14 Feb 2017

§1.8 Linear Transformations — the soul of linear algebra

We want a more geometric picture of what A is "doing to" $\vec{x}$ in the matrix-vector product $A\vec{x}$. We will ultimately take the view that this geometric action is what matrices are really all about.

Let's consider the 2×2 case first we can think of A as a function that takes a vector $\vec{x} \in \mathbb{R}^2$ and returns another vector $A\vec{x} \in \mathbb{R}^2$. We write $A: \underbrace{\mathbb{R}^2}_{\text{domain}} \to \underbrace{\mathbb{R}^2}_{\text{target, or codomain}}$.

We can't graph this because that would require $2+2=4$ spatial dimensions. However, we can make a movie where each vector (which we'll think of as just a point) starts at its location $\vec{x}$ and moves to its destination $A\vec{x}$.

Example Picture $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

<u>Solution</u> Let's consider a few points: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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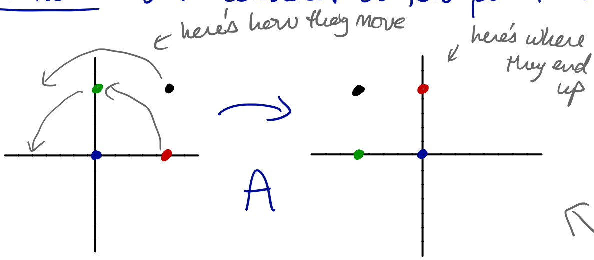


Note: the actual "motion" in between is just for fun. The function only actually tells us the <u>ending</u> point

So it looks like a <u>rotation</u>.

We refer to A, regarded as a function, as a **linear transformation**. The following theorem gives us some geometric intuition for linear transformations:

"the following are equivalent." I.e., they're all true or all false

Theorem Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Then TFAE.

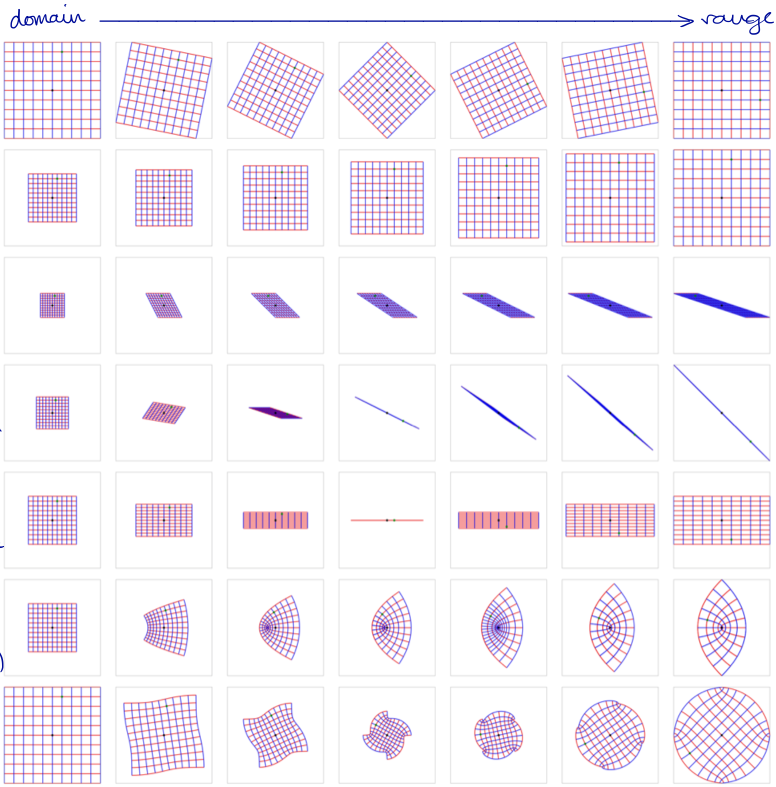
not in your book, or any linear algebra text I could find. Proved on last page

① T is <u>linear</u>, meaning $T(c\vec{x}+\vec{y}) = cT(\vec{x}) + T(\vec{y})$ for all $c, \vec{x}, \vec{y}$

② there is some 2×2 matrix A so that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$

☆ Further, if T is bijective: "T maps lines to lines and $\vec{0}$ to $\vec{0}$" $\Leftrightarrow$ "T is linear"

Examples & Nonexamples



	matrix
rotation	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Scale	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
"Shear"	$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$
projection (squishes down to one line)	$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$
reflection (reflect in y direction, scale in x)	$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$
nonlinear (complex squaring, if you're interested)	none
nonlinear a conformal map, if you're interested)	none

The same stuff applies to $m \times n$ matrices but is a bit trickier to visualize: An $m \times n$ matrix A takes a vector $\vec{x} \in \mathbb{R}^n$ and returns a vector $A\vec{x} \in \mathbb{R}^m$, so it represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

so these kinda flip around compared to $m \times n$

Exercise Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 6 \\ 11 \end{bmatrix}$. Find $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ for all x, y.

