MATH 19 NOTES
FALL 2016
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Please do not hesitate to contact me about any mistakes you find in these notes. Some of the exercises herein are adapted from Gilbert Strang's Calculus. *This document is hyperlinked, meaning that references to examples, theorems, etc. can be clicked for convenient navigation within the pdf. This file also includes a table of contents in its metadata, accessible in most pdf viewers.*

1 Integration by parts

07 September

Many integration techniques may be viewed as the inverse of some differentiation rule. For example, substitution is the integration counterpart of the chain rule:

$$\frac{d}{dx}[e^{5x}] = 5e^{5x} \qquad \text{Substitution: } \int 5e^{5x}\,dx \overset{u=5x}{=} \int e^u\,du = e^{5x} + C.$$

Integration by parts is the reverse of the product rule. The situation is somewhat more complicated than substitution because the product rule increases the number of terms. Let's consider an example.

Example 1.1

Find $\int xe^{2x}\,dx$

Solution

This kind of expression tends to show up when differentiating functions like xe^{2x}. Let's investigate:

$$\frac{d}{dx}[xe^{2x}] = (x)'e^{2x} + x(e^{2x})' = e^{2x} + 2xe^{2x}.$$

Integrating both sides gives $xe^{2x} = \int e^{2x}\,dx + \int 2xe^{2x}\,dx$. This means that $xe^{2x} = 2\int xe^{2x}\,dx + \frac{1}{2}e^{2x}$, so $\int xe^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

This solution required a stroke of luck, namely our ability to recall how the integrand tends to show up in differentiation problems. Let's come up with a technique. The product rule says

$$(fg)' = f'g + fg'.$$

Integrating both sides tells us that

$$fg = \int f'g + \int fg'.$$

Solving for $\int f'g$, we obtain

Theorem 1.1: Integration by parts

$$\int f'g = fg - \int fg'$$

In other words, when integrating $f'g$, we may "pass the prime" to g at the cost of a factor of -1 and an extra term fg. This is called **integration by parts**

Since our original integrand will seldom have a prime already in it, we will need to introduce one by writing one factor as the derivative of something.

Example 1.2

Find $\int x \cos x \, dx$.

Solution

We would like to pass the prime onto x, since that would yield 1, which is simpler than x. So we write $\cos(x) = (\sin(x))'$ to get

$$\int x \cos x \, dx = \int x (\sin x)' \, dx = x \sin x - \int x' \sin x \, dx = x \sin x + \cos x + C.$$

Example 1.3

Find $\int \ln x \, dx$.

Solution

This integrand only has one factor, which makes it harder to recognize as an integration by parts problem. However, we can always write an expression as 1 times itself, and in this case that is helpful:

$$\int 1 \cdot \ln x \, dx = \int x' \cdot \ln x \, dx = x \ln x - \int x \left(\tfrac{1}{x}\right) \, dx = x \ln x - x + C.$$

09 September

Example 1.4

Evaluate $\int e^x \sin x \, dx$.

Solution

Neither factor simplifies much when differentiated, but e^x is easier to integrate, so let's try passing the prime from it onto $\sin x$:

$$\int e^x \sin x \, dx = \int (e^x)' \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

It isn't clear we've made much progress at this point. This integral looks just as difficult as the original one, and if we do it again we'll get $e^x \sin x$ again, which feels like we're right back where we've started. Amazingly, however, this actually works:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left[e^x \cos x - \int e^x (-\sin x) \, dx \right].$$

You'll notice that if we collect the two $\int e^x \sin x\, dx$ terms, they don't cancel! So we can solve this equation for $\int e^x \sin x\, dx$ to find $\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.

Here's a helpful mnemonic for an order of priority for which factor the derivative should be passed to.

I	inverse trig
L	logarithms
A	algebraic ($\sqrt{x}, x^2$, etc.)
T	trig
E	exponential

Example 1.5

Find $\int_0^1 \arctan x\, dx$.

Solution

Seeing an inverse trig function, we pass the prime to it:

$$\int_0^1 (x)' \arctan x\, dx = x \arctan x \Big]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}\, dx = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Note that we combined the fundamental theorem of calculus with integration by parts here; the effect is to evaluate the fg term at the two endpoints and subtract. For this reason, the fg term is sometimes called the **boundary term**.

Example 1.6

Find the area under the graph of $f(x) = x^2 \sin x$ over the interval $[0, \pi]$

Solution

Applying integration by parts, we get

$$\begin{aligned}
\int_0^\pi x^2 \sin x\, dx &= \int_0^\pi x^2(-\cos x)'\, dx \\
&= (x^2)(-\cos x)\Big]_0^\pi - \int_0^\pi (2x)(-\cos x)\, dx \\
&= \pi^2 + 2\int_0^\pi x \cos x\, dx.
\end{aligned}$$

We need to apply integration by parts again to get

$$\int_0^\pi x^2 \sin x\, dx = \pi^2 + 2x \sin x\Big]_0^\pi - 2\int_0^\pi \sin x\, dx = \pi^2 + 0 - 2(2) = \pi^2 - 4.$$

Exercise 1.7

Find the following integrals.

(a) $\int x \arcsin x\, dx$

(b) $\int_0^1 x^2 e^x\, dx$

(c) $\int t(t+6)^{21}\, dt$

(d) $\int \frac{\ln x}{x^5}\, dx$

(e) $\int \theta \sin\theta \cos\theta\, d\theta$

(f) $\int e^{6s} \sin e^{3s}\, ds$

2 Resonance integrals

12 September

We call these functions **basic waves**

In this section, we'll learn to integrate products of functions*

$$\sin x, \sin 2x, \sin 3x, \ldots, \cos x, \cos 2x, \cos 3x, \ldots$$

over the interval $[0, 2\pi]$. These functions and their integrals are the foundation of Fourier analysis, which we will explore later in the course.

Interlude 1 (Trig Review)

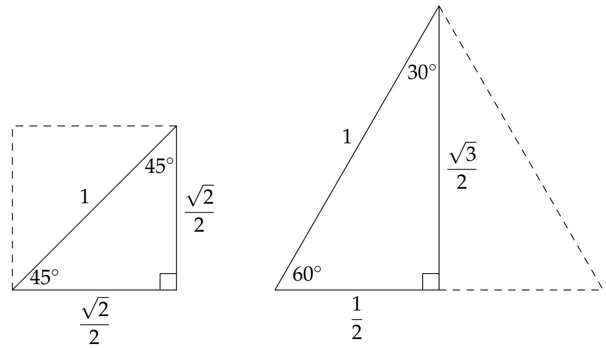
Trig classes give you a lot of random stuff which is hard to remember. This section will include a streamlined presentation of trig which is intended to provide enough starting off points to recover everything else you need.

1. **Sine and cosine**. The basic trig functions are $\cos\theta$ and $\sin\theta$. The most important definition of these functions is the following: the cosine of an angle θ is equal to the **x-coordinate** of the point on the unit circle corresponding to the angle θ. Sine is the same, but with the y-coordinate instead of x.

2. **The other ones**. The other four trig functions are simply abbreviations for various combinations of sine and cosine:

$$\begin{aligned} \sin\theta &= \sin\theta & \sec\theta &= \tfrac{1}{\cos\theta} \\ \cos\theta &= \cos\theta & \csc\theta &= \tfrac{1}{\sin\theta} \\ \tan\theta &= \tfrac{\sin\theta}{\cos\theta} & \cot\theta &= \tfrac{\cos\theta}{\sin\theta} \end{aligned}$$

3. **Special right triangles**. The following two triangles, each half of a regular polygon, can be hand for evaluating trig functions at special angles.



4. **Pythagorean identities** The famous identity $\sin^2\theta + \cos^2\theta = 1$ follows from the definition of sine and cosine combined with the Pythagorean theorem. Dividing both sides of this equation by $\sin^2\theta$ or $\cos^2\theta$, we get

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{and} \qquad 1 + \cot^2\theta = \csc^2\theta.$$

5. **Sum-angle formulas**. The sine sum-angle formula is worth memorizing: for all α and β,

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha}$$

The cosine sum-angle formula is worth memorizing too, although it can be derived fairly easily from the sine formula by substituting $\frac{\pi}{2} - \alpha$ for α and $-\beta$ for β. We get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

From these two, we can derive many other identities. For example, setting $\alpha = \beta$ in the cosine sum-angle formula, we get

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$$

Substituting $\cos^2\alpha = 1 - \sin^2\alpha$, we find that

$$\cos 2\alpha = 1 - 2\sin^2\alpha.$$

which can be solved to express $\sin^2\alpha$ in terms of $\cos 2\alpha$.

Example 2.2

Suppose that $p > 0$ and $q > 0$ are integers. Find $\displaystyle\int_0^{2\pi} \sin px \cos qx \, dx$.

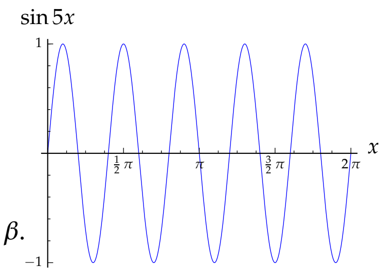
Solution

We're going to use a trig identity which is not in the review section above, so we'll derive it. Recall the sine sum-angle formula:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

The first term on the right-hand side looks helpful. To get rid of the second term on the right-hand side, we substitute $-\beta$ for β to get the difference-angle formula

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\end{aligned}$$



If we add these equations, the $\cos\alpha\sin\beta$ terms cancel to give

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta.$$

We can use this identity in the original integral to get

$$\int_0^{2\pi}\sin px\cos qx\,dx=\frac{1}{2}\int_0^{2\pi}\sin(p+q)x+\sin(p-q)x\,dx.$$

This integral equals $\boxed{0}$ if $p\neq q$, since both terms of the integrand are periodic and integrate to zero over each interval. The integral is also equal to 0 if $p=q$, since $\sin(p-q)x=0$ in that case.

Example 2.3

Suppose that $p>0$ and $q>0$ are integers. Find $\int_0^{2\pi}\cos px\cos qx\,dx$

Solution

We can do this one similarly to the last one. We work with the cosine sum-angle formula instead of the product sum-angle formula to figure out that

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$$

So we get

$$\int_0^{2\pi}\cos px\cos qx\,dx=\frac{1}{2}\int_0^{2\pi}\cos(p+q)x+\cos(p-q)x\,dx.$$

Once again, this integral evaluates to 0 if $p\neq q$ since cosine integrates to zero over each period, as in the preceding example. If $p=q$, then $\cos(p-q)x=1$. Therefore,

$$\int_0^{2\pi}\cos px\cos qx\,dx=\begin{cases}\pi & \text{if } p=q\\ 0 & \text{if } p\neq q.\end{cases}$$

The following example shows why these integrals are useful.

Example 2.4

Suppose that $f(x)=A\sin x+B\sin 2x+C\cos 3x$ for some constants A, B, and C. Use the previous two exercises to show how to express C in terms of f.

Solution

In this section, we have seen that multiplying basic waves by other basic waves and integrating over

$[0, 2\pi]$ tends to give zero. The only time it doesn't give zero is when the two functions match. So let's multiply both sides of $f(x) = A\sin x + B\sin 2x + C\cos 3x$ by $\cos(3x)$ and integrate to get

$$\int_0^{2\pi} f(x)\cos 3x\,dx = A\int_0^{2\pi} \sin x\cos 3x\,dx + B\int_0^{2\pi} \sin 2x\cos 3x\,dx + C\int_0^{2\pi} \cos^2 3x\,dx.$$

The first two terms are zero, while the third works out to πC, by Examples 2.2 and 2.3. Therefore,

$$\int_0^{2\pi} f(x)\cos 3x\,dx = \pi C,$$

and $C = \boxed{\frac{1}{\pi}\int_0^{2\pi} f(x)\cos 3x\,dx}$.

The idea of the title of this section is that the $\cos 3x$ wave "resonates" with the $\cos 3x$ term in the definition of f and creates cancellation with the other terms, allowing us to "pick out" just the $\cos 3x$ term.

Computational Investigation 1

Suppose we run*

```
f(x) = random()*sin(x) + random()*sin(2*x)
```

which defines a function `f` to be a random constant times $\sin x$ plus a random constant times $\sin(2x)$. Suppose the computer system no longer directly stores these two random coefficients, but it knows what `f` is an how to do stuff with it (like integrate*). By the previous example, we can run

```
(1/pi)*integrate(f(x)*sin(x),x,0,2*pi)
```

which integrates `f` times $\sin x$ over the interval from 0 to 2π. The result of this calculation will be equal to the coefficient of $\sin x$. Similarly,

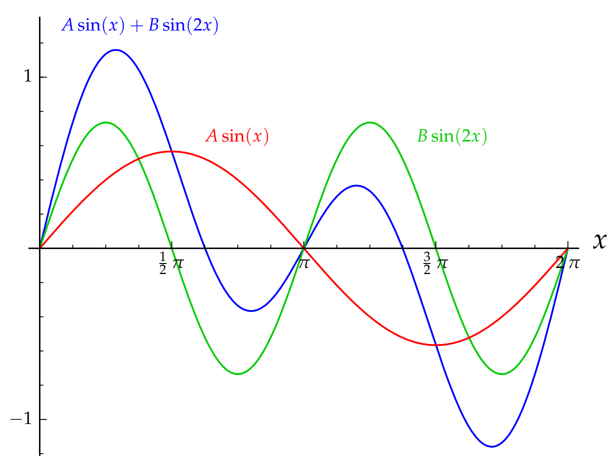
```
(1/pi)*integrate(f(x)*sin(2*x),x,0,2*pi)
```

will give us the second coefficient.

The SageMath code in this course can be run at sagecell.sagemath.org

Suppose the values of f at 1000 equally-spaced points from 0 to 2π are stored. Then the coefficients aren't visible, but we can approximate integrals involving f using the trapezoid rule.

The following figure illustrates the concept of resonance integrals: they allow us to break a combination of basic waves (shown in blue) down into its constituent basic waves (red and green)

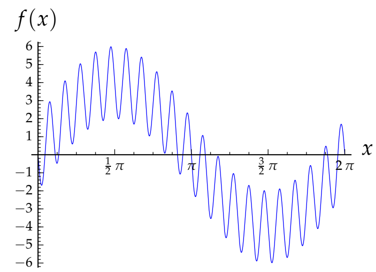


Exercise 2.5

(a) Find $\int_0^{2\pi} 3\sin x\cos x - 2\sin^2 x + 11\sin x - (\cos x - \sin x)^2\,dx$ without doing much work.

Exercise 2.6

(a) The graph of the function $f(x) = A\sin x + B\sin 20x$ is shown to the right, where A and B are integers. Find A and B by using trial and error and a computer algebra system. Here's an example to get you started.




```
plot(-1*sin(x)+3*sin(20*x),x,0,2*pi)
```

How could you have estimated A and B directly from the graph, without trial and error?

3 Trig integrals

14 September

In this section, we learn how to do integrals of products of $\sin x$ and $\cos x$. The basic idea is to try $u = \sin x$ or $u = \cos x$, using $\sin^2 x + \cos^2 x = 1$ to convert even powers of sine to cosines and vice versa. Let's start with an example.

Example 3.1

Find $\displaystyle\int \cos^7 x \sin^4 x\,dx$.

Solution

If we make a substitution with $u = \sin x$, then we'd have $du = \cos x\,dx$. So we split off one factor of $\cos x$ to put with the dx, and we can use $\cos^2 x = 1 - \sin^2 x$ to convert the rest of the cosines to sines. So we get

$$\int \cos^7 x \sin^4 x\,dx = \int (\cos^2 x)^3 \sin^4 x(\cos x\,dx) = \int (1-\sin^2 x)^3 \sin^4 x(\cos x\,dx) = \int (1-u^2)^3 u^4\,du.$$

From here it's a matter of cubing out the binomial and integrating:

$$\int (1-3u^2+3u^4-u^6)u^4\,du = \int u^4 - 3u^6 + 3u^8 - u^{10}\,du = \frac{u^5}{5} - \frac{3}{7}u^7 + \frac{3}{9}u^9 - \frac{1}{11}u^{11} + C.$$

So the desired anti-derivative is $\boxed{\frac{1}{5}\sin^5 x - \frac{3}{7}\sin^7 x + \frac{1}{3}\sin^9 x - \frac{1}{11}\sin^{11} x + C}$.

Observe that this trick will work to integrate $\sin^m x \cos^n x$ whenever m or n is odd: split off one factor of sine or cosine (whichever one is odd), and let u be the other one (in other words, $\cos x$ or $\sin x$, respectively). So it remains to handle the case where m and n are both even.

Example 3.2

Find $\int \cos^2 x \, dx$.

Solution

Substitution won't work in this case. However, recall from the trig review section that

$$\cos 2x = 2\cos^2 x - 1.$$

Solving this equation for $\cos^2 x$, we get

$$\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx.$$

This integral can be solved by substituting for $2x$; we get

$$\int \cos^2 x \, dx = \boxed{\frac{x}{2} + \frac{\sin 2x}{4} + C}.$$

With enough patience, this method works for all integrals of the form $\int \sin^m x \cos^n x \, dx$ where m and n are even, since we can convert all the sines to cosines (or vice versa) and substitute $\frac{1 \pm \cos 2x}{2}$ for $\cos^2 x$ or $\frac{1-\cos 2x}{2}$ for $\sin^2 x$ to reduce the exponent.

Example 3.3

Find $\int \sec^4 x \tan^2 x \, dx$.

Solution

Recall that $(\sec x)' = \sec x \tan x$, and $(\tan x)' = \sec^2 x$. The substitution $u = \tan x$ gives $du = \sec^2 x \, dx$. Reserving a factor of $\sec^2 x$ for du leaves behind $\sec^2 x \tan^2 x$, which we can rewrite so that it only involves $\tan x$ by using the relation $\sec^2 x = 1 + \tan^2 x$. We get

$$\begin{aligned}
\int \sec^2 x \tan^2 x (\sec^2 x \, dx) &= \int (1+\tan^2 x) \tan^2 x (\sec^2 x \, dx) \\
&= \int (\tan^2 x + \tan^4 x)(\sec^2 x \, dx) \\
&= \boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}.
\end{aligned}$$

In the last step we made the substitution $u = \tan x$, but without explicitly putting the u in.

The approach of Example 3.3 works whenever the power of secant is even*. Substituting for $\sec x$ works whenever the power of $\tan x$ is odd:

...unless the power of secant is zero, in which case you can convert all or all but one of the tangents to secants using $\tan^2 x = \sec^2 x - 1$.

Example 3.4

Find $\int \sec^3 x \tan^3 x \, dx$.

Solution

The substitution $u = \sec x$ gives $du = \sec x \tan x \, dx$:

$$\begin{aligned}\int \sec^3 x \tan^3 x \, dx &= \int \sec^2 x \tan^2 x (\sec x \tan x \, dx) \\ &= \int \sec^2 x (\sec^2 x - 1)(\sec x \tan x \, dx) \\ &= \int (\sec^4 x - \sec^2 x)(\sec x \tan x \, dx) \\ &= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}.\end{aligned}$$

We'll omit the remaining case (secant to an odd power times tangent to an even power).

Exercise 3.5

Evaluate the following integrals.

(a) $\int \sin^2 x \cos^2 x \, dx$

(b) $\int \sin x \cos^2 x \, dx$

(c) $\int \sec \theta \tan^3 \theta \, d\theta$

(d) $\int \sec^{10} y \tan^3 y \, dy$

4 Trig substitution

16 September

It is often possible to simplify an integral by making a substitution involving a trig function, even when the original integrand doesn't involve any trig functions. You can recognize these situations when you see *Pythagorean combinations* of variables and constants. For example, the hypotenuse of a right triangle with side lengths x and 4 is $\sqrt{x^2 + 16}$; we call this expression a Pythagorean combination of x and 4.

To figure out the appropriate substitution, we draw the triangle suggested by the Pythagorean combination, and we express the variable in terms of one of the angles of the right triangle. Let's do an example.

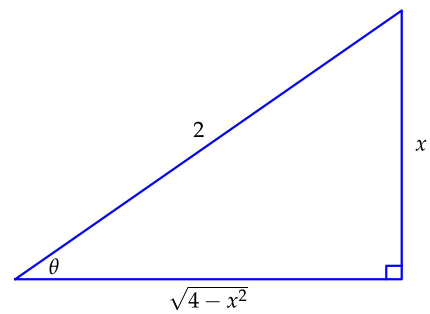
Example 4.1

Find $\int_0^1 \sqrt{4-x^2}\,dx$.

Solution

We begin by sketching a right triangle involving 2, x, and $\sqrt{4-x^2}$ as shown. In this figure, we have $x = 2\sin\theta$, so we make that substitution:

$$\int_0^1 \sqrt{4-x^2}\,dx = \int_0^{\pi/6} \sqrt{4-(2\sin\theta)^2}(2\cos\theta\,d\theta).$$



Note that we substituted for the limits of integration as well. We can simplify the radial expression using $1-\sin^2\theta = \cos^2\theta$ to get (remember that $\sqrt{A^2} = |A|$, not A!)

$$\cdots = 4\int_0^{\pi/6} |\cos\theta|\cos\theta\,d\theta.$$

Since $\cos\theta$ is positive over the interval of θ values we're integrating over, we get

$$\cdots = 4\int_0^{\pi/6} \cos^2\theta\,d\theta.$$

Substituting $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ gives

$$\cdots = 2\int_0^{\pi/6} 1+\cos 2\theta\,d\theta = \boxed{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}.$$

Example 4.2

Find $\int \frac{1}{\sqrt{x^2+9}}\,dx$.

Solution

The triangle with legs 3 and x and hypotenuse $\sqrt{x^2+9}$ suggests the substitution $x = 3\tan\theta$. We get

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{\sqrt{9+(3\tan\theta)^2}}(3\sec^2\theta\,d\theta) = \int \sec\theta\,d\theta.$$

This integral is $\ln|\sec\theta+\tan\theta|+C$, which after substituting becomes*

$$\ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|+C.$$

This answer remains correct if you replace the 3's in the denominators with 1's. Why?

