Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

Find $\iint_R \frac{y}{x} \, dA$ where R is the region bounded by the curves $y = 0$, $y = x/2$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$, by changing coordinates. (Hint: try letting y/x be one of your new coordinates.)

Solution

If we let $u = y/x$ and $v = x^2 - y^2$, then the four boundary arcs of the given region are $u = 0$, $u = \frac{1}{2}$, $v = 1$ and $v = 4$. We can see that the map $(x,y) \mapsto (u,v)$ sends R bijectively to the rectangle $[0, \frac{1}{2}] \times [1,4]$. We also have

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1} = \left| \det \begin{bmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 2x & -2y \end{bmatrix} \right|^{-1} = |2u^2 - 2|^{-1} = \frac{1}{2 - 2u^2}.$$

since $u \in [0, \frac{1}{2}]$ implies $2u^2 - 2 < 0$. Therefore, the change of variable formula implies that

$$\iint_R \frac{y}{x} \, dA = \int_0^{\frac{1}{2}} \int_1^4 \frac{u}{2 - 2u^2} \, dv \, du = \boxed{\frac{3}{4} \ln \frac{4}{3}}.$$

Problem 2

Find
$$\iint_R \frac{x - 2y}{3x - y} \, dA,$$
where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$.

Solution

The clear candidates for u and v are $x - 2y$ and $3x - y$. This is a linear transformation, so its Jacobian will be a constant. We could solve for x and y, but we also know that the determinant of a linear transformation is the reciprocal of the determinant of its inverse, so the Jacobian is
$$\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}^{-1} = \frac{1}{5}.$$

So we have
$$\iint_R \frac{x - 2y}{3x - y} \, dA = \int_1^8 \int_0^4 \frac{u}{v} \left(\frac{1}{5} \right) du \, dv = \boxed{\frac{8 \log 8}{5}}.$$

Problem 3

Consider the functions f, w_1, and w_2 from $[1, 2]^2 \to \mathbb{R}$ defined by

$$f(x, y) = xy,$$
$$w_1(x, y) = x + y,$$
$$w_2(x, y) = 4 - x - y.$$

(a) Describe qualitatively how you can guess which of the following is greater, before doing any calculation (i) the w_1-weighed average of f, or (ii) the w_2-weighted average of f.

(b) Calculate the w_1-weighted average of f and the w_2-weighted average of f.

(c) Confirm that both weighted averages of f computed above lie between the maximum and minimum values of f.

Solution

(a) We see that f is larger in the top right part of the square and smaller as you move to the bottom left. Therefore, we expect a weighted average which weights the top right values more to be larger than a weighted average which weights the bottom left values more. So the w_1 weighted average should be larger.

(b) We have

$$\int_1^2 \int_1^2 xy \, (x + y) \, dx \, dy = \int_1^2 \left(\frac{3y^2}{2} + \frac{7y}{3} \right) dy = 7,$$

and

$$\int_1^2 \int_1^2 xy \, (-x - y + 4) \, dx \, dy = \int_1^2 \left(-\frac{3y^2}{2} + \frac{11y}{3} \right) dy = 2,$$

and

$$\int_1^2 \int_1^2 (x + y) \, dx \, dy = \int_1^2 \left(y + \frac{3}{2} \right) dy = 3,$$

and

$$\int_1^2 \int_1^2 (-x - y + 4) \, dx \, dy = \int_1^2 \left(-y + \frac{5}{2} \right) dy = 1.$$

So the w_1-weighted average of f is $\frac{7}{3}$ and the w_2-weighted average of f is $\frac{2}{1} = 2$.

(c) The maximum and minimum values of f are clearly 1 and 4. Indeed, both $\frac{7}{3} = 2.\overline{3}$ and 2 lie between these values.

