1. (a) (8 points) Suppose that p is a real number. Find the fourth-order Taylor polynomial of $f(x) = (1+x)^p$ centered at $x = 0$. Express your answer in terms of p.

The first four derivatives of $f(x) = (1+x)^p$ are $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}$, $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$, $f^{(4)}(x) = p(p-1)(p-2)(p-3)(1+x)^{p-4}$. Evaluating these expressions at $x = 0$ and substituting into the Taylor Series formula, we get

$$P_4(x) = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \frac{p(p-1)(p-2)(p-3)}{24}x^4.$$

(b) (6 points) Use your answer to part (a) to find the fourth-order Taylor polynomial of $g(x) = \frac{1}{\sqrt{1-x^2}}$ centered at $x = 0$. (Hint: first find the Taylor series for h where $h(y) = 1/\sqrt{1-y}$ and then substitute $y = x^2$.)

We substitute into the above expression to find that

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \approx 1 + (-1/2)(-x^2) + \frac{(-1/2)(-3/2)}{2}(-x^2)^2 + \ldots = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \ldots$$

2. (10 points) Find $\int \dfrac{1}{x^2(1+x)}\,dx$

Partial fraction decomposition gives $\dfrac{1}{x^2(1+x)} = -\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{1+x}$. Integrating, we get

$$-\ln|x| - \frac{1}{x} + \ln|1+x|.$$

(b) (5 **bonus** points) Find

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{19\cdot 20}$$

(Note: your answer will be simpler than $\displaystyle\sum_{n=1}^{20} \frac{1}{n^2} = 17299975731542641/10838475198270720$!)

Using partial fractions, we get

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{19\cdot 20} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{19} - \frac{1}{20}$$
$$= 1 - \frac{1}{20}$$
$$= \frac{19}{20}.$$

3. (15 points) Consider a pentagon P with vertices $(0,0)$, (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4), listed in clockwise order. Use vectors to find the area of P. Express your answer as a quadratic polynomial in the variables $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$.

4. (a) (5 points) Let S_1 be the surface defined by the spherical-coordinate equation $\theta = \pi/4$. Let S_2 be the solid defined by the spherical-coordinate inequality $1 \leq \rho \leq 4$. Sketch the intersection of S_1 and S_2.

(b) (5 points) Write the **spherical-coordinate** equation of the infinite cylinder with unit radius whose axis is the z-axis.

In Cartesian coordinates, the equation of the cylinder is $x^2 + y^2 = 1$. Substituting $x = \rho \cos \theta \sin \phi$ and $y = \rho \sin \theta \sin \phi$, we get $\rho^2 \sin^2 \phi = 1$.

(c) (5 points) Write systems of inequalities describing each of the following solids. In each case, use whatever coordinate system is most suitable.
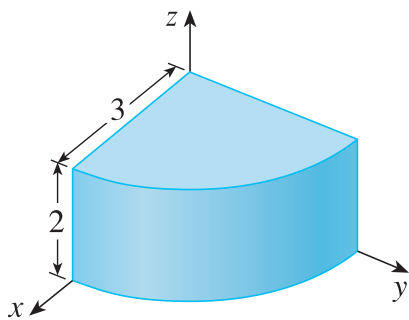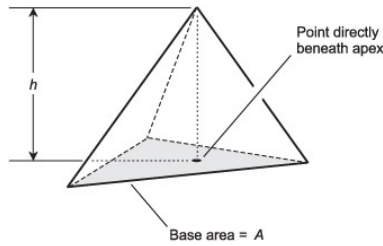

$$0 \leq r \leq 3, \quad 0 \leq \theta \leq \pi/2, \quad 0 \leq z \leq 2$$
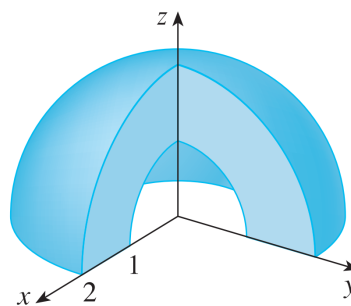
$$1 \leq \rho \leq 2, \quad \pi/2 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/2$$

5. The volume of a tetrahedron is equal to 1/3 times the area A of its base times its height h (see the figure below).

The four planes $x + y = 1$, $x + z = 1$, $y + z = 1$, and $x + y + z = 1$ divide $\mathbb{R}^3$ up into several regions, one of which is a tetrahedron T.