Exercise 4.3

Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^3}{(1-x^2)^3}\,dx.$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{(1+x^2)^2}\,dx$

(c) $\displaystyle\int \frac{x^3}{(1-x^2)^3}\,dx.$

(d) $\displaystyle\int_{\sqrt{2}}^{2} \frac{(x^2-2)^{3/2}}{x}\,dx$

5 Force, work, and energy

If a force applied to an object displaces that object in the direction of the force, then the force is said to be doing **work** on that object. The amount of work done is equal to the product of the magnitude of the force (measured in Newtons) times the distance moved in the direction of the force (measured in meters). Since work is force times distance, the unit of work is the Newton-meter, also known as the *joule*.

Example 5.1

Hooke's law states that the force applied by a spring is equal to $-kx$, where k is a constant depending on the spring (called *stiffness*), and x is the displacement (the distance of stretching or compression) of the spring from its resting state. Calculate the amount of work required to stretch a spring with stiffness k from rest to a total displacement of x_0.

Solution

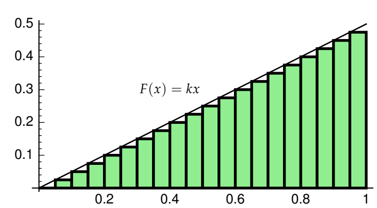
We can't solve this problem by simply multiplying force times distance, because the force changes continuously as we stretch the spring. What we can do is imagine first stretching the spring from 0 to $x_0/100$, then from $x_0/100$ to $2x_0/100$, and so on. Over each of these very short intervals, the force will be approximately constant, and then we can add up the amount of work corresponding to each small step to get the total work.

Let $F(x) = kx$ be the amount of force we have to exert to counter the force of the spring at displacement x. Then the work required to move the spring from rest to $\frac{x_0}{100}$ is approximately* $F(0)\cdot(\frac{x_0}{100})$.

We approximate F using its value at the left endpoint 0 of the interval $[0, \frac{x_0}{100}]$, but it would also have been fine to take the midpoint or the right endpoint.

The force required to move from $\frac{x_0}{100}$ to $2\frac{x_0}{100}$ is approximately $F(\frac{x_0}{100})\cdot\frac{x_0}{100}$, and so on. Altogether, the total force is approximately

$$F(0\cdot\tfrac{x_0}{100})\cdot\tfrac{x_0}{100} + F(1\cdot\tfrac{x_0}{100})\cdot\tfrac{x_0}{100} + \cdots + F(99\cdot\tfrac{x_0}{100})\cdot\tfrac{x_0}{100}. \tag{5.1}$$



Before we try to add this up, we should reflect on the fact that our choice of 100 was arbitrary—our answer would be more accurate if we took 1000 or 10,000 instead. In fact, if we send this number to ∞, we recognize (5.1) as a Riemann sum converging to

$$\int_0^{x_0} F(x)\,dx.$$

In other words, we can interpret the terms in (5.1) as the areas of small rectangles which can be assembled to fit nicely just under the graph of F, as shown in the figure.

Finally, we calculate the integral

$$\int_0^{x_0} kx\,dx = \boxed{\tfrac{1}{2}kx_0^2}.$$

19 September

When work is done to overcome a force, like gravity or a spring, we refer to the amount of work that the force has the potential to do as **potential energy**. By this definition, the amount of work it takes to move an object from one location to another in the presence of a force is equal to the difference in the potential energy corresponding to those two locations. Furthermore, Example 5.1 shows that the amount of work it takes to move an object from a to b is equal to

$$\boxed{W = \int_a^b F(x)\,dx,}$$

where $F(x)$ is the force the object is subjected to when it's at location x. Let's do another example.

Example 5.2

Calculate the potential energy of a 10 kg object which is 100 m above the surface of the earth. Recall that the force exerted on an object by earth is equal to $-\dfrac{GMm}{x^2}$, where G is a universal constant, M is mass of the earth, m is the mass of the object, and x is the distance from the object to the center of the earth.

Solution

We integrate

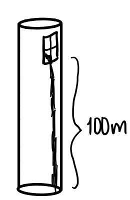
$$\int_R^{R+100} F(x)\,dx = \int_R^{R+100} \frac{GMm}{x^2}dx = -\left.\frac{GMm}{x}\right]_R^{R+100} = GMm\left(\frac{1}{R} - \frac{1}{R+100}\right)$$

If we simplify (and approximate) the expression in parentheses, we get $\dfrac{100}{R(R+100)} \approx \dfrac{100}{R^2}$. So the potential energy is approximately $\dfrac{GMm(100)}{R^2}$. For earth*, $g := \dfrac{GM}{R^2} \approx 9.8$ meters per second squared, so the potential energy is about $g(100)(10) = 9800$ joules.

:= means "is defined to be".

Example 5.3

How much work does it take Rapunzel to lift her hair, which has a density of 500 g per meter, into her 100-meter tall castle?

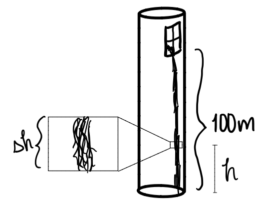


Solution

Each segment of hair requires a different amount of work to lift: hair closer to the ground takes more work because it has to be lifted farther. If n is a large integer, then $\Delta h = \frac{100}{n}$ is small, and the hair between height h and $h + \Delta h$ has mass $\left(0.5\,\frac{\text{kg}}{\text{m}}\right)(\Delta h\ \text{m}) = 0.5\Delta h$ kilograms and takes approximately

$$(0.5\Delta h)(g)(100 - h)$$

joules of work to lift, by Example 5.2.



So the total amount of work is approximately

$$\left(100 - \frac{100}{n}\right)\frac{g\Delta h}{2} + \left(100 - 2\cdot\frac{100}{n}\right)\frac{g\Delta h}{2} + \ldots + \left(100 - n\cdot\frac{100}{n}\right)\frac{g\Delta h}{2}. \tag{5.2}$$

As $n \to \infty$, this expression converges to $\int_0^{100} \frac{g}{2}(100 - h)\,dh$. We evaluate

$$\int_0^{100} \frac{g}{2}\,(100 - h)\,dh = -\frac{g}{4}\,(100 - h)^2\Big]_0^{100} = 24{,}500 \text{ joules.}$$

Once you get the hang of physics or geometry problems requiring calculus, you'll be able to skip the somewhat tedious step (5.2) and jump straight from "the sum of expressions of the form $\frac{1}{2}(100 - h)\frac{g\Delta h}{2}$" to "the integral of $\frac{1}{2}g(100 - h)\,dh$ over the appropriate range of h values". Replacing a sum with an integral and Δ with d is a common pattern in applications of calculus.

Exercise 5.4

It takes 20 joules of work to stretch a spring two meters. How much work will it take to stretch it one more meter?

Exercise 5.5

Suppose the great pyramid has base 800 feet by 800 feet and height 500 feet, and suppose the rock weighs 100 pounds per cubic foot. Find the cross-sectional area of the pyramid at height h. How much work did it take to lift all the rock?

Exercise 5.6

A rocket burns 100 kg of fuel at a constant rate and reaches a height of 25 km. How much work does it take to lift the fuel?

6 Arclength

Example 6.1

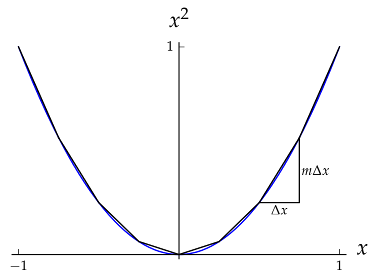
How long is the portion of the parabola $y = x^2$ between $(-1, 1)$ and $(1, 1)$?

Solution

We approximate the parabola as a union of short line segments, as shown.

The length of any particular line segment can be written in terms of the horizontal displacement Δx between its endpoints and the slope m: the Pythagorean theorem gives

$$\sqrt{(\Delta x)^2 + (m\Delta x)^2} = \Delta x\sqrt{1+m^2}.$$



So the total length of the parabola is approximately equal to a sum of expressions of the form $\Delta x\sqrt{1+m^2}$, where Δx is the horizontal displacement and m is the slope of each line segment. When the number of line segments is very large, then the slope of each segment over $[x, x+\Delta x]$ is well approximated by the derivative $f'(x)$. As we let the number of segments go to ∞, we recognize the sum of expressions of the form $\Delta x\sqrt{1+m^2}$ as a Riemann sum converging to

$$\int_{-1}^{1} \sqrt{1+f'(x)^2}\, dx = \int_{-1}^{1} \sqrt{1+4x^2}\, dx.$$

To solve this integral, we use a trig sub $x = \frac{1}{2}\tan\theta$, which gives

$$\cdots = \int_{-\arctan 2}^{\arctan 2} \sqrt{1+\tan^2\theta}\,\sec^2\theta\, d\theta = \frac{1}{2}\int_{-\arctan 2}^{\arctan 2} \sec^3\theta\, d\theta.$$

We can say $|\sec\theta| = \sec\theta$ because arctan takes values in $(-\pi/2, \pi/2)$, where sec is positive.

This doesn't fall into the category we solved in the trig integrals section, so we'll have to do it by hand. Integration by parts works:

$$\int \sec^3\theta = \int (\sec\theta)(\sec^2\theta)\, d\theta = \int \sec\theta(\tan\theta)'\, d\theta.$$

Applying integration by parts gives

$$\cdots = \sec\theta\tan\theta - \int (\sec\theta)'\tan\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta.$$

Applying a Pythagorean identity:

$$\cdots = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta + \int \sec\theta\, d\theta.$$

Solving for $\int \sec^3\theta$ gives

$$\int \sec^3\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|.$$

Substituting $-\arctan 2$ and $\arctan 2$ for θ and subtracting gives

$$\int_{-\arctan 2}^{\arctan 2} \sec^3\theta \, d\theta = \frac{1}{4}(2\sqrt{5} + \ln|\sqrt{5}+2|) - \frac{1}{4}(-2\sqrt{5} + \ln|\sqrt{5}-2|) = \sqrt{5} + \ln\left(\frac{\sqrt{5}+2}{\sqrt{5}-2}\right).$$

One handy way to evaluate these trig functions is to sketch a 1-2-$\sqrt{5}$ triangle to find the absolute values of $\tan\theta$ and $\sec\theta$, and sketch graphs of $\sec\theta$ and $\tan\theta$ to think through what the signs are.

Let's generalize what we learned in this example:

Theorem 6.1

The arc length of the graph of a differentiable function f over the interval $[a, b]$ is equal to

$$\int_a^b \sqrt{1 + f'(x)^2} \, dx.$$

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Exercise 6.2

(a) Find the arc length of the graph of the function $f(x) = \ln\sec x$ over $[0, \pi/4]$.

(b) Determine the length of the graph of $f(x) = \frac{2}{3}(x-1)^{3/2}$ over the interval from 1 to 4.

7 Polar coordinates

We begin by defining the functions

$$r(x, y) = \text{distance from the origin to } (x, y)$$

and, for all (x, y) other than the origin,

$$\theta(x, y) = \text{the angle in } [0, 2\pi) \text{ between the positive } x\text{-axis and the line segment from the origin to } (x, y).$$

The function r is defined on the whole plane, and θ is defined everywhere in the plane except the origin. The key fact about these two functions is that if you know the values of r and θ at a point, you know exactly where the point is: rotate an angle θ from the positive x-axis and walk r units in that direction. In equation form, we can recover x and y from r and θ using*

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

This follows from the definitions of sine and cosine given in the trig interlude.

Since idea of coordinates in the plane is "a pair of numbers whose values specify a location", it is reasonable to describe the pair (r, θ) as a coordinate system. It is called the **polar** coordinate system.

You might not be used to thinking of coordinates as functions on the plane, but consider the x-coordinate of a point P is given by

$$x = \text{the signed distance from } P \text{ to the } y\text{-axis.}$$

and the y-coordinate by

$$y = \text{the signed distance from } P \text{ to the } x\text{-axis.}$$

In this way, polar coordinates are entirely analogous to Cartesian coordinates. However, there are some differences. First, the coordinate θ is not defined at the origin, whereas x and y are both defined everywhere. Second, it is often convenient to let θ range over $(-\infty, \infty)$ rather than just $[0, 2\pi)$. If this is permitted, then each point other than the origin is represented by infinitely many coordinate pairs: (r, θ), $(r, \theta + 2\pi)$, $(r, \theta + 4\pi)$, ... all represent the same point.

Example 7.1

Find all the points in the plane whose r and θ values satisfy* $r = \cos\theta$.

It doesn't matter which value of θ we use to represent a point in this equation, because we get the same value of $\cos\theta$ for all of them.

Solution

Multiplying both sides by r, we get $r^2 = r\cos\theta$. We know that $x^2 + y^2 = r^2$ from the definition of r, and we also know $x = r\cos\theta$. So converting our equation to Cartesian coordinates, we get

$$x^2 + y^2 = x.$$

Converting to standard form of a circle by completing the square, we get

$$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \implies \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$$

So the graph of the polar equation $r = \cos\theta$ is a circle of radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, 0\right)$.

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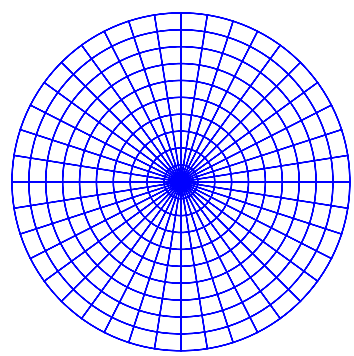
Example 7.2

Find the area of the region in the plane whose r value lies between r_0 and $r_0 + \Delta r$ and with a θ value lying between θ_0 and $\theta_0 + \Delta\theta$.

Solution



The region in question is a sector of the annulus of points outside the circle of radius r_0 and inside the circle of radius $r_0 + \Delta r$. This sector constitutes a $\Delta\theta/(2\pi)$ fraction of the total annulus. The area of the total annulus is

$$\pi(r_0 + \Delta r)^2 - \pi r_0^2 = 2\pi r_0 \Delta r + \pi \Delta r^2.$$

Multiplying by $\frac{\Delta\theta}{2\pi}$ gives

$$\boxed{r_0 \Delta r \Delta\theta + \frac{\Delta r^2 \Delta\theta}{2}}.$$

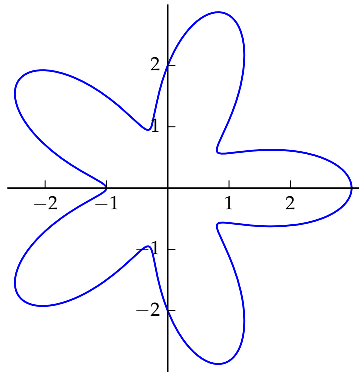
The area of the Cartesian-coordinate rectangle of points whose x-coordinate lies between some values x and $x + \Delta x$ and whose y-coordinate lies between y and $y + \Delta y$ is equal to $\Delta x \Delta y$. Polar coordinates are different: this example shows that the curvy "polar coordinate rectangle", which we'll call a *polar coordinate patch* has an area which depends on the value of the r coordinate. This makes sense if you look at the diagram: the curvy rectangles near the edge are larger than the ones in the middle.

Computational Investigation 2

You can make some nice pictures by plotting $r = f(\theta)$ for some very simple functions f. Like this starfish*:

or gingerbread man, take your pick

```
polar_plot(2+1*cos(5*x),x,0,2*pi)
```

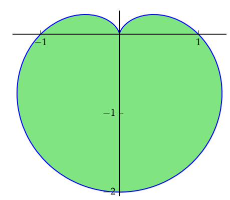


What happens when you change the coefficient of cosine from 1 to 2 to 3? What happens when you change the coefficient of x from 5 to 10 to 20?

Example 7.3

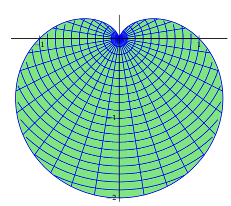
Find the area enclosed by the illustrated *cardioid*, whose equation is

$$r = 1 - \sin\theta.$$



Solution

Since the cardioid is specified in polar coordinates, we divide it up using polar coordinate patches, as shown. We define Δr and $\Delta\theta$ to be the mesh* of our division in the radial and angular directions, respectively.



To find the total area, we add up the areas of these small patches. The area of a patch whose location (r, θ) is given by approximately* $r\Delta r\Delta\theta$ by Example 7.2. Adding up these areas for all the patches between θ and $\theta + \Delta\theta$ and taking $\Delta r \to 0$ gives

$$\int_0^{1-\sin\theta} r\,dr\Delta\theta = \frac{1}{2}(1-\sin\theta)^2\,\Delta\theta$$

for the total area of the slice between θ and $\theta + \Delta\theta$. Adding up this expression over all the θ-slices and taking $\Delta\theta \to 0$ gives

$$\int_0^{2\pi} \frac{1}{2}(1-\sin\theta)^2\,d\theta.$$

This integral, which we'll leave as an exercise, works out to $\boxed{\dfrac{3\pi}{2}}$.

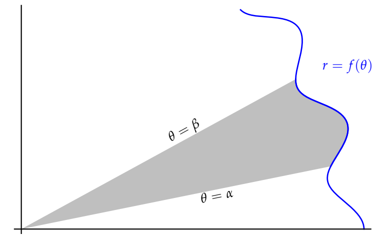
*This means that the angular separation between consecutive straight cuts is $\Delta\theta$, and the radial separation between consecutive curvy cuts is Δr.

*We're dropping the second term from Example 7.2 because its total contribution even after summing over all the patches goes to 0.

Theorem 7.1

The area of the region between the graph of $r = f(\theta)$ and the origin between the rays $\theta = \alpha$ and $\theta = \beta$ is equal to

$$\int_\alpha^\beta \frac{1}{2} f(\theta)^2 \, d\theta.$$



Exercise 7.4

Sketch a graph of $r = \theta$.

Exercise 7.5

(a) Find the area inside the graph of $r = 3 + 2\sin\theta$.

(b) Find the area of the set of points inside both $r = 3 + 2\sin\theta$ and $r = 2$.

8 Complex numbers

26 September

A complex number is an expression of the form $a + bi$, where a and b are real numbers. The symbol i is called the **imaginary* unit**. Complex numbers are plotted in the obvious way: $a + bi$ is located at the point whose Cartesian coordinates are (a, b) in the complex plane. We call a the **real part** of $z = a + bi$ and b the **imaginary part** of z. These are sometimes denoted $\operatorname{Re} z$ and $\operatorname{Im} z$. The set of complex numbers is denoted $\mathbb{C}$, and we sometimes refer to $\mathbb{C}$ as the **complex plane**.

Don't be put off by the term imaginary. These numbers are just as philosophically real as real numbers, with more than enough applications outside mathematics to be real in that sense too.

We do addition, subtraction, and multiplication with complex numbers using all your standard algebraic manipulations, except that whenever we encounter powers of i higher than i^1, we reduce them using the rule $i^2 = -1$.

Example 8.1

Write $(4i - 1)^2 + 6 - i + i^3$ in the form $a + bi$.

Solution

We expand out the square and combine like terms to get

$$16i^2 - 8i + 1 + 6 - i + i^3 = i^3 + 16i^2 - 9i + 7.$$

We substitute -1 for i^2 to get $i^3 = (i^2)(i) = -i$, which implies $\boxed{-9 - 10i}$.

Example 8.2

Find the complex number whose product with $2 - i$ equals 1.

Solution

We write the desired number as $\frac{1}{2-i}$ and multiply* numerator and denominator by $2 + i$.

$$\frac{1}{2-i} \cdot \frac{2+i}{2+i} = \frac{2+i}{4-i^2} = \frac{2+i}{5} = \frac{2}{5} + \frac{i}{5}.$$

Since some of these intermediate steps involved a bit of faith (for example, even writing down the expression $\frac{1}{2-i}$ before we know that $2 - i$ has a multiplicative inverse), we should check that indeed $(2 - i)(\frac{2}{5} + \frac{i}{5}) = 1$.

This technique is the complex version of multiplying by a radical conjugate to rationalize a denominator

The techniques of Examples 8.1 and 8.2 are sufficient to reduce any arithmetic combination (involving the four basic operations and integer exponents) of complex numbers to the form $a + bi$.

Polar coordinates and complex numbers play very nicely together. For a complex number $z = a + bi$, we define $|z| = \sqrt{a^2 + b^2}$. This is called the **modulus** or **norm** or **absolute value** of z. One handy fact is that the **modulus distributes across complex multiplication**:

Example 8.3

Show that if z and w are complex numbers, then $|zw| = |z||w|$.

Solution

We write z as $a + bi$ and w as $c + di$ so we can just work out both sides:

$|zw|^2 = |(a + bi)(c + di)|^2 = |ac - bd + (bc + ad)i|^2 = (ac - bd)^2 + (bc + ad)^2 = a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2.$

And
$$|z|^2|w|^2 = (a^2 + b^2)(c^2 + d^2) = a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2.$$

We also define the notation*
$$\operatorname{cis}\theta = \cos\theta + i\sin\theta.$$

pronounced like 'sis'

The point of this function is that it gives us a complex number on the unit circle corresponding to angle θ. Any complex number $z = a + bi$ can be written in polar form $r \operatorname{cis}\theta$, where r and θ are the polar coordinates of the point (a, b). To see this, recall that $(a, b) = (r\cos\theta, r\sin\theta)$ if r and θ are the polar coordinates of (a, b), so $a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$.

The following example is arguably the source of all the amazingness of complex numbers.

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Example 8.4

Show that $\operatorname{cis}(\alpha+\beta) = \operatorname{cis}\alpha \operatorname{cis}\beta$.

Solution

We can do this by direct calculation:

$$\begin{aligned}\operatorname{cis}(\alpha+\beta) &= \cos(\alpha+\beta) + i\sin(\alpha+\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i\sin\alpha\cos\beta + i\sin\beta\cos\alpha.\end{aligned}$$

Meanwhile,

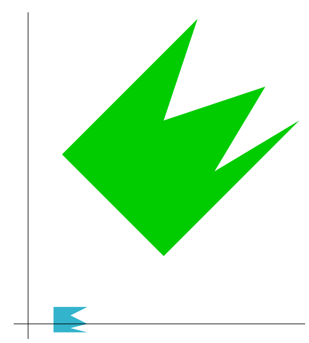
$$\operatorname{cis}\alpha\operatorname{cis}\beta = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta).$$

Expanding out the right hand side gives an expression which matches the one we derived for $\operatorname{cis}(\alpha+\beta)$.

