23 Mar 2017

§4.5 Dimension of a Vector space

We have already defined the terms 'basis' and 'dimension' of a vector space because we did last week's lessons a bit differently from the book. However, let's prove:

Theorem Any linearly independent list $\{\vec{v}_1, \ldots, \vec{v}_n\}$ in a finite-dimensional vector space V can be extended to form a basis for V.

Proof If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is not already a spanning list, then that means there is some vector — call it $\vec{v}_{n+1}$ — in V not in $\mathrm{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\})$. Then either $\{\vec{v}_1, \ldots, \vec{v}_{n+1}\}$ spans V, in which case we're since $\{\vec{v}_1, \ldots, \vec{v}_{n+1}\}$ is linearly independent, by the linear dependence lemma

done, or there is some $\vec{v}_{n+2} \in V$ not in $\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_{n+1}\})$. Appending that vector to our list, we get another linearly independent list, and so on. Eventually we must find a spanning list, because otherwise V would be infinite dimensional, since it would have arbitrarily long linearly independent lists. ∎

Corollary If H is a subspace of V, then $\dim H \leq \dim V$.

§4.6 Rank

The rank of a linear transformation $T: V \to W$ is the dimension of the range $T(V)$.

Example $T(p) = p'$ ← derivative, $T: \mathbb{P}_5 \to \mathbb{P}_5$. $T(\mathbb{P}_5) = \mathbb{P}_4$, so $\text{rank}\, T = \boxed{5}$

The rank of an $m \times n$ matrix A is the dimension of the column space of A.

Defn The **row space** of A is the span of A's rows. It is a subspace of $\mathbb{R}^n$.

The **row rank** of A is the dimension of the row space of A.

theorem The row rank of A equals the (column) rank of A, for all $m \times n$ matrices A.

Example: $A = \underbrace{\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}}_{\substack{\text{row rank} = 1 \\ \text{row space} = \text{Span}(\{(1\ 2\ 4)\})}}$ $\Big\}$ rank $= 1$, $\text{Col}\, A = \text{Span}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right)$

~~Proof~~ We'll do this differently from the book.

See Mathematics Magazine, 316-318, Wardlaw 2005.

Consider the <u>least</u> integer r so that

$\overset{m\times n}{A} = \overset{m\times r}{B}\,\overset{r\times n}{C}$ for B an $m\times r$ matrix and C an $r\times n$ matrix. For example,

$$\underset{3\times 4}{\begin{pmatrix} 11 & 14 & 17 & 20 \\ 23 & 30 & 37 & 44 \\ 35 & 46 & 57 & 68 \end{pmatrix}} = \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}} \underset{2\times 4}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}}$$

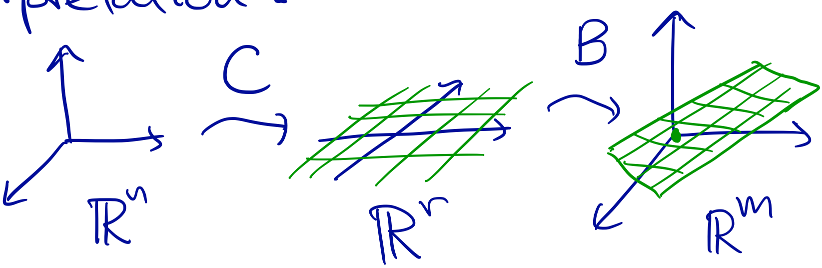
then the rows of C form a spanning list for the rowspace of A, because the rows of A are linear combinations of the rows of C, with weights given by the rows of A (this is how matrix multiplication works!). Furthermore, if there were a shorter spanning list, we could assemble those vectors into an $\overset{\text{here } r'<r}{r'}\times n$ matrix C' and write each row of A as a linear combination

of the rows of C', write those weights into an $m \times r'$ matrix B', & get $A = B'C'$ with a smaller r-value.

So the rows of C form a <u>minimal</u> spanning list of the row space of A. Thus they form a basis, and it follows that the row rank of A is r.

