PSet 11

$\boxed{1}$ If f is odd, we have, for $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0} f(x)\cos(nx)\,dx}_{u=-x} + \int_0^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{\pi}^{0} f(-u)\cos(-nu)\,du + \int_0^{\pi} f(x)\cos nx\,dx\right]$$

↓ flipping costs a factor of -1, & so does replacing $f(-u)$ with $f(u)$

$$= \frac{1}{\pi}\left[-\int_0^{\pi} f(u)\cos(nu)\,du + \int_0^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\cdot 0 = 0,$$

& $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= 0.$$

Similarly, if f is even,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{\pi}^{0} f(-u)\sin(-nu)(-du) + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{\pi}^{0} f(u)(-\sin nu)(-du) + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= 0.$$

$\boxed{2}$ We calculate $a_0 = \frac{1}{2\pi}\int_0^{2\pi} (5 + 2\sin x + 3\cos 2x)\, dx$

$$= \frac{1}{2\pi}\left[10\pi + 0 + 0\right]$$

$$= 5.$$

Also, $a_n = \frac{1}{\pi}\int_0^{2\pi} (5 + 2\sin x + 3\cos 2x) \cos nx \, dx$

$$= \frac{1}{\pi}\left[0 + 0 + \begin{cases} 3\pi & \text{if } n=2 \\ 0 & \text{if } n \neq 2 \end{cases}\right]$$

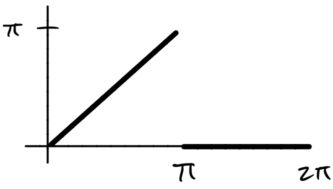
$$= 3 \text{ if } n=2, \text{ else } 0.$$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} (5 + 2\sin x + 3\cos 2x) \sin nx \, dx$

$$= \frac{1}{\pi}\left[0 + \begin{cases} 2\pi & \text{if } n=1 \\ 0 & \text{if } n \neq 1 \end{cases} + 0\right]$$

$$= 2 \text{ if } n=1, \text{ else } 0.$$

$\boxed{3}$



We get: $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{2\pi}\int_0^{\pi} x \, dx = \frac{1}{2\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{4}$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$

$$= \frac{1}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} x \left(\frac{\sin nx}{n}\right)' dx$$

$$= \frac{1}{\pi}\left[x \frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx\right] = \frac{1}{\pi}\cdot\frac{\cos nx}{n^2}\Big|_0^{\pi} = \begin{cases} \frac{-2}{\pi n^2} & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\left(-\frac{\cos nx}{n}\right)' dx$$

$$= \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right]$$

$$= \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ -\frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

So the Fourier series is

$$\frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{-2}{\pi}\cdot\frac{1}{(2k+1)^2}\cos((2k+1)x) + \frac{1}{2k}\sin((2k+1)x) - \frac{1}{2k+2}\sin((2k+2)x)$$

The Fourier series converges to $\frac{1}{2}\left[\lim_{x\to\pi^+} f(x) + \lim_{x\to\pi^-} f(x)\right] = \frac{\pi}{2}$, so we'd need $f(\pi) = \boxed{\pi/2}$ to have convergence of the Fourier series everywhere.

4 We calculate $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx$

$$= \frac{1}{2\pi}\int_0^{\pi} e^{-inx}dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-e^{-in\pi}+1}{in}\right] = \begin{cases} \frac{1}{\pi i n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Reading off $a_n = 2\,\mathrm{Re}\, c_n$, we get $a_n = 0$, and

$$\begin{aligned} b_n &= -\mathrm{Im}(2c_n) \\ &= \frac{2}{\pi n} \text{ if } n \text{ is odd, else } 0. \end{aligned}$$

We see that this agrees with the example on page 69.

Also, $c_{-n} = \dfrac{1}{\pi i(-n)} = \overline{\left(\dfrac{1}{i\pi n}\right)} = \overline{c_n}$, as indicated in theorem 13.5.

$\boxed{5}$ We calculate

$$c_n = \frac{1}{2\pi}\int_0^{\pi} x(1-x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} -x^2 e^{-inx}\,dx + \int_0^{\pi} x e^{-inx}\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\left(\frac{x^2}{-in} - \frac{2x}{(in)^2} - \frac{2}{(in)^3}\right)e^{-inx} + \left(\frac{x}{-in} - \frac{1}{(in)^2}\right)e^{-inx}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\Bigg[\left(\frac{-\pi^2}{-in} + \frac{2\pi}{(in)^2} + \frac{2}{(in)^3} + \frac{\pi}{-in} - \frac{1}{(in)^2}\right)e^{-in\pi} - \frac{2}{(in)^3} + \frac{1}{(in)^2}\Bigg]$$

$$= \begin{cases} \dfrac{1}{2\pi}\left[\dfrac{i\pi^2}{n} + \dfrac{2\pi}{n^2} + \dfrac{2i}{n^3} - \dfrac{\pi i}{n} - \dfrac{1}{n^2} - \dfrac{2i}{n} - \dfrac{1}{n^2}\right] & n \text{ odd} \\[2ex] \dfrac{1}{2\pi}\left[\dfrac{-i\pi^2}{n} - \dfrac{2\pi}{n^2} - \dfrac{2i}{n^3} + \dfrac{\pi i}{n} + \cancel{\dfrac{1}{n^2}} - \dfrac{2i}{n} - \cancel{\dfrac{1}{n^2}}\right] & n \text{ even} \end{cases}$$

