Name: Key

MATH 19 MIDTERM II
FALL 2016
BROWN UNIVERSITY
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$\boxed{1}$ (a) (5 points) What expression would you substitute for y when using the method of undetermined coefficients to find a particular solution of the differential equation $y'(t) + y(t) = t^3 - t$? You do not actually have to solve for the coefficients. In other words, do the first step of solving this DE and stop *before* you substitute into the differential equation to find the coefficients.

$$at^3 + bt^2 + ct + d$$

(b) (10 points) Find the general solution of the differential equation $y''(t) - 2y'(t) - 3y(t) = 6e^{2t}$.

homogeneous solution:

$$y''(t) - 2y'(t) - 3y(t) = 0 \Rightarrow$$

$$y(t) = Ae^{3t} + Be^{-t}$$

since $\lambda^2 - 2\lambda - 3 \Rightarrow \lambda \in \{3, -1\}$.

particular solution:

$$y(t) = Ce^{2t} \Rightarrow$$

$$y''(t) - 2y'(t) - 3y(t)$$

$$= 4Ce^{2t} - 2(2Ce^{2t}) - 3Ce^{2t}$$

$$= -3Ce^{2t}$$

this equals $6e^{2t}$ if $C = -2$, so we get

$$y(t) = Ae^{3t} + Be^{-t} - 2e^{2t}$$

as our general solution

$\boxed{2}$ Determine whether the following series converge absolutely, converge conditionally, or diverge. For each example, clearly state which convergence test or tests you are using. Be sure to address the hypotheses of that test, if the test has any hypotheses.

(a) (8 points) $\sum_{n=1}^{\infty} \arccos\left(\frac{1}{n!}\right)$

As $n \to \infty$, $\frac{1}{n!} \to 0$. And $\arccos(0) = \pi/2$ and $\arccos$ is continuous. So $\arccos\left(\frac{1}{n!}\right) \not\to 0$, so the series fails the n^{th} term test and therefore diverges.

(b) (8 points) $\sum_{n=1}^{\infty} \frac{7(n!)^3(2n-1)!}{(4n)!}$

All terms are positive, so we may apply the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{7((n+1)!)^3(2(n+1)-1)!}{(4(n+1))!} \cdot \frac{(4n)!}{7(n!)^3(2n-1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3(2n+1)(2n)}{(4n+4)(4n+3)(4n+2)(4n+1)}$$

$$= \infty,$$

So the series diverges.

(c) (8 points) $\sum_{n=2}^{\infty} \frac{3}{n(\ln(n))^4}$

$f(x) = \frac{3}{x(\ln x)^4}$ is positive & decreasing, so we may apply the integral test:

$$\int_2^\infty \frac{3}{x(\ln x)^4}\,dx \xrightarrow{u=\ln x} = \left.\frac{-1}{(\ln x)^3}\right|_2^\infty \xrightarrow{\text{since } (\ln x)^3 \to \infty \text{ as } x \to \infty} < \infty.$$

Thus $\sum_{n=2}^{\infty} \frac{3}{n(\ln n)^4}$ converges.

(d) (8 points) $\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{1}{\sqrt{n}}\right)$

The series converges by the alternating series test: $\frac{1}{\sqrt{n}}$ is decreasing and sin is an increasing, positive function on $[0, \pi/2]$. So $\sin(1/\sqrt{n})$ is positive & decreasing. Also $\sin(1/\sqrt{n}) \to 0$ since sine is continuous and $\sin(0) = 0$.

$\boxed{3}$ (15 points) Determine whether the following improper integral converges, and if so determine its value:

$$\int_0^1 \left(\frac{1}{\sqrt{x}} + \frac{1}{1-x} \right) dx.$$

$$\int_0^1 \left(\frac{1}{\sqrt{x}} + \frac{1}{1-x}\right)dx = \lim_{a\to 0^+} \int_a^{1/2} \left(\frac{1}{\sqrt{x}} + \frac{1}{1-x}\right) dx + \lim_{b\to 1^-} \int_{1/2}^{b} \left(\frac{1}{\sqrt{x}} + \frac{1}{1-x}\right)dx$$