Example 8.4 says that if you start at a location $\operatorname{cis}\theta$ on the unit circle and you want to change your location by an angle-β rotation, you can do that by multiplying by $\operatorname{cis}\beta$.

Example 8.5

This figure illustrates a blue region B as well the result (shown in green) obtained by multiplying each of the points in B by the complex number $4+4i$.



Explain why the green and blue regions have the same shape.

Solution

Let's write $4+4i$ in the form $r\operatorname{cis}\theta$, so we can take advantage of Example 8.4. The radial polar coordinate r is given by $|4+4i| = 4\sqrt{2}$. That leaves

$$\operatorname{cis}\theta = \frac{4+4i}{4\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

We can recognize* these values as the cosine and sine of $\pi/4$, so we get

$$4+4i = 4\sqrt{2}\operatorname{cis}\frac{\pi}{4}.$$

In this case, we also could have just plotted the point $(4,4)$ and discern its angle θ from the graph.

Now, if $z = r \operatorname{cis} \theta$ is some point in B, then the result of multiplying z by $4\sqrt{2} \operatorname{cis} \frac{\pi}{4}$ is

$$(r \operatorname{cis} \theta) \left(4\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right) = (4\sqrt{2} \cdot r) \operatorname{cis} \left(\theta + \tfrac{\pi}{4}\right),$$

where we combined the angles using Example 8.4. From this expression we can read off the polar coordinates $4\sqrt{2} \cdot r$ and $\theta + \pi/4$ of this new point.

Geometrically, this means that the new point is $4\sqrt{2}$ times as far away from the origin as the original point, and its angle is $\pi/4$ greater. In other words, multiplying by $4 + 4i$ scales by $4\sqrt{2}$ and rotates by 45 degrees.

So this explains why the shape of the green and blue regions is the same: scaling and rotation do not change shape of a figure. Furthermore, the green figure looks roughly $4\sqrt{2} \approx 5.66$ times larger than the blue one, and it's rotated 45 degrees, so these findings make sense.

The lesson of this example is super important:

COMPLEX MULTIPLICATION IS A ROTATION/SCALING

In other words, every complex number can be written in polar form $r \operatorname{cis} \theta$, and multiplying by that number scales by a factor of r and rotates by an angle of θ.

What happens when we raise a complex number to a positive integer exponent n? We get

$$(r \operatorname{cis} \theta)^n = (r \operatorname{cis} \theta) \cdots (r \operatorname{cis} \theta) = r^n \operatorname{cis}(n\theta).$$

So an exponent of n operates on r the same way as for real numbers, and they multiply the angle by n.

Example 8.6

Find all complex numbers z such that $z^3 = 1$.

Solution

We write $z = r \operatorname{cis} \theta$ and $1 = 1 \operatorname{cis}(0)$ so we can compare both sides of the equation in polar coordinates. The equation becomes

$$r^3 \operatorname{cis}(3\theta) = 1 \operatorname{cis} 0$$

Clearly we need $r = 1$ for the moduli of the two sides to match up. Furthermore, we need to choose an angle θ such that when we triple it, we're back around to an angle of 0. Clearly $0°$ and $120°$ work. But notice that $240°$ works too, since tripling $2 \cdot 120°$ gives us two full rotations around the unit circle. We can make sure we've found them all by solving

$$3\theta = 360° k$$

where k is any integer. This gives $\theta = 120° k$, so the solutions are

$$\dots, -360°, -240°, -120°, 0°, 120°, 240°, 360°, 480°, \dots$$

We can trim this last down to $\{0°, 120°, 240°\}$, since the rest of the angles are redundant. So all the solutions are

$$\operatorname{cis}(0°) = 1, \quad \operatorname{cis}(120°) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \operatorname{cis}(240°) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

30 September

Example 8.7

Solve $z^{10} = 1$.

Solution

There was nothing particularly special about 3 in the previous example. We can take the same approach:

$$r^{10} \operatorname{cis}(10\,\theta) = 1.$$

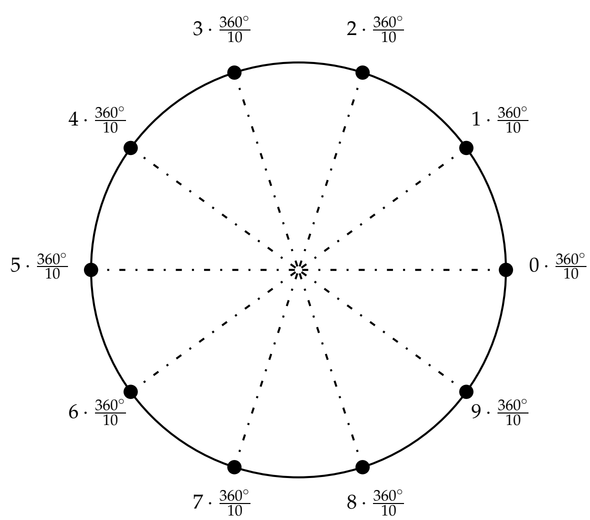
This implies $r = 1$ and $10\theta = 360°k$ for some integer k. So θ takes one of the values

$$0°, \frac{360°}{10}, \frac{2 \cdot 360°}{10}, \ldots, \frac{9 \cdot 360°}{10}.$$

so the solutions are

$$\operatorname{cis} 0°, \operatorname{cis}\left(\frac{360°}{10}\right), \operatorname{cis}\left(\frac{2 \cdot 360°}{10}\right), \ldots, \operatorname{cis}\left(\frac{9 \cdot 360°}{10}\right).$$

These can't be conveniently expressed in Cartesian coordinates, but we can try to plot them all to try to gain some insight:



The solutions are the 10 points equally spaced around the circle starting from $1 = \operatorname{cis} 0$. These numbers are called the tenth **roots of unity**.

Exercise 8.8

(a) The **complex conjugate** of a complex number $z = a + bi$ is defined to be $\bar{z} = a - bi$. Show that $z\bar{z} = |z|^2$.

(b) Show that $z + \overline{z} = 2\,\mathrm{Re}\,z$

(c) Show that $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{\overline{z}} = z$

(d) Using (a), (b), and (c), show that $|z+w|^2 = |z|^2 + 2\,\mathrm{Re}(z\overline{w}) + |w|^2$.

Exercise 8.9

The set of complex numbers is a *field*, meaning that we can divide by any nonzero complex number $z = a + bi$. Find the multiplicative inverse of z, that is, the complex number whose product with z equals 1. Hints: you can use the polar representation of z, or consider Exercise 8.8 part (a). See also Example 8.2.

Exercise 8.10

By definition, the distance from z to 0 in the complex plane is given by $|z - 0|$. Show that if z and w are points in the complex plane, then the distance from z to w is given by $|z - w|$.

Exercise 8.11

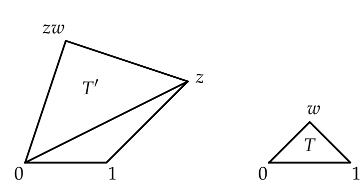
(a) Find all complex solutions of $z^3 = 8$.

(b) Find all complex solutions of $z^4 = i$.

Exercise 8.12

We proved that $\mathrm{cis}(\theta + \phi) = \mathrm{cis}(\theta)\,\mathrm{cis}(\phi)$ using the cosine and sine sum-angle formulas. However, the identity $|zw| = |z||w|$ is actually powerful enough to prove $\mathrm{cis}(\theta+\phi) = \mathrm{cis}(\theta)\,\mathrm{cis}(\phi)$ without the sum-angle formulas.

(a) Consider the triangle T with vertices 0, 1, and w in the complex plane. Show that the triangle T' with vertices 0, z, and zw is similar to T.



(b) Use the figure above and the result of (a) to show that if $z = r\,\mathrm{cis}(\theta)$ and $w = s\,\mathrm{cis}(\phi)$, then $zw = rs\,\mathrm{cis}(\theta + \phi)$.

(c) Use the result of (b) with $r = s = 1$ to prove the sine and cosine sum angle formulas (!).

2 October

Exercise 8.13

In this exercise, we develop a beautiful geometric proof of one of the most famous formulas in mathematics: Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $e \approx 2.718$ is Euler's constant. We define the exponential function for complex numbers z using the formula*

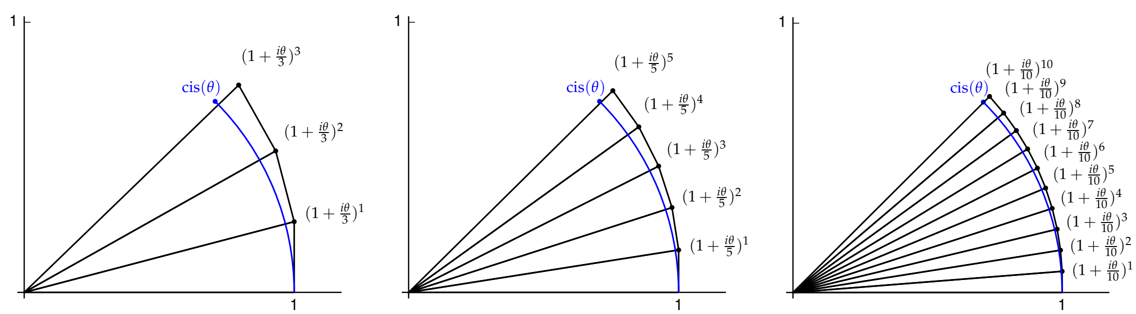
$$e^z = \lim_{p\to\infty}\left(1+\frac{z}{p}\right)^p, \tag{8.1}$$

We're taking this limit considering only integer values of p here; see Section 11.1 for more discussion of limits of sequences.

which agrees with the exponential function you're already familiar with whenever z is real*, and it makes sense for all $z \in \mathbb{C}$, because the right-hand side only involves addition, multiplication, division, and raising a complex number to a power.

To prove this, write the limit as $e^{\lim_{p\to\infty}\ln\left(1+\frac{z}{p}\right)^p}$, bring the exponent p down, and use L'Hospital's rule to evaluate the limit of $\frac{\ln(1+z/p)}{1/p}$ as $p\to\infty$.

The following sequence of figures uses the geometric picture of complex multiplication explained in Exercise 8.12 to illustrate what happens when $\left(1+\frac{i\theta}{p}\right)^p$ is evaluated:



You can see that $\left(1+\frac{i\theta}{p}\right)^p$ is getting increasingly close to $\operatorname{cis}\theta$ as p gets larger. Supply the relevant calculations necessary to show that $\left(1+\frac{i\theta}{p}\right)^p$ really does converge to $\operatorname{cis}\theta$. (Hint: you can write $1+\frac{i\theta}{p}$ in polar form so it can be conveniently be raised to the pth power; then you need to show that the radial and angular polar coordinates of the result converge to 1 and θ respectively.)

9 Differential equations

9.1 Introduction and examples

An algebraic equation is an equation involving algebra. The unknown quantities in the equation are numbers, and the goal is to find all the numbers that satisfy the equation. For example,

$$x^2 - 16 = 0$$

is an equation whose solution is $x = -4$ or $x = 4$.

A *differential* equation is an equation involving *derivatives*. The unknown quantities are *functions*, and the goal is to find all the functions that satisfy the equation. You are already in a position to solve some differential equations. For example, the equation

$$f'(x) = x^2$$

is satisfied by any function of the form $f(x) = \frac{1}{3}x^3 + C$, where C is a real number. Furthermore, those are the only solutions*. Of course, solving differential equations with only an f' in them amounts to taking anti-derivatives; we can make differential equations more interesting by involving both f and f'.

Do you recall how to show that if $f'(x) = x^2$, then $f(x) - \frac{1}{3}x^3$ is a constant function?

Example 9.1

Find a solution to the differential equation $f'(x) = f(x)$.

Solution

We recall from differential calculus that there is a function which is equal to its own derivative, namely $f(x) = e^x$. Actually, any function of the form $f(x) = Ce^x$ works, where C is constant.

We say that a differential equation is **linear** if it can be written in a form where the left-hand side is a sum of terms each of which is some particular function of x (not involving f, in other words) times a derivative* of f, and the right-hand side is a function of x. For example*,

The function itself counts as its own zeroth derivative.

We may write y synonymously with $f(x)$.

$$-2f'''''(x) - 3f'(x) + f(x) = -x^2$$

is a linear differential equation. A linear differential equation is called **homogeneous** if the term on the right-hand side is 0. If all the functions multiplying the derivatives of f on the left-hand side are constant, we say the differential equation has *constant coefficients*. The variable playing the role of the argument of the unknown function, in this case x, is called the *independent variable*.

Linear homogeneous differential equations are important because they show up in many applications, including physics.

Example 9.2

Find the differential equation governing the motion of an object attached to a spring, and find some of its solutions.

4 October 2016

Solution

Hooke's law tells us that the force being applied to the spring when the mass is at position y is given by $F = -ky$, where k is the stiffness of the spring. Furthermore, Newton's second law tells us that the object moves in response to this force with an acceleration which is equal to the force divided by the mass m of the object. Acceleration is equal to the second derivative of position, so this tells us that the trajectory of the object is given by*

$$y''(t) = -\frac{k}{m}y(t).$$

We use t for the independent variable instead of x, because this variable represents time.

Since this equation can be written as $y''(t) + \frac{k}{m}y(t) = 0$, it is a linear homogeneous differential equation.

If we recall from differential calculus that the second derivative of $\sin t$ is equal to $-\sin t$, we can figure out* that $\sin\left(\sqrt{\frac{k}{m}}t\right)$ is a solution to this differential equation. The same can be said of cosine, so $\cos\left(\sqrt{\frac{k}{m}}t\right)$ is also a solution. In fact, we can check that if A and B are constants, then*

$$y(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right).$$

solves the differential equation.

"Figure out" means try $y(t) = \sin(ct)$ where c needs to be determined, and then substitute into the differential equation to find $c = \sqrt{k/m}$.

Does this solution match your intuition for how an object attached to a spring moves?

Example 9.3

A dampener is attached to the spring system from the previous example, and it has the effect of applying a force which proportional to the velocity of the object. How does this change the differential equation and its solution?

Solution

We replace $F = -ky(t)$ in the equation

$$y''(t) = \frac{F}{m}$$

with $F = -ky(t) - dy'(t)$, where $d > 0$ is the constant of proportionality of the dampening force. We can rearrange the resulting equation to get

$$my''(t) + dy'(t) + ky(t) = 0.$$

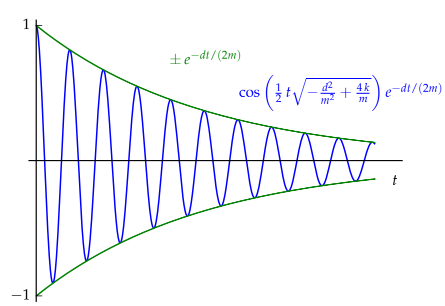
We will learn how to solve this differential equation soon, but for now, let's get Sage to do it for us:

```
var("t m d k") # declare these four letters to be mathematical symbols
y = function('y')(t) # declare y to be a function of t
assume(4*k*m-d^2>0) # if you omit this assumption, Sage will tell you it needs it
desolve( m*diff(y,t,2) + d*diff(y,t) + k*y, y, ivar=t) # solve, declaring the independent
                                                          # variable to be t
```

This code block returns

$$\left(A\cos\left(\frac{1}{2}t\sqrt{-\frac{d^2}{m^2}+\frac{4\,k}{m}}\right) + B\sin\left(\frac{1}{2}t\sqrt{-\frac{d^2}{m^2}+\frac{4\,k}{m}}\right)\right)e^{-\frac{dt}{2\,m}},$$

where A and B are arbitrary constants. If we look past all the constant coefficients, this expression tells us something physically reasonable: you get a sine/cosine factor accounting for the spring oscillation and a dampening exponential factor. Here's an example of what the graph looks like:



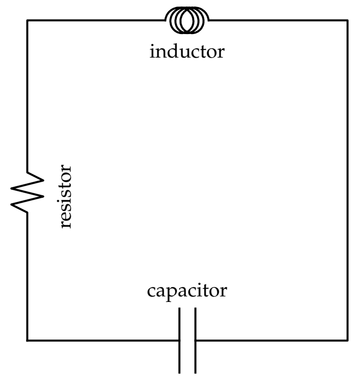
The **order** of a linear differential equation is the highest derivative of the unknown that appears. Note that the dampened spring differential equation ends up taking a very general form: by adjusting the coefficients, we can model *any* second-order linear homogeneous differential equation whose coefficients happen to be positive.

The order is analogous to the degree of a polynomial equation.

Physical examples of differential equations come in the electricity-and-magnetism flavor too:

Example 9.4

The figure to the right illustrates an RLC circuit. The capacitor stores some charge $Q(t)$ and has voltage $Q(t)/C$ across it, where C is a constant particular to the capacitor, called the *capacitance*. As the charge flows through the system, the change in current $I(t) = Q'(t)$ induces a voltage of $LI'(t)$ across the inductor (where L is the *inductance* of the inductor) and the current induces a voltage of $RI(t)$ across the resistor (where R is the *resistance* of the resistor). Use Kirchhoff's law to write down a differential equation governing the flow of charge in the circuit.



The hydraulic analogy might be helpful: the circuit is a pipe, charge is water, a capacitor is a flexible rubber membrane across the pipe, an inductor is a heavy paddle wheel whose momentum slows the current as it speeds up and sustains it as it slows down, and a resistor is a constriction in the pipe.

Solution

Kirchhoff's law says that the sum of the voltage differences around the circuit is equal to 0:

$$LI'(t) + RI(t) + \frac{1}{C}Q(t) = 0.$$

We should express I in terms of Q so the DE doesn't look like it has two unknowns in it:

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0.$$

Exercise 9.5

The circuit and dampened spring examples turn out to be mathematically identical, in the sense that they model any second order linear, homogeneous differential equation with constant, positive coefficients. Furthermore, the five physical features *spring, dampener, capacitor, inductor,* and *resistor* correspond to specific terms in their respective differential equations. Match up these features of according to the order of the term they correspond to (note that one of them won't match up).

9.2 Solving linear homogeneous differential equations with constant coefficients

07 October 2016

Now we begin actually developing techniques for solving differential equations. The key idea for a linear, homogeneous differential equation with constant coefficients is to guess that the solution takes the form $y(x) = e^{\lambda x}$.

Example 9.6

Find all values of λ for which $y(x) = e^{\lambda x}$ solves the differential equation $y''(x) + 4y'(x) - 5y(x) = 0$.

Solution

Let's substitute: $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$, so $y(x) = e^{\lambda x}$ satisfies the equation if and only if

$$\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} - 5e^{\lambda x} = 0.$$

Recalling that $e^{\lambda x}$ is never zero, we can multiply through by $e^{-\lambda x}$ and get

$$\lambda^2 + 4\lambda - 5 = 0.$$

We solve this equation to find that $\lambda = -5$ or $\lambda = 1$. So e^{-5x} and e^x work.

If a linear differential equation is homogeneous, then any linear combination* of two solutions is also a solution* . So

$$y(x) = Ae^{-5x} + Be^x$$

satisfies $y''(x) + 4y(x) - 5y(x) = 0$ for any constants A and B. Actually, this expression represents all the solutions:

Linear combination means some constant times the first one plus some constant times the second one.

Work this out to check that it's true

Theorem 9.1

A degree-n linear homogeneous differential equation has n solutions, called **fundamental solutions**, such that every solution of the differential equation is a linear combination of the fundamental solutions.

You can learn how to prove this theorem in a differential equations course.

This theorem is telling us that if we find two different* solutions to a second-order differential equation, then we can write down an arbitrary linear combination of these functions and declare the differential equation solved. This arbitrary linear combination is called the **general solution** of the differential equation.

Different here means that one is not a constant multiple of the other

The equation $\lambda^2 + 4\lambda - 5$ is called the **characteristic equation** of the differential equation $y''(x) + 4y'(x) - 5y(x) = 0$. Let's see an example where the roots of the characteristic equation are not real.

Example 9.7

Solve $y''(x) + 4y(x) = 0$.

Solution

We again guess $y(x) = e^{\lambda x}$ and find that this function solves the differential equation if and only if

$$\lambda^2 + 4 = 0,$$

which means $\lambda = 2i$ or $\lambda = -2i$. So we get e^{2ix} and e^{-2ix} as our two solutions. More generally,

$$y(x) = Ae^{2ix} + Be^{-2ix}$$

solves the differential equation, where A and B are arbitrary constants.

However, we are generally interested in real-valued solutions, not complex-valued ones. We will handle this by choosing A and B to be complex numbers that make $Ae^{2ix} + Be^{-2ix}$ real. Applying Euler's formula gives

$$Ae^{2ix} + Be^{-2ix} = (A + B)\cos(2x) + i(A - B)\sin(2x)$$

If we choose $A = \frac{1}{2}(a - bi)$ and $B = \frac{1}{2}(a + bi)$, where a and b are arbitrary real numbers, then $A + B = a$ and $i(A - B) = b$, so our solution becomes

$$y(x) = a\cos(2x) + b\sin(2x).$$

So we've handled the case where the roots are real or purely imaginary; what happens if the roots have nonzero real part and nonzero imaginary part?

Example 9.8

Solve $y''(x) + 2y'(x) + 2y(x) = 0$.

Solution

The roots of the characteristic polynomial are $-1 \pm i$. So we get a solution of

$$y(x) = Ae^{(-1-i)x} + Be^{(-1+i)x}.$$

The reasoning we applied in Example 9.7 allows us to "decomplexify" this solution; it works the same except that we have an extra factor of e^{-x} in this case. We end up with

$$y(x) = ae^{-x}\cos x + be^{-x}\sin x$$

as our final answer.

Exercise 9.9

Consider a differential equation of the form $y''(x) + by'(x) + cy(x) = 0$ for which the characteristic equation has a repeated root λ. Show that

$$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$$

is the general solution of the differential equation.

Exercise 9.10

Show that the general solution of Example 9.7 can be written in the form $C\cos(2x + \alpha)$ for some constants C and α. So the solutions all have the same shape as $\cos(2x)$, just arbitrarily scaled vertically and shifted horizontally.