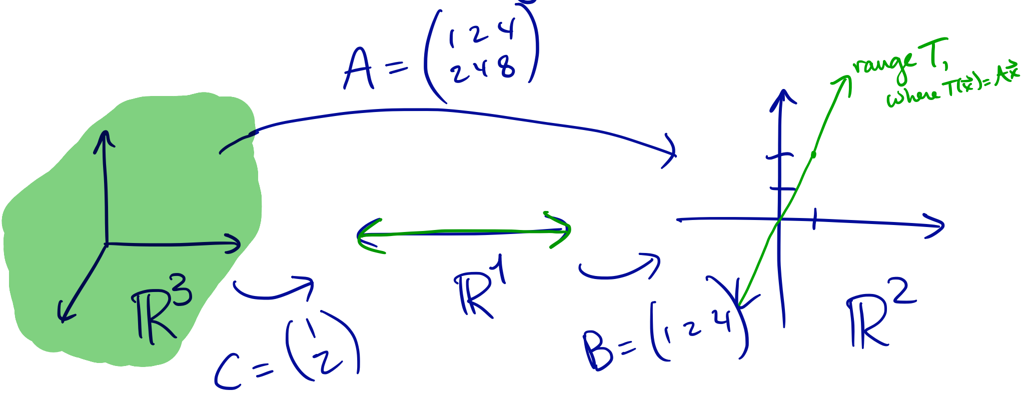
<u>But</u>, the r columns of B form a basis for $\operatorname{Col} A$, for the same reasons. So $\operatorname{rank} A = r$ too. $\blacksquare$

This proof has a natural geometric interpretation:



If $A = BC$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ has to "squeeze" through $\mathbb{R}^r$, then its rank can be no larger than r. In fact, the rank is the <u>smallest</u> dimension of Euclidean space that A can squeeze through in this way.

to prove this, note that $C\vec{x}=0 \Rightarrow BC\vec{x}=0$, so nullity $C \le$ nullity BC



Non-examinable

Alternate proof that row rank equals (column) rank:

We claim that when we remove a redundant row from A, say row k, the column rank of A does not change.

Let $T: \mathbb{R}^m \to \mathbb{R}^{m-1}$ be the linear transformation "drop the k^{th} component":

$$T(x_1, \dots, x_m) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m),$$

and consider a sublist L of the columns of A which spans the column space of A [to make L, you can go through the columns of A and discard any vector in the span of the ones before it]. Also, let $c_1, \dots, c_m$ be so that $\vec{w}_k = c_1\vec{w}_1 + \dots + c_{k-1}\vec{w}_{k-1} + c_{k+1}\vec{w}_{k+1} + c_m\vec{w}_m$

where $A = \begin{pmatrix} \vec{w}_1 \\ \vdots \\ \vec{w}_k \end{pmatrix}$ ← so, $\vec{w}$'s are the rows of A., and consider the subspace W of $\mathbb{R}^m$ defined by

$$W = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} : x_k = c_1 x_1 + \cdots + c_{k-1} x_{k-1} + c_{k+1} x_{k+1} + \cdots + c_m x_m \right\}.$$

Then Col A is a subspace of W, because the columns of A have to be in W in order for row k to be redundant. Now restricting T to W, we get that $T: W \to \mathbb{R}^{m-1}$ is injective since $T(\vec{w}) = \vec{0} \Rightarrow \vec{w} = 0$ (exercise!). The image of a linearly independent list under an injective map is linearly independent (exercise!), so $\{T(\vec{v}) : \vec{v} \in L\}$ is a basis for the column space of A with the k^{th} row removed.

the spanning list of columns for A

Similarly, deleting a redundant column does not change the row rank. So we can delete redundant rows and columns until the rows are linearly independent and the columns are linearly independent & row rank and column rank are the same as for A. If this reduced matrix is $p \times q$, then the p rows form a linearly independent list in $\mathbb{R}^q$, so $p \leq q$. By symmetry, $q \leq p$. So $p = q$! Therefore, the row rank and column rank of the reduced matrix are both p. ∎