspace (since it is a solution set to a homogeneous system), called the eigenspace corresponding to that λ value. We can say that A acts on this eigenspace by scaling every vector by λ. This is <u>super handy</u> if we can find a basis of $\mathbb{R}^n$ consisting of eigenvectors of $A: \mathbb{R}^n \to \mathbb{R}^n$:



Because then if we represent an arbitrary vector $\vec{v}$ in terms of $\vec{v}_1$ and $\vec{v}_2$, we know A acts on $\vec{v}$ by scaling its $\vec{v}_1$ and $\vec{v}_2$ components:

$$A(\vec{v}) = A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 A(v_1) + c_2 A(v_2)$$
$$= c_1\lambda\vec{v}_1 + c_2\lambda_2\vec{v}_2.$$

Let's see, before we go any further with computation, how cool this can be.

Example Find a formula for the n^{th} Fibonacci number (!!)

Solution Let us frame the Fibonacci iteration in linear algebra terms:

$$\overset{F_1}{1},\ \overset{F_2}{1},\ \overset{F_3}{2},\ \overset{F_4}{3},\ \overset{F_5}{5},\ \overset{F_6}{8},\ \overset{F_7}{13},\ \overset{F_8}{21},\ \ldots$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x+y \end{pmatrix}$$

We see that the update rule takes us from (x,y) as our last two terms to $(y, x+y)$. In other words:

$$\underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{\substack{\text{where we are} \\ \text{after } n-1 \text{ updates}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1}}_{\substack{\text{update} \\ n-1 \text{ times}}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{starting point}}$$

So if we could raise $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ to the 1, we'd have it made.

Since this the eigenvector section, let's try that. The eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are $\overbrace{\phi}^{\text{"golden ratio"}} = \frac{1+\sqrt{5}}{2}$ and $1-\phi = \frac{1-\sqrt{5}}{2}$, [we'll talk about how to get that later], with eigenvectors $\begin{bmatrix} \phi \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1-\phi \\ 1 \end{bmatrix}$.

We've already seen that A acts very simply on a vector expressed in terms of A's eigenvectors.

So we look for c_1, c_2 so that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \phi \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1-\phi \\ 1 \end{bmatrix}$$

the solution here is $c_1 = \dfrac{\phi}{\sqrt{5}}$, $c_2 = -\dfrac{1-\phi}{\sqrt{5}}$

So we have, with $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$,

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^{n-1}\left(c_1 \begin{pmatrix} \phi \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1-\phi \\ 1 \end{pmatrix} \right)$$

$$= c_1 A^{n-1} \begin{pmatrix} \phi \\ 1 \end{pmatrix} + c_2 A^{n-1} \quad ,$$

$$= c_1 \phi^{n-1} \begin{pmatrix} \phi \\ 1 \end{pmatrix} + c_2 (1-\phi)^{n-1} \begin{pmatrix} 1-\phi \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} c_1 \phi^n + c_2 (1-\phi)^n \\ c_2 \phi^{n-1} + c_2 \phi^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}}\left(\phi^{n+1} - (1-\phi)^{n+1}\right) \\ \frac{1}{\sqrt{5}}\left(\phi^{n} - (1-\phi)^{n+1}\right) \end{bmatrix}$$

So $F_n = \frac{1}{\sqrt{5}}\left(\phi^n - (1-\phi)^n\right)$. (!!!)

For fun:

Here's the formula in action

it works!

```
julia> [(ϕ^n - (1-ϕ)^n)/√(5) for n=1:10]
10-element Array{Float64,1}:
  1.0
  1.0
  2.0
  3.0
  5.0
  8.0
 13.0
 21.0
 34.0
 55.0
```

$(1-\phi)^n \to 0$ fast, because $|1-\phi| < 1$. So $\frac{1}{\sqrt{5}}\phi^n$ and the integer closest to it differ by an amount tending to 0 as $n \to \infty$. As you can see here, the same is true of ϕ^n [challenge problem!]

```
julia> [ϕ^n for n=1:10]
10-element Array{Float64,1}:
   1.61803
   2.61803
   4.23607
   6.8541
  11.0902
  17.9443
  29.0344
  46.9787
  76.0132
 122.992
```

This is a special property of ϕ. Typically, powers of an irrational number have $\overbrace{\text{fractional part}}^{\text{what you get when you throw away the digits before the decimal point}}$ jumping around. Like π.

```
julia> [π^n for n=1:10]
10-element Array{Float64,1}:
     3.14159
     9.8696
    31.0063
    97.4091
   306.02
   961.389
  3020.29
  9488.53
 29809.1
 93648.0
```