MATH 520 PRACTICE FINAL
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BROWN UNIVERSITY
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Name: Solutions

Allowed materials are pen, pencil, and straightedge. You have three hours.

Problem 1

Find t such that the solution set of $A\mathbf{x} = \mathbf{0}$ has 8 free variables, where

$$A = \begin{bmatrix} -4 & 3 & 4 & 1 & 4 & 4 & 3 & -3 & -5 & 1 \\ 4 & 4 & 6 & -6 & 1 & 4 & 1 & -3 & -2 & -4t+2 \\ 0 & 7 & 10 & -5 & 5 & 8 & 4 & -6 & -7 & t \end{bmatrix}.$$

Solution

8 free variables $\Rightarrow$ 2 pivot rows $\Rightarrow$ rows of A are linearly independent. Since row 3 = row 1 + row 2 if $1+(-4t+2)=t$, we see that $3-4t=t \Rightarrow t=\frac{3}{5}$.

Final answer:

$t = \frac{3}{5}$

Problem 2

Suppose that A is an $m \times 5$ matrix for which the equation $A\mathbf{x} = \mathbf{0}$ has

$$\mathbf{x} = \begin{bmatrix} 8 \\ 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

in its solution set. Are the first 4 columns of A linearly independent? Explain your reasoning.

Solution

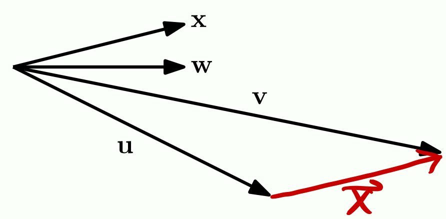
these columns are <u>not</u> linearly independent, because

$$8\vec{a}_1 + 3\vec{a}_2 + 2\vec{a}_3 - \vec{a}_4 = \vec{0},$$

[if $\vec{a}_1, \ldots, \vec{a}_4$ are the first four columns].

Problem 3

Two of the following four vectors sum to one of the other two vectors. Write an equation expressing this relationship.



Solution

Final answer:

$\vec{u} + \vec{x} = \vec{v}$

Problem 4

Find the least number M such that every 3×4 matrix can be reduced to its reduced row echelon form with no more than M row operations. Explain your reasoning.

Solution

(*) If diagonal entries are nonzero:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Each of these entries requires a row operation to eliminate if they're nonzero

Each of these entries requires one row operation to normalize

So $\boxed{9}$ operations.

(*) If some zeros on diagonal, but no zero columns:
it costs one extra row switch to get a nonzero entry to the right row, but then saves one operation because we have one fewer entry to zero out

(*) zero column: saves operations; don't have to deal w/ that column

Problem 5

Suppose that A is an 5×7 matrix of rank 3, and that B is a 7×4 matrix of rank 2. Show that the rank of AB is no greater than 2.

Solution

The range of B is a two-dimensional subspace of $\mathbb{R}^7$; suppose $\{\vec{b}_1, \vec{b}_2\}$ is a basis of range B. Then $\{A\vec{b}_1, A\vec{b}_2\}$ is a spanning set for range AB, since $\vec{y} = AB\vec{x} = A(\alpha\vec{b}_1 + \beta\vec{b}_2) = \alpha A\vec{b}_1 + \beta A\vec{b}_2$ for some α, β. Since dim $\leq$ length of spanning list, we get rank $AB \leq 2$.

Problem 6

Solve the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix}$ two ways: (i) using the 2×2 matrix inversion formula, and (ii) using Cramer's rule. Show that you get the same answer either way.

Solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} de-bf \\ af-ce \end{bmatrix} \leftarrow 2\times 2 \text{ inversion}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{ad-bc} = \frac{de-bf}{ad-bc}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{ad-bc} = \frac{af-ce}{ad-bc}$$

Cramer

Problem 7

Find c and d so that $A^2 = I$, where $A = \begin{bmatrix} 1 & \frac{1}{3} \\ c & d \end{bmatrix}$.

Solution

$$A^2 = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1+\frac{1}{3}c & \frac{1}{3}+\frac{1}{3}d \\ c+cd & \frac{1}{3}c+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if $1+\frac{1}{3}c = 1 \Rightarrow \boxed{c = 0}$

$\frac{1}{3}+\frac{1}{3}d = 0 \Rightarrow \boxed{d = -1}$

Problem 8

Suppose that A is a square matrix. Show that if A^2 is invertible, then A is also invertible.

Solution

If A were noninvertible, then there would be nonzero $\vec{x}$ so that

$$A\vec{x} = \vec{0}.$$

then $A(A\vec{x}) = A\vec{0} \Rightarrow A^2\vec{x} = \vec{0}$, so $A^2\vec{x} = \vec{0}$ would have nontrivial solutions too. so A^2 would be noninvertible.

