PSet 8

① (a) The series converges by the ratio test.

(b) The sequence converges to 0, by the n^{th} term test, since the series converges

② (a) $\sum_{n=1}^{\infty} \frac{n^3}{10^n}$ converges because

$$\frac{(n+1)^3}{10^{n+1}} \cdot \frac{10^n}{n^3} \to \frac{1}{10} \text{ as } n \to \infty$$

(b) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ diverges because

$$\frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \frac{n+1}{100} \to \infty \notin [-1,1]$$

(c) $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ diverges because

$$\frac{2^{(n+1)^2}}{(n+1)!} \cdot \frac{n!}{2^{n^2}} = \frac{2^{n^2+2n+1-n^2}}{n+1} = \frac{2^{2n+1}}{n+1} \to \infty$$

(d) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges:

$$\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{1}{\cancel{(n+1)}} \cdot \frac{n^n \cancel{(n+1)}}{(n+1)^n}$$

$$= \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e}.$$

(3.) the ratio test gives

$$\frac{a_{n+1}}{a_n} = \underbrace{\frac{[4(n+1)]!}{(4n)!}}_{=(4n+4)(4n+3)(4n+2)(4n+1)} \cdot \underbrace{\frac{n!^4}{(n+1)!^4}}_{=\frac{1}{(n+1)^4}} \cdot \underbrace{\frac{396^{4n}}{396^{4n+4}}}_{=\frac{1}{396^4}} \cdot \underbrace{\frac{1103+26390(n+1)}{1103+26390n}}_{\text{converges to } 1}$$

$$\text{So } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4^4}{396^4} = \left(\frac{4}{396}\right)^4 < 1$$

So the sum converges.

(4.) $$\lim_{n\to\infty} \frac{\frac{1}{3^n-2^n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{1}{1-\left(\frac{2}{3}\right)^n} = \frac{1}{1-0} = 1,$$

$$\text{So } \sum \frac{1}{3^n} < \infty \Rightarrow \sum \frac{1}{3^n-2^n} < \infty.$$

(5.) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \infty$, but $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} < \infty$ by the alternating series test ($\frac{1}{\sqrt{n}} > 0$, and $\frac{1}{\sqrt{n}}$ decreases). So conditionally convergent.

(b) the series is absolutely convergent, by the integral test:

$$\int_1^\infty x e^{-x}\,dx \underset{\text{IBP}}{=} -(x+1)e^{-x}\Big|_1^\infty \underset{\text{L'Hopital}}{<} \infty$$

(c) Divergent — fails the n^{th} term test

(d) Conditionally convergent, because

$$\lim_{n\to\infty} \frac{\sin(\pi/n)}{1/n} \to \pi, \quad \sum \frac{1}{n} = \infty \underset{\text{limit comparison}}{\Rightarrow} \sum \sin(\pi/n) = \infty$$

But $\sin(\pi/n)$ is decreasing because sin is increasing on $[0, \pi/2]$, so the alternating series test implies that

$\sum_{n=1}^\infty (-1)^n \sin(\pi/n)$ converges

(6) (a) $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

(b) S_n is an overestimate if n is odd and an underestimate otherwise. The reason is that S_{2n} is increasing:

$$S_{2n} = \underbrace{1 - \frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{\frac{1}{12}} + \cdots + \underbrace{\frac{1}{2n-1} + \frac{1}{2n}}_{\frac{1}{(2n-1)(2n)}}$$

all terms are positive!

while S_{2n+1} is decreasing:

$$S_{2n+1} = 1 \underbrace{- \frac{1}{2} + \frac{1}{3}}_{-\frac{1}{6}} \underbrace{- \frac{1}{4} +}_{-\frac{1}{20}} \cdots \underbrace{- \frac{1}{2n} + \frac{1}{2n+1}}_{-\frac{1}{2n(2n+1)}}$$

all negative!

Since $S_{2n} \to \ln 2$ and S_{2n} is increasing, all the S_n's have to be less than $\ln 2$. Same idea for the odd ones.

⑦ (a) $f(x) = \frac{1}{1+x}$

$f'(x) = \frac{-1}{(1+x)^2}$

$f''(x) = \frac{2}{(1+x)^3}$

$L(x) = f(0) + f'(0)(x-0)$

$= 1 + (-1)x$

$= 1 - x$

$Q(x) = L(x) + \frac{1}{2} f''(0)(x-c)^2$

$= 1 - x + x^2$

(b) $f(x) = \ln x$

$f'(x) = 1/x$

$f''(x) = -1/x^2$

$L(x) = \ln 1 + \frac{1}{1}(x-1)$

$= x - 1$

$Q(x) = L(x) + \frac{f''(x)}{2}(x-1)^2$

$= (x-1) - \frac{1}{2}(x-1)^2.$

(8). (a) $Q(x) = f(0) + f'(0)\,x + \frac{1}{2} f''(0)\,x^2$

$= 1 + 0x - \frac{1}{2}x^2$

$= 1 - x^2/2.$

$Q(0.1) = 0.995$

$\cos(0.1) = 0.99500416\ldots$

So the difference is about $4.16 \times 10^{-6} < 10^{-5}$.

(b) $\frac{1 - Q(x)}{x^2} = 1/2$, so $\lim_{x \to 0} \frac{1 - Q(x)}{x^2} = 1/2$

$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$ 2 ✓

(9) Note that $2^{1/n} - 1$ converges to 0 as $n \to \infty$. In particular $2^{1/n} - 1 < \frac{1}{2}$ for large enough n. So $(2^{1/n} - 1)^n < \left(\frac{1}{2}\right)^n$ for large enough n. By the comparison test, $\sum (2^{1/n} - 1)^n$ converges.

(10) Following the hint, the desired sum is

$$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{1}{2k+2}\right)$$

$$=\sum_{k=0}^{\infty}\frac{(4k+3)(2k+2)+(4k+1)(2k+2)-(4k+1)(4k+3)}{(4k+1)(4k+3)(2k+2)}$$

$$=\sum_{k=0}^{\infty}\frac{\cancel{8k^2}+14k+6+\cancel{8k^2}+2k+8k+2-\cancel{16k^2}-16k-3}{(4k+1)(4k+3)(2k+2)}$$

$$=\sum_{k=0}^{\infty}\frac{8k+5}{(4k+1)(4k+3)(2k+2)}$$

These terms are all positive, so any partial sum is less than the limit. But even the zeroth term is $5/6 = 0.8\overline{3} > 0.69\ldots$.

☆ Fun fact: the limit is actually $\frac{3}{2}\ln 2$. ☆