DATA 1010
Problem Set 2
Due 21 September 2018 at 11 PM

Problem 1

Each of 68 people is interviewed and scored on a scale from 0 to 10 in three different categories. A composite score is obtained for each person by averaging the person's category scores. These data are arranged into a 68×4 matrix X, so that each row consists of a particular interviewee's category scores and composite score.

Find the determinant of $X'X$.

Solution

The rank of X is at most 3, since the fourth column is in the span of the first three columns. Therefore, the rank of $X'X$ (which is same as the rank of X, for *any* matrix X), is at most 3. Since $X'X$ is a 4×4 matrix, this means that it is not full rank. The determinant of a rank-deficient square matrix is 0, so the determinant of $X'X$ is zero.

Problem 2

Use matrix differentiation to find the vector $\mathbf{x} \in \mathbb{R}^n$ which minimizes the expression $|W(A\mathbf{x} - \mathbf{b})|^2$, where A is an $m \times n$ matrix and W is an $m \times m$ matrix. You may assume that WA is full-rank.

Solution

We write the given expression as $(W(A\mathbf{x} - \mathbf{b}))'W(A\mathbf{x} - \mathbf{b})$, which expands to

$$\mathbf{x}'AW'WA\mathbf{x} - \mathbf{b}W'WA\mathbf{x} - \mathbf{x}'A'W'W\mathbf{b} + \mathbf{b}'W'W\mathbf{b}.$$

Differentiating with respect to $\mathbf{x}$, we get

$$2\mathbf{x}'A'W'WA - \mathbf{b}'W'WA - \mathbf{b}'W'WA.$$

Setting this equal to 0, we get

$$\mathbf{x}'A'W'WA = \mathbf{b}'W'WA.$$

Transposing both sides and solving for $\mathbf{x}$, we find that $\mathbf{x} = (A'W'WA)^{-1}AW'W\mathbf{b}$. The step of inverting $A'W'WA = (WA)'(WA)$ is valid because it has the same rank as WA, which assumed to be m.

Problem 3

Find the derivative of $|\mathbf{x}|$ with respect to $\mathbf{x}$. Hint: write $|\mathbf{x}|$ as $\sqrt{\mathbf{x}'\mathbf{x}}$ and use the chain rule, which says that if $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, then

$$\frac{\partial}{\partial \mathbf{x}} f(g(\mathbf{x})) = f'(g(\mathbf{x}))\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}).$$

Interpret your answer geometrically and explain why it makes sense.

Solution

We have

$$\frac{\partial}{\partial \mathbf{x}}\left(\sqrt{\mathbf{x}'\mathbf{x}}\right) = \frac{1}{2|\mathbf{x}|}2\mathbf{x}' = \frac{\mathbf{x}'}{|\mathbf{x}|}$$

This is the unit vector in the direction of $\mathbf{x}$. This makes sense because the direction in which the length of the vector from 0 to $\mathbf{x}$ increases the fastest is directly away from the origin, and the rate of increase in that direction is 1 distance unit per distance unit.

Problem 4

(i) Find the line through the origin for which the sum of squared distances from the line to points in the set

$$\{(3,-1),(2,4),(-1,-1),(-2,2),(-3,1),(5,-1),(-2,4)\}$$

is as small as possible.

(ii) Find the slope of the zero-intercept line of best fit for these points using the formula $m = (A'A)^{-1}A'\mathbf{b}$, where A is a column vector whose entries are the x coordinates of the points and where $\mathbf{b}$ is a column vector whose components are the y-components of the points (in the same order). Recall that this is the line which minimizes $\sum_i (mx_i - y_i)^2$ where (x_i, y_i) ranges over the given points.