The map below was obtained by placing one dot at the residence of each person in the contiguous United States in the 2010 US Census (with color indicating race identification) and then representing each of many small squares in a grid with the average of the colors of the dots in the square.
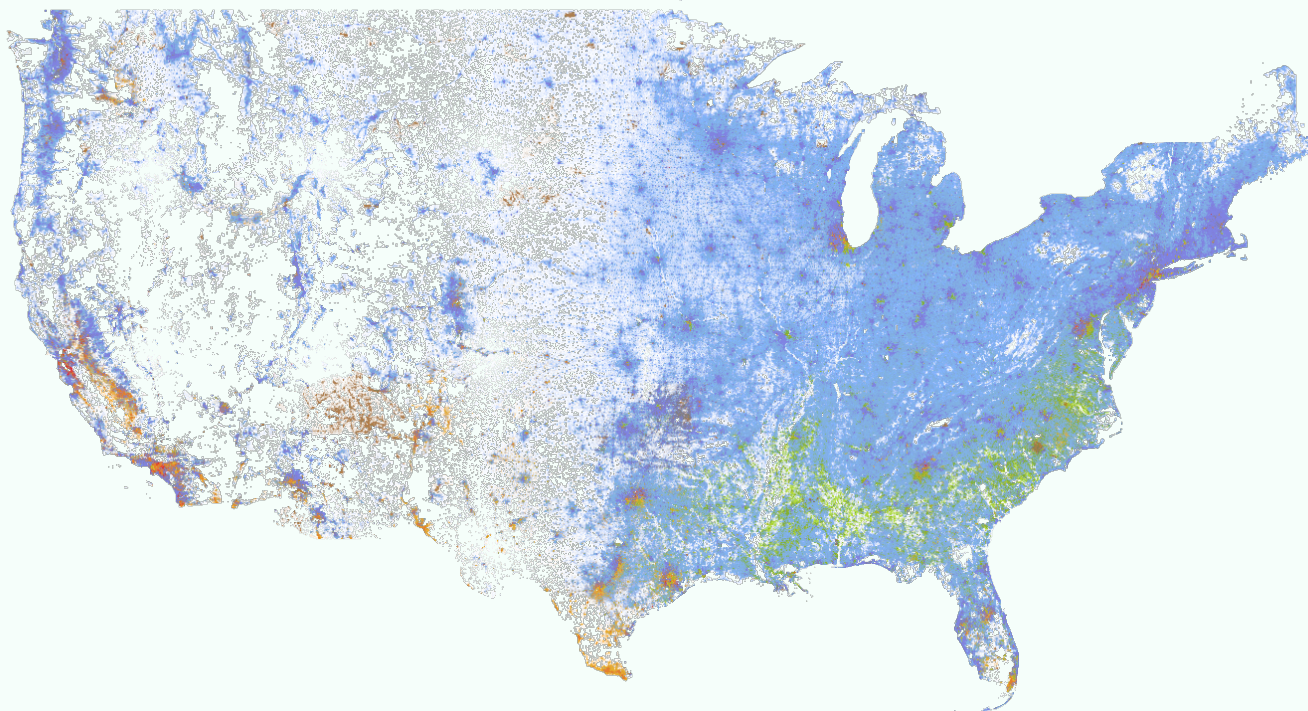
You can interact with the underlying data *at the level of individual dots* at `https://demographics.virginia.edu/DotMap/`

Suppose you're given the 764×1366 grid of numbers which represent how darkly colored in each little square is (see the figure to the right for a small piece of this grid). These numbers are roughly proportional to the number of people living in each square.

(a) Using the center-of-mass formula and Riemann approximations thereof, explain what calculation you could perform with these numbers to approximate the population-weighted center of mass of the United States. Apply your procedure to the small 3×3 portion of the grid below to show how it works.

0.35	0.33	0.3
0.0	0.3	0.3
0.02	0.0	0.18

(b) In what state would you guess the population-weighted center of mass is?

Note: click here to access the full matrix if you want to do some of these computations yourself.

(a) To find x-coordinate of the center of mass, we find the population-weighted average of the value of the function x over our rectangle. We have to choose distance units, which we might as well choose to be one pixel width. So for each pixel, we multiple its darkness value by the index of its column minus $\frac{1}{2}$, total up the results, and divide by the sum of all the darkness values over the whole grid. We do the same thing except with the index of the row to find the y-coordinate of the center of mass.

