BROWN UNIVERSITY

Probability Math 1610
Lead instructor: Samuel S. Watson
Problem Set 4
Due: 29 October 2015 at 11:59 PM

Problem numbers refer to Grinstead & Snell.

Recommended problems (not required): 18, 19, 23 on p. 249ff, 4, 10, 14 on p. 263ff, 1, 3, 20, 23 on p. 278ff

1. (from #20 on p. 249) Consider a game in which you receive 2^N dollars, where N is the index of the first appearance of heads in a sequence of fair, independent coin flips.

(a) Show that the expected winnings of this game is $+\infty$.

(b) It seems reasonably safe to assume that if you were entitled to more than 2^{30} dollars from the result of this game, you would not be paid the full amount. Find the expected value of the game assuming that you receive only 2^{30} dollars whenever $N \geq 30$.

(c) Assume that your utility for n dollars is $\sqrt{n}$. Find the expected utility for this game.

2. (#21 on p. 250) Find the expected value of a Poiss(λ) random variable. (Hint: there are two approaches: an elementary one that involves re-indexing the sum, and a second which involves differentiating a Taylor series.)

3. (from #22 on p. 267) Show that if X, Y, $X + Y$, and $X - Y$ are random variables with the same distribution, then $P(X = Y = 0) = 1$. (Note: your book includes an assumption of discreteness, but the problem can be solved with only fundamental properties of expectation and variance, so I prefer to state the problem with no assumptions on X and Y.)

4. (#24 on p. 267) A professor wishes to make up a true-false exam with n questions. She assumes that she can design the problems in such a way that a student will answer the jth problem correctly with probability p_j, and that the answers to the various problems may be considered independent experiments. Let S_n be the number of problems that a student will get correct. The professor wishes to choose p_j so that $E(S_n) = 0.7n$ and so that the variance of S_n is as large as possible. Show that, to achieve this, she should choose $p_j = 0.7$ for all j; that is, she should make all the problems have the same difficulty.

5. (see #26 on p. 267) Consider successive flips of a coin which has probability $p > 0$ of turning up heads. Let $n \geq 1$ be an integer, and find the expected value and variance of the index of the flip on which the nth head appears.

6. (#7 on p. 279) Let X be a random variable with density function f_X . Show, using elementary calculus, that the function $\phi(a) = E((X - a)^2)$ takes its minimum value when $a = \mu(X)$, and in that case $\phi(a) = \sigma^2(X)$.

7. (from #12 on p. 280) Find $E(X^Y)$ where X and Y are independent Unif$([0, 1])$ random variables.

8. (from #17 on p. 281) Let X and Y be random variables. The covariance cov(X, Y) is defined by

$$\text{cov}(X, Y) = E((X - \mu(X))(Y - \mu(Y)))$$

(a) Show that $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$.

(b) Show that $\text{cov}(X, Y) = 0$, if X and Y are independent. (Caution: the converse is not always true.)

(c) Show that $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(X,Y)$.

9. (from #18 on p. 281) Let X and Y be random variables with positive variance. The correlation of X and Y is defined as

$$\rho(X,Y) = \mathrm{cov}(X,Y)/\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}.$$

(a) Show that

$$0 \le \mathrm{Var}(X/\sigma(X) + Y/\sigma(Y)) = 2(1+\rho(X,Y)).$$

(b) Show that

$$0 \le \mathrm{Var}(X/\sigma(X) - Y/\sigma(Y)) = 2(1-\rho(X,Y)).$$

(c) Using (a) and (b), show that

$$-1 \le \rho(X,Y) \le 1.$$

10. (#22 on p. 282) A point Y is chosen at random from $[0,1]$. A second point X is then chosen from the interval $[0,Y]$. Find the density of X.