Side calculations:

$$\int x e^{-inx}\,dx = \int x\left(\frac{e^{-inx}}{-in}\right)' = x\frac{e^{-inx}}{-in} - \int \frac{e^{-inx}}{-in} = \frac{xe^{-inx}}{-in} - \frac{e^{-inx}}{(in)^2}$$

$$\int x^2 e^{-inx}\,dx = \int x^2\left(\frac{e^{-inx}}{-in}\right)' = \frac{x^2e^{-inx}}{-in} - 2\int x\left(\frac{e^{-inx}}{-in}\right) = \frac{x^2e^{-inx}}{-in} + \frac{2}{in}\int xe^{-inx}$$

$$= \frac{x^2e^{-inx}}{-in} + \frac{2}{in}\left(\frac{xe^{-inx}}{-in} - \frac{e^{-inx}}{(in)^2}\right) = \frac{x^2e^{-inx}}{-in} - \frac{2xe^{-inx}}{(in)^2} - \frac{2e^{-inx}}{(in)^3}$$

$\boxed{6}$ (a) $2 - \sin 3x - \sin 5x = 2 - \dfrac{e^{3ix} - e^{-3ix}}{2i} - \dfrac{e^{5ix} - e^{-5ix}}{2i}$

$$= 2 + \frac{i}{2}e^{3ix} - \frac{i}{2}e^{-3ix} + \frac{ie^{5ix}}{2} - \frac{ie^{-5ix}}{2}$$

(b) $\frac{1}{2}\cos x - \frac{1}{2}\sin x$

$$= \frac{1}{2} \cdot \frac{e^{ix} + e^{-ix}}{2} - \frac{1}{2}\,\frac{e^{ix} - e^{-ix}}{2i}$$

$$= \frac{1}{4}(1+i)e^{ix} + \frac{1}{4}(1-i)e^{-ix}.$$

$\boxed{7}$ (a) $3e^{-ix} + 3e^{ix} = 3\cos x - 3i\sin x$

$$+ 3\cos x + 3i\sin x$$

$$= 6\cos x$$

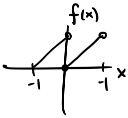
(b) $(1+i)e^{3ix} + (1-i)e^{-3ix} + ie^{5ix} - ie^{-5ix}$

$$= (1+i)(\cos 3x + i\sin 3x) + (1-i)(\cos 3x - i\sin 3x)$$
$$+ i(\cos 5x + i\sin 5x) - i(\cos 5x - i\sin 5x)$$

$$= \cos 3x + i\cos 3x + i\sin 3x - \sin 3x$$
$$+ \cos 3x - i\cos 3x - i\sin 3x - \sin 3x$$
$$+ i\cos 5x - \sin 5x - i\cos 5x - \sin 5x$$

$$= 2\cos 3x - 2\sin 3x - 2\sin 5x.$$

$\boxed{8}$ $$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx$$



$$= \frac{1}{2}\cdot 2\cdot \int_0^1 x\,dx$$
$$= \frac{1}{2}\cdot 2\cdot \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

For $n \geq 1$, $$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx$$
$$= \int_0^1 x \overbrace{\cos(n\pi x)}^{\left(\frac{\sin n\pi x}{n\pi}\right)'}\,dx + \int_{-1}^{0}(1+x)\overbrace{\cos(n\pi x)}^{\left(\frac{\sin n\pi x}{n\pi}\right)'}\,dx$$
$$= \frac{x\sin n\pi x}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx + \frac{(1+x)\sin n\pi x}{n\pi}\Big|_{-1}^{0} + \int_{-1}^{0}\frac{\sin n\pi x}{n\pi}\,dx$$
$$= \cancel{\frac{\sin n\pi}{n\pi}} - \cancel{\frac{0\cdot\sin(0)}{n\pi}} + \frac{\cos n\pi x}{(n\pi)^2}\Big|_0^1 + \cancel{0} - \cancel{0} + \frac{-\cos n\pi x}{(n\pi)^2}\Big|_{-1}^{0}$$
$$= \frac{\cos n\pi - \cos 0}{n^2\pi^2} + \frac{-\cos 0 + \cos(-n\pi)}{n^2\pi^2}$$
$$= \frac{2(\cos n\pi - 1)}{n^2\pi^2} = \begin{cases} \frac{-4}{\pi^2 n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

