Brown University
DATA 1010
Fall 2018: Midterm I
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Name:

You will have three hours to complete this exam. The exam consists of 12 written questions and one separate computational problem. You will hand in your answers to the first 12 questions and then get out your laptop to submit a solution to the last question electronically.

For the written part of the exam, no calculators or other materials are allowed, except the Julia-Python-R reference sheet. For the computational part of the exam, you may use any internet technologies which do not involve active communication with another person.

You are responsible for explaining your answer to **every** *question. Your explanations do not have to be any longer than necessary to convince the reader that your answer is correct.*

I verify that I have read the instructions and will abide by the rules of the exam: ______________________________

Problem 1 [SETFUN]

Suppose that Ω is a set and that A, B, and C are subsets of Ω. Which of the following is necessarily equal to

$$(A \cap B \cap C)^{\mathsf{c}}?$$

(a) $A \cup B \cup C$
(b) $A \cap B \cap C$
(c) $A^{\mathsf{c}} \cup B^{\mathsf{c}} \cup C^{\mathsf{c}}$
(d) $A^{\mathsf{c}} \cap B^{\mathsf{c}} \cap C^{\mathsf{c}}$

Solution

Final answer:

Problem 2 [LINALG]

(i) (Multiple choice) Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and $\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly dependent. Then

(a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly independent
(b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly dependent
(c) It is impossible to determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly independent based on the information given.

(ii) (Multiple choice) Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and $\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly independent. Then

(a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly independent
(b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly dependent
(c) It is impossible to determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is linearly independent based on the information given.

Solution

Problem 3 **[MATALG]**

(a) Suppose that A and B are 4×4 matrices with the property that $\mathbf{a}'\mathbf{b} = 7$ whenever $\mathbf{a}'$ is a row of A and $\mathbf{b}$ is a column of B. Find the matrix product AB.

(b) Suppose that A and B are 4×4 matrices with the property that $\mathbf{a}\mathbf{b}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ whenever $\mathbf{a}$ is a column of A and $\mathbf{b}'$ is a row of B. Find the matrix product AB.

Solution

Problem 4 **[EIGEN]**

The matrix $A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$ diagonalizes as **??** $\begin{bmatrix} \frac{11}{25} & \frac{6}{125} & -\frac{8}{125} \\ \frac{6}{125} & \frac{977}{1250} & \frac{182}{625} \\ -\frac{8}{125} & \frac{182}{625} & \frac{1147}{1875} \end{bmatrix} \begin{bmatrix} 0 & -\frac{4}{5} & -\frac{3}{5} \\ \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ \frac{3}{5} & \frac{12}{25} & -\frac{16}{25} \end{bmatrix}$.

Which of the following is closest to A^{100}?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (b) $\begin{bmatrix} 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ -\frac{3}{5} & \frac{12}{25} & -\frac{16}{25} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution

Final answer:

Problem 5 **[OPT]**

The code below represents an attempt to implement gradient descent for the function $f(x,y) = (x-1)^4 + x(y-2)^2$.

```
f(x,y) = (x-1)^4 + x*(y-2)^2
df(x,y) = [4(x-1)^3 + (y-2)^2, 2x*(y-2)]
function gradient_descent(f,df,start,ϵ,threshold)
    (x,y) = start
    while norm(df(x,y)) > threshold
        x,y = [x,y] + ϵ*df(x,y)
    end
    x,y
end
gradient_descent(f,df,(1,1),0.1,1e-5)
```

(a) Find the bug. (Note: all of the syntax is correct. The mistake has to do with what is being calculated in the body of the function).

(b) Even after the bug you found in (a) is fixed, the algorithm still doesn't get anywhere close to the global minimum at $(1,2)$ if you start at $(0,0)$. However, it does find the global minimum if you start at $(1/4,1/4)$ instead. Explain what goes wrong with the $(0,0)$ starting point.

Solution

Problem 6 **[MATDIFF]**

Suppose that A and C are $m \times n$ matrices and $\mathbf{b}$ and $\mathbf{d}$ are vectors in $\mathbb{R}^m$ such that there is a unique vector $\mathbf{x}$ which minimizes the expression

$$|A\mathbf{x} - \mathbf{b}|^2 + |C\mathbf{x} - \mathbf{d}|^2.$$

Find the minimizing vector $\mathbf{x}$ (in terms of A, $\mathbf{b}$, C, and $\mathbf{d}$). You may assume invertibility wherever convenient.

Solution

Final answer:

Problem 7 **[MACHARITH]**

Find the largest possible value of the quotient $\frac{|a-b|}{b}$, where a is a real number between 2^{-1000} and 2^{1000} and b is the `Float64` value to which a is rounded.

