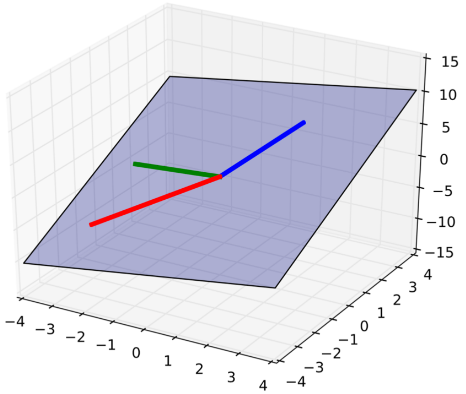
① Clarification from last time: we showed that $\mathrm{Span}\left(\left\{\overset{\vec{u}}{\begin{pmatrix}1\\-4\\-3\end{pmatrix}}, \overset{\vec{v}}{\begin{pmatrix}3\\2\\2\end{pmatrix}}, \overset{\vec{w}}{\begin{pmatrix}4\\-6\\-7\end{pmatrix}}\right\}\right)$ is equal to a plane in $\mathbb{R}^3$, by row reducing $\left(\begin{array}{ccc|c}1 & 3 & 4 & x\\ -4 & 2 & -6 & y\\ -3 & 2 & -7 & z\end{array}\right)$ and showing that the system is only consistent if $\overbrace{x - \frac{1}{2}y + z = 0}^{\text{this is the equation of a plane}}$. This implies that the three vectors $\vec{u}, \vec{v}, \vec{w}$ all lie in the same plane:



This is because $\vec{u} \in \mathrm{Span}(\{\vec{u}, \vec{v}, \vec{w}\})$ & same for $\vec{v}, \vec{w}$.

Further, since $\vec{u}, \vec{v}$ are not parallel (i.e. one is a scalar times the other), $\mathrm{Span}(\{\vec{u}, \vec{v}\})$ is already a plane. So we can say 'throwing $\vec{w}$ in there doesn't make the span any bigger.'

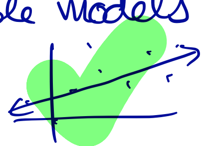
NOT EXAMINABLE CONTENT (whole page)

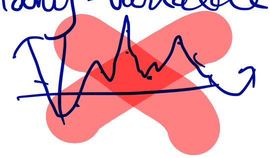
② Why should you care about linear algebra?

A: L.A. is foundational to pure math (both algebra and analysis) as well as physics (both theoretical & applied) and basically all applied science: statistics, biology, chemistry, economics, computer science, etc.

Examples: modeling heat flow (like in Pset 1), machine learning (like AlphaGo), studying relationships between observed variables (like Pre-K opportunities and educational outcomes), all image processing (an image is a triple of matrices: how much red, green, blue for each pixel)

The ubiquity of linear algebra owes to ① the linearity of some important equations governing physical phenomena, and ② the need for simple models in many-variable contexts.





§1.5 Solution Sets of Linear Systems

$$\begin{pmatrix} 3 & 2 & -1 \\ 4 & 6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

recall: this is a shorthand for something like this

We call a system $A\vec{x} = \vec{b}$ **homogeneous** if $\vec{b} = \vec{0}$. (vector of all zeros)

Observation 1 $\vec{x} = 0$ is always a solution of $A\vec{x} = \vec{0}$.

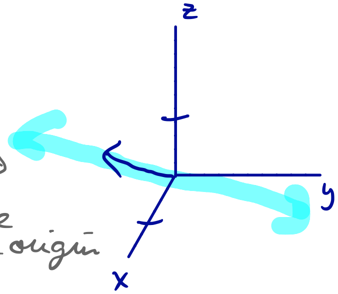
Observation 2 $\vec{x} = 0$ is the <u>only</u> solution of $A\vec{x} = \vec{0}$ if every column of A is a pivot column. [Why? Backsolving shows that x_n has to be zero, then x_{n-1}, etc.]

