Brown University
Problem Set 4
Instructor: Samuel S. Watson
Due: 6 October 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

Find the limit, if it exists, or show that the limit does not exist, for each of the following functions:

(a) $\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$ (b) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$ (c) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$

Solution

(a) The function $(x,y) \mapsto x^2y^3 - 4y^2$ is a sum of products of continuous functions, and is thus itself continuous. Therefore,

$$\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2) = (3)^2(2)^3 - 4(2)^2 = \boxed{56}.$$

(b) Fix $b \in \mathbb{R}$, and consider points (x,y) satisfying $y = bx$. We then have that

$$\lim_{\substack{(x,y)\,\to\,(0,0)\\ \text{along } y=bx}} \frac{x^2y}{x^3+y^2} = \frac{bx^3}{x^3+b^3x^3} = \frac{b}{1+b^3},$$

for $x \neq 0$. Since this limit depends on b, it follows that there are straight line paths through the origin along which the limits of f are different. Therefore, the limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist.

(c) We begin by converting to polar coordinates.

$$\frac{xy^4}{x^4+y^4} = \frac{r^5\cos(\theta)\sin^4(\theta)}{r^4\cos^4(\theta)+r^4\sin^4(\theta)} = r\left(\frac{\cos(\theta)\sin^4(\theta)}{\cos^4(\theta)+\sin^4(\theta)}\right).$$

Since $\left|\frac{\cos(\theta)\sin^4(\theta)}{\cos^4(\theta)+\sin^4(\theta)}\right|$ is bounded, we conclude that this expression can be made as close to 0 as desired by making r suitably small. Therefore, the limit exists and is equal to $\boxed{0}$.

Problem 2

Which of the following are true? Explain carefully.

I. If $f(x, y)$ is not defined at the point $(7, 5)$, then the limit as $(x, y) \to (7, 5)$ of $f(x, y)$ does not exist.

II. If $f(1.99, 3.01) = 105$ and f is continuous, then $\lim_{(x,y)\to(2,3)} f(x, y)$ must be at least 100.

III. If $\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$, then f is continuous at the origin.

Solution

I. This is not necessarily true. The limit may exist at this point even if the function is not defined there. Consider the function f on $\mathbb{R}^2 \setminus \{(7, 5)\}$ which returns the output 6 for every (x, y) in its domain.
Then the limit exists and equals 6, but the function does not have a value at the point $(7, 5)$.

II. This is not true. The function might have a very steep slope around the point $(2, 3)$. For example, consider a plane which passes through $(1.99, 3.01, 105)$ and $(2, 3, 99)$. This plane is the graph of some function, and that function gives a counterexample to the given statement.

III. This is true. This is the definition of what it means for a function to be continuous at the origin and equal to 0 there.

Problem 3

A function $u(x, y)$ is said to satisfy *Laplace's equation* if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Show that $u(x, y) = e^x \sin y$ satisfies Laplace's equation.

Solution

We begin by first finding the first and second partial derivatives of u with respect to x. We get

$$\frac{\partial u}{\partial x} = e^x \sin y, \text{ and}$$
$$\frac{\partial^2 u}{\partial x^2} = e^x \sin y.$$

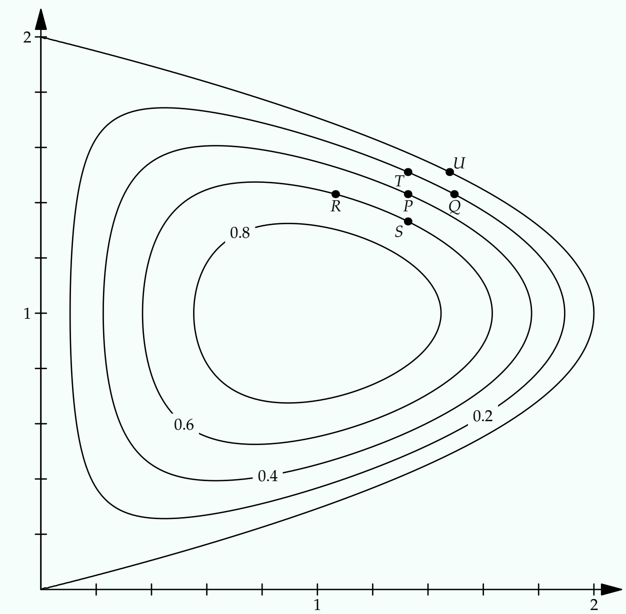
Similarly, we have $\frac{\partial^2 u}{\partial y^2} = -e^x \sin y$. And finally,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y + -e^x \sin y = 0.$$

Problem 4

A contour plot of a funtion f is shown to the right. Determine the signs of f_x, f_y, f_{xx}, f_{xy}, and f_{yy} at the point P.

The points Q, R, S, T, and U are labeled on the diagram for your convenience: R and Q are due west and east of P, respectively, while S and T are due south and north, respectively. Point U is due east of T. You will want to consider the slopes of secant lines passing through various pairs of these points.



Solution

(a) $f_x < 0$, since the value of f decreases from 0.4 to 0.2 as you move slightly to the right from P.

(b) $f_y < 0$, since the value of f decreases from 0.4 to 0.2 as you move slightly upward from P.

(c) $f_{xx} < 0$, because the slope of the secant line passing through $(R, f(R))$ and $(P, f(P))$ is smaller in absolute value than the slope of the secant line passing through $(P, f(P))$ and $(Q, f(Q))$ (note that the numerators are the same, while the denominator is larger for the first secant line than for the second). Since both are negative, this means that f_x is increasing as x increases, which in turn means that $f_{xx} < 0$.

(d) $f_{yy} < 0$ for the same reason as f_{xx}, with points S and T replacing R and Q.

(e) $f_{xy} < 0$, because the slope of the secant line through $(T, f(T))$ and $(U, f(U))$ is larger in absolute value than the one passing through $(P, f(P))$ and $(Q, f(Q))$ (again, same numerator and smaller denominator). Since both quantities are negative, this means that f_x is decreasing as y increases. Thus $f_{xy} < 0$.

Problem 5

Consider the equation $z = x^2 - xy + 3y^2$. As (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, find the change in the value of z. Repeat the exercise with $z = x^2 - xy + 3y^2$ replaced by the plane tangent to $z = x^2 - xy + 3y^2$ at the point $(3, -1)$.

Solution

Let's define $f(x, y) = x^2 - xy + 3y^2$, so that the graph of the given equation is equal to the graph of f. We substitute $x = 3$ and $y = -1$ into $f(x, y)$ to get $f(3, -1) = 15$. We also substitute $x = 2.96$ and $y = -0.95$ into the expression for f to get 14.2811. So z changes by -0.7189.

The equation for the plane tangent to the graph of f at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0), \tag{1}$$

where $z_0 = f(x_0, y_0)$. With $(x_0, y_0, z_0) = (3, -1, 15)$ and $f_x = 2x - y$, $f_y = 6y - x$, we get

$$z - z_0 = 7(x - 3) - 9(y + 1).$$

Substituting $x = 2.96$ and $y = -0.95$ into the right-hand side of this equation, we get -0.73. This is reasonably close to the exact value of -0.7189.

Additional space