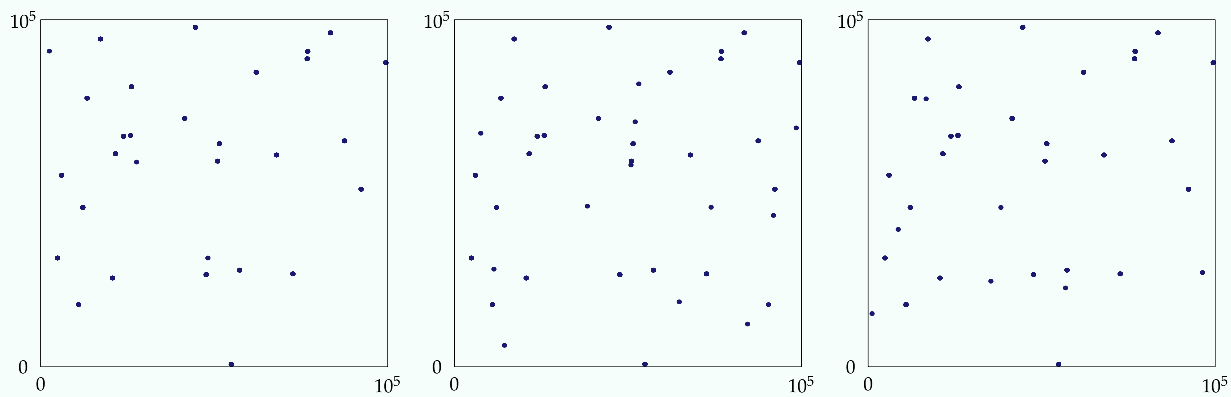
Solution

Final answer:

Problem 8 **[PRNG]**

Consider the following pseudorandom number generator: begin with a four-digit number s, square s, and take second through fifth digits of the result (if the result has fewer than 5 digits, fill in with zeros on the right). For example, if we start with 2431, then the next number would be 9097, since $2431^2 = 5909761$.

If you generate the first 10,000 terms of this sequence starting from the seed 2431, split the terms into blocks of 2, and plot the resulting 5,000 pairs in the plane, you get the first figure shown below. Repeating with the seeds 7532 and 1074, you get the second and third plots.



Using appropriate terminology, discuss one or more major weaknesses of this pseudorandom number generator.

Solution

Problem 9 **[NUMERROR]**

(a) Explain why the last line in the code block below does not return 0, even though the functions f and g are mathematically equivalent for $x \neq 2$.

```
f(x) = (x^2 - 4) / (x-2)
g(x) = x + 2
f(2.002) - g(2.002) # returns -1.6431300764452317e-13
```

(b) Consider the problem of evaluating the (mathematical) function $f(x) = (x^2 - 4)/(x - 2)$ for values of x near 2 (but not equal to 2). This problem is

(i) well-conditioned
(ii) ill-conditioned
(iii) stable
(iv) unstable

(c) The algorithm executed by the Julia function `f` is

(i) well-conditioned
(ii) ill-conditioned
(iii) stable
(iv) unstable

(d) The algorithm executed by the Julia function `g` is

(i) well-conditioned
(ii) ill-conditioned
(iii) stable
(iv) unstable

Solution

Problem 10 **[COUNTING]**

You have 4 mystery novels and 6 textbooks, and you want to arrange 3 of the mystery novels and 4 of the textbooks in order on your bookshelf. If you want the mystery novels to appear all together and the textbooks to appear all together, how many ways are there to do this?

(For example, $T_6T_1T_3T_4M_1M_4M_2$ be an acceptable arrangement, while $T_3T_1M_3M_4M_1T_5T_4$ would not be. Furthermore, $T_6T_1T_3T_4M_1M_2M_3$ and $M_1M_2M_3T_6T_1T_3T_4$ count as different arrangements.)

Solution

Problem 11 **[PROBSPACE]**

Suppose that E and F are events with the property that if E does not occur, then F occurs.

(a) Translate the real-world statement "if E does not occur, then F occurs" into a statement about E and F as subsets of the sample space Ω.

(b) Show that $\mathbb{P}(E) + \mathbb{P}(F) \geq 1$ using the probability space axioms and subadditivity.

Solution

Problem 12 **[JULIA]**

Write a Julia function `zigzag` which accepts a vector as an argument and returns `true` if the second entry is greater than the first, the third is *less* than the second, the fourth is greater than the third, and so on.

```
@assert zigzag([1,4,3,7,-1,5,2]) == true
@assert zigzag([1,4,0,1,-2,0,16,3]) == false
@assert zigzag([7,1,7,1]) == false
@assert zigzag([4,4,2,3,0,10]) == false
```

Solution

Problem 13 **[JULIA]**

The Babylonian method for approximating $\sqrt{x}$ works as follows. We begin with $t_0 = 1$, and for $n \geq 1$, we define $t_n = \frac{1}{2}(t_{n-1} + x/t_{n-1})$. As $n \to \infty$, t_n converges to $\sqrt{x}$ quite quickly.

(a) Write a Julia function `babylonsqrt` which takes `x` as an argument and computes the 20th iterate of the above sequence (in other words, t_{20}) for the given value of `x`.

```
@assert isapprox(babylonsqrt(5),sqrt(5))
```

(b) Show that if you apply the Babylon square root algorithm with 20 iterations to `[5 1; 0 5]`, you get

$$\begin{bmatrix} 2.2361 & 0.2236 \\ 0.0 & 2.2361 \end{bmatrix}.$$

You will have to alter your function a bit so that it works for matrices. For example, the iteration should start at `I` instead of 1, and x/t should be replaced with `x*inv(t)`.

(c) Interpret the top-right entry 0.2236.

Additional space

Additional space