§4.4 Coordinates in a vector space

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One of the most important features of a basis is that it allows us to represent an arbitrary vector (that is, element of our vector space) uniquely as a linear combination of the vectors in the basis.

Proposition If $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is a basis of a vector space V and if $\vec{x} \in V$, there exist unique $c_1, c_2, \ldots, c_n \in \mathbb{R}$ so that $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$.

Why? We know $c_1, \ldots, c_n$ exist because the basis spans V.

We know they're unique by linear independence of the basis: $c_1\vec{b}_1 + \cdots + c_n\vec{b}_n = d_1\vec{b}_1 + \cdots + d_n\vec{b}_n \Rightarrow (c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n = \vec{0}$

$\Rightarrow c_1 = d_1, \ldots, c_n = d_n.$ ■

here we mean "ordered tuple of real numbers," not "element of V"

We write $[\vec{x}]_{\mathcal{B}}$ to mean the vector of weights that represent $\vec{x}$ as a linear combination of the vectors in the basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$.

These are called the **coordinates** of $\vec{x}$ with respect to $\mathcal{B}$.

the vector space here is $\mathbb{P}_2$

Example Find $[\vec{x}]_{\mathcal{B}}$, where $\vec{x} = 4-t+2t^2$ and $\mathcal{B} = \{1, t, t^2\}$.

Solution Easy: $4(1) + (-1)t + (2)t^2 = \vec{x}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

Example Find $[\vec{x}]_{\mathcal{B}'}$, where $\vec{x} = 4-t+2t^2$ and $\mathcal{B}' = \{1+t, t, t^2\}$.

Solution Requires a bit more thought:

$$\vec{x} = 4(1+t) - 5t + 2t^3 \Rightarrow$$

$$[\vec{x}]_{\mathcal{B}'} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}.$$

These examples illustrate an important point:

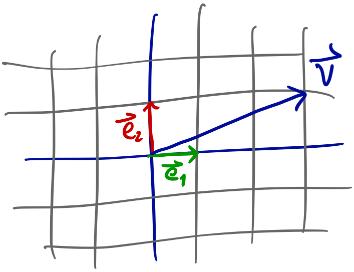
> *** Coordinates of a vector with respect to a basis depend on the basis! ***

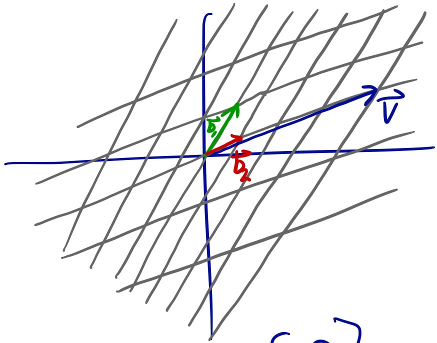
So, strictly speaking, we can't talk about the coordinates of a vector without specifying our basis. We do this in $\mathbb{R}^n$ (e.g., "6 is the 2nd coord. of $\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$") only because of a tacit agreement to use $\{\vec{e}_1, \dots, \vec{e}_n\}$ when no other basis is named.

"standard basis"

Visualizing coordinates

In $\mathbb{R}^2$, changing coordinates from the standard ones to those represented by the basis $\{\vec{b}_1, \vec{b}_2\}$ amounts to replacing the usual grid with a different parallelogram tiling of the plane:





$$[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$\mathcal{E}$: standard basis $\qquad$ $\mathcal{B}$: $\{\vec{b}_1, \vec{b}_2\}$

How does this change of coordinates work computationally? If $\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2$, how can we find d_1, d_2 so that $\vec{v} = d_1\vec{b}_1 + d_2\vec{b}_2$?

Just set up a system of equations

$$\overbrace{\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}}^{B}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = B^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

pop quiz: why is B invertible?

That's pretty tidy: to convert $\vec{v}$'s standard-basis coordinates to coordinates with respect to a basis $\mathcal{B}$, we assemble the vectors in $\mathcal{B}$ into a matrix B; then left-multiplying by B^{-1} does the trick.

Going the other way is even easier: to convert $[\vec{v}]_{\mathcal{B}}$ to $[\vec{v}]_{\mathcal{E}}$, we just multiply by B — no inverse.

the coordinate mapping

Consider a vector space V with a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. We define the coordinate mapping $T: V \to \mathbb{R}^n$ by $T(\vec{x}) = [\vec{x}]_{\mathcal{B}}$.

I.e., send each vector to its coordinates w.r.t. (with respect to) $\mathcal{B}$.

Claim T is linear.

Proof Suppose $\vec{x}, \vec{y} \in V$. Then $\vec{x} = x_1\vec{b}_1 + \cdots + y_n\vec{b}_n$, where $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, & $\vec{y} = y_1\vec{b}_1 + \cdots + y_n\vec{b}_n$.

So $T(\vec{x}+\vec{y}) = T((x_1+y_1)\vec{b}_1 + \cdots + (x_n+y_n)\vec{b}_n)$

$= \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$. But that's equal to $T(\vec{x}) + T(\vec{y})$!

Same idea for $T(c\vec{x}) = cT(\vec{x})$. $\blacksquare$

Claim T is onto (surjective) and one-to-one (injective).

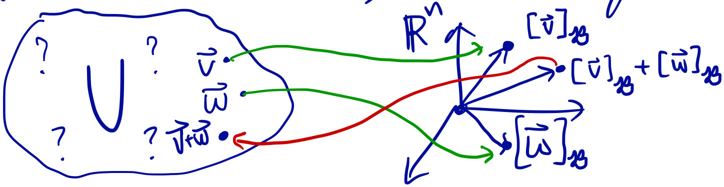
Proof If $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, then it gets mapped to from $x_1\vec{b}_1 + \cdots + x_n\vec{b}_n$. So T is onto. If $T(\vec{x}) = T(\vec{y})$, then $\vec{x}=\vec{y}$ by the uniqueness of coordinates.

Def$^{\underline{n}}$ A bijective linear map from a vector space V to a vector space W is called a vector space **isomorphism**.

"same"

We have have shown that the coordinate mapping is an isomorphism from V to $\mathbb{R}^n$.

This means V and $\mathbb{R}^n$ are "the same" in the following sense: don't like performing a vector space operation in V for some reason? No problem: map your vectors over to $\mathbb{R}^n$, do the desired operation over there, and map back.



Example Show that $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$

Solution Let's use the basis $\{1, t, t^2, t^3\}$ and consider the coordinate mapping:

$$4 - t^2 + 6t^3 \longmapsto (4, 0, -1, 6) \in \mathbb{R}^4$$

We already know the coordinate mapping is an isomorphism, so we're done. ∎

Actually, the previous example makes sense directly: the 'slots' in front of $1, t, t^2, t^3$ are really just places to stick a real number, just like the 'slots' in $(\_, \_, \_, \_) \in \mathbb{R}^4$. And addition/multiplication work the same in both cases.

This example is not that special: it shows that any n-dimensional vector space is isomorphic to $\mathbb{R}^n$!

However, it remains useful to consider abstract vector spaces, because this isomorphism is a little arbitrary in the sense that it <u>depends on a choice of basis</u>.