Name (same as in Gradescope): Solutions

MATH 520 MIDTERM I
SPRING 2017
BROWN UNIVERSITY
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This is a pencil-and-paper-only exam.

If you need additional space, you may use the pages at the end of the packet. Indicate where your additional work may be found by clearly writing "continued on page n" for an appropriate value of n. Please remove your staple immediately before handing in your exam (if you bend it in the middle first, it comes out quite easily). If you don't want to handle the staple, please request that your proctor remove it for you.

You only need to justify your answers when it is explicitly asked for. "Justify" means "provide an explanation for your answer which is clear and complete". You may assume facts we have learned in this course unless you are directly being asked to explain such a fact.

$\boxed{1}$ Company A manufactures a pencils per day, Company B manufactures b pencils per day, and Company C manufactures c pencils per day. If Company A operates for 3 days, Company B for 2 days, and Company C for 1 day, then a total of 21,000 pencils are manufactured. If A and B both operate for 4 days while Company C is on vacation, then a total of 28,000 pencils are manufactured.

(a) (5 points) Find a system of equations whose solution set consists of all possible values for the triple (a, b, c). Write your answer in the box. (Throughout this problem, don't worry about the fact that you can't have a negative or fractional number of pencils; just translate the above scenario into a system of linear equations.)

$$\boxed{\begin{aligned} 3a+2b+c &= 21000 \\ 4a+4b \quad\; &= 28000 \end{aligned}}$$

(b) (4 points) Write the system of equations from (a) in augmented matrix form. Write your answer in the box.

$$\boxed{\left[\begin{array}{ccc|c} 3 & 2 & 1 & 21000 \\ 4 & 4 & 0 & 28000 \end{array}\right]}$$

(c) (5 points) Solve the system (note: there may be more than one solution; in that case express your answer using a free variable). Write your final answer in the box.

$$\boxed{\left\{ \begin{pmatrix} 7000-c \\ c \\ c \end{pmatrix} : c \in \mathbb{R} \right\}}$$

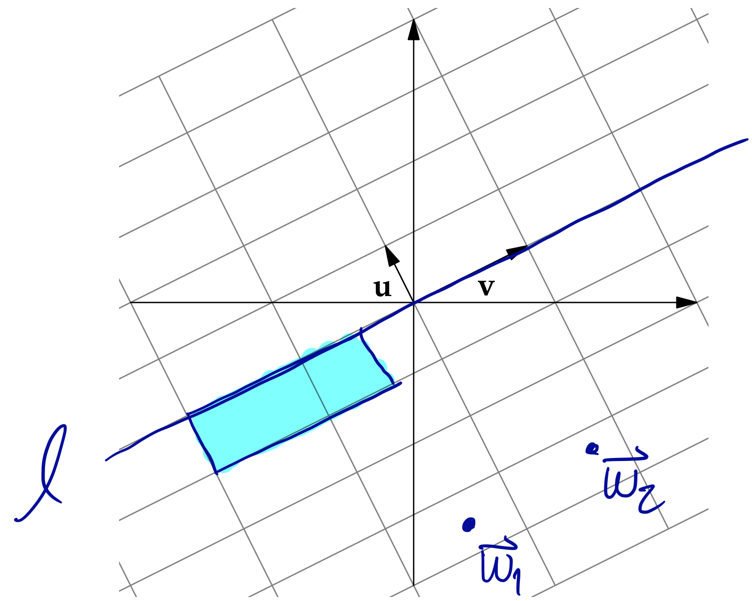
$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 21000 \\ 4 & 4 & 0 & 28000 \end{array}\right] \sim \left[\begin{array}{ccc|c} 12 & 8 & 4 & 84000 \\ 12 & 12 & 0 & 84000 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 12 & 8 & 4 & 84000 \\ 0 & 4 & -4 & 0 \end{array}\right]$$

$$\Rightarrow b = c$$

$$a = \frac{84000 - 12c}{12} = 7000 - c$$

$\boxed{2}$ Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ shown.



(a) (4 points) Is the list $\{\mathbf{u}, \mathbf{v}\}$ linearly dependent? Write yes or no in the box and explain your reasoning.

$\boxed{\text{N}}$

$\vec{u}$ & $\vec{v}$ are not parallel, & for two vectors that's all you have to check

(b) (4 points) In the figure above, shade the region consisting of all points which can be written as $a\mathbf{u} + b\mathbf{v}$ where a is a real number between -1 and 0 and b is a real number between -2 and -0.5.