Exercise 9.11

Solve $y'''(x) + 6y''(x) + 11y'(x) + 6y(x) = 0$.

9.3 Initial value problems

12 October

We learned in the previous section that the general solution to a differential equation is not a single function but rather a family of functions. For example, the solution of $f''(x) = f(x)$ is the set of all functions of the form $f(x) = Ae^x + Be^{-x}$, where A and B are any real numbers. It is often desirable, especially in applications of differential equations, to specify additional information about the function which allows these constants to be determined.

Example 9.12

Consider an object with mass m attached to a spring with stiffness k and no dampener. The spring is stretched to a displacement of 2 meters, at which point the object is held stationary and then released. Find the subsequent displacement $y(t)$ as a function of time.

Solution

Since a force $F = -ky(t)$ acting on an object of mass m induces an acceleration of F/m, the differential equation governing the motion of the object is

$$y''(t) = -\frac{k}{m}y(t).$$

This differential equation linear and homogeneous with constant coefficients, and its characteristic equation is $\lambda^2 + \frac{k}{m} = 0$. The roots of this equation are $\pm i\sqrt{\frac{k}{m}}$, so the general solution is

$$y(t) = Ae^{i\sqrt{\frac{k}{m}}t} + Be^{-i\sqrt{\frac{k}{m}}t} = C\sin\left(\sqrt{\frac{k}{m}}t\right) + D\cos\left(\sqrt{\frac{k}{m}}t\right),$$

where C and D are arbitrary constants.

However, the problem statement gives us some additional information: the displacement at time $t = 0$ (we will always assume 'initial' means $t = 0$ unless specified otherwise) is equal to 2. This means

$$2 = y(0) = C\sin 0 + D\cos 0,$$

which implies that $D = 2$. Similarly, the object starts at rest, meaning that initial velocity is zero. So we get

$$0 = y'(0) = C\sqrt{k/m}\cos 0 - D\sqrt{k/m}\sin 0,$$

which shows that $C = 0$. Therefore, the position of the object at time t is given by

$$y(t) = 2\cos\left(\sqrt{\frac{k}{m}}t\right).$$

If the problem gives us enough additional data to solve for the constants in the general solution, we call it an **initial value problem**. Physically, the differential equation describes how a system evolves in time, while the initial value data describe how the system begins.

Exercise 9.13

Find a function f which satisfies $f'(x) = 7f(x)$ and $f(0) = 2$.

Exercise 9.14

Find a function f which satisfies $f''(x) = f'(x) + 6f(x)$, $f(0) = 2$, and $f'(0) = 1$.

Exercise 9.15

In Exercise 9.13 the problem specified the zeroth* derivative of f at 0, and in Exercise 9.14, the zeroth and the first derivative of f at 0 are specified. How many derivatives of f should be specified to uniquely determine a solution to a differential equation of order n?

Recall that the zeroth derivative of a function is just the function itself.

Computational Investigation 3

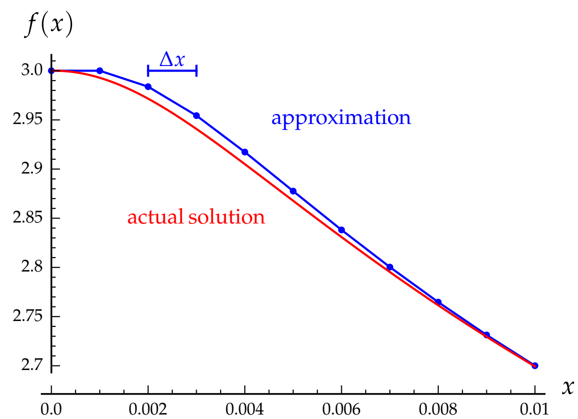
Many differential equations that arise in real life do not have elementary solutions. In other words, there's no way to put together polynomials, logs, exponentials, and trig functions to get a function that satisfies the differential equation. However, such differential equations can nevertheless be solved *numerically*, in the sense that we can find and plot a bunch of points which are on or very near the graph of the solution*. Furthermore, these numerical methods illustrate rather vividly what we mean when we say that the initial conditions determine function starts and the differential equation determines how it evolves. Let's look at an example.

Note that a solution can exist even though it can't be represented using elementary functions.

Suppose we want to solve $\boxed{f'(x) = -2xe^{f(x)^2}}$ with $f(0) = 3$. If we try our methods on this differential equation, we'll get stuck.

However, we do know already the value of f at 0: it's 3. What is the value of f, approximately, just to the right of 0? The differential equation tells us that the slope of the function at 0 is equal to $-2xe^{f(x)^2} = -2(0)e^{3^2} = 0$, so the value of the function at some small number, which we'll call Δx, is approximately the same as it was at 0 (see the figure below). So $f(\Delta x) \approx 3$.

From there we can scoot to the right on the graph of f again. The DE tells us that the slope of the graph of f at Δx is approximately $-2\Delta x e^{f(\Delta x)^2} \approx -2\Delta x e^9$, so we can multiply the slope by the length of our small rightward step Δx and find that we move approximately $-2(\Delta x)^2 e^9$ downward to $f(2\Delta x) \approx 3 - 2(\Delta x)^2 e^9$.



Continuing in this way would be very tedious, of course, and increasingly so as Δx gets smaller. However, the calculations we need to carry out this procedure are just arithmetic (and evaluating the exponential function), which computers can do very fast. Here's an example (click here for this in SageMathCell).

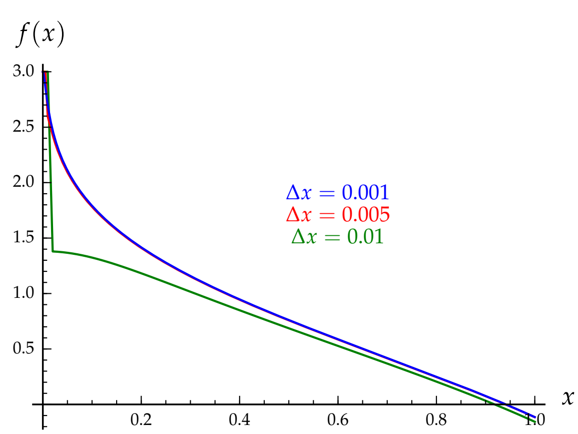
```
pairs = []  # pairs is an empty list, for storing our x and y values
dx = 0.001  # Delta x is a small number
y = 3       # we store our starting y-value as y
for x in [0,dx,2*dx,..,1.0]:  # the indented code below is run
                              # for all the x values 0,dx,2dx,...,1.0

    pairs.append((x,y))       # store the current point (x,y)

    y = y - 2*x*exp(y^2)*dx   # update the value of y using the DE --
                              # the equals sign means calculate the right-
                              # hand side and store the result as y

line(pairs) # draw a curve through all the pairs
```

The figure below shows the resulting graph for various values of Δx.



We see that 0.01 is not small enough, since our numerical solution misbehaved around $x = 0$. However, once we got below 0.001 or so, the graph we get is pretty similar regardless of whether Δx is super small or merely small. This is a good indication that our numerical solution is close to the actual solution.

Here's an extension to this investigation: above we solved over $[0, 1]$, and in fact the solution to this differential equation blows up around 1.33 or so. Try replacing `1.0` in the fourth line to explore this behavior (note that if you go past the blow-up point, your graph will look awful, because it's trying to plot some huge numbers).

9.4 Separable differential equations

In this section we'll learn a new technique for solving differential equations. Let's start with an example.

Example 9.16

Solve the differential equation $f'(x)\sqrt{f(x)} = 1$.

Solution

This differential equation is not linear, so the techniques we have developed so far will not help. However, note that if we integrate both sides, we can make a substitution $u = f(x)$:

$$\begin{aligned}
\int \sqrt{f(x)}\, f'(x)\, dx &= \int 1\, dx \\
\int \sqrt{u}\, du &= x + C \\
\frac{2}{3}u^{3/2} &= x + C \\
u &= \left(\frac{3}{2}x + C\right)^{2/3}.
\end{aligned}$$

We rewrote $\frac{3}{2}C$ as C from the next-to-last line to the last. This is because C represents an arbitrary constant, and 3/2 times an arbitrary constant is just some other arbitrary constant. This "mutable constant" notation seems a bit odd at first but is quite common and very handy.

14 October

Sometimes we have to do a bit of rearrangement before integrating. The goal is to get everything involving the unknown f on the left-hand side along with f', and get everything involving the independent variable (usually x or t) on the right-hand side.

Example 9.17

Find a function f which satisfies $f'(x) = \frac{x(e^{x^2}+2)}{6f(x)^2}$ and $f(0) = 1$.

Solution

Moving the denominator over by multiplying both sides by it, we get

$$6f(x)^2 f'(x) = x(e^{x^2} + 2).$$

Integrating both sides, we have

$$2f(x)^3 = \frac{1}{2}e^{x^2} + x^2 + C.$$

Setting $x = 0$ we find

$$2(1)^3 = \frac{1}{2}e^0 + 0 + C,$$

so $C = 3/2$. So the solution is $f(x) = \sqrt[3]{\frac{e^{x2}}{4} + \frac{x2}{2} + \frac{3}{4}}$.

Exercise 9.18

Find the general solution of the differential equation $f'(x) = e^{-f(x)}(2x - 4)$.

Exercise 9.19

Solve the initial value problem $\frac{f''(x)}{(f'(x))^2} = 2$, $f(1) = -\frac{\ln 2}{2}$, $f'(1) = -\frac{1}{2}$.

9.5 Nonhomogeneous differential equations

What happens if we have a linear differential equation with constant coefficients which is *not* homogeneous? For example:

$$f''(x) - f'(x) - 6f(x) = -12e^x \tag{9.1}$$

Theorem 9.2

If f_P is any particular solution of a degree-n nonhomogeneous linear differential equation with constant coefficients and $f_1, \ldots, f_n$ are a collection of fundamental solutions for the corresponding homogeneous equation*, then every solution of the original differential equation is of the form

$$f(x) = f_P(x) + A_1 f_1(x) + \cdots + A_n f_1(x),$$

where $A_1, \ldots, A_n$ are constants.

*The one you get if you replace the right-hand side with zero.

Theorem 9.2 tells us, in other words, that "general homogeneous solution plus particular solution equals general nonhomogeneous solution". This suggests a trial-and-error type approach to find a particular solution. Let's solve (9.1).

Example 9.20

Find the general solution of the differential equation $f''(x) - f'(x) - 6f(x) = 12e^{4x}$.

Solution

Seeing e^{4x} on the right-hand side and bearing in mind that derivatives of e^{4x} are constant multiples of e^{4x}, we figure that some function like $f(x) = Ce^{4x}$ might work. So we substitute $f(x) = Ce^{4x}$ and find that this function works if and only if

$$16Ce^{4x} - 4Ce^{4x} - 6Ce^{4x} = 12e^{4x}.$$

Collecting terms on the left-hand side, we get $6Ce^{4x} = 12e^{4x}$, so $C = 2$ works.

Solving the homogeneous equation $f''(x) - f'(x) - 6f(x) = 0$ gives us $f(x) = Ae^{3x} + Be^{-2x}$. So Theorem 9.2 implies that the general solution of the non-homogeneous equation is

$$f(x) = 2e^{4x} + Ae^{3x} + Be^{-2x}.$$

Let's try another one.

17 October

Example 9.21

Solve $f''(x) - 4f'(x) - 12f(x) = \sin 2x$ subject to $f(0) = 0$ and $f'(0) = 0$.

Solution

This is an initial value problem, so we begin by finding the general solution of the differential equation. We try the same idea of substituting $f(x) = C \sin 2x$. After a few steps we get

$$-16C \sin 2x - 8C \cos 2x = \sin 2x.$$

Matching up coefficients on the two sides, we would have to have $C = 0$ to get the cosine coefficients to be the same, but we'd have to have $C = -1/16$ to get the sine coefficients to be the same. So this didn't work. The key is to throw in an extra cosine term, so that we have all the different functions that appear when we successively take derivatives of $\sin 2x$. We try

$$f(x) = C \cos 2x + D \sin 2x,$$

which, after a few steps, gives us

$$(-16C - 8D) \cos 2x + (8C - 16D) \sin 2x = \sin 2x.$$

Now we can match up coefficients on both sides, and we find that $C = 1/40$ and $D = -1/20$. So our particular solution is

$$f_P(x) = \frac{1}{40} \cos 2x - \frac{1}{20} \sin 2x.$$

The general solution of the homogeneous equation is $Ae^{-2x} + Be^{6x}$, because the characteristic equation has roots -2 and 6. So our general solution is

$$f(x) = \frac{1}{40} \cos 2x - \frac{1}{20} \sin 2x + Ae^{-2x} + Be^{6x}.$$

We can substitute $x = 0$ to get $0 = f(0) = \frac{1}{40} + A + B$, and differentiate both sides and then substitute $x = 0$ to get

This notation means "substitute $x = 0$ into the expression".

$$0 = f'(0) = \left(-\frac{2}{40} \sin 2x - \frac{2}{20} \cos 2x - 2Ae^{-2x} + 6Be^{6x} \right) \Bigg|_{x=0} = -\frac{2}{20} - 2A + 6B.$$

Solving this system of equations for A and B gives $A = -\frac{1}{32}$ and $B = \frac{1}{160}$. So our final answer is

$$\frac{1}{40}\cos 2x - \frac{1}{20}\sin 2x - \frac{1}{32}e^{-2x} + \frac{1}{160}e^{6x}.$$

In summary, to find a particular solution, we form an arbitrary linear combination of the functions we see on the right-hand side along with some other functions that show up when we differentiate those functions some number of times, and we try to solve for the coefficients.

Exercise 9.22

Find the general solution of the differential equation

$$f''(x) - 4f'(x) - 12f(x) = 2x^3 - x + 3.$$

(Hint: try $f(x) = Ax^3 + Bx^2 + Cx + D$.)

Exercise 9.23

Find a particular solution to

$$f''(x) - 4f'(x) - 12f(x) = xe^{4x}.$$

Exercise 9.24

Show that if f_P is a particular solution to

$$f''(x) + P(x)f'(x) + Q(x)f(x) = R_1(x)$$

and g_P is a particular solution to

$$f''(x) + P(x)f'(x) + Q(x)f(x) = R_2(x),$$

then $f_P + g_P$ is a particular solution

$$f''(x) + P(x)f'(x) + Q(x)f(x) = R_1(x) + R_2(x).$$

In other words, if the function on the right-hand side of a non-homogeneous differential equation is a sum of several terms, then we can split the original problem into several sub-problems, one for each of those terms.

10 Improper integrals

21 October

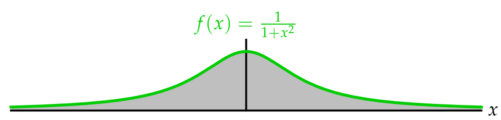
Recall that the integral of a function $f(x)$ over an interval is the signed area of the region between the graph of the function and the x-axis. The term *signed* means that area counts as negative if it is below the x-axis. You already know how to integrate a continuous function over a bounded interval like $[a, b]$, so let's consider intervals with one or both endpoints infinite.

Example 10.1

Find the total area under the graph of $f(x) = \dfrac{1}{1+x^2}$.

Solution

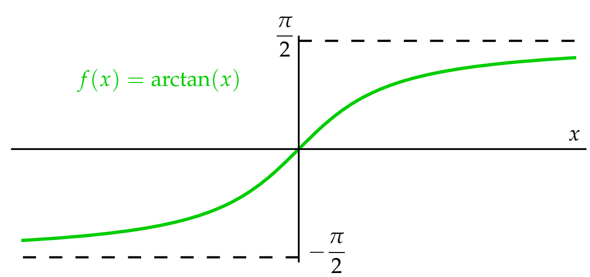
The graph of the function is shown to the right—our goal is to find the shaded area (continued out to $-\infty$ and to $+\infty$). Recall that the fundamental theorem of calculus only applies over a bounded interval $[a, b]$, not over the whole real number line. However, it makes sense that we could integrate $f(x)$ over $[-b, b]$, where b is very large, to closely approximate the total shaded area. The actual value of the total shaded area will then be the limit as $b \to \infty$ of $\int_{-b}^{b} f(x)\,dx$.



We calculate

$$\int_{-b}^{b} \frac{1}{1+x^2}\,dx = \arctan(b) - \arctan(-b).$$

Recalling the graph of arctan,



we get

$$\lim_{b\to\infty} [\arctan(b) - \arctan(-b)] = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Example 10.2

Find the total signed area under the graph of $f(x) = x$.

Solution

It's clear from looking at the graph of $f(x) = x$ the total area above the x-axis is ∞ and the total area below is ∞, which means that the signed area is $\infty - \infty$, which is $\boxed{\text{undefined}}$.

Note that if we integrate from $-b$ to b and take $b \to \infty$, we get

$$\lim_{b\to\infty} \int_{-b}^{b} x\,dx = \lim_{b\to\infty} \left(\frac{b^2}{2} - \frac{b^2}{2} \right) = \lim_{b\to\infty} 0 = 0.$$

Therefore, we can see that our method from the first example requires some caution. We can avoid this pitfall by integrating from 0 to ∞ and from $-\infty$ to 0 separately, adding the results if both converge.

Let's consolidate what we figured out in Examples 10.1 and 10.2 as a definition:

Definition 10.1: Improper integration, unbounded intervals

We define the improper integral of a function f over (a, ∞) to be

$$\lim_{b\to\infty} \int_{a}^{b} f(x)\,dx$$

if this limit exists. If the limit does not exist, we say that the improper integral **diverges**, and if it does exist, we say that the improper integral **converges**. Similarly, we define

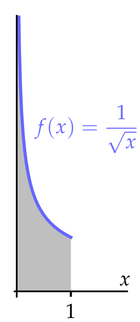
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx.$$

We define the improper integral of f over $(-\infty, \infty)$ to be the sum of the integral of f over $(-\infty, 0]$ and the integral of f over $[0, \infty)$, assuming both of these integrals converge. If either diverges, then we say that $\int_{-\infty}^{\infty} f(x)\,dx$ diverges too.

We can also have improper integrals over bounded intervals, if the function we're integrating isn't continuous at one endpoint.

Example 10.3

Find the area of the shaded region.



Solution

We can't do this one directly using the fundamental theorem of calculus, because that theorem only applies to a function which is continuous over the desired interval of integration. In this case, $1/\sqrt{x}$ blows up as $x \to 0$.

However, we can find the area from a to 1, where $a > 0$, and then we can take $a \to 0$ to find the total area from 0 to 1. We get

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0+} 2\sqrt{x}\Big|_a^1 = \lim_{a\to 0+} 2\sqrt{x}\Big|_a^1 = \lim_{a\to 0^+} (2 - 2\sqrt{a}) = 2.$$

Therefore, the area of the shaded region is 2.

Definition 10.2: Improper integration, bounded intervals

If f is a function which is continuous on $(a, b]$ but does not have a right limit at a, then the improper integral $\int_a^b f(x)\,dx$ is defined by

$$\lim_{c\to a^+} \int_c^b f(x)\,dx,$$

if the integral exists. If the limit does not exist, then we say the improper integral **diverges**. If it does exist, we say the integral **converges**. Similarly, we define

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

if f is continuous on $[a, b)$ but does not have a left limit at b.

In Definition 10.2, we have omitted the case where the function does not have a limit at either endpoint, but the idea is the same: choose a point c between a and b and break up $\int_a^b$ as $\int_a^c + \int_c^b$. The same goes if the two types of improperness are mixed, for example, if a function does not have a right limit at a and is being integrated from a to ∞.

Let's conclude with a physics application.

Example 10.4

Find the amount of work it takes to move an object of mass m which is at a distance of R from a planet with mass M to a point which is extremely far away from the planet.

Solution

Recall that the gravitational force to which the mass m is subjected when it is at distance r is GMm/r^2. Therefore, the total work to move the object from distance R to distance D is

$$\int_R^D \frac{GMm}{r^2}\,dr = -\frac{GMm}{r}\Big|_R^D = \frac{GMm}{R} - \frac{GMm}{D}.$$

Note that as distance D goes to ∞, this amount of work converges to

$$\int_R^\infty \frac{GMm}{r^2}\,dr = \frac{GMm}{R}.$$

This is called the *escape energy* of the object in the gravitational field.

Exercise 10.5

Show that $\int_{-\infty}^0 e^x\,dx = -\int_0^1 \ln x\,dx$ in two ways: (i) calculate both integrals, and (ii) geometrically, by graphing both functions and shading the regions whose areas are represented by the two integrals.

Exercise 10.6

a) Find which values of $p \in \mathbb{R}$ have the property that $\int_1^\infty x^p\,dx$ converges.

(b) Find which values of $p \in \mathbb{R}$ have the property that $\int_0^1 x^p\,dx$ converges.

(c) Are there any values of p for which $\int_0^\infty x^p\,dx$ converges?

Exercise 10.7

Determine whether $\int_{-\infty}^0 x^{-2}e^{1/x}\,dx$ converges, and if it does, determine its value.

Exercise 10.8

Find $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2, 3, 4$. Formulate a conjecture about how this sequence would continue if evaluated for larger values of n.

11 Sequences and series

24 October 2016

11.1 Sequences

The perfect squares, arranged in increasing order, form a **sequence**:

$$0, 1, 4, 9, 16, \ldots.$$

We say that 0 is the *zeroth term* of this sequence, 1 is the *first* term, and so on.

We could also denote this sequence by $(n^2)_{n=0}^\infty$, which we read as "the sequence whose nth term is equal to n^2, where n ranges from 0 to ∞." We could also write the sequence as $(a_n)_{n=0}^\infty$, where $a_n = n^2$.

Specifying a particular sequence amounts to saying for every index n what the nth term of the sequence. In other words, a sequence may be thought of as a *function* which takes integers as inputs and gives real or complex numbers as outputs.