$$= \lim_{a\to 0^+} \left[2\sqrt{x} - \ln(1-x) \Big|_a^{1/2} \right] + \lim_{b\to 1^-} \left(2\sqrt{x} - \ln(1-x) \Big|_{1/2}^{b} \right)$$

$$= \lim_{a\to 0} \left[\frac{2}{\sqrt{2}} - \ln(1/2) - 2\sqrt{a} + \ln(1-a) \right]$$

$$+ \lim_{b\to 1^-} \left[2\sqrt{b} - \ln(1-b) - 2/\sqrt{2} + \ln(1/2) \right]$$

$$= \frac{2}{\sqrt{2}} + \ln(1/2) + \infty$$

$$= \infty$$

So the improper integral diverges.

$\boxed{4}$ Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers with the property that

$$a_1 + a_2 + \cdots + a_n = \frac{n-1}{n}$$

for all $n = 1, 2, 3, \ldots$.

(a) (8 points) Determine whether the infinite series $\sum_{n=1}^{\infty} a_n$ converges, and if so determine its value.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

$$= \lim_{N\to\infty} \frac{N-1}{N} = 1$$

(b) (4 points) Find a_n. Express your answer as a single, simplified fraction in terms of n.

$$a_n = a_n + a_{n-1} + \cdots + 1 - (a_{n-1} + \cdots + 1)$$

$$= \frac{n-1}{n} - \frac{(n-1)-1}{n-1}$$

$$= \frac{(n-1)^2 - n(n-2)}{n(n-1)}$$

$$= \frac{n^2 - 2n + 1 - n^2 + 2n}{n(n-1)}$$

$$= \frac{1}{n(n-1)}$$

$\boxed{5}$ (12 points) By calculating derivatives of f and using the formula for a Taylor polynomial, calculate the second-order Taylor polynomial of the function $f(x) = x^2$ centered at $c = 1$.

$$\begin{cases} f(x) = x^2 & f(1) = 1 \\ f'(x) = 2x & f'(1) = 2 \\ f''(x) = 2 & f''(1) = 2 \end{cases}$$

the Taylor series is

$$P_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(1)}{k!} x^k$$

$$= 1 + \frac{2}{1!}(x-1) + \frac{2}{2!}(x-1)^2$$

$$= 1 + 2(x-1) + (x-1)^2$$

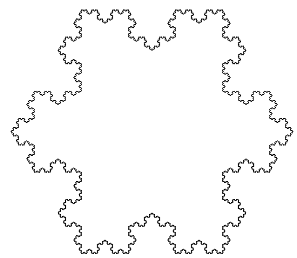
(b) (2 points) How well does this polynomial approximate the original function? (This seems like a vague question, but it does have a clear answer.)

exactly: $(x-1)^2 + 2(x-1) + 1$

$$= x^2 - 2x + 1 + 2x - 2 + 1$$

$$= x^2$$

$\boxed{6}$ The Koch Snowflake is a *fractal curve*, meaning that it continues to look wiggly no matter how far you zoom in on it:



It is created by starting with an equilateral triangle with side-length 1 and by repeating the following process:

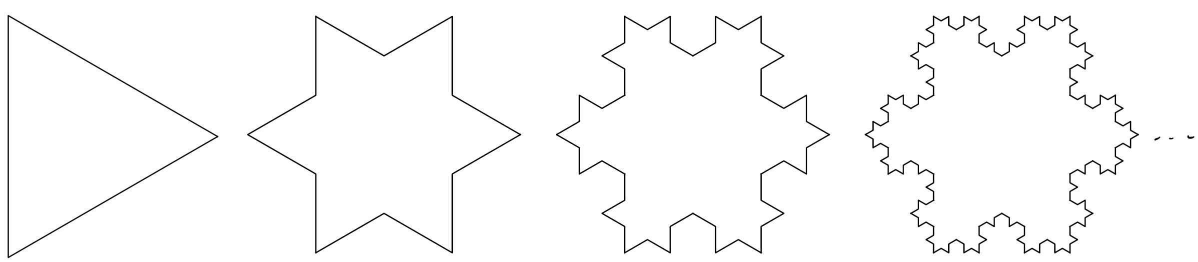
Step 1: Divide each line segment in the figure into three equal-length pieces.

