§2.2 Matrix inverses

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Many linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ are bijective [every vector is mapped to, exactly once], meaning that they have an inverse function.

Example Find T^{-1} where $T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution T sends $\vec{e}_1$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{e}_2$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so T is a $90°$ ccw rotation. The inverse of such a rotation is the $\boxed{90° \text{ cw rotation}}$. In matrix form $T^{-1}(\vec{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ image of $\vec{e}_2$ image of $\vec{e}_1$. ∎

Note that the inverse of T is linear; this is a general fact:

Theorem The inverse of a bijective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is linear.

Proof The only linear transformation from $\mathbb{R}^1$ to $\mathbb{R}^1$ are the ones of the form $T(x) = ax$

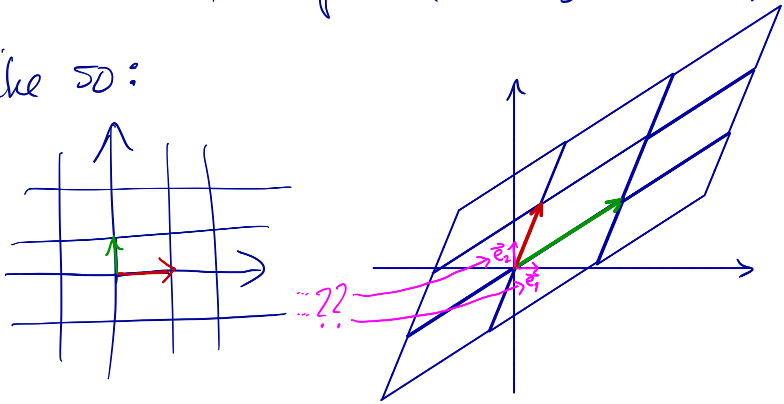
linearity — this is our a

for some $a \in \mathbb{R}$ [since $T(x) = T(x \cdot 1) = x\,T(1)$ for all $x \in \mathbb{R}$]. So $T^{-1}(x) = \frac{1}{a}x$. In higher dimension, T^{-1} must map lines to lines if T does, so T^{-1} is linear in that case too.

this takes a little thinking, actually

Example Find $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1}$. ← means inverse of corresponding transformation

Solution This matrix maps $\vec{e}_1$ to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{e}_2$ to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, like so:



To find the inverse, we just need to figure out what vectors $\vec{v}_1, \vec{v}_2$ map to $\vec{e}_1$ and $\vec{e}_2$. Then the inverse will be the linear map sending $\vec{e}_1, \vec{e}_2$ to $\vec{v}_1, \vec{v}_2$, i.e. $[\vec{v}_1\ \vec{v}_2]$. So we solve $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, & $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We can solve these simultaneously by admitting multiple columns right of the augmented matrix bar:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \end{array}\right].$$

So, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ maps to $\vec{e}_1$ and $\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$ to $\vec{e}_2$. So the inverse sends $\vec{e}_1$ to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\vec{e}_2$ to $\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$, i.e., the inverse is $\begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$. ∎

All that reasoning is general! Recall $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$.

Theorem If $[\underbrace{A}_{\text{any } n\times n \text{ matrix}} \ \underbrace{I}_{\text{the } n\times n \text{ identity matrix}}]$ row reduces to $[I \ B]$, then $B = A^{-1}$! Otherwise A is not bijective (and therefore not invertible).

Example Show that $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is not invertible.

Solution

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In [4]: rref([1 2 3 1 0 0; 4 5 6 0 1 0; 7 8 9 0 0 1])
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Out[4]:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -\frac{8}{3} & \frac{5}{3} \\ 0 & 1 & 2 & 0 & \frac{7}{3} & -\frac{4}{3} \\ \boxed{0 \quad 0 \quad 0} & & & 1 & -2 & 1 \end{bmatrix}$$

oops! non-pivot row in $A \implies A$ not surjective $\Rightarrow A$ not invertible

Theorem If A, B are invertible $n \times n$ matrices, then:

(0) $AB = I \Rightarrow B = A^{-1}$ [uniqueness of inverse]

(1) $AA^{-1} = A^{-1}A = I$

(2) $(A^{-1})^{-1} = A$

(3) $(AB)^{-1} = B^{-1}A^{-1}$

(4) $(A^{-1})^{T} = (A^{T})^{-1}$ transpose

Comments: (0), (1), & (2) are just properties of inverse functions in general.

For (3): $(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$. So $B^{-1}A^{-1}$ is a matrix that multiplies AB to give I. By (0), $B^{-1}A^{-1} = (AB)^{-1}$.

For (4) $(A^{-1})^{T}A^{T} = (AA^{-1})^{T} = I^{T} = I$. So $(A^{T})^{-1} = (A^{-1})^{T}$.

Moral: to check an inverse relationship, just multiply & see that you get I.

Exercise Find $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

maybe don't worry too much about all the details of this calculation

Solution
$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d-\frac{c}{a}b & -\frac{c}{a} & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}-\frac{b}{a}\left(\frac{-c}{ad-bc}\right) & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

So: $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \boxed{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.}$$

§2.3 "Invertible Matrix Theorem"

This is really just a long list of characterizations of invertibility:

An $n \times n$ matrix A is invertible iff:

1. A is row equivalent to I
2. A has n pivot positions
3. $A\vec{x} = \vec{0}$ has only the trivial solution
4. The columns of A are linearly independent
5. (The transformation corresponding to) A is one-to-one
6. $A\vec{x} = \vec{b}$ is consistent, for all $\vec{b} \in \mathbb{R}^n$
7. The columns of A span $\mathbb{R}^n$
8. (The transformation corresponding to) A is onto
9. There is an $n \times n$ matrix C for which $CA = I$
10. There is an $n \times n$ matrix D for which $AD = I$
11. A^T is invertible.