18 Apr

§5.5 complex eigenvalues

Example $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a $90°$ ccw rotation and therefore has no eigenvectors. But its characteristic polynomial is quadratic and therefore has two roots. What gives?

Solution The char poly is $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+1$, so its roots are $\pm\sqrt{-1} = \pm i$, where i is the imaginary unit. So while it has no real e'vals, it does have complex ones:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

complex eigenvectors

We will see that these eigenvectors do contain geometric information about how $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ acts on the plane, even though vectors of the form $\begin{bmatrix} a+bi \\ c+di \end{bmatrix}$

can't be plotted in the plane.

Example Let $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$. Find the eigenvalues & eigenvectors of A.

do w/ computer:

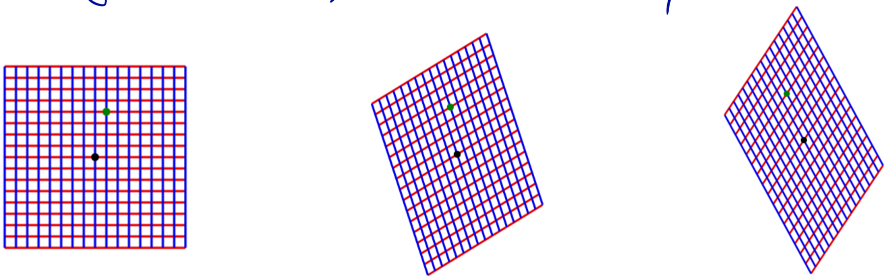
```
julia> A = [0.5 -0.6; 0.75 1.1]
2×2 Array{Float64,2}:
 0.5   -0.6
 0.75   1.1

julia> eigvals(A)
2-element Array{Complex{Float64},1}:
 0.8+0.6im
 0.8-0.6im
```

<u>Solution</u>: $\det(A-\lambda I) = 0 \Rightarrow \lambda = 0.8 \pm 0.6i$. Then solving $(A-\lambda I)\vec{v}$ for each of these λ values gives

$$\vec{v} = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix} \quad \text{for} \quad \lambda = 0.8 - 0.6i$$

$$\vec{v} = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix} \quad \text{for} \quad \lambda = 0.8 + 0.6i.$$

If we look at a movie of how A acts (e.g. in Julia) it's kind of a rotation, but not exactly:



$\text{id} \longrightarrow A$

what's going on?

Proposition If A is an $n \times n$ matrix with real entries, & $\vec{x}$ is an eigenvector w/ e'val λ, then $\overline{\vec{x}}$ is an eigenvector with eigenvalue $\overline{\lambda}$. ← bars denote complex conjugate: $\overline{a+bi} = a-bi$

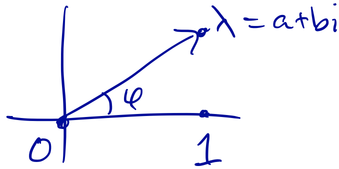
Proof

$$A\overline{\vec{x}} = \overline{A}\,\overline{\vec{x}} = \overline{A\vec{x}} = \overline{\lambda\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}.$$

check that "bars break across matrix mult."

Proposition The eigenvalues of $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ are $\lambda = a \pm bi$.

Also, $C = r\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ where $r = |\lambda| = \sqrt{a^2+b^2}$, and φ is the angle between λ, 0, and 1:



Example Let $P = [\mathrm{Re}\,\vec{v}_1 \ \ \mathrm{Im}\,\vec{v}_1]$ where $\vec{v}_1$ is $\begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ is the e'vector w/ e'val $\lambda = 0.8-0.6i$ for the matrix $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$. Show that $P^{-1}AP$ is a pure rotation matrix.

Solution $P^{-1}AP = \frac{1}{20}\begin{bmatrix} 0 & 4 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}\begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$

This is $\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ if we let $\varphi = \cos^{-1}(0.8)$. ▨

So we see that we can change basis via P^{-1}, rotate by φ in the new basis, and change basis back. This always works:

Theorem If A is a 2×2 matrix with eigenvalue $\lambda = a + bi$ with $b \neq 0$, then $A = PCP^{-1}$ where $P = [\operatorname{Re} v \;\; \operatorname{Im} v]$ & $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.