Step 2: For each line segment from Step 1, draw an equilateral triangle whose base is the middle segment from Step 1.

Step 3: For each triangle you draw in Step 2, erase its base.

The result of the nth iteration of this procedure is shown below for $n = 0, 1, 2, 3$. The Koch Snowflake is defined to be the set of all points which are eventually drawn and then never erased as this procedure is repeated for all $n = 0, 1, 2, \ldots$.

n: 0 1 2 3 n



segments 3 $\quad 3\cdot 4$ $\quad 3\cdot 4^2$ $\quad 3\cdot 4^3$ $\quad 3\cdot 4^n$

(a) (3 points) What is the length of the curve (that is, the sum of the lengths of all the line segments) after the nth iteration? Express your answer in terms of n.

# of segments is $3\cdot 4^n$ as shown above.

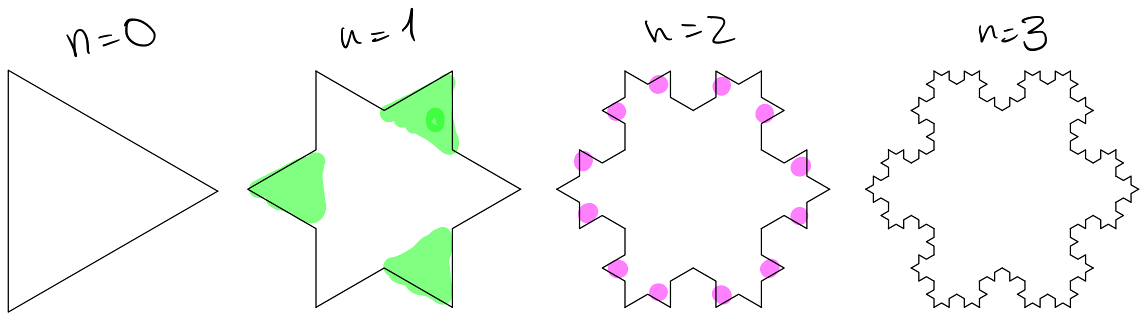
length of each segment is 3^{-n}. So

the total is $3\cdot 4^n \cdot 3^{-n} = 3\left(\frac{4}{3}\right)^n$.

(b) (3 points) What is the length of the snowflake itself? In other words, what is the limit as $n \to \infty$ of the length of the curve after n iterations? Express your answer as an element of the interval $(0, +\infty]$.

$$\lim_{n\to\infty} 3\cdot\left(\frac{4}{3}\right)^n = \infty$$

(c) (3 points) What is the area inside the curve after the nth iteration? You may express your answer as an unsimplified sum of $n+1$ terms, using either summation notation or ellipsis notation. Hint: You may want to count the number of new triangles added at the nth iteration and figure out the area of each of these triangles to calculate the area added at the nth step.



total area is $\frac{\sqrt{3}}{4}$ (original triangle)

$+ \frac{1}{9} \cdot 3 \cdot \frac{\sqrt{3}}{4}$ (3 green triangles)

$+ \frac{1}{81} \cdot 3 \cdot 4^1 \cdot \frac{\sqrt{3}}{4}$ (12 pink triangles)

$+ \cdots$

$+ \frac{1}{9^n} 3 \cdot 4^{(n-1)} \frac{\sqrt{3}}{4}$

$$= \frac{\sqrt{3}}{4} + \sum_{k=1}^{n} \left(\frac{1}{9}\right)^k 3 \cdot 4^{k-1} \frac{\sqrt{3}}{4}$$

(d) (3 points) What is the area inside the snowflake? (same idea as for (b), except with area instead of length)

the limit as $n \to \infty$ of the area of the n^{th} iterate is

$$= \frac{\sqrt{3}}{4} + \sum_{k=1}^{\infty} \left(\frac{1}{9}\right)^k 3 \cdot 4^{k-1} \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \sum_{k=1}^{\infty} \left(\frac{4}{9}\right)^k$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \left(\frac{4/9}{1-4/9}\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \cdot \frac{4}{9} \cdot \frac{9}{5}$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20}$$

$$= \frac{8\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$$

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