Loosely speaking, we say that a sequence $(a_n)_{n=0}^{\infty}$ **converges** if there is some number L such that a_n is very close to L whenever n is very large. More precisely:

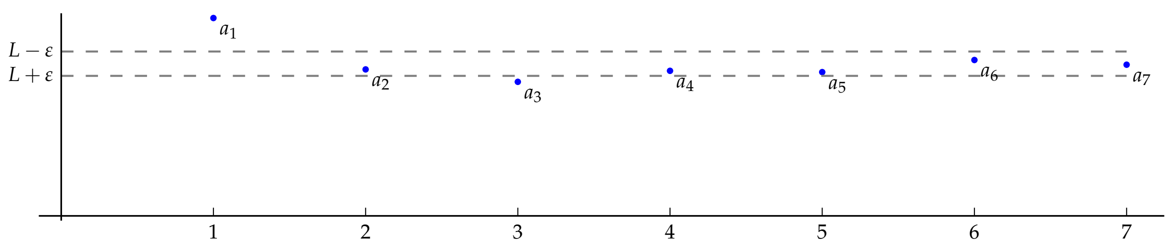
Definition 11.1

The sequence $(a_n)_{n=0}^{\infty}$ converges to L if for all $\varepsilon > 0$, there exists N such that a_n is within a distance of ϵ from L (in other words, $|a_n - L| < \epsilon$) for all $n \geq N$.

You can think of Definition 11.1 as a game: an adversary chooses a tolerance ϵ as small as they like, and after seeing their choice of ϵ you have to choose N so that all the terms from the Nth term onward are within that tolerance ϵ of L. If $(a_n)_{n=0}^{\infty}$ converges to L, we write "$a_n \to L$ as $n \to \infty$" or $\lim_{n\to\infty} a_n = L$.

Example 11.1

In the context of Definition 11.1, find a value of N which works for the sequence $(a_n)_{n=1}^{\infty}$ and the particular value of ε illustrated in the figure below.



Solution

We see that a_1 and a_3 are not in the interval $(L - \varepsilon, L + \varepsilon)$, but a_2, a_4, a_5, a_6 and a_7 are. The figure does not show the rest of the sequence (a_8, a_9, and so on), but assuming that they also fall between the two dashed gray lines, we would be able to say that a_n is within ε of L for all $n \geq N = \boxed{4}$*.

4 is the least value of N that works, since a_3 is not within ϵ of L. But any value of N larger than 4 would also work.

Example 11.2

Show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Solution

We write $\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$. Therefore, the distance from $a_n = \frac{n}{n+1}$ to its purported limit is equal to $\frac{1}{n+1}$. This expression can clearly be made as small as desired by taking n large enough; for example, if we let N be any integer larger than $1/\epsilon$, then $\frac{1}{n+1} < \epsilon$ for all $n \geq N$.

The following theorem gives a simple condition which implies convergence.

Theorem 11.2: (Bounded monotone sequence theorem)

If $(a_n)_{n=1}^{\infty}$ is **bounded** (meaning that there exists B so that $-B \leq a_n \leq B$ for all n) and **monotone** (meaning that either $a_{n+1} \geq a_n$ for all n, or else $a_{n+1} \leq a_n$ for all n), then $(a_n)_{n=1}^{\infty}$ converges.

We say "increasing" to mean $a_{n+1} \geq a_n$ for all n; sometimes this is called "weakly increasing" since we use the weak inequality $\geq$ rather than its strict version $>$. If we want to require $a_{n+1} > a_n$ for all n, we will say "strictly increasing".

The idea of the proof is to take L to be the least number which is greater than or equal to all the terms a_n (let's assume the sequence is increasing*; the decreasing case is similar). Then the sequence terms can't exceed L by definition, and it also can't leave a gap below L which is free of a_n's, because otherwise we could have chosen L a bit smaller. So the sequence converges to L.

The bounded monotone sequence theorem tells us that an increasing sequence must either diverge to $+\infty$ or converge. The first case occurs if the sequence is not bounded, and the second occurs if the sequence is bounded.

Exercise 11.3

Write out the first few terms of each sequence. For (a), (b), and (c), determine whether the sequence converges.

(a) $\left(\dfrac{n}{n^2+1}\right)_{n=1}^{\infty}$

(b) $(k^2+k)_{k=3}^{\infty}$

(c) $(2^{-n})_{n=1}^{\infty}$

(d) $\left(\dfrac{p^3}{3^p}\right)_{p=0}^{\infty}$

Interlude 2 (Summation Notation)

Suppose we want to write an expression like

$$1+4+9+16+25+36+49+64+81+100$$

in a less tedious way. We could write $1+4+\cdots+100$, but that solution is not ideal because it leaves some ambiguity in the omitted terms.

A more robust solution is to find an expression that generates the sequence and use sequence notation: the above sum can be represented as "the sum of $(n^2)_{n=1}^{10}$". We can abbreviate even further by changing "sum" to Σ and decorating the Σ with the sequence bounds, like this:

$$1+4+9+16+25+36+49+64+81+100 = \sum_{n=1}^{10} n^2.$$

Exercise 11.5

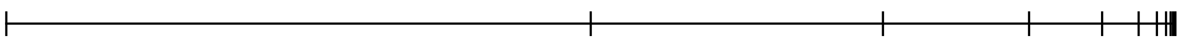
Evaluate each of the following sums. Hint for (b): add the first and last terms, the second and next-to-last terms, and so on.

(a) $\displaystyle\sum_{n=1}^{4} \frac{1}{n}$ (b) $\displaystyle\sum_{n=1}^{100} n$

11.2 Geometric series

“That which is in locomotion must arrive at the half-way stage before it arrives at the goal.”’—Aristotle

Zeno’s paradox argues that motion is impossible, because to move from A to B one must first move halfway from A to B, then halfway from that point to B, and so on.



In other words, motion requires the completion of infinitely many tasks. Since this is presumed to be impossible, then motion itself is impossible.

Let’s frame this supposed paradox more precisely. Suppose that the distance from A to B is 1 meter, and that we move at a rate of 1 meter per second. Then the amount of time it takes to go halfway from A to B is $\frac{1}{2}$ seconds, the amount of time to go half the remaining distance is $\frac{1}{4}$ seconds, then $\frac{1}{8}$ seconds, and so on.

So the total amount of time it takes from A to B should be the *sum* of all these times:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

What does it mean to add infinitely many positive quantities together? Zeno’s argument might be that it’s impossible, or perhaps that the question doesn’t even make sense. But we have a way to make this concept precise, using convergence of sequences discussed in the previous section.

The sum of the first two terms in the sequence $(2^{-n})_{n=1}^{\infty}$ is $\frac{3}{4}$, the sum of the first three terms is $\frac{7}{8}$ the sum of the first four terms is $\frac{15}{16}$, and so on. Generally, the sum of the first n terms* is $1 - \frac{1}{2^n}$. It makes sense, then, to say that the sum of all the terms is

We call the sum of the first n terms the **nth partial sum**

$$\lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

So even though there are infinitely many steps, the sum of their durations can be finite. We call a sum involving infinitely many terms an **infinite series**. Let’s record this idea as a definition.

Definition 11.3

We say that the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges** if and only if the sequence of partial sums $S_N = \displaystyle\sum_{n=1}^{N} a_n$ converges as $N \to \infty$.

Example 11.6

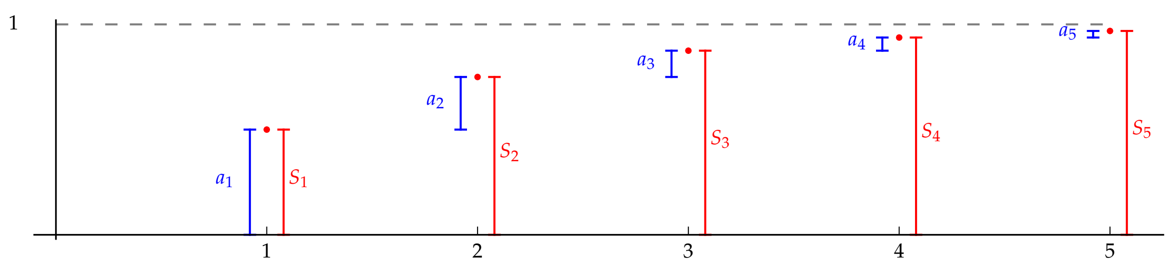
Draw a graph of the sequence of partial sums of the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, and draw line segments in the resulting figure whose lengths are equal to the *terms* of the series.

Solution

We observe that the difference between two consecutive partial sums is

$$S_n - S_{n-1} = (a_1 + \cdots + a_{n-1} + a_n) - (a_1 + \cdots + a_{n-1}) = a_n,$$

so we can recognize the terms of a series as the **differences between successive partial sums**. So our graph looks like this:



Example 11.7

Suppose that Achilles is running at a speed of 1 meter per second and chasing a tortoise which runs at $r < 1$ meters per second and which received a head start of a meters. Express the amount of time it takes Achilles to catch the tortoise as an infinite series, and find its value.

Solution

We want to find the time t such that Achilles' location at time t is equal to the tortoise's location at time t. Achilles' distance from his starting point is equal to 1 meter per second times t, so just t. The tortoise's distance from Achilles starting point is equal to

$$a + rt.$$

Solving $t = a + rt$ gives $t = \frac{a}{1-r}$. Intuitively, this formula makes sense, because if r is very small then Achilles should catch the tortoise in about a seconds, whereas if r is very close to 1, then it will take a long time.

On the other hand, it takes Achilles a seconds to reach the tortoise's original location. At this point, the distance between Achilles and the tortoise will be ar meters. It takes Achilles ar seconds to cover this distance, at which point the tortoise will have moved an additional $(ar)(r) = ar^2$ meters, and so on. So the total time to catch the tortoise is

$$a + ar + ar^2 + \cdots,$$

and we have already figured out this equals $\frac{a}{1-r}$.

Exercise 11.8

(a) Show that $a + ar + ar^2 + \cdots = \frac{a}{1-r}$ whenever $-1 < r < 1$ by showing that

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{ar^n - a}{r-1}. \tag{11.1}$$

for all $r \neq 1$. Hint: set the left-hand side of (11.1) equal to S_n, multiply both sides of the resulting equation by r, and subtract to get a lot of cancellation.

(b) Determine whether $\sum_{n=0}^{\infty} ar^n$ converges in the case where $r \leq -1$ or $r \geq 1$.

Computational Investigation 4

We saw in Exercise 11.8 that when $-1 < x < 1$, we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots. \tag{11.2}$$

If we integrate both sides* of (11.2) from 0 to t, we get

Let's assume for now that the integral distributes across the infinite sum on the right-hand side.

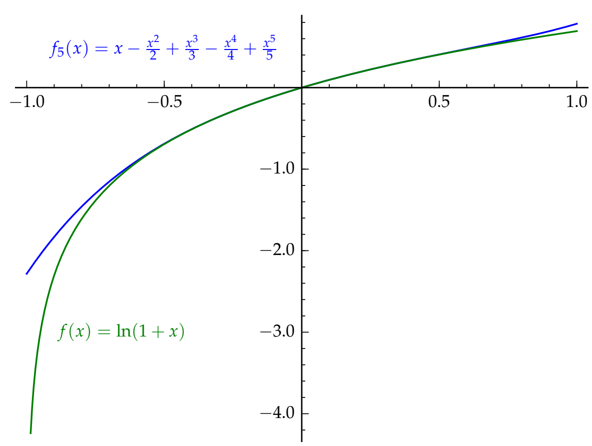
$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}. \tag{11.3}$$

This suggests that if we want to approximate the natural logarithm of a number close to 1, we can evaluate a few terms on the right-hand side and get pretty close. Here's an example:

```
var("x n") # declare x and n to be variables
ln(1.1), sum( (-1)^(n+1)*x^n/n, n, 1, 5).subs(x=0.1)
```

This returns $(0.0953101798043249, 0.0953103333333333)$, which suggests that the first five terms do a pretty good job of approximating $\ln(1+x)$ around $x = 0$. We can get a better sense of this by plotting the sum of the first five terms along with the original function $\ln(1+x)$, like this:

```
plot(sum((-1)^(n+1)*x^n/n, n, 1, 5),x,-1,1) + plot(ln(1+x),x,-1,1,color='green')
```



Exercise 11.9

(a) Differentiate both sides of (11.2) to come up with an infinite series representation of $\frac{1}{(1+x)^2}$.

(b) Square both sides of (11.2) (the right-hand side will be a mess, but just work out the first few terms).

(c) Comment on the relationship between your answers to (a) and (b).

Exercise 11.10

Substitute x^2 for x in (11.2) and integrate the resulting expression to show that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Substitute $x = 1$ into this equation to derive a neat identity involving π.

11.3 Convergence Tests

In the following sections we will develop a variety of techniques for determining whether a series converges. Sometimes, however, a problem is easy enough that we don't need any special techniques:

Example 11.11

Determine whether each series converges.

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n}$

Solution

The partial sums of (a) are

$$1, 0, 1, 0, 1, 0, \ldots.$$

This sequence obviously does not converge. This is an example of an **oscillatory** divergence: there are numbers* $a < b$ such that the sequence has infinitely many terms less than a and infinitely many terms greater than b.

Like $a = 1/3$ and $b = 2/3$ in this case; all the zero terms are less than $1/3$ and all the 1 terms are greater than $2/3$

The partial sums of (b) are

$$2, \frac{7}{2}, \frac{29}{6}, \frac{73}{12}, \frac{437}{60} \ldots$$

These numbers are clearly growing without bound; in fact, the number we add each time is greater than 1. This means that the nth partial sum is greater than n, which implies that the series diverges* to $+\infty$.

These are the two ways a real series can diverge: (i) oscillation, or (ii) divergence to $+\infty$ or to $-\infty$

Let's record this observation as a theorem:

Theorem 11.4: (nth term test)

If a_n does not converge to zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

11.3.1 Integral test

In Example 11.11, the series diverged by virtue of failing the nth term test. Is it possible for a series to pass the nth term test and still diverge?

Example 11.12

Determine whether $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

Solution

If we work out the first few partial sums of this series, the calculations become a mess and no discernible pattern emerges. We could try using a computer to get a sense of what's going on:

```
var("n"); [sum(1.0/n,n,1,N) for N in (100,1000,10000,100000)]
```

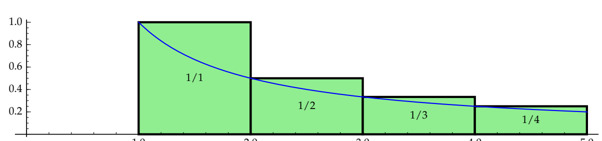
This returns

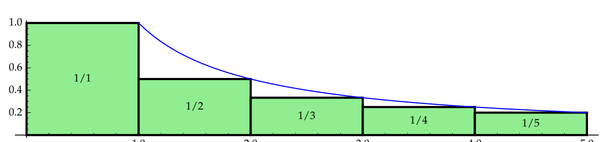
```
[5.187377517639621,
 7.485470860550343,
 9.787606036044348,
 12.090146129863335]
```

which gives us a hint that the series does not converge but is not conclusive*.

For contrast, replacing `1.0/n` with `1.0/n^2` gives 1.634983900184, 1.643934566681, 1.6448340718480, 1.644924066898. This is what convergence looks like.

An important observation is that the sequence of partial sums is increasing (since we're always adding positive terms), and therefore it can only fail to converge by diverging to ∞. Another thing we might recognize is that $f(x) = 1/x$ is much easier to integrate than $1/n$ is to sum, and we would expect them to behave similarly. Consider the following two figures:





The graph of $f(x) = 1/x$ is shown in blue in both pictures, and the desired sum is represented as a sum of areas of the green rectangles. If we draw the rectangles starting from $x = 1$ as in the first figure, we see that the sum of all the green areas exceeds the area under the blue curve. If we draw the rectangles starting from $x = 0$, the we see that the sum of the green areas *minus one* (for the first block) is less than the area under the blue curve.

So if $\int_1^\infty (1/x)\,dx$ converges, then the second figure shows that $\sum_{n=1}^\infty 1/n$ also converges. If $\int_1^\infty (1/x)\,dx$ diverges, then the first figure shows that $\sum_{n=1}^\infty 1/n$ also diverges.

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With this observation, the problem becomes easy. We have $\int_1^\infty (1/x)\,dx = \lim_{b\to\infty} \ln b = \infty$, so $\sum_{n=1}^\infty \frac{1}{n} = \infty$ too.

Let's record this observation as a theorem.

Theorem 11.5: (Integral test)

If $f(x)$ is a positive decreasing function defined on $[1, \infty)$, then $\int_1^\infty f(x)\,dx$ and $\sum_{n=1}^\infty f(n)$ either both converge or both diverge.

Exercise 11.13

Determine the convergence or divergence of $\sum_{n=1}^\infty n^p$, where p a real number. In other words, determine which values of p have the property that $\sum_{n=1}^\infty n^p$ converges.

Exercise 11.14

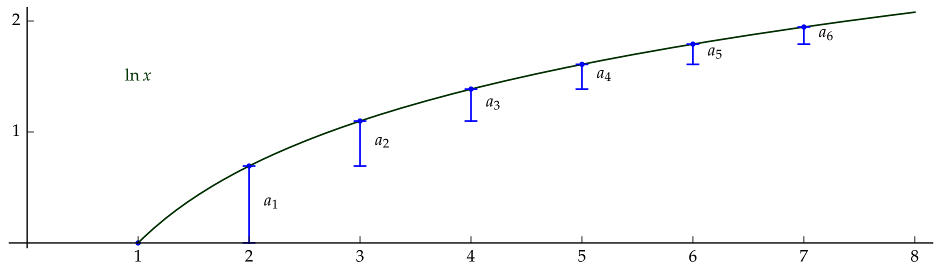
To see why we must assume that f is decreasing in Theorem 11.5, consider the function $f(x) = 1 - \cos(2\pi x)$ and show that $\int_1^\infty f(x)\,dx$ diverges while $\sum_{n=1}^\infty f(n) = 0$.

Exercise 11.15

Show that $\sum \frac{1}{n \ln n}$ diverges, while $\sum \frac{1}{n(\ln n)^2}$ converges.

Interlude 3 (Sequences can converge to zero but sum to infinity)

It might be surprising that even though the individual terms $\frac{1}{n}$ converge to 0 as $n \to \infty$, the sum $\sum_{n=1}^N \frac{1}{n}$ grows without bound as $N \to \infty$. However, it is possible to come up with a more intuitively clear example of a divergent series whose terms converge to zero.



Consider the graph of the natural logarithm function, shown above. A couple of features of this graph are already familiar to you:

1. **The graph of the natural logarithm eventually crosses any horizontal line**. In other words, $\ln x$ grows without bound as $x \to \infty$. This is true because the logarithm is the inverse of e^x, which is defined for all real numbers x (in other words, the graph of the exponential function includes points with arbitrarily large x-coordinates, so the graph of the logarithm includes points with arbitrary large y-coordinates).

2. **The "jumps" $a_n = \ln(n+1) - \ln n$ shown in the figure tend to zero as $n \to \infty$**. This can be seen by writing $a_n = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right)$, which converges to $\ln 1 = 0$ by the continuity of the natural logarithm. Alternatively, observe that the derivative* of $\ln x$ is $1/x$, which tends to zero as $x \to \infty$.

You would apply the mean-value theorem to make this line of reasoning rigorous.

So the second observation tells us $a_n \to 0$, while the first observation tells us that $\sum_{n=1}^{N} a_n$ nevertheless increases without bound as $N \to \infty$ (in other words, $\sum_{n=1}^{\infty} a_n$ diverges to ∞).

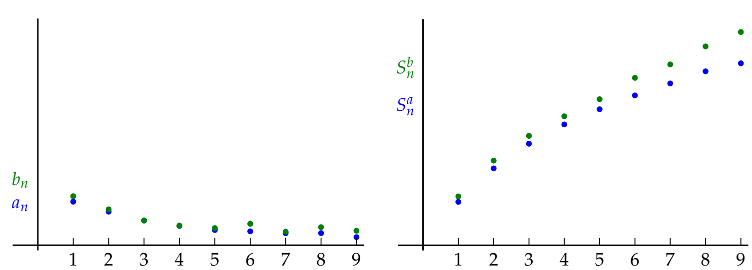
11.3.2 Comparison test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series for which (i) all the terms a_n and b_n are positive, and (ii) $a_n \leq b_n$ for all n. Then we have

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n, \tag{11.4}$$

Informal language like "smaller" should be taken to mean in the weak sense, namely "less than or equal to", rather than "strictly less than".

for the straightforward reason that if you add up smaller numbers you get a smaller result*:



Note that (11.4) implies the following result about the convergence of $\sum a_n$ and $\sum b_n$.

Theorem 11.6: Comparison test

Suppose that $0 \leq a_n \leq b_n$ for all n. Then* $\sum_{n=1}^{\infty} a_n = \infty$ implies $\sum_{n=1}^{\infty} b_n = \infty$, and $\sum_{n=1}^{\infty} b_n < \infty$ implies $\sum_{n=1}^{\infty} a_n < \infty$.

When the terms a_n of an infinite series are nonnegative, the sequence of partial sums is nondecreasing and therefore either (i) convergent or (ii) divergent to ∞ (which we may write as (i) $\sum a_n < \infty$, or (ii) $\sum a_n = \infty$).

Example 11.17

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - e^{-n}}$ diverges.

Solution

Since we already know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we'd like to compare these two series. Since e^{-n} is always positive, we have $n - e^{-n} < n$. This means that $\frac{1}{n-e^{-n}} > \frac{1}{n}$ for all n.

By Theorem 11.6, then, the divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ implies that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - e^{-n}}$ diverges too.

Example 11.18

Use the comparison test to show that $\displaystyle\sum_{n=11}^{\infty} \frac{1}{n^2 - 100}$ converges.

Solution

We suspect that the 100 term is irrelevant to convergence, since it's being subtracted from a much larger n^2 term which dominates the denominator. However, if we try to drop the -100 term, that makes the denominator larger, which makes the overall sum smaller. So if we try compare to $\sum_{n=11}^{\infty} \frac{1}{n^2}$, we find that the original sum is *greater* than a convergent series, which tells us nothing.

