Pset 11 Solutions

1 Gram Schmidt:

$$\vec{b}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \text{proj}_{\vec{b}_1} \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{b}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \text{proj}_{b_1} \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \text{proj}_{b_2} \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

then $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is an orthogonal basis for the column space of $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

2

$$\vec{u}\,\vec{u}^T = \frac{1}{31} \begin{bmatrix} 16 & -4 & 8 & 12 & 4 \\ -4 & 1 & -2 & -3 & -1 \\ 8 & -2 & 4 & 6 & 2 \\ 12 & -3 & 6 & 9 & 3 \\ 4 & -1 & 2 & 3 & 1 \end{bmatrix}.$$

← (we learned uu^T is the projection onto Col U)

$\boxed{3}$ True If $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$ and for all $\vec{w} \in W^\perp$, then $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in \mathbb{R}^n$ and therefore $\vec{v} = \vec{0}$.

False $U^T U = I$; UU^T is the proj onto Col U.

True we learned this in class.

True Projecting to a space you're already in doesn't change you.

$\boxed{4}$ Let $\{b_1, \ldots, b_p\}$ be a basis of W, and extend it to a basis $\{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$. Then Gram-Schmidt that basis to get an ONB $\{\vec{u}_1, \ldots, \vec{u}_n\}$ of $\mathbb{R}^n$.

then $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$, and every vector in $\text{span}\{\vec{v}_{p+1}, \dots, \vec{v}_n\}$ is orthogonal to every vector in W. So $\text{span}\{\vec{v}_{p+1}, \dots, \vec{v}_n\}$ is a subset of $W^\perp$. But any vector in $W^\perp$, written as $a_1\vec{v}_1 + \dots + a_n\vec{v}_n$, must have $a_1 = \dots = a_p = 0$, because

$$\vec{v}_1 \cdot (a_1\vec{v}_1 + \dots + a_n\vec{v}_n) = a_1|\vec{v}_1|^2 = 0 \Rightarrow$$

$$a_1 = 0$$

& same for $a_2, \dots, a_p$. So $W^\perp = \text{span}(v_{p+1}, \dots, v_n)$.

thus $\dim W + \dim W^\perp = p + (n - (p+1) + 1) = n$.

$\boxed{5}$ Let $v_1, \dots, v_p$ be a basis for Row A, and $v_{p+1}, \dots, v_n$ a basis for Nul A. Then $T: \text{Row } A \to \text{Col } A$ is surjective because if $\vec{y} \in \text{Col } A$ then there exists $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{y}$.

Writing this $\vec{x}$ as $\vec{x}_1 + \vec{x}_2$ where $\vec{x}_1 \in \text{Row } A$, $x_2 \in \text{Nul } A$ [we can do this by expanding $\vec{x}$ in the basis $\{v_1, \dots, v_n\}$], we get $T(\vec{x}_1) = T(\vec{x}) - T(\vec{x}_2)$

$$= \vec{y} - \vec{0}$$
$$= \vec{y}.$$

So $T: \text{Row } A \to \text{Col } A$ is surjective.

Also, if $T(\vec{x}) = T(\vec{x}')$ for $\vec{x}, \vec{x}' \in \text{Row } A$, then $T(\vec{x}) - T(\vec{x}') = 0 \Rightarrow T(\vec{x} - \vec{x}') = 0 \Rightarrow$ $\vec{x} - \vec{x}' \in \text{Nul } A$. <u>But</u>, $\text{Nul } A = (\text{Row } A)^{\perp}$, so $\vec{x} - \vec{x}' \in \text{Nul } A \cap \text{Row } A \Rightarrow \vec{x} - \vec{x}' = 0$. So $\vec{x} = \vec{x}'$, & thus T is injective.