Solution We know $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Since T is linear, then

$$T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x\begin{bmatrix} -2 \\ 5 \end{bmatrix} + y\begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} -2x + 6y \\ 5x + 11y \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 5 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

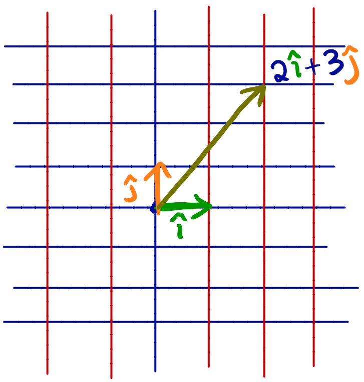
[Segue to §1.9]

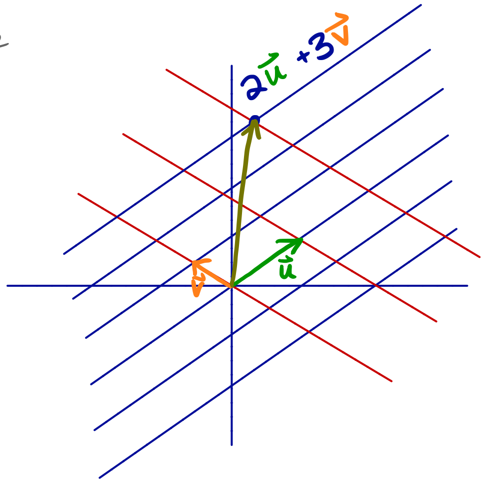
we call $\hat{i}$ and $\hat{j}$ "standard basis vectors"

"i hat"

Visually, knowing that T maps $\hat{i} = \begin{bmatrix}1\\0\end{bmatrix}$ to $\vec{u}$ and $\hat{j} = \begin{bmatrix}0\\1\end{bmatrix}$ to $\vec{v}$ tells us where T maps everything else:

$\hat{i}, \hat{j}$ also called $\vec{e}_1, \vec{e}_2$





Further, the vectors that $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ map to can be read off A: they're its columns!

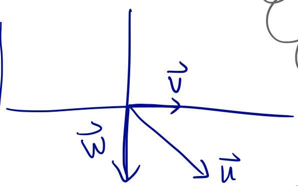
Exercise Locate $T(\vec{w})$.

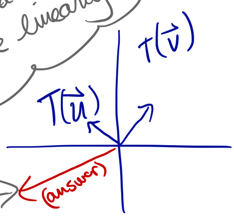
Key idea: use linearity

Solution $\vec{w} = \vec{u} - 2\vec{v}$,

so $T(\vec{w}) = T(\vec{u}) - 2T(\vec{v})$

$=$





Exercise Show that linearly dependent lists map to linearly dependent lists under linear transformations

Solution Suppose $c_1, \ldots, c_n$, not all zero, make $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$. Taking T of both sides,

$$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = T(\vec{0})$$

$$c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) = \vec{0}.$$ ✓ exercise

So $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is linearly dependent.

!! Achtung: $y = mx + b$ is not linear in the sense of the term we're using here, unless $b = 0$; if $b \neq 0$ we call it "affine". So:

Algebra-1-linear = affine,

LinAlg-linear = Alg-1-linear with 0 intercept.

JUST FOR FUN

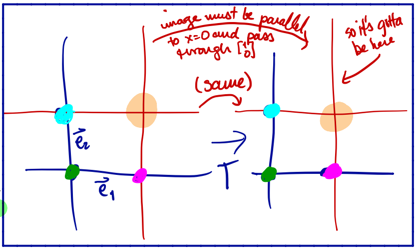
Appendix (non-examinable)

we add this hypothesis for simplicity. It can be dropped.