Instead, what we can do is (1) throw away the first few terms* and study convergence of $\sum_{n=15}^{\infty} \frac{1}{n^2-100}$ instead, and (2) replace 100 with an expression like $\frac{1}{2}n^2$. That way we're subtracting *more* in the denominator, which makes the overall sum larger*. So we get

$$\sum_{n=15}^{\infty} \frac{1}{n^2 - 100} < \sum_{n=15}^{\infty} \frac{1}{n^2 - \frac{1}{2}n^2} = \sum_{n=15}^{\infty} \frac{1}{\frac{1}{2}n^2} = \frac{1}{\frac{1}{2}} \sum_{n=15}^{\infty} \frac{1}{n^2} < \infty.$$

For the last step, recall that $\sum \frac{1}{n^2} < \infty$ by the integral test.

The first several terms add up to something finite; therefore, the convergence or divergence of the sum does not depend on whether they are included.

Note that $\frac{1}{2}n^2 > 100$ only when $n \geq 15$. That is why we threw away the first few terms.

Exercise 11.19

The comparison test for sums has a straightforward analogue for integrals: if $0 \leq f(x) \leq g(x)$, then convergence of $\int_1^{\infty} g(x)\,dx$ implies convergence of $\int_1^{\infty} f(x)\,dx$.

Use this integral comparison test to show that $\int_1^\infty e^{-x^2}\,dx$ converges.

Exercise 11.20

Show that $\sum_{n=1}^\infty \frac{1}{n^{n/20}}$ converges.

11.3.3 Ratio test

30 October

Recall factorial notion: if we put an exclamation point after a nonnegative integer, that means multiply all the positive integers less than or equal to that integer*. For example,

0! = 1, since 1 is the "empty product"

$$3! = 3 \cdot 2 \cdot 1 = 6 \qquad \text{and} \qquad 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

Example 11.21

Determine whether $\displaystyle\sum_{k=1}^\infty \frac{10^k}{k!}$ converges.

Solution

We might guess that $\frac{10^k}{k!}$ goes to zero fast enough for this series to converge, because, for example:

$$\frac{10^{1000}}{1000!} = \frac{10 \cdot 10 \cdot \cdots \cdot 10}{1000 \cdot 999 \cdot \cdots \cdot 1},$$

and the most of the factors in the denominator are much larger than 10.

Said another way, the factor we need to multiply by to get from one term to the next (let's say from the $(k-1)$st term to the kth) is $10/k$. We already know from our study of geometric series that a series converges if each term is obtained from the previous one by multiplying by a *constant* factor, as long as that factor is less than 1. Since $10/k$ is much less than 1 when k is large, it should be pretty easy for us to show that this series converges. Let's do this carefully.

Since $10/k$ is less than $1/2$ for all $k \geq 21$, we can start at the 20th term and multiply by $1/2$ each time instead of $10/k$, and the resulting sequence will be greater than the original one: for all $k \geq 20$, we have

$$\frac{10^k}{k!} \leq \frac{10^{20}}{20!}\left(\frac{1}{2}\right)^{k-20}.$$

So,

$$\sum_{k=21}^\infty \frac{10^k}{k!} \leq \frac{10^{20}}{20!}\sum_{k=21}^\infty \left(\frac{1}{2}\right)^{k-20} < \infty,$$

since a geometric series with common ratio $\frac{1}{2}$ converges. By the comparison test, then, we conclude that $\sum_{k=1}^\infty \frac{10^k}{k!}$ converges.

Let's organize our ideas from this example and come up with a general test we can use.

Theorem 11.7: Ratio test

Suppose that $a_n \geq 0$ for all n, and suppose that $\dfrac{a_{n+1}}{a_n}$ converges to r as $n \to \infty$. Then
(i) if $0 \leq r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges
(ii) if $1 < r \leq \infty$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

The ratio test tells us nothing new if $r > 1$, since in that case the series already fails the nth term test. When $r < 1$, the proof works like our solution to Example 11.21: choose any value r' strictly between r and 1, go far enough into the series that the ratio of successive terms is less than r' from that point onward, and apply the comparison test with a geometric series whose common ratio is r'.

If $r = 1$, the ratio tests tells us nothing *at all*:

Exercise 11.22

Show that the divergent series $\sum \frac{1}{n}$ and the convergent series $\sum \frac{1}{n^2}$ both give $r = 1$ in the ratio test.

It is convenient sometimes to omit the summation limits when discussing questions of convergence, with the understanding that the upper limit is ∞ and the lower limit is irrelevant to convergence

Example 11.23

Redo Example 11.21 using the Ratio Test.

Solution

We calculate*

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k} = \lim_{k\to\infty} \frac{10}{k+1} = 0.$$

Since $0 < 1$, the ratio test tells us that the series converges.

Note that $(k+1)! = (k+1) \cdot k!$. This technique of peeling off one or more factors from a factorial expression is very common in applications of the ratio test.

Exercise 11.24

Apply the ratio test to determine convergence or divergence of each of the following series.

(a) $\displaystyle\sum \frac{n^{100}}{2^n}$ (b) $\displaystyle\sum \frac{n^3 - 3n + 1}{3^n - n^2 - 1}$

(c) $\displaystyle\sum \frac{n!}{n^n}$ (d) $\displaystyle\sum \frac{(n!)^2}{(2n)!}$

Hint for (c): when dealing with $(n+1)^{n+1}$, first split one off factor of $n+1$. Then use (8.1) in Exercise 8.13.

Think "ratio test" when you see an infinite series involving factorials.

11.3.4 Alternating series test

2 November

So far we have mostly dealt with sums involving positive terms. Let's introduce some terminology for dealing with negative terms:

Definition 11.8

If a series $\sum a_n$ has the property that $\sum |a_n|$ converges, then we say that $\sum a_n$ **converges absolutely**, or that it is **absolutely convergent**. A convergent series which is not absolutely convergent is said to be **conditionally convergent**.

The following theorem justifies the terminology "absolutely convergent".

Theorem 11.9: Absolute convergence test

If a series converges absolutely, then it converges.

The idea* of the proof of this theorem is that since $-|a_n| \le a_n \le |a_n|$, we know that $\sum a_n$ is between $-\sum |a_n|$ and $\sum |a_n|$ and therefore can't go off to ∞ or $-\infty$. Furthermore, if for some $b < c$ the sequence of partial sums of $\sum a_n$ oscillates below b and above c a total of T times, then the sum of the "jump sizes" $|a_n|$ has to be at least $(c-b)T$. So any interval $[b, c]$ can be crossed no more than $\frac{\sum_{n=1}^{\infty} |a_n|}{c-b}$ times (in particular, finitely many times), which means $\sum a_n$ can't have an oscillatory divergence either. This paragraph is optional.

Example 11.25

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges.

Solution

The sequence

$$\left(\frac{(-1)^{n+1}}{n^2}\right)_{n=1}^{\infty} = +1, \quad -\frac{1}{4}, \quad +\frac{1}{9}, \quad -\frac{1}{16}, \quad +\frac{1}{25}, \quad \cdots$$

is an example of an *alternating sequence*, meaning that every other term is positive and the rest are negative. Since there are negative terms, let's check whether the series is absolutely convergent, with hopes of applying Theorem 11.9. Turning every term positive gives us

$$\sum \left|\frac{(-1)^{n+1}}{n^2}\right| = \sum \frac{1}{n^2}.$$

Using the integral test, we see that this series converges. So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges too.

Note that Theorem 11.9 does *not* say that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge to the same value. In fact, if any of the a_n terms are negative, the sum $\sum_{n=1}^{\infty} a_n$ is smaller than $\sum_{n=1}^{\infty} |a_n|$. For example, it turns out that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

Wait, how'd π get in there?

while

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}.$$

The following theorem gives a simple test for convergence of an **alternating series** (a series whose terms' signs alternate):

Theorem 11.10: Alternating series test

If $(a_n)_{n=1}^{\infty}$ is a positive, decreasing sequence converging to zero, then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

The idea of the proof of this theorem is to note that the kth partial sums $S_k = \sum_n^k a_n$ have the property that

(i) the even-indexed partial sums S_{2k} form an increasing sequence, and

(ii) the odd-indexed partial sums S_{2k+1} form a decreasing sequence.

By the bounded monotone sequence theorem (Theorem 11.2), the even partial sums converge to some limit L, and the odd ones converge to some limit L'. Furthermore,

$$L' - L = \lim_{k\to\infty} S_{2k+1} - \lim_{k\to\infty} S_{2k} = \lim_{k\to\infty} (S_{2k+1} - S_{2k}) = \lim_{k\to\infty} a_{2k+1} = 0.$$

(Said another way, if $L \neq L'$, then the partial sums couldn't keep crossing the gap between L and L' with jump sizes $|a_n|$ converging to 0). So $L = L'$, and that means that $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges to L.

Example 11.26

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

Solution

Recall that conditional convergence means that the series converges and that it does not converge absolutely. To check that the series does not converge absolutely, we note that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

To show that the series does converge, we check the conditions of the alternating series test: (i) the series is alternating, (ii) the terms $\frac{(-1)^{n+1}}{n}$ go to 0, and (iii) the terms are decreasing in absolute value (in other words, $1/n$ is a decreasing sequence). These conditions hold, so Theorem 11.10 tells us that the series does converge.

Example 11.27

Show that the alternating series test does not imply that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n}{n}$ converges.

Solution

At first glance, this series seems to be ideal for the alternating series test. It includes the factor $(-1)^{n+1}$, and

$$(-1)^{n+1}\frac{\sin n}{n} \to 0,$$

since $(-1)^{n+1}\sin n$ is bounded between -1 and 1. However, let's check the values of the sequence with Sage:

```
[(-1.0)^(n+1)*sin(n)/n for n in range(10)]
```

This returns

```
[0.841470984807897,
 -0.454648713412841,
 0.0470400026866224,
 0.189200623826982,
 -0.191784854932628,
 0.0465692496998210,
 0.0938552283883984,
 -0.123669780827923,
 0.0457909428046396]
```

We see that the signs don't alternate. Actually we could have recognized that right away: the factor $\sin n$ is not always positive. So the alternating series test does not apply.

Techniques we will learn later in the course can be used to show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n}{n}$ does converge. Prior to developing those techniques, though, we can still sum the series using Sage:

```
var("n"); sum((-1)^(n+1)*sin(n)/n,n,1,oo)
```

returns

```
1/2*arctan(sin(1)/cos(1)).
```

Since $\sin 1/\cos 1 = \tan 1$, we see that the series converges to $1/2$.

I'm not sure why Sage doesn't make this simplification on its own.

Exercise 11.28

Determine whether each series is conditionally convergent, absolutely convergent, or divergent.

(a) $\sum_{n=1}^{\infty}(-1)^n n^{1/n}$

(b) $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{\sqrt{n}}$

(c) $\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{1}{n^2}\right)$

(d) $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n^2}$

Hint: recall that to show that $f(x)$ is a decreasing function over a particular interval, it suffices to show that $f'(x) < 0$ for all x in that interval.

12 Taylor Series

4 November

12.1 Linear and quadratic approximation

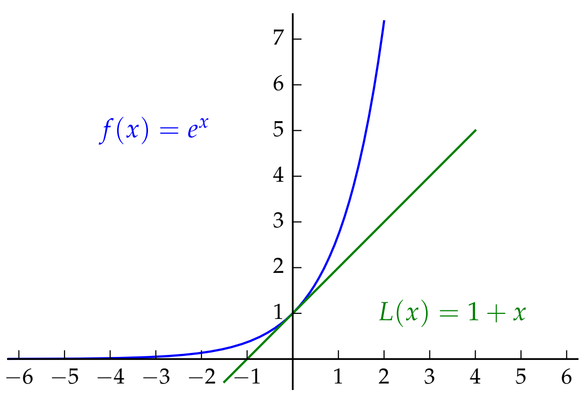
It is often useful to approximate a function with a *linear* function, since linear functions are very simple. For example, when we came up with a formula for arc length in Section 6, we did it by approximating the graph with a bunch of short line segments. Similarly, the trapezoid rule for approximating the integral of a function works by approximating the function using many linear functions. Many of the principles and formulas developed in a multivariable calculus class are also based on this idea.

Note that if the graph of our function is very curvy, it will be impossible to approximate it well everywhere at once using a line. So linear approximation is *local*, meaning that we have to choose a particular x-value, called the *center*, for which our approximation will be good near x.

The linear approximation L to a differentiable function f with center c must pass through the point $(c, f(c))$, and for the best approximation we should choose the slope to be equal to the derivative of f.

Example 12.1

Find the function $L(x)$ which best approximates $f(x) = e^x$ near $x = 0$.



Solution

Clearly the y-intercept of the approximating line should be $f(0) = 1$. Furthermore, the slope of the line is $f'(0) = 1$, so $L(x) = 1 + x$.

7 November

Generalizing Example 12.1, we get

$$\boxed{L(x) = f(c) + f'(c)(x - c).} \tag{12.1}$$

One way to view this formula is to note that $L(c) = f(c)$ and $L'(c) = f'(c)$; in other words, the **zeroth and first derivatives of L and f match up at c**. The second derivatives are not necessarily equal, since the second derivative of a linear function is zero everywhere while $f''(c)$ may be nonzero. So we came up with the best linear approximation by matching up as many derivatives at c as we could.

Let's use this idea to step up our game.

Example 12.2

Find the *quadratic* function $Q(x)$ which best approximates $f(x) = e^x$ near $x = 0$.

Solution

Let's interpret "best approximates" to mean "matches as many derivatives as possible". We can start with the general form of a quadratic:

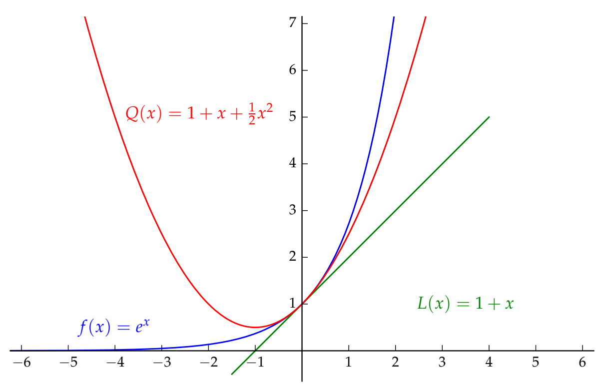
$$Q(x) = a_0 + a_1 x + a_2 x^2.$$

We see that $Q(0) = a_0$, so if we want $Q(0) = f(0) = 1$, we must set $a_0 = 1$. Similarly, $Q'(0) = a_1$, so if we want $Q'(0) = f'(0) = 1$, have to choose $a_1 = 1$ as well.

For a_2, we calculate $Q''(x) = (a_1 + 2a_2 x)' = 2a_2$, so to get $Q''(0) = f''(0) = 1$, we have to let $a_2 = \frac{1}{2}$. So

$$Q(x) = 1 + x + \tfrac{1}{2}x^2$$

is the best we can do. Looking at the inset figure, we set Q does indeed do a better job of 'hugging' the graph of f near $x = 0$ than L does.



Let's put these ideas together in a definition.

Definition 12.1: (Linear and quadratic approximation)

If f is a twice-differentiable function, then we define the linear approximation of f at $x = c$ to be

$$L(x) = f(c) + f'(c)(x - c)$$

and the quadratic approximation of f at c to be

$$Q(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2.$$

Exercise 12.3

Use a calculator to check that the $Q(0.01)$ is closer to $f(0.01)$ than $L(0.01)$ is, in the context of Example 12.2.

Exercise 12.4

Find the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 2$.

Exercise 12.5

For each of the following functions f, find the best linear and quadratic approximations of f near the given value of c.

(a) $f(x) = \ln x; \quad c = 1$

(b) $f(x) = \sin x; \quad c = 0$

(c) $f(x) = \cos x; \quad c = \pi$

(d) $f(x) = \dfrac{1}{x}; \quad c = 1$

12.2 Taylor approximations

A natural follow-up question to the previous section on linear and quadratic approximation is "what about cubic approximation?" And what about fourth-order approximations and so on? We will see that the ideas behind linear and quadratic approximation extend in a straightforward way to higher order terms.

Example 12.6

Find the best cubic approximation to $f(x) = e^x$ near $x = 0$.

Solution

We can do this problem the same way we did Example 12.2. We consider a general cubic polynomial:

$$P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3,$$

and just as before we find that $a_0 = 1$, $a_1 = 1$, and $a_2 = 1/2$. To find a_3, we differentiate three times and get

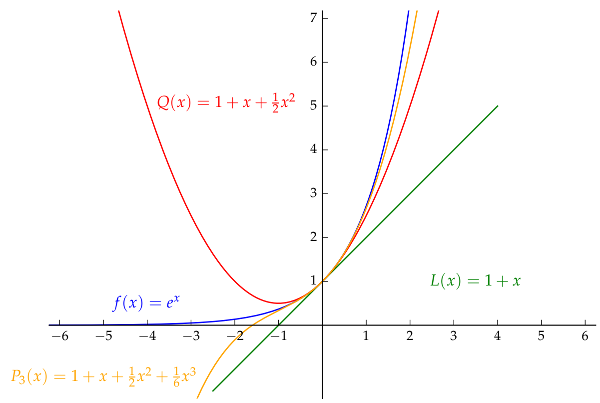
$$P_3'''(0) = 6a_3.$$

If this is to equal $f'''(0) = 1$, then a_3 must be $1/6$. So

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

is the best cubic approximation of $f(x) = e^x$.

The graph below shows the cubic approximation of e^x. Note that it hugs the graph of e^x better than the linear and quadratic approximations.



In full generality, we want to approximate a function f with a polynomial P_n of degree n near $x = c$. In other words, we want to find

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$$

so that all derivatives of P_n at c, from the zeroth up to the nth, match those of f.

Exercise 12.7

Find the value of the kth derivative of P_n, evaluated at $x = c$.

The result of Exercise 12.7 shows that the values of $(a_k)_{k=0}^n$ which make all derivatives of P and f up to order n equal at c are

$$a_k = \frac{f^{(k)}(c)}{k!},$$

where $f^{(k)}$ denotes the kth derivative of f. Putting it all together:

Definition 12.2

The nth-order Taylor approximation, centered at c, of an n-times differentiable function f is defined to be

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

09 November

In summation notation, the nth order Taylor approximation of f centered at c is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Exercise 12.8

Find the fifth-order Taylor approximation of $\ln x$ centered at $x = 1$. Compare your answer to (11.3) on page 46.

Exercise 12.9

Find the nth order Taylor approximations of $\sin x$, $\cos x$, and e^x. You may express your answer either in summation notation or using an ellipsis*.

Substitute $x = i\theta$ in the Taylor approximation for e^x, add the Taylor approximation for $\cos x$ to i times the Taylor approximation for $\sin x$. Comment on how your answer relates to Euler's formula (Exercise 8.13).

An ellipsis is three-dot notation which indicate terms continue according to the specified pattern, like $1+2+\cdots+10$

12.3 Taylor series

Infinitely differentiable means that for all n, the nth derivative exists

Since the nth order Taylor approximation of an infinitely differentiable function f approximates the function increasingly well as n increases, we might be inspired to consider what happens if we try setting $n = \infty$. In other words, what if we start writing down Taylor approximation terms and don't stop? This infinite series is called a **Taylor series**. Since the sum is infinite, we have to be concerned about whether the series converges.

Definition 12.3

The Taylor series, centered at c, of an infinitely differentiable function f is defined to be

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

Example 12.10

Find the Taylor series $P_\infty(x)$ for $f(x) = e^x$ centered at $x = 0$, and show that for any fixed real number x, this series converges.

Solution

We use the formula in Definition 12.3:

$$P_\infty(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This expression has an x in it, but if we choose any particular number for x and substitute, it becomes an infinite series of plain numbers. So we can investigate its convergence using our tests from Section 11.3.

Seeing factorials, we think to apply the ratio test. The ratio of the $(k+1)$st term to the kth is $\frac{x}{k+1}$, and as $k \to \infty$ (again, for any *fixed* x), this sequence converges to 0. Therefore, the ratio test tells us that the Taylor series converges.

Taylor series do not always converge for all x:

Example 12.11

Show that the Taylor series for $f(x) = \ln x$ centered at $x = 1$ only converges for x between 0 and 2.

Solution

Using the calculation we did in Exercise 12.8, we find that the Taylor series for $\ln x$ centered at $x = 1$ is

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$$

The ratio of the $(n+1)$st term to the nth term is $\frac{(x-1)(n)}{n+1}$. Taking $n \to \infty$, we get a limiting ratio of $x-1$. The infinite series converges if this ratio is strictly between -1 and 1, which means

$$-1 < x - 1 < 1 \implies 0 < x < 2.$$

Also by the ratio test, the series diverges whenever $x < 0$ or $x > 2$.

The only two values of x where we have not determined convergence are $x = 0$ and $x = 2$. If we substitute $x = 0$, we get

$$-1 - \frac{1}{2} - \frac{1}{3} - \cdots,$$

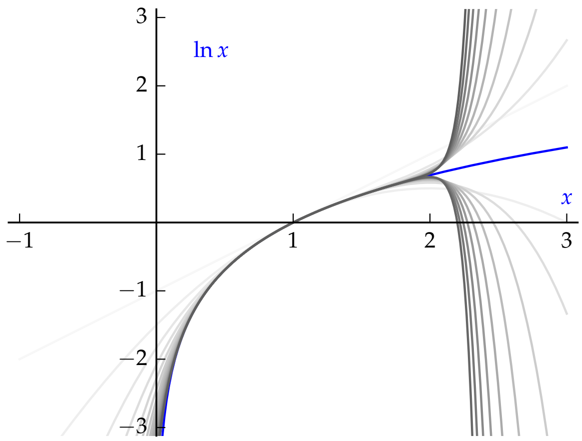
which diverges to $-\infty$ by the integral test. If we substitute $x = 2$, we get

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots,$$

which converges by the alternating series test.

So, in summary, the Taylor series for $\ln x$ converges when $0 < x \leq 2$ and diverges for all other values of x.

Let's investigate the convergence of the Taylor series for $\ln x$ using a graph. The figure below shows Taylor approximations of $\ln x$, in an increasingly dark shade of gray as the order of the approximation ranges from 1 to 20.