(iii) Draw both of these lines and explain why they are not the same even though they both minimize a sum of squared distances.

```
using Plots, LinearAlgebra
A = [3 2 -1 -2 -3 5 -2; -1 4 -1 2 1 -1 4]
scatter(A[1,:], A[2,:])
```

Solution

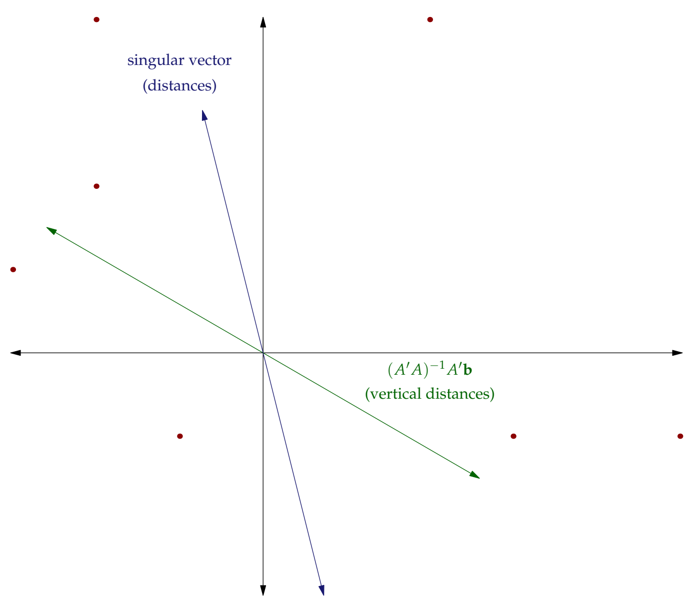
(i) The line through the origin which gets closest to the given points (in the sum-of-squared-distances sense) is the one running along the first column of U in the SVD of

$$\begin{bmatrix} 3 & 2 & -1 & -2 & -3 & 5 & -2 \\ -1 & 4 & -1 & 2 & 1 & -1 & 4 \end{bmatrix}$$

(Or, equivalently, the first column of V in the SVD of the transpose of this matrix). The unit vector representing this line is $[-0.865, 0.502]$.

(ii) Using the given formula, we find that the line of best fit through the origin has slope $-\frac{1}{4}$.

These are not the same because the singular vector minimizes the sum of the squared (perpendicular) distances from the line to the points, while the the best-fit minimizes the sum of *vertical* squared distances.



Problem 5

Find a value of x which is less than 1 and for which `1 + x + x + x > 1 + 3x` returns `true`. Explain this behavior.

Solution

Let x be a number slightly larger than the gap between 1 and the first representable value greater than 1, like $2^{-53} + 2^{-57}$. This number is a bit larger than half the gap ϵ between representable values between 1 and 2. Then each addition of x takes us up to the next representable float value. Meanwhile, $3x$ is less than 2ϵ and will therefore be smaller than the result of successively adding x three times.

Problem 6

Explain why the following function returns a value rather than running forever. Explain why it returns the particular value that it returns.

```
function countdown()
    x = 1.0
    ctr = 0
    while x > 0.0
        x /= 2
        ctr += 1
    end
    ctr
end
```

Solution

The function returns 1075. The reason it does not run forever is that eventually `x` reaches the smallest representable number (2^{-1074}), at which point halving results in rounding to zero. It takes 1074 steps to get to 2^{-1074} and then one more to get to a number which rounds to zero, for a total of 1075 steps.

Problem 7

Show that an invertible, square matrix and its inverse have the same condition number.

Solution

If $A = U\Sigma V'$ is the SVD of A, then the SVD of the inverse of A is $V\Sigma^{-1}U'$. Therefore, the singular values of the inverse of A are the reciprocals of the singular values of A. Thus the largest singular value of the inverse of A is the reciprocal of the smallest singular value of A, and the smallest singular value is the reciprocal of A's largest. Thus the largest-to-smallest ratio of A's singular values and is equal to the largest-to-smallest ratio of A^{-1}.