Example Describe the solution set of $A\vec{x} = \vec{0}$ where $A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$.

<u>Solution</u> $A \sim \begin{pmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, [row reduction steps omitted] so x_3 is going to be a free variable. Row reducing further, we

get $\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So we have x_3 free, $x_2 = 0$, $x_1 = \frac{4}{3}x_3$. Thus the solution set is $\left\{ \begin{pmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$. We can also write this as $\left\{ x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$, or even more succintly as $\mathrm{Span}\left(\left\{ \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix} \right\} \right)$.

this is a line through the origin



Example Solve the 1-equation system $10x - 3y - 2z = 0$.

Solution: We may regard y, z as free and then $x = \frac{3}{10}y + \frac{1}{5}z$. So $\begin{pmatrix} \frac{3}{10}y + \frac{1}{5}z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 3/10 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1/5 \\ 0 \\ 1 \end{pmatrix}$, which as y & z range over $\mathbb{R}$ gives $\mathrm{Span}\left(\begin{pmatrix} 3/10 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/10 \\ 0 \\ 1 \end{pmatrix} \right)$.

So:

# free variables	solution set		Shape
0	$\mathrm{Span}(\{\vec{0}\})$	for some $\vec{u}$	point
1	$\mathrm{Span}(\{\vec{u}\})$	for some $\vec{u}, \vec{v}$	line
2	$\mathrm{Span}(\{\vec{u}, \vec{v}\})$		plane

all through the origin

Nonhomogeneous Systems

Now let's consider $\vec{b} \neq \vec{0}$. We've done this before:

Example Solve $A\vec{x} = \vec{b}$ where $A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$.

<u>Solution</u> Row reduce, etc. (steps omitted) to get

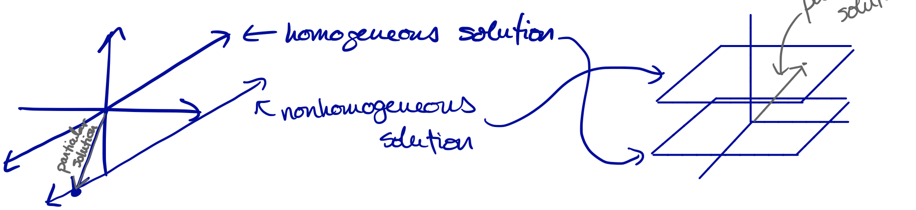
$$\vec{x} = \begin{pmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}.$$

Note that $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ is a solution.

This situation is general:

nonhomogeneous solution = particular solution + solution of homogeneous system

Geometrically, adding the particular solution to the homogeneous solution <u>shifts</u> it:



§ 1.6 Applications

Example Consider a 3-industry economy:

	coal	electric	steel
coal	0.0	0.4	0.6
electric	0.6	0.1	0.2
steel	0.4	0.5	0.2

means that 20% of the output of the electric industry is purchased by the steel industry

Let p_C, p_E, p_S be the output (in dollars) of each industry.

What values can $\vec{p} = (p_C, p_E, p_S)$ have so that each industry balances its budget.

<u>Solution</u> We need $\underset{\text{coal income}}{p_C} = \underbrace{0.0p_C + 0.6p_E + 0.4p_S}_{\text{coal expenditures}}$ and

similarly for E & S. So we want to solve:

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.6 & 0 \\ -0.6 & 0.9 & -0.2 & 0 \\ -0.4 & -0.5 & 0.8 & 0 \end{array}\right)$$

which row reduces to $\begin{pmatrix} 1 & 0 & -\frac{31}{33} & 0 \\ 0 & 1 & -\frac{28}{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So the

solution set is $\text{Span}\left(\begin{pmatrix} 31/33 \\ 28/33 \\ 1 \end{pmatrix}\right) = \text{Span}\begin{pmatrix} 31 \\ 28 \\ 33 \end{pmatrix}$. This makes sense:

why does this work?

if everyone's budget balanced and all outputs multiplied by c, then budgets would still balance.