This figure makes the divergence at $x = 0$ clear: $\ln x$ diverges to $-\infty$ as x goes to 0, so if the Taylor series is approximating the function increasingly well, it makes sense that it would also go to $-\infty$. On the other hand, the divergence beyond $x = 2$ is a bit mysterious: the function is not doing anything weird around $x = 2$, yet the Taylor approximations are nevertheless approximating the function *worse* as the order increases.

It turns out that the important feature of the value $x = 2$ is that it's the same distance from the center $x = 1$ as the vertical asymptote $x = 0$. Let's discuss how this works a bit more generally:

We define the **interval of convergence** for a Taylor series centered at c of an infinitely differentiable function f to be the set of all x values such that that Taylor series converges. To justify our terminology, we need to show that (i) the interval of convergence is actually an interval, and (ii) it is centered at c.

Theorem 12.4

The interval of convergence of a Taylor series centered at c is an interval and is symmetric about c, with the possible exception* of the endpoints.

The caveat about the endpoints is essential, as we saw for $\ln x$, whose interval of convergence about 1 is $(0, 2]$.

Proof

We will show that if the Taylor series* $\sum a_n(x-c)^n$ converges for any particular value $x = x_0$, then it also converges for any value of x which is closer to c than x_0 is.

It won't matter for this proof that $a_n = f^{(n)}(c)/n!$, so we'll just call it a_n

So suppose x is closer to c than x_0 is; in other words suppose $|x - c| < |x_0 - c|$. By the nth term test, convergence of $\sum a_n(x_0 - c)^n$ implies that $a_n(x_0 - c)^n \to 0$ as $n \to \infty$. So for all n large enough that $|a_n(x_0 - c)^n| < 1$, we have

$$|a_n(x-c)^n| = |a_n(x_0-c)^n| \frac{|x-c|^n}{|x_0-c|^n} < \frac{|x-c|^n}{|x_0-c|^n},$$

which is a convergent geometric series, since $|x - c| < |x_0 - c|$. Therefore, by the comparison test, the Taylor series converges absolutely at x.

We define the **radius of convergence** of a Taylor series to be half the length of the interval of convergence. Based on the examples we've done so far, we can say that the radius of convergence for e^x centered at 0 is ∞, and the radius of convergence for $\ln x$ centered at $x = 1$ is 1.

12.4 Manipulating Taylor series

Suppose we know that a Taylor series P_∞ for a function f converges in some interval (a, b). Does it follow that that $P_\infty(x)$ converges to $f(x)$ for all $x \in (a, b)$? Remarkably, the answer is "not always".

Example 12.12

Show that the function with $f(0) = 0$ and $f(x) = e^{-1/|x|}$ for $x \neq 0$ has a zero Taylor series.

This example is something you should be aware of, not necessarily something you should work through the details of.

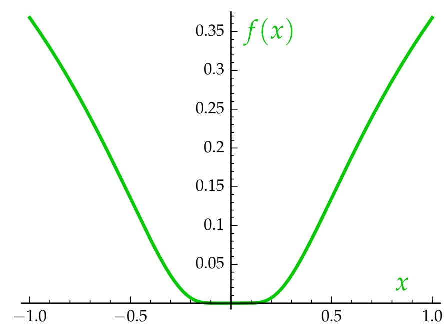
Solution

It is tedious but possible to show that every derivative of f is of the form

$$\begin{cases} p(1/x)e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ p(-1/x)e^{1/x} & \text{if } x < 0, \end{cases}$$

where p is some polynomial. So all the derivatives of f at 0 exist and are 0, and therefore the Taylor series centered at 0 for this function is the zero function.

Graphically, what's going on is that f is coming in toward the origin so incredibly flat that we cannot detect from the derivatives of f at the origin that f is anything different from zero. Since the function is not the zero function (in fact, it's only zero at $x = 0$), we conclude that the Taylor series of this function *does not converge* to the function.



The following theorem, whose proof we omit, gives us a way to say that a Taylor series does indeed converge to a function f.

Theorem 12.5

Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers for which $f(x) = \sum_{n=1}^{\infty} a_n(x-c)^n$ for all x in some interval $(c - R, c + R)$. Then $a_n = \frac{f^{(n)}(c)}{n!}$. In other words, if f can be written in the form $\sum_{n=1}^{\infty} a_n(x-c)^n$, then the a_n's have to be the Taylor coefficients and f's Taylor series converges to f inside the interval of convergence.

An infinite series of the form $\sum_{n=1}^{\infty} a_n(x-c)^n$ is called a **power series**.

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Example 12.13

Find the radius of convergence of $\frac{1}{1+x^2}$ centered at $x = 0$.

Solution

We use the formula for the sum of a geometric series (in reverse) to calculate

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 - \cdots.$$

Furthermore, we know by Theorem 12.5 that this is in fact the Taylor series of $\frac{1}{1+x^2}$.

The common ratio of terms in this series is $-x^2$, so it converges if and only if $-1 < x < 1$. Therefore, the radius of convergence is $\boxed{1}$.

Interlude 4 (Radius of convergence and the complex plane)

There is no vertical asymptote forcing the series for $\frac{1}{1+x^2}$ to have a finite radius of convergence. With other examples without vertical asymptotes, like e^x and $\sin x$, the interval of convergence is the whole real line. So what's going on?

The answer is that $\frac{1}{1+x^2}$ *does* have discontinuities, sort of—the denominator is zero at $x = \pm i$ in the *complex plane*. Those points are a distance of 1 unit from 0, hence the radius of convergence of 1. So here, as we also saw when solving differential equations, complex numbers are peeking in even though we are primarily concerned with real numbers.

It's handy to know that the Taylor series for most common elementary functions f do converge to f. We refer to the Taylor series centered at $x = 0$ as the **Maclaurin series**.

Theorem 12.6

If $f(x)$ is e^x, $\sin x$, or $\cos x$, then the Maclaurin series for f converges to $f(x)$ for all $x \in \mathbb{R}$.

The trick we used in Example 12.13, namely finding the Taylor series without calculating derivatives, is extremely useful. Here are some basic tools for manipulating Taylor series, starting from ones we already know, like those for e^x, $\ln x$, or $\frac{1}{1-x}$:

1. We can multiply or add Taylor series term-by-term.
2. We can integrate or differentiate a Taylor series term-by-term.
3. We can substitute one Taylor series into another to obtain a Taylor series for the composition.

Actually, we want to exclude endpoints of intervals of convergence from this theorem, so "wherever" should be interpreted to mean "inside the intervals of convergence"

Theorem 12.7

All the operations described above may be applied wherever* all the series in question are convergent. In other words, f and g have Taylor series P and Q converging to f and g in some open interval, then the Taylor series for fg, $f+g$, f', and $\int f$ converge in that interval and are given by PQ, $P+Q$, P', and $\int P$, respectively. If P has an infinite radius of convergence*, then the Taylor series for $f \circ g$ is given by $P \circ Q$.

We have to add some hypothesis for P, because we need to be sure we can substitute g's *output* values into P.

Example 12.15

Find the Maclaurin series for $f(x) = \cos x + xe^{x^2}$.

Solution

Taking derivatives is going to be no fun, especially with that second term. What we can do, however, is just substitute x^2 into the Taylor series for the exponential function, multiply that by x, and add the Taylor series for cosine:

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + x\left(1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots\right) = 1 + x - \frac{x^2}{2!} + x^3 + \frac{x^4}{4!} + \frac{x^5}{2!} + \cdots.$$

In summation notation, we could write this series as $\sum_{n=0}^{\infty} a_n x^n$ where a_n is equal to $(-1)^{n/2}/n!$ if n is even and $x^n/((n-1)/2)!$ if n is odd.

Example 12.16

Find the Maclaurin series of $\arctan x$.

Solution

We use arctan's relationship with $\frac{1}{1+x^2}$ and Theorem 12.7:

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + \cdots\,dx = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \cdots.$$

Example 12.17

Use a Taylor series to find $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$.

Solution

We start with*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

and differentiate both sides with respect to x to get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots \tag{12.2}$$

Substituting $x = 1/2$ into both sides, we get $4 = 1 + 2/2 + 3/4 + 4/8 + \cdots$, so $\boxed{2} = 1/2 + 2/4 + 3/8 + 4/16 + \cdots$.

*This seems like a *deus ex machina* [link], but even a small handful of Taylor series (like this one and e^x) and the tools of Theorem 12.7 can help you evaluate a variety of sums.

Exercise 12.18

What does Theorem 12.5 imply about the function in Example 12.12?

Exercise 12.19

Substitute Maclaurin series and simplify to evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2}$ (b) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$

Exercise 12.20

(a) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies the DE $f''(x) = -f(x)$, with $f(0) = 0$ and $f'(0) = 1$. Use the initial data to find a_0 and a_1, then use the differential equation to find a_2. Use the differential equation once again to find a_3, and continue until you recognize the pattern. What is the name of this function f?

Exercise 12.21

Calculate the third derivative of $\frac{1}{1-\sin x}$ at $x = 0$ without taking derivatives (!) by: (i) substituting the Taylor series for $\sin x$ into the Taylor series for $\frac{1}{1-x}$, and (ii) figuring out the coefficient of x^3 for the resulting expression.

13 Fourier series

18 November

13.1 Introduction and examples

We might summarize Taylor series as an answer to the question "how do you approximate a function using* powers of x?" The story has two components: using finitely many powers of x to *approximate* the function (Taylor approximations), and using infinitely many powers of x to represent the function *exactly* (Taylor series). Taylor approximations are partial sums of Taylor series.

"Using" here means "as a linear combination of". A linear combination of a set of functions is a sum of finitely many constant multiples of those functions.

The idea behind Fourier approximations and Fourier series is similar, except that instead of using powers of x, we use *basic waves*. As we discussed in Section 2, the basic waves are the functions

$$\sin x, \sin 2x, \sin 3x, \ldots, \cos x, \cos 2x, \cos 3x, \ldots$$

Since all these functions are 2π-periodic, they cannot possibly do a good job of approximating functions which are not 2π-periodic (see Exercise 13.1 below). Therefore, we will assume through this section that the function f we're trying to approximate is 2π-periodic.*

Contrast this with Taylor series, where we approximated many functions which are not periodic.

Exercise 13.1

Recall that a function f defined on the real number line is said to be 2π-periodic if $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$. Show that every linear combination of basic waves is 2π-periodic.

Fourier series have many applications: if we can decompose a complex sound wave into its constituent frequencies, we can do things like (i) cut out the frequencies that are inaudible to the human ear anyway, enabling compressing of music files. Or (ii) cancel the sound wave by reproducing its opposite using a speaker—this is how noise-cancelling headphones work. Similarly, we can break an image into 8 pixel by 8 pixel blocks, decompose each block into basic waves, and keep only the first few basic waves for each block. This is how the JPEG file format works. Lastly, Fourier series are important for engineering robust bridges, since large coefficients a_n or b_n for some terms indicate that the bridge might resonate and self-destruct in the presence of forces acting on it with frequencies corresponding to those high a_n or b_n values (click here for a video whose entertainment value extends beyond the resonating bridges in it). We will learn more about this in Section 13.3.

We have already learned the most important fact about basic waves, which is important enough to repeat here:

Theorem 13.1

The integral over $[0, 2\pi]$ of any basic wave is zero. Furthermore, the integral over $[0, 2\pi]$ of the product of two basic waves is zero, unless the two waves are the same, in which case the integral is π.

We can use this theorem to derive the formula for Fourier coefficients: let's imagine that the coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ have been selected so that f is represented exactly as an infinite sum of basic waves with those coefficients, namely:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \cdots$$

We want to use this equation to come up with a formula for each coefficient, and our main tool is Theorem 13.1. We can multiply both sides of this equation by $\cos nx$, where n is a positive integer, and integrate over $[0, 2\pi]$ to get*

$$\begin{aligned}\int_0^{2\pi} f(x) \cos nx \, dx = \int_0^{2\pi} a_0 \cos nx \, dx + \int_0^{2\pi} a_1 \cos x \cos nx \, dx + \int_0^{2\pi} a_2 \cos 2x \cos nx \, dx + \cdots \\ + \int_0^{2\pi} b_1 \sin x \cos nx \, dx + \int_0^{2\pi} b_2 \sin 2x \cos nx \, dx + \cdots\end{aligned}$$

*We're assuming here that the integral distributes across the infinite sum on the right-hand side.

By Theorem 13.1, all these integrals on the right-hand side are zero except the term $\int_0^{2\pi} a_n \cos nx \cos nx \, dx$, which is equal to πa_n. So, for all $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx. \tag{13.1}$$

Similarly, we have*

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx, \tag{13.2}$$

for all $n \geq 1$, and simply integrating f over $[0, 2\pi]$ gives us

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx. \tag{13.3}$$

*The 2π-periodicity of all the functions involved means that we can replace the interval of integration with any interval of length 2π, such as $[-\pi, \pi]$.

Equations (13.1), (13.2), and (13.3) tell us how much of each basic wave we should include* if we want to approximate f as a linear combination of basic waves.

There is a recipe analogy being used here: the basic waves are ingredients and the coefficient of each basic wave specifies the amount to include.

Definition 13.2

The ***N*th order Fourier approximation** of a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ is defined to be

$$a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_N \cos Nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_N \sin Nx,$$

where the coefficients $(a_n)_{n=0}^{N}$ and $(b_n)_{n=1}^{N}$ are defined by (13.1), (13.2), and (13.3).

Let's choose a function to approximate and see what its Fourier approximations look like.

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Computational Investigation 5: Fourier approximations

Consider the square-wave function, which is defined to be 1 on $[0, \pi)$ and 0 on $[\pi, 2\pi)$ (and extended to the rest of the real line by periodicity), and plot some of its Fourier approximations.

We calculate a_0 to be* $1/2$, and we calculate

Note that the formula for a_0 is the formula for the average value of f over $[0, 2\pi]$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{\pi} \cos nx \, dx = 0.$$

Similarly,

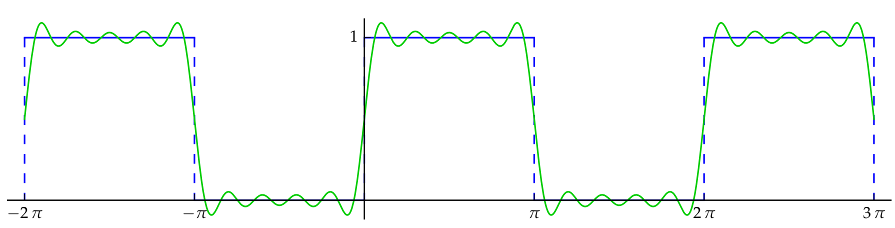
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{\cos 0 - \cos n\pi}{n\pi}.$$

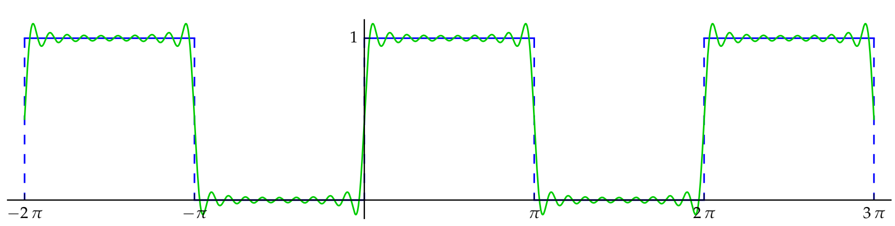
This means that b_n is zero if n is even and $\frac{2}{n\pi}$ if n is odd. Therefore, the Nth-order Fourier approximation of f is

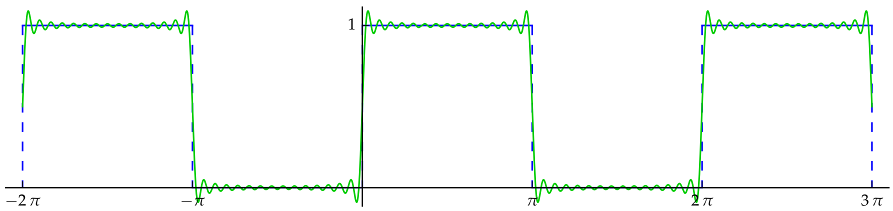
$$\frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \cdots + \frac{2}{N\pi} \sin Nx$$

if N is odd. If N is even, then the last term is $\frac{2}{(N-1)\pi} \sin(N-1)x$.

The three figures below the Nth order approximation for $N = 5, 10$, and 15, respectively.







These functions can be plotted in Sage:

```
var("j x") # declare j and x to be variables
n = 10 # set n to be 10
s(x) = 1/2 + sum(2/(pi*(2*j-1))*sin((2*j-1)*x),j,1,n) # define s to be the desired sum
plot(s(x), x,0,-2*pi,3*pi) # plot s(x)
```

We can see from these figures that if we include more terms, we do an increasingly good job of approximating the square wave. So, although Taylor polynomials can approximate non-periodic functions, Fourier approximations are better at handling discontinuities like the one exhibited by this square wave function.

Just as we defined Taylor series by taking $N \to \infty$ in the definition of the Nth order Taylor approximation, we define a function's Fourier series as a limit as $N \to \infty$ of its Nth order Fourier approximation.

Definition 13.3

The **Fourier series** of a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ is defined to be

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

where the coefficients $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are defined by (13.1), (13.2), and (13.3).

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The Nth-order Fourier approximations of the square-wave function above appear to converge $N \to \infty$ to the value of the function, except at the jumps, where the function appears to converge to $1/2$. This guess is correct, and it generalizes to many more functions than just the square wave. We say that a 2π-periodic function satisfies the **Dirichlet conditions** if it has finitely many jumps and extrema over $[0, 2\pi]$. More precisely, we require that (i) f is continuous except possibly at finitely many points over each period, (ii) any discontinuities of f are jump discontinuities or holes*, and (iii) there are not infinitely many points at which f has a strict local maximum or minimum. For a proof, click here to see Section 11.5.6 of Giaquinta and Modica's *Mathematical analysis: linear and metric structures and continuity*.

Recall that f has a jump discontinuity at x if both one-sided limits at x exist and are unequal, and f has a hole at x if the one-sided limits are equal to each other but are different from $f(x)$.

Theorem 13.4: (Dirichlet-Jordan convergence theorem)

If $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function which satisfies the Dirichlet conditions, then the Fourier series for f converges for all x. Furthermore, the limit of the Fourier series is equal to $f(x)$ for any point x where f is continuous. If f is discontinuous at x, the sum of the Fourier series is the average of the right and left limits at x

$$\frac{1}{2}\left[\lim_{t \to x^+} f(t) + \lim_{t \to x^-} f(t)\right].$$

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Exercise 13.2

(a) Find the Fourier series for the 2π-periodic *triangle wave*, whose graph over $[0, 2\pi]$ consists of straight lines connecting the origin, the point (π, π), and the point $(2\pi, 0)$. Does this Fourier series converge to the triangle wave? (b) Substitute $x = 0$ to show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Exercise 13.3

Find the Fourier series for the 2π-periodic function which is equal to x^2 on $[-\pi, \pi)$. Substitute a suitable value for x into the resulting equation to find

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

Interlude 5 (Fourier series and music)

Your ears detect small changes in air pressure, and your brain interprets that signal as sound. A sound may therefore be completely encoded by a function P whose value $P(t)$ at each time t tells us the air pressure at that time. The Fourier coefficients of P are musically meaningful: a sound wave of the form $\sin(kt)$ is perceived by the ear as a pure note of frequency* k. So if we play the sound wave

$$P(t) = 2\sin(k_1 t) + \sin(k_2 t) + \sin(k_3 t), \tag{13.4}$$

we will hear a chord (click here for audio; the three notes enter at different times so you can hear them better) with three notes whose frequencies are k_1, k_2, k_3, with the first note twice as loud as each of the others.

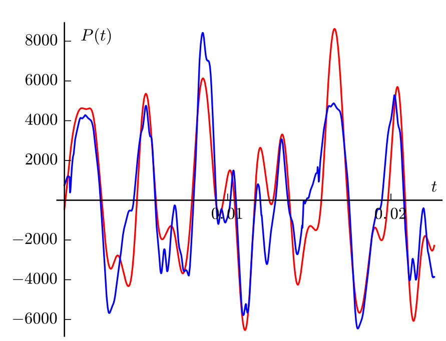
The frequency of a basic wave is normally defined as the number of periods per unit time, but we will define it here to mean the number of periods per 2π units of time, for consistency with our convention to focus on 2π-periodic functions.

From this point of view, the calculation of Fourier coefficients has a concrete and illuminating interpretation: it is a way for us to reverse engineer the notes that went into making the sound. Decomposing a wave into its constituent frequencies has applications: as mentioned in the introduction to this section, we can compress an audio file by storing Fourier coefficients rather than samples of the sound wave itself. Even though there are infinitely many Fourier coefficients, this method nevertheless allows us to save space, because we can store only those frequencies which are in the range of human hearing.

The following graph* (the blue one) shows $P(t)$ for a brief snippet of a 9-second recording of a cello playing a few notes. The red graph shows the result of keeping only the 2000 terms with the largest coefficients in the Fourier series of the full 9-second clip. Click here for the original clip and here for the highly compressed version.*

The units for the x-axis are seconds, and the units for the y-axis are whatever air pressure unit was in the .wav file I used (it doesn't matter because they're rescaled before they're played anyway).

It's compressed enough for you to be able to tell the difference. If you wanted a high-fidelity version, you would need to include more frequencies.



13.2 Complex Fourier series

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Euler's formula tells us that sines and cosines are essentially the same as complex exponentials. So rather than representing a 2π-periodic function using basic sine and cosine waves, we can instead represent it using functions of the form e^{inx}, where n is an integer. It will be convenient to use negative as well as positive values of n, because any linear combination of $\cos(nx)$ and $\sin(nx)$ can be written as a linear combination of e^{inx} and e^{-inx} (see the solution of Example 9.7, as well as Exercise 13.5 below). The same is not true if we try to use just e^{inx} instead of e^{inx} and e^{-inx}.