Problem 8

Consider the $n \times n$ Frank matrix F_n, defined as shown in the code block below.

```
function frankmatrix(n)
    A = zeros(n,n)
    for i=1:n
        for j=1:n
            if j == i-1
                A[i,j] = n + 1 - i
            elseif j ≥ i
                A[i,j] = n + 1 - j
            end
        end
    end
    A
end
```

Find $F_n^{-1}\mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^n$ has all components equal to 1, by inspection. (Generate F_n for some small values of n and look at it).

Evaluate `frankmatrix(n) \ ones(n)` for $n \in \{10, 15, 20, 25, 30\}$ and calculate the norm of the difference between this numerical solution and the true solution. Compare your result to the product of `eps()` (which equals 2^{-52}, the gap between 1 and the nearest representable 64-bit floating point) and the condition number of F_n (which can be calculated using the function `cond`). Hint: a good way to do this comparison is to plot the log of each of these quantities over the specified range of n values.

Based on your findings, comment on whether the algorithm being used for `\` is stable.

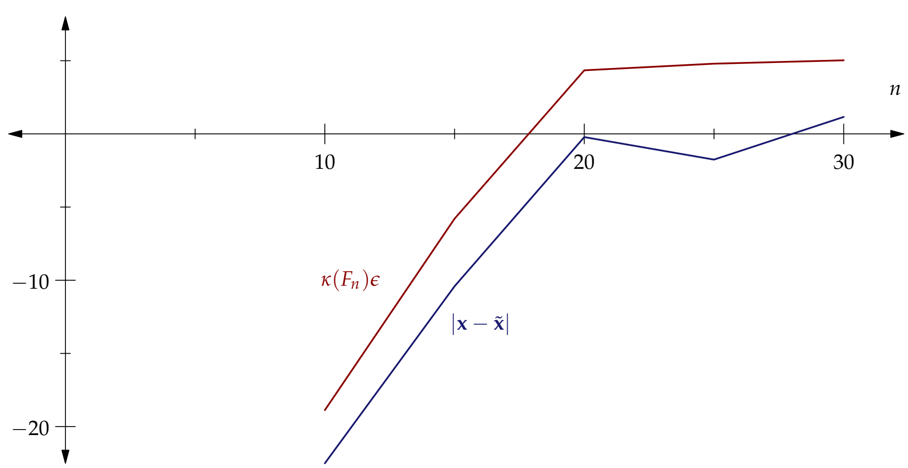
Solution

We calculate the error and plot it as a function of n, as well as computing the condition number and plotting that:

```
function solve_error(n)
    A = frankmatrix(n)
    b = ones(n)
    x = A \ b
    norm(x - [zeros(n-1);[1]])
end

r = 10:5:30
plot(r,[log(solve_error(k)) for k=r];dpi=300)
plot!(r,[log(cond(frankmatrix(k))*eps()) for k=r])
```

We see that the error is actually less than $\kappa(F_n)\epsilon$. Even though the error is large when n is large, it is not large compared to the condition number of the matrix. Therefore, the algorithm used for `\` does appear to be stable.



Problem 9

Consider the following PRNG (which was actually widely used in the early 1970s): we begin with an odd positive integer a_1 less than 2^{31} and for all $n \geq 2$, we define a_n to be the remainder when dividing $65539a_{n-1}$ by 2^{31}.

Use Julia to calculate $9a_{3n+1} - 6a_{3n+2} + a_{3n+3}$ for the first 10^6 values of n, and show that there are only *15* unique values in the resulting list (!). Explain what you would see if you plotted many points of the form $(a_{3n+1}, a_{3n+2}, a_{3n+3})$ in three-dimensional space.

Solution

We generate the first 3,000,003 elements of the sequence and convert the suggested linear combination into set to see how many distinct elements it contains. Indeed, there are only 15:

```
A = [seed]
for i=1:3*10^6+2
    push!(A,mod(65539*A[end],2^31))
end
length(Set([[9,-6,1]⋅A[3n+1:3n+3] for n=0:10^6]))
# returns 15
```

In other words, splitting the sequence into blocks of three and plotting the points in 3D space shows us a collection of 15 planes such that every point lies on one of those planes.