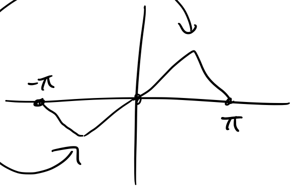
For $n \geq 1$, we have
$$b_n = \int_{-1}^{0}(1+x)\overbrace{\sin n\pi x}^{\left(\frac{-\cos n\pi x}{n\pi}\right)'}\,dx + \int_0^1 x\overbrace{\sin n\pi x}^{\left(\frac{-\cos n\pi x}{n\pi}\right)'}\,dx$$
$$= -(1+x)\frac{\cos n\pi x}{n\pi}\Big|_{-1}^{0} - \int_{-1}^{0}\frac{-\cos(n\pi x)}{n\pi}\,dx - \frac{x\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{-\cos n\pi x}{n\pi}\,dx$$
$$= -(1+0)\frac{\cos 0}{n\pi} + \cancel{(1+(-1))\cdot \text{stuff}} - \cancel{\frac{\sin(n\pi x)}{(n\pi)^2}\Big|_{-1}^{0}} - \frac{\cos n\pi}{n\pi} - \cancel{\frac{\sin n\pi x}{n\pi}\Big|_0^1}$$
$$= \frac{-1-\cos n\pi}{n\pi} = \begin{cases} 0 & n \text{ odd} \\ \frac{-2}{n\pi} & n \text{ even} \end{cases}$$

So the Fourier series is

$$\frac{1}{2} - \frac{4}{\pi^2}\cos x - \frac{4}{9\pi^2}\cos 3x - \cdots$$
$$- \frac{2}{2\pi}\sin 2x - \frac{2}{4\pi}\sin 4x - \frac{2}{6\pi}\sin 6x - \cdots$$

9

The graph of f is



(a) and we can extend it to an odd function like

(b) The Fourier series of the new function has $a_n = 0$ for all n since f is odd, and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,dx$$

since $\int_{-\pi}^{0} = \int_{0}^{\pi}$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(n\,x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\,\underbrace{\sin(n\,x)}_{\left(-\frac{\cos(n\,x)}{n}\right)'}\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin(n\,x)\,dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{-x\cos(n\,x)}{n}\right]_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos(n\,x)}{n}\,dx - \left[\frac{(\pi - x)\cos nx}{n}\right]\Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi}\left(-\frac{\cos nx}{n}\right)dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{\sin(n\pi/2)}{n^2} + \frac{\pi\cos(\pi n/2)}{2n} - \frac{\sin nx}{n^2}\Big|_{\pi/2}^{\pi}\right\}$$

$$= \frac{4}{\pi n^2}\sin(n\pi/2) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 4/\pi n^2 & \text{if } n = 4k+1 \text{ for some } k \\ -4/\pi n^2 & \text{if } n = 4k+3 \end{cases}$$

(c) We have

$$s(1) = \frac{4}{\pi} e^{-\gamma} \cos(2\pi f_1 t)$$
$$- \frac{4}{9\pi} e^{-3\gamma} \cos(6\pi f_1 t)$$
$$+ \frac{4}{25\pi} e^{-5\gamma} \cos(10\pi f_1 t)$$
$$- \cdots$$

The coefficients are $0.232\ldots, -0.00086\ldots,$ 1.03×10^{-5}, etc. So there are only $\boxed{2}$ terms with coefficients at least $10^{-4} \cdot 0.232\ldots$.

$\boxed{10}$ We have coefficients $c_n = \frac{a_n - ib_n}{2} = \frac{-i}{2}\frac{2}{n\pi}$ if n is odd, $= \frac{-i}{n\pi}$ $n \geq 1$

and 0 if n is even, from p.69 & Theorem 13.5, and $c_{-n} = \frac{i}{n\pi}$ for $n \geq 1$.

Then the Fourier coefficients of the steady-state solution are $C_n = \frac{-i}{\pi n((in)^2 + in + 4)}$ and $c_0 = \frac{1}{2\pi}\int_0^\pi dx = \frac{1}{2}$

$$= \frac{-i}{\pi n}\left(\frac{1}{4 - n^2 + in}\right) = \frac{-i(4 - n^2 - in)}{\pi n((4-n^2)^2 + n^2)} = \frac{-n - i(4-n^2)}{\pi n((4-n^2)^2 + n^2)}.$$

$$\frac{-n - i(4-n^2)}{\pi n\left((4-n^2)^2 + n^2\right)}.$$

Thus $a_n = \dfrac{-1}{\pi\ ((4-n^2)^2+n^2)}$ and $b_n = \dfrac{4-n^2}{\pi n\left[(4-n^2)^2+n^2\right]}$ for $n \geq 1$,

and the steady-state solution is

$$\frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{-n}{\pi\left((4-n^2)^2+n^2\right)}\cos(nx) + \frac{4-n^2}{\pi n\left((4-n^2)^2+n^2\right)}\sin nx\right].$$