(c) (4 points) If $\mathbf{w}$ is a point in the fourth quadrant (below the x-axis and to the right of the y-axis) and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ then what can you deduce about a and b? Circle one and explain your answer.

(a) a and b must both be negative

(b) a must be negative but b can be positive or negative

(c) b must be negative but a can be positive or negative

(d) both a and b can be either positive or negative

If $a \geq 0$, then $a\vec{u} + b\vec{v}$ would lie above the line l marked in the picture, so not in Q4. However, $\vec{w}_1$ & $\vec{w}_2$ have $b < 0$ & $b > 0$, resp., so we don't know the sign of b.

$\boxed{3}$ For each of the following statements, determine whether it is true or false. If it is false, give a counterexample demonstrating that it is false and write a big capital F in the box. If it is true, just write a big capital T in the box.

(a) (4 points) If the columns of a matrix A are linearly dependent, then the system of equations $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

$\boxed{T}$

(b) (4 points) If T is a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^2$, then T is surjective.

$\boxed{F}$ $T(\vec{x}) = \vec{0}$ is not surjective

(c) (4 points) If A and B are matrices and $\mathbf{x}$ is a vector, then $A(B\mathbf{x})$ and $(AB)\mathbf{x}$ are equal.

$\boxed{T}$

(d) (4 points) If Q is a square in the plane and T is a linear transformation, then $T(Q)$ has the same area as Q.

$\boxed{F}$ $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ quadruples area

(e) (4 points) There are infinitely many solutions to the system of equations represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

$\boxed{T}$

4 (a) (10 points) Balance the chemical reaction

$$(x_1)C_3H_8 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O.$$

In other words, find integers x_1, x_2, x_3, x_4 such that the number of atoms of each type on each side of the reaction are equal. Justify your answer.

$$\begin{cases} C: & 3x_1 = x_3 \\ H: & 8x_1 = 2x_4 \\ O: & 2x_2 = 2x_3 + x_4 \end{cases}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 8/3 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 8 & -6 & 0 \end{bmatrix}$$

So x is free. Letting $x_4 = 1$, we get $\begin{bmatrix} 1/4 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}$. So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 4 \end{bmatrix}$$ is an integer solution.

(b) (2 points) Is there a unique answer to part (a)? Explain.

No, any positive integer multiple of the above solution is also a solution.

$\boxed{5}$ (12 points) Determine whether the columns of the following matrix are linearly independent. Justify your answer.

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 6 \\ -3 & 2 \end{bmatrix}$$

Yes; they are not multiples of each other since

$$\frac{2}{3} \neq \frac{0}{-1}$$

$\boxed{6}$ (a) (6 points) Show by example that $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{b}$ can have different solution sets even if A and B are row equivalent. Here you may take A and B to be 2×2 matrices if you want, and $\mathbf{b}$ to be a 2×1 vector.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

(b) (6 points) If A and B are row equivalent and $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, then $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions. Explain why this is true.

Because $A\vec{x} = \vec{0}$ has nontrivial solutions iff rref(A) has non-pivot columns. So rref(A) = rref(B) implies either both or neither of A, B has nontrivial kernel.

$\boxed{7}$ (8 points) Using the definition of linear dependence (namely, that a list of vectors is linearly dependent if $\mathbf{0}$ can be written as a linear combination of the vectors with a list of weights which are not all zero), show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, T is a linear transformation, and $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is linearly dependent, then $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

$\{T(v_1), \ldots, T(\vec{v}_n)\}$ lin. ind. $\Rightarrow$

$c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) = \vec{0}$ for $c_1, \ldots, c_n$ not all zero

$\Rightarrow T(\underbrace{c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n}) = \vec{0}$ for $c_1, \ldots, c_n$ not all zero.

But this vector is nonzero since $\{v_1, \ldots, v_n\}$ is l.i. So its a nontrivial solution of $T(\vec{x}) = \vec{0}$.

$\boxed{8}$ (10 points) Any matrix of the form

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

is in reduced row echelon form, where the asterisk denotes an entry that can be any number. Using this notation, write down all possible reduced row echelon forms of a 3×3 matrix. There are ~~X~~ 8 in total, including the one above.

Note: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ does not count as a new case, separate from $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$. In other words, if a particular entry can be any number, we insist on putting an asterisk in that position.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{3 pivots}$$

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{2 pivots}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{1 pivot}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{0 pivots}$$

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