The key to Fourier analysis in the real case was Theorem 13.1. Let's look for an analogue. If $m \neq n$, then

$$\int_0^{2\pi} e^{i(m-n)x}\,dx = \left.\frac{e^{i(m-n)x}}{m-n}\right|_0^{2\pi} = 0,$$

since $e^{i(m-n)x}$ is 2π-periodic. If $m = n$, then

$$\int_0^{2\pi} e^{i(m-n)x}\,dx = \int_0^{2\pi} 1\,dx = 2\pi.$$

So the complex waves e^{inx} have a similar property to the real ones: the integral of the product of any two of them over $[0, 2\pi]$ is equal to 0 unless the corresponding values of n are additive opposites, in which case the integral of the product is 2π.

Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic function, and suppose that*

If $(z_n)_{-\infty}^{\infty}$ is a doubly-infinite sequence of complex numbers, we define $\sum_{n=-\infty}^{\infty} z_n$ to be the sum of $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$, provided both converge.

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imx}.$$

If we want to pick out the coefficient c_n, we can multiply both sides by e^{-inx} and integrate to obtain (again assuming that the interchange of sum and integral is correct)

$$\int_0^{2\pi} f(x)e^{-inx}\,dx = \sum_{m=-\infty}^{\infty} \int_0^{2\pi} c_m e^{imx} e^{-inx}\,dx. \tag{13.5}$$

We see that all the terms but one on the right-hand are 0, and we're left with

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

Theorem 13.5: (Fourier convergence, complex version)

The Fourier convergence theorem (Theorem 13.4) holds for complex Fourier series as well. Furthermore, if f is a real-valued function with real Fourier coefficients $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ and complex coefficients $(c_n)_{n=-\infty}^{\infty}$, then

$$c_n = \begin{cases} a_0 & \text{if } n = 0 \\ \dfrac{a_n - ib_n}{2} & \text{if } n \geq 1 \\ \dfrac{a_{-n} + ib_{-n}}{2} & \text{if } n \leq -1. \end{cases}$$

Theorem 13.5 provides us with a dictionary for translating back and forth between real and complex coefficients: real coefficients give complex ones using the given formula, and also complex coefficients give real ones by reading off the real and imaginary parts of c_n. This dictionary is simple enough to derive from scratch if you need to (see Exercise 13.5). But it's probably worth memorizing as a labor-saving device.

Exercise 13.5

Use Euler's formula to show that if c_n is defined as in Theorem 13.5 and $n \geq 1$, we have

$$c_n e^{inx} + c_{-n} e^{-inx} = a_n \cos nx + b_n \sin nx.$$

One advantage of using complex series is the convenience of taking derivatives. It works term-by-term, as you would expect. We say a 2π-periodic function f is **piecewise smooth** if $[0, 2\pi]$ can be subdivided into finitely many intervals in such a way that f and f' are continuous inside* each interval, with each having left and right one-sided limits everywhere. The graph of such a function looks smooth except for finitely many jumps and finitely many "corners", where the derivative jumps. For a proof of the following theorem, click here to see Theorem 2.2 of Gerald Folland's *Fourier Analysis and its Applications*.

"Inside" here means that the endpoints of each interval are excluded from consideration.

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Theorem 13.6: (Differentiation of complex Fourier series)

Suppose $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, piecewise smooth, **and continuous**. Let $(c_n)_{n=-\infty}^{\infty}$ be the Fourier coefficients of f. Then* $f(x) = \sum\limits_{n=-\infty}^{\infty} c_n e^{inx}$ and

$$f'(x) = \sum_{n=-\infty}^{\infty} in\, c_n e^{inx}.$$

This means that the infinite series on the right-hand side converges for all x, and that the limit is $f(x)$.

Example 13.6

(a) Find the complex Fourier series for the 2π-periodic function which is defined by $f(x) = x$ for all $x \in (-\pi, \pi]$. Show that term-by-term differentiation fails for this function.

(b) Find the complex Fourier series for the 2π-periodic function which is defined by $g(x) = \pi^2 - x^2$ for all $x \in (-\pi, \pi]$. Verify Theorem 13.6 for g.

Solution

(a) We calculate $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = 0$ and for $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx \\ &= \frac{1}{2\pi} \left[\left. \frac{x e^{-inx}}{-in} \right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in}\, dx \right] \\ &= \frac{1}{2\pi} \left[\frac{\pi e^{-in\pi}}{-in} - \frac{-\pi e^{in\pi}}{-in} - 0 \right]. \end{aligned}$$

Since* $e^{-in\pi} = \operatorname{cis}(-n\pi) = (-1)^n$ and similarly for $e^{in\pi}$, we get $c_n = (-1)^n \frac{1}{-in} = (-1)^n \frac{i}{n}$.

Think about rotating an integer multiple of π radians on the unit circle in the complex plane.

So the Fourier series of f is $\sum_{n=-\infty}^{\infty}(-1)^n \frac{i}{n} e^{inx}$. Differentiating term-by-term yields $\sum_{n=-\infty}^{\infty}(-1)^n \frac{i(in)}{n} e^{inx}$, whereas the derivative of f is the constant function 1 on $(-\pi, \pi)$ and therefore has a complex Fourier series with only one term: $c_0 = 1$ and $c_n = 0$ for all $n \neq 0$.

(b) For g, we calculate $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi^2 - x^2\,dx = \frac{2}{3}\pi^2$, and for $n \neq 0$, we have

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi^2 - x^2 e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi^2 - \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx \\
&= 0 - \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx \\
&= -\frac{1}{2\pi}\left[\frac{x^2 e^{-inx}}{-in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{e^{-inx}}{-in}\,dx\right]^* \\
&= -\frac{1}{2\pi}\left[0 - \frac{2}{-in}(-1)^n\frac{2\pi i}{n}\right] \\
&= (-1)^{n+1}\frac{2}{n^2}.
\end{aligned}$$

Here we're reusing our calculation from (a).

So the Fourier series of g is $\frac{2}{3}\pi^2 + \sum_{n=-\infty}^{\infty}(-1)^{n+1}\frac{2}{n^2}e^{inx}$. Differentiating term-by-term gives $\sum_{n=-\infty}^{\infty}(-1)^{n+1}\frac{2i}{n}e^{inx}$, which is indeed the Fourier series for the derivative of $\pi^2 - x^2$, namely $-2x$, by (a).

Exercise 13.7

Compute the complex Fourier coefficients for the square wave in Computational Investigation 5 and verify the formula in Theorem 13.5 in this case.

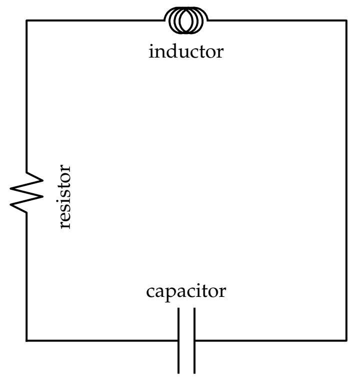
A series of the form $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $(c_n)_{n=-\infty}^{\infty}$ is a sequence of complex numbers, is called a *complex trigonometric series*. The following theorem tells us that for any 2π-periodic function f, there is at most one way to write f as a convergent complex trigonometric series. Click here to see J. Marshall's article "Uniqueness of representation by trigonometric series" (American Mathematical Monthly 96.10 (1989): 873-885) for more discussion and a proof.

Theorem 13.7: (Cantor's uniqueness theorem)

If $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function and $(c_n)_{n=-\infty}^{\infty}$ and $(d_n)_{n=-\infty}^{\infty}$ are sequences such that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} d_n e^{inx}$ for all $x \in \mathbb{R}$, then $c_n = d_n$ for all n.

13.3 Fourier series and differential equations

Consider the following circuit, with a 0.25 farad capacitor, a 1.0 ohm resistor, and a 1.0 henry inductor.



We discussed in Example 9.4 that the charge $Q(t)$ on the capacitor at time t satisfies the differential equation

$$Q''(t) + Q'(t) + 4Q(t) = 0.$$

Now, let's add an alternating current source to the circuit, which applies a voltage of $V(t) = \sin t + \cos 2t$ to the circuit. Then applying Kirchhoff's law, as we did in Example 9.4, gives us an extra term, which we put on the right hand side:

$$Q''(t) + Q'(t) + 4Q(t) = V(t). \tag{13.6}$$

We recognize this differential equation as a linear, non-homogeneous differential equation. We call the term $V(t)$ on the right-hand side the *driving* function.

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Usually we would supply some initial conditions or search for a general solution, but when the driving function is periodic, the solution will converge to a unique, periodic **steady state** solution with the same period as the driving function*. So, if we are interested in the long-term behavior of the system, we can just search for the periodic solution to the differential equation.

You can show this yourself by finding the solution of the homogeneous version and seeing that the terms with the free constants go to 0.

We could solve (13.6) using the method of undetermined coefficients (Section 9.5), but we will instead do it using Fourier series, which handles a wider variety of driving functions $V(t)$ and provides greater physical insight.

Example 13.8

Use Fourier series to find the steady state solution of the DE

$$Q''(t) + Q'(t) + 4Q(t) = V(t),$$

where $V(t) = \sin t + \cos 2t$.

Solution

We begin by expressing the solution Q and the voltage V in terms of their complex Fourier coefficients:

$$Q(t) = \sum_{n=-\infty}^{\infty} q_n e^{int} \qquad V(t) = \sum_{n=-\infty}^{\infty} v_n e^{int}.$$

Let's save ourselves some work in calculating v_n. We can read off the real coefficients $a_2 = 1$ and $b_1 = 1$ (all the rest are zero). So by Theorem 13.5, we have $v_1 = (a_1 - ib_1)/2 = -i/2$, and similarly $v_{-1} = i/2$, $v_2 = 1/2$, and $v_{-2} = 1/2$.

The left-hand side is also easy to work out using term-by-term differentiation (Theorem 13.6). We get

$$Q''(t) + Q'(t) + 4Q(t) = \sum_{n=-\infty}^{\infty} [(in)^2 + in + 4] q_n e^{int}.$$

Two trigonometric series can only be equal if all their coefficients are equal (Theorem 13.7), so we can set coefficients equal to get

$$[(in)^2 + in + 4] q_n = v_n$$

Substituting $n = -2, -1, 1,$ and 2, we get

$$q_1 = -\frac{1}{20} - \frac{3}{20} i$$
$$q_2 = -\frac{1}{4} i,$$

and q_{-1} and q_{-2} will have to be complex conjugates of q_1 and q_2, respectively, by Theorem 13.5.

Translating these complex coefficients back to real ones again using Theorem 13.5, we get

$$Q(t) = -\frac{1}{10} \cos t + \frac{3}{10} \sin t + \frac{1}{2} \sin 2t.$$

Let's reflect. First, we see that the circuit effectively transforms the driving function $V(t)$ into a response function $Q(t)$ by dividing all its complex Fourier coefficients by $(in)^2 + in + 4$. This expression we're dividing by is the characteristic polynomial of the left-hand side, evaluated at in. So if the characteristic polynomial happens to be very close to 0 at some integer multiple of i, then even if the driving function includes a modest amount of the corresponding basic wave, that wave will still be very large in the response function $Q(t)$. We call such a frequency, which is amplified more than others, **resonant**.

The places where the characteristic polynomial is zero are called the **spectrum** of the DE. Note that the spectrum is a finite collection of points in the complex plane. If we want to find the values of n for which $p(in)$ is large, we should look at the points on the imaginary axis which are close to points in the spectrum, because if $p(in)$ is close to zero, then $1/p(in)$ is large.

Exercise 13.9

Find the periodic solution of $f''(x) + 2f'(x) + 2f(x) = \sin x$.

Exercise 13.10

Show that the method we developed in this section breaks down for the differential equation

$$f''(x) + f(x) = \sin x$$

Show that in fact this DE does not have a periodic solution by showing that

$$A \cos x + B \sin x - \frac{x \cos x}{2}$$

solves the DE and is therefore the general solution. Interpret the term $-\frac{x \cos x}{2}$ physically in light of our observation that if $p(in)$ is *close* to 0 for some n, then the basic waves $\sin nx$ and $\cos nx$ are amplified.

APPENDIX I: BACKGROUND KNOWLEDGE ASSUMED IN THIS COURSE

1. **Basic algebra skills**
   (a) Addition and multiplication of fractions
   (b) Expanding and factoring expressions involving variables
   (c) Simplifying expressions involving variables, by applying additive or multiplicative cancellation
   (d) Solving linear equations and systems of linear equations in one and two variables
   (e) Solving polynomial equations using the zero product property
   (f) Exponent rules, square roots
   (g) Inverse functions and logarithms, properties of logarithms
   (h) Absolute values, distributive property of absolute value across multiplication
2. **Geometry**
   (a) Basic area/volume formulas, like one-half base times height for a triangle, π times radius squared for a disk, or volume equals length times width times height for a rectangular prism
   (b) Pythagorean theorem, distance formula
   (c) The length of an arc of a circle equals the arc's angle measure in radians times the radius of the circle
   (d) Equations of lines, particularly slope-intercept and point-slope forms
3. **Trigonometry**
   (a) The contents of the trig interlude in Section 2: definition of cosine and sine as the x and y coordinates of a point on the unit circle, definition of other trig functions in terms of cosine and sine, trig functions evaluated at multiples of 45° and 30° using special right triangles, Pythagorean identities, and the sum-angle formulas.
4. **Calculus**
   (a) Basic limits, like $\lim_{x\to 0} \frac{\sin x}{x}$, $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$, $\lim_{x\to 0^+} \ln x$, $\lim_{x\to 1} \frac{x}{(1-x)^2}$, behavior at infinity of common functions such as polynomials, ratios of polynomials, logs, exponentials, and L'Hôpital's rule
   (b) Comparison of growth at infinity: a^x (for $a > 1$) grows faster than x^t (for $t > 0$), which in turn grows faster than $\log_b x$ (for $b > 1$). In other words, exponential > power > log
   (c) Differentiation rules: linearity of differentiation, power rule, product rule, quotient rule, chain rule, derivatives of trig functions, inverse trig functions, exponentials, logs
   (d) Showing that a function is increasing/decreasing over an interval by showing that its derivative is positive/negative there
   (e) Mean-value theorem
   (f) Definition of the integral as a limit of Riemann sums, interpretation of an integral as a signed area, the fundamental theorem of calculus
   (g) Integration by substitution (a.k.a. u-substitution)

Differentiation rules

power rule	$(x^n)' = nx^{n-1}$
product rule	$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
chain rule	$(f(g(x))' = f'(g(x))g'(x)$
exponential	$(a^x)' = (\ln a)a^x$
logarithm	$(\ln x)' = \frac{1}{x}$.

Trig identities

Pythagorean identities	$\sin^2\theta + \cos^2\theta = 1$
	$\tan^2\theta + 1 = \sec^2\theta$
Sum-angle identities	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Half-angle identities	$\cos^2\theta = \frac{1+\cos 2\theta}{2}$
	$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

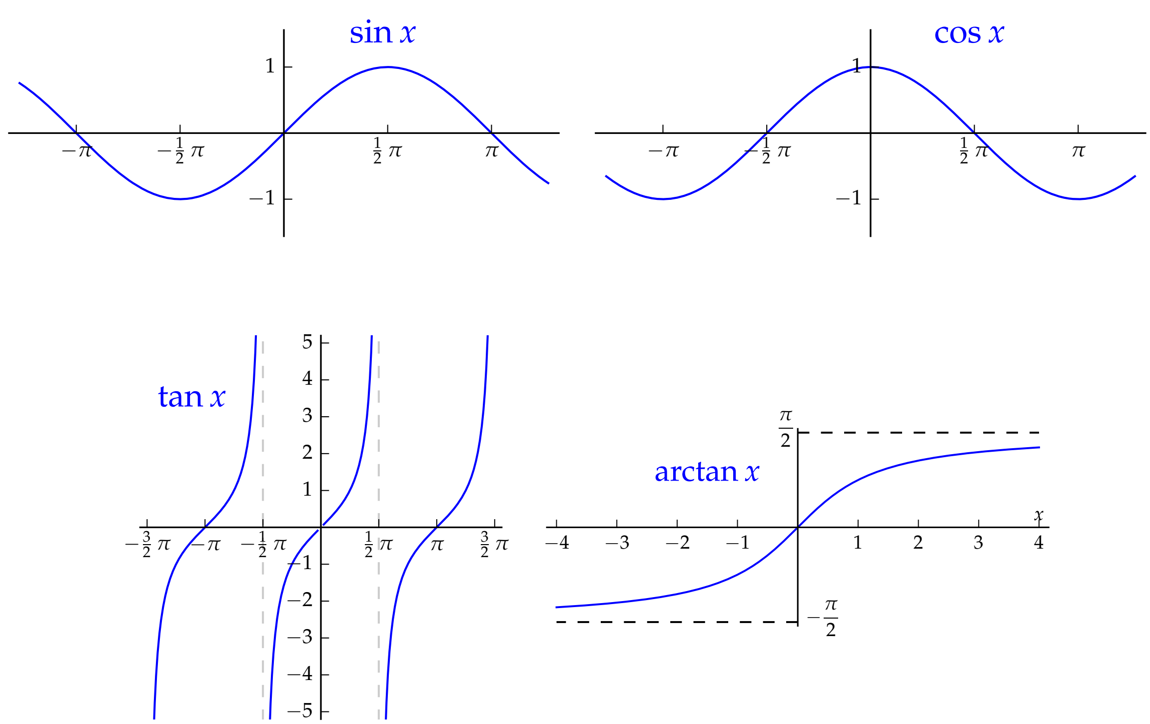
Trig derivatives and integrals

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$\int \sin x\,dx = -\cos x$	$\int \cos x\,dx = \sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$	$\int \tan x\,dx = -\ln\lvert\cos x\rvert$	$\int \cot x\,dx = \ln\lvert\sin x\rvert$
$(\sec x)' = \sec x\tan x$	$(\csc x)' = -\csc x\cot x$	$\int \sec x = \ln\lvert\sec x + \tan x\rvert$	$\int \csc x = -\ln\lvert\csc x + \cot x\rvert$

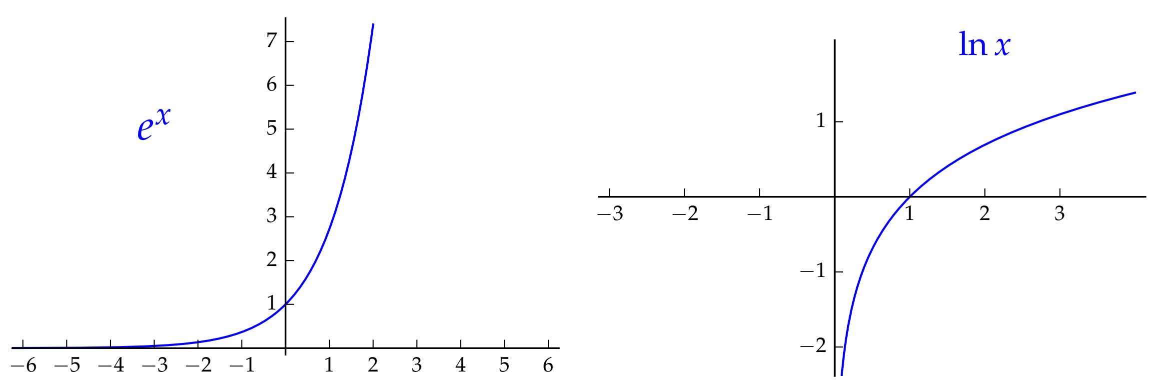
Derivatives of inverse trig functions

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $(\arctan x)' = \frac{1}{1+x^2}$ $\qquad$ $(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$

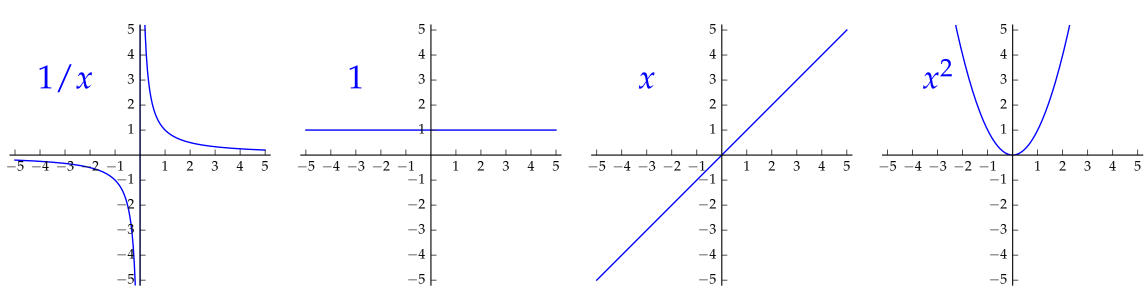
Graphs of trig and inverse trig functions



Graphs of exponential and logarithm functions



Graphs of powers of x



Appendix III: Tips

Convergence tests for infinite series

1. nth **term test**: always start with this one
2. Tests for series of nonnegative terms:
   (a) **Integral test**: try if the series looks like $\sum f(n)$ for some function f you know how to integrate
   (b) **Comparison test**: good for getting rid of pesky terms that you know intuitively shouldn't affect convergence or divergence of the series
   (c) **Ratio test**: try when it's clear approximately what you multiply each term by to get the next one, for example if factorials are involved
3. Tests for series which include negative terms:
   (a) **Absolute convergence test**: a good first check if negative terms are involved: just flip every term positive and check the convergence of the resulting series using the nonnegative-term tests above.
   (b) **Alternating series test**: try if the absolute convergence test fails

Calculating Fourier series

1. Many 2π-periodic functions you'll calculate Fourier series for are specified as piecewise functions. If you begin by **sketching a graph** of the given function f, this will help you avoid the mistake of using one of f's defining expressions over an interval for which it does not apply
2. When integrating a function f which is defined in a piecewise manner, **split your interval of integration** at the same places where the definition of f is split.
3. Bear in mind that **trig expressions simplify** at integer multiples of π: $\sin \pi n = 0$ for all integers n, and $\cos \pi n = (-1)^n$ for all integers n.
4. You may **change the interval of integration** from $[0, 2\pi]$ to $[-\pi, \pi]$ in the definition of the Fourier coefficients, since the integral of a given 2π-periodic function over any interval of length 2π is the same.
5. If f is **even** (meaning that its graph stays the same if you reflect it across the y axis), then its Fourier series will not have any sine terms. If f is **odd** (meaning that its graph stays the same if you rotate it 180 degrees about the origin), then its Fourier series will not have any cosine terms.