(a) (5 points) Find the vertices of T.

The vertices of T are the points at which three of the four planes intersect. We solve the system $x + y = 1$, $x + z = 1$, $y + z = 1$ to find the first point $(x, y, z) = (1/2, 1/2, 1/2)$. Solving the system $x + y = 1$, $x + z = 1$, $x + y + z = 1$, we get $(1, 0, 0)$. Similarly, the other two points are $(0, 1, 0)$, and $(0, 0, 1)$.

(b) (7 points) Find the area of any one the faces of T. (Note: if you are unable to solve part (a), you may solve (b) and (c) in terms of the answer to (a).)

The face with vertices $A = (1,0,0)$, $B = (0,1,0)$, and $C = (0,0,1)$ is given by

$$\frac{1}{2}|(B - A) \times (C - A)| = \sqrt{3}/2.$$

(c) (8 points) Find the distance from the area you found in part (b) to the vertex which is not contained in that face. Use your answer and the answer to (b) to find the volume of T.

To find the distance from $D = (1/2, 1/2, 1/2)$ to the plane containing A, B, and C, we define the normal vector $\mathbf{n} = (B - A) \times (C - A)$ and the angle θ between $\mathbf{n}$ and $\overrightarrow{AD}$. Applying right-triangle trigonometry to the diagram below reveals that the distance from D to the plane equals

$$
\begin{aligned}
|\overrightarrow{AD}| \cos\theta &= \frac{|\overrightarrow{AD}||\mathbf{n}| \cos\theta}{|\mathbf{n}|} = \frac{\mathbf{n} \cdot \overrightarrow{AD}}{|\mathbf{n}|} \\
&= \left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}\right) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3} \\
&= \frac{1}{2\sqrt{3}}.
\end{aligned}
$$
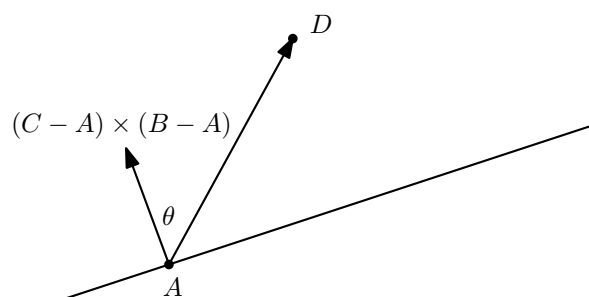
6. The matrix

$$M = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{2} & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{2} \end{pmatrix}$$

represents a three-dimensional rotation.

(a) (7 points) Based on the fact that M represents a rotation, what is det M? Explain in words.

The determinant of a linear map represents the factor by which the map transforms volumes or areas. Since rotations do not change volumes, this means that volumes transform by a factor of $\boxed{1}$ under the map M.

(b) (6 points) Confirm your answer to part (a) by directly calculating det M.

Expanding by minors along the first row, we get

$$\det M = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \cdot \frac{1}{2} - \left(-\frac{1}{2} \cdot \frac{1}{\sqrt{2}} \right) \right) + 0 \cdot (\text{some stuff}) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \cdot \frac{1}{2} - \left(-\frac{1}{2} \cdot \frac{1}{\sqrt{2}} \right) \right) = 1.$$

7. The location of particle A at time $t \in \mathbb{R}$ is given in Cartesian coordinates by (t, t^2, t^3). The location of particle B at time t is given in Cartesian coordinates by $(1 + 2t, 1 + 6t, 1 + 14t)$.