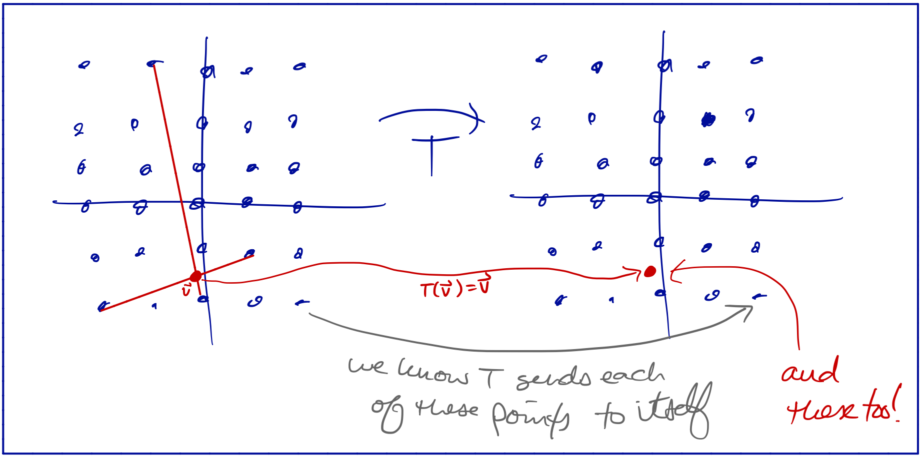
Theorem If $T:\mathbb{R}^2 \to \mathbb{R}^2$ is continuous and bijective and if $T(L)$ is a line for every line L in $\mathbb{R}^2$, then there exist A and $\vec{b}$ so that $T(\vec{x}) = A\vec{x} + \vec{b}$ for all $\vec{x} \in \mathbb{R}^2$.

Proof If L and M are parallel lines, then $T(L)$ and $T(M)$ are parallel too, because if they intersected, then T would map distinct points in L and M to the same point (the intersection of $T(L)$ and $T(M)$). Let's assume for now that $T(\vec{0}) = \vec{0}$, $T(\vec{e}_1) = \vec{e}_1$ and $T(\vec{e}_2) = \vec{e}_2$.

Then $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$ has to lie on the lines $x=1$ and $y=1$; therefore $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$. ← We say T "fixes" $\begin{bmatrix}1\\1\end{bmatrix}$



Similarly, we can reason that $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ and continue in this way to fill out the whole integer grid:

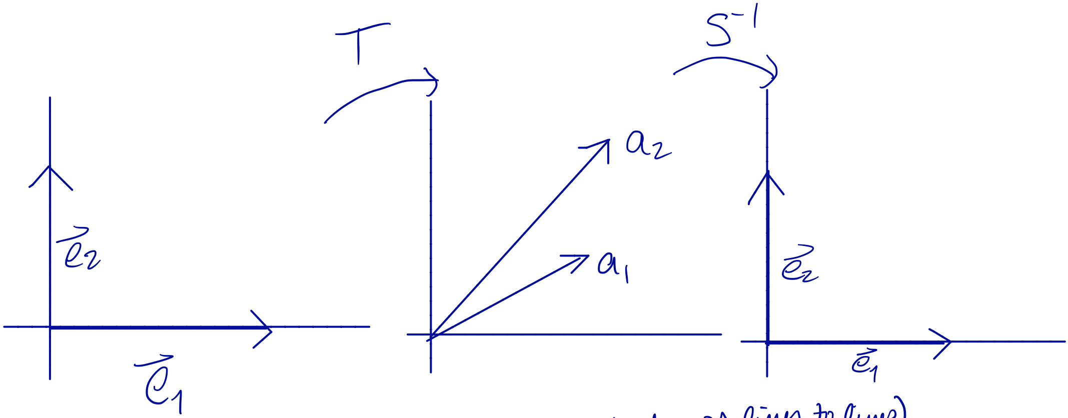


this means that T also fixes any intersection point $\vec{v}$ of two line segments with endpoints on the grid. But this includes all points with rational coordinates! [left as a not-tricky exercise]

Since $T(\vec{x}) = \vec{x}$ for all $\vec{x}$ with rational coordinates and T is continuous, $T(\vec{x}) = \vec{x}$ for all $\vec{x}$.

To handle the general case, suppose $T(\vec{0}) = \vec{b}$, $T(\vec{e}_1) = \vec{a}_1$ and $T(\vec{e}_2) = \vec{a}_2$. Define the map

$S(\vec{x}) = A\vec{x} + \vec{b}$ where the columns of A are $\vec{a_1} - \vec{b}$ and $\vec{a_2} - \vec{b}$ (so S sends $\vec{0}, \vec{e_1}, \vec{e_2}$ to the same places T does). Then



$T \circ S^{-1}$ fixes $\vec{0}, \vec{e_1}, \vec{e_2}$ (and maps lines to lines). We have already shown that this implies $T \circ S^{-1}$ is the identity map, that is $T = S$ $\blacksquare$