To apply this procedure to the 3×3 grid given, we see that the average x-value is

$$\tfrac{1}{2} \cdot 0.035 + \tfrac{1}{2} \cdot 0.0 + \tfrac{1}{2} \cdot 0.02 + \tfrac{3}{2} \cdot 0.33 + \tfrac{3}{2} \cdot 0.3 + \tfrac{3}{2} \cdot 0.0 + \tfrac{5}{2} \cdot 0.3 + \tfrac{5}{2} \cdot 0.3 + \tfrac{5}{2} \cdot 0.18 = 2.9225$$

divided by $0.35 + 0.33 + 0.3 + 0.0 + 0.3 + 0.3 + 0.02 + 0.0 + 0.18 = 1.78$, which works out to $2.922/1.78 \approx 1.64$. For the y-coordinate, we get

$$\tfrac{1}{2} \cdot 0.02 + \tfrac{1}{2} \cdot 0.0 + \tfrac{1}{2} \cdot 0.18 + \tfrac{3}{2} \cdot 0.0 + \tfrac{3}{2} \cdot 0.3 + \tfrac{3}{2} \cdot 0.3 + \tfrac{5}{2} \cdot 0.35 + \tfrac{5}{2} \cdot 0.33 + \tfrac{5}{2} \cdot 0.3 = 3.45.$$

divided by 1.78, which is $3.45/1.78 \approx 1.94$. So the center of mass is $(1.64, 1.94)$, which is shown as a red dot above. This point is to the right and above the center of the square, which makes sense since the values are larger on the top and right sides.

(b) The mean population center of the United States is in southern central Missouri. I applied the method above to the image and got a point in Missouri, although it was a bit too far north. So the method ends up being reasonably accurate (the inaccuracy comes from the assumption that the color of each pixel in the screenshot I took is actually proportional to the population).

0.35	0.33	0.3
0.0	0.3	0.3
0.02	0.0	0.18

Problem 5

The dart thrower in Example 5.5.3 in the book is terribly unlikely to hit the triple-20. What a shame. Let's see how they can increase their chances by becoming less accurate.

(a) Suppose that the probability density function of the dart's location is given by

$$f_\alpha(x,y) = \frac{1}{\pi\alpha}e^{-\frac{x^2+y^2}{\alpha}},$$

where $\alpha > 0$ is an accuracy parameter. If a player becomes more accurate, does their α value increase or decrease? (You'll need to explain how a change in α affects the shape of the above density).

(b) Explain in intuitive terms why a thrower with accuracy α is extremely unlikely to hit the triple-20 either when α is very small or when α is very large.

(c) Find the value of α that maximizes the probability of hitting the triple-20.

Solution

(a) Increasing α corresponds to decreasing accuracy. To see this, note that if $\alpha = 100$ (say), then $e^{-\frac{x^2+y^2}{\alpha}}$ is still reasonably large even when $x^2 + y^2 = 100$. This corresponds to a decent probability of hitting around 10 inches from the bullseye. Meanwhile, if $\alpha = 1/100$, then $e^{-\frac{x^2+y^2}{\alpha}}$ is Avogadro-reciprocal-level small even when $x^2 + y^2 = 1$.

(b) An extremely inaccurate thrower is likely to miss the triple 20 because they aren't even particularly likely to hit the board at all. More precisely, their probability of hitting the board tends to zero as $\alpha \to \infty$. An extremely accurate thrower hits the bullseye with probability tending to 1 (and therefore hits the triple-20 with probability tending to 0).

(c) To find the value of α that maximizes the chances of hitting the triple-20, we work out that

$$\int_{\frac{9\pi}{20}}^{\frac{11\pi}{20}} \int_{3.85}^{4.2} \frac{re^{-\frac{r^2}{\alpha}}}{\pi\alpha}\, dr\, d\theta = \int_{\frac{9\pi}{20}}^{\frac{11\pi}{20}} \left(-\frac{e^{-\frac{17.64}{\alpha}}}{2\pi} + \frac{e^{-\frac{14.8225}{\alpha}}}{2\pi} \right) d\theta$$

$$= \frac{\pi}{10}\left(-\frac{e^{-\frac{17.64}{\alpha}}}{2\pi} + \frac{e^{-\frac{14.8225}{\alpha}}}{2\pi} \right)$$

This function is maximized when

$$\frac{d}{d\alpha}\left[e^{-17.64/\alpha} - e^{-14.8225/\alpha} \right] = 0.$$

Solving this equation for α, we get $\alpha = \dfrac{14.8225 - 17.64}{\ln(14.8225/17.64)} \approx \boxed{16.19}$. The corresponding probability is still only about 0.32%. Well, we tried.

Final answer:

16.19