(a) (6 points) Do particles A and B collide? That is, is there a time $t \in \mathbb{R}$ at which the two particles occupy the same point in space?

The two points would collide if there exists a time t such that each coordinate of A equals the corresponding coordinate of B. Setting $t = 1 + 2t$, we find that this would have to happen at time $t = -1$. However, substituting -1 into t^2 and into $1 + 6t$, we see that they are not equal. Therefore, there is no point in time at which the two particles occupy the same position.

(b) (6 points) Do the paths of particles A and B intersect? That is, are there any points in space which are visited by both particles?

Yes. The first particle visits the point $(1, 1, 1)$ at the time $t = 1$, and the second particle visits the same point at time $t = 0$. To find this point, we could solve the system $(s, s^2, s^3) = (1 + 2t, 1 + 6t, 1 + 14t)$ for s and t—we use two different variable names to account for the possibility that the points we find are visited at different times by the two particles.
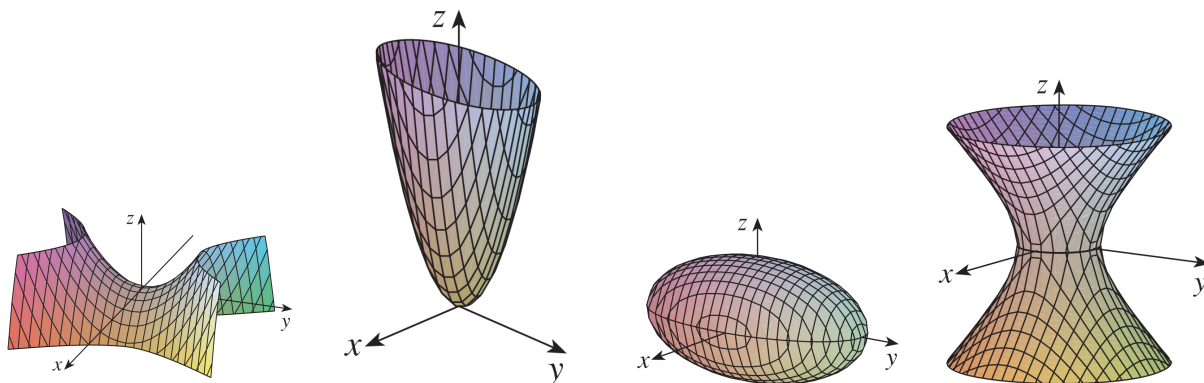
8. (a) (8 points) Matching: draw lines connecting each equation to its graph.

$$x^2 + \frac{y^2}{9} + z^2 = 1 \qquad\qquad x^2 + y^2 - z^2 = 1 \qquad\qquad z = -x^2 + y^2 \qquad\qquad z = x^2 + \frac{y^2}{4}$$

If the equations are numbered 1 through 4 left to right, then the graphs are 3, 4, 1, 2

(b) (8 points) Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the line with parametric equation $(a + t, b + t, c + 2(b - a)t)$ lies on the hyperbolic paraboloid. (Thus, even though the paraboloid is curvy, it contains lots of lines!)

If (a, b, c) lies on the paraboloid, then by definition this means that $c = b^2 - a^2$. To check whether $(a + t, b + t, c + 2(b - a)t)$ lies on the paraboloid, we $a + t$ for x, $b + t$ for y, and $c + 2(b - a)t$ for z into the equation defining the paraboloid. We get

$$c + 2(b - a)t \stackrel{?}{=} (b + t)^2 - (a + t)^2.$$

Factoring using difference of squares on the right-hand side, we get $(b + a + 2t)(b - a)$ on the right-hand side. Substituting $b^2 - a^2$ for c on the left-hand side and factoring that as $(b - a)(b + a)$, we get $(b + a + 2t)(b - a)$ for the left-hand side as well. Thus both sides are in fact equal, and that means the point $(a + t, b + t, c + 2(b - a)t)$ lies on the hyperboloid.

(Note: it is also possible to solve this problem by expanding both sides out rather than factoring.)