OR

A^2 invertible $\Rightarrow$ A maps range A onto $\mathbb{R}^n$

$\Rightarrow$ A is surjective

Problem 9

Find det $\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$. $= A$

Solution

$$|A| = \begin{vmatrix} 2 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$

(row $j \to$ row j − row$(j-1)$ for $j = 2, \dots, 5$)

column operation

$$= \begin{vmatrix} 2 & 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

more column operations, executed right to left

$$= \begin{vmatrix} 6 & 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \boxed{6}$$

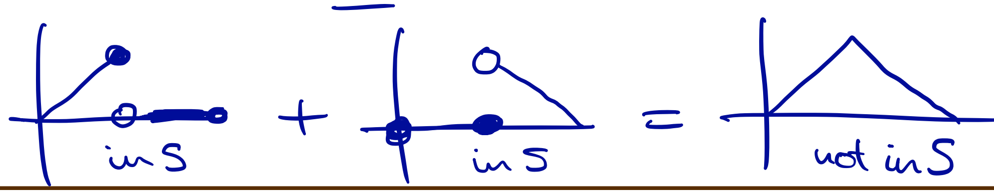
Problem 10(a)

The set V of all functions from $[0,1]$ to $\mathbb{R}$, equipped with the usual notions of function addition and scalar multiplication, is a vector space. Consider the subset S of V consisting of those functions which are discontinuous at one or more values of x. Show that S is not a subspace of V.

Solution

S does not contain the zero vector

OR



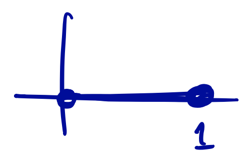
not closed under addition

Problem 10(b)

Consider the set S' of all elements of V which are discontinuous at at most finitely many points (that is, $f \in S'$ if and only if the discontinuity set of f is empty or finite). Show that S' *is* a subspace of $\mathbb{R}^n$. (Note: recall from calculus that if two functions f and g are both continuous at $x \in [0,1]$, then $f + g$ is continuous at x).

Solution

☑ 0 is discontinuous nowhere:



so $0 \in S$

☑ $f, g \in S \Rightarrow f$ discontinuous at some finite set D_1

g discontinuous at some finite set D_2

$\Rightarrow f+g$ discontinuous at some subset of $D_1 \cup D_2$, since $f+g$ can only be discontinuous where either f or g is

☑ $f \in S \Rightarrow cf \in S$ because cf has the same discontinuity set as f if $c \neq 0$ and $cf = 0$ if $c = 0$.

Problem 11

Consider the linear transformation $T : \mathbb{P}_{100} \to \mathbb{P}_{100}$ defined by $T(p) = p + p''$. Show that the rank of T is 101. Hint: start by considering the nullity of T.

Solution

If p has leading term ax^k, then $p+p''$ has leading term ax^k as well. So $p+p''=0 \Rightarrow p=0$. Thus nullity $T = 0 \Rightarrow$ rank $T = \dim \mathbb{P}_{100} = 101$.

Problem 12

Suppose that $A = \begin{bmatrix} 1 & 0 & -1 \\ \sqrt{3} & a & 17 \\ 2 & 0 & b \end{bmatrix}$ has eigenvalues 2 and 3, and that the eigenvalue 2 has multiplicity 2. Find a and b.

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ \sqrt{3} & a-\lambda & 17 \\ 2 & 0 & b-\lambda \end{vmatrix}$$

$$= (a-\lambda)\left[(1-\lambda)(b-\lambda)+2\right]$$

$$= -\lambda^3 + \lambda^2(a+b+1) + \lambda(-2-a-b-ab) + a(2+b)$$

We also have $\det(A-\lambda I) = -(\lambda-2)^2(\lambda-3)$

$$= -\lambda^3 + 7\lambda^2 - 16\lambda + 12$$

So $a+b=6$ & $-2-(a+b)-ab = -16 \Rightarrow ab = 8 \Rightarrow (a,b)=(4,2)$ or $(a,b)=(2,4)$.

But $a(2+b) = 2(2+4) = 12$ only if $\boxed{(a,b)=2,4}$

Problem 13

The *spectral theorem*, probably the most important thorem not covered in the course, states that if A is an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n, \tag{1}$$

where P_k is a matrix representing a projection onto the eigenspace corresponding to λ_k, for each $k = 1, 2, \ldots, n$.

To find P_k satisfying (1), we let U be the matrix whose columns are linearly independent eigenvectors of A (in the same order as their corresponding λ's), and we obtain U_k from U by replacing all but the kth column with zeros. Then we let $P_k = U_k U^{-1}$.

Verify the spectral theorem in the case $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. (Hint: this problem might seem intimidating, but it's purely a matter of following the instructions and doing some matrix calculations.)

Solution

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda + 6 - 2 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 1, 4.$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$ (or some multiple thereof)

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ (or some multiple thereof)

then $P_1 = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{3}$

$$= \frac{1}{3} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

and $P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{3} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

So $\lambda_1 P_1 + \lambda_2 P_2 = \frac{1}{3} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 4 & 8 \\ 4 & 8 \end{bmatrix}$

$$= \frac{1}{3} \begin{bmatrix} 6 & 6 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} = A$$:)

Problem 14

Use linear algebra to find real numbers α and β which minimize

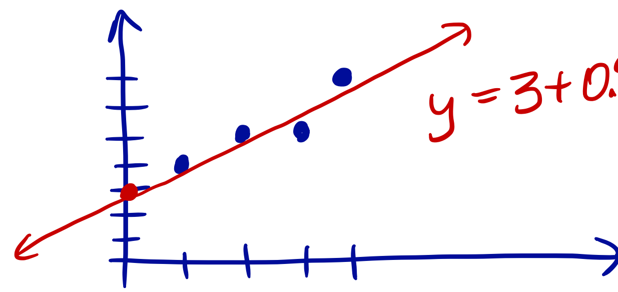
$$(y_1 - \alpha - \beta x_1)^2 + (y_2 - \alpha - \beta x_2)^2 + (y_3 - \alpha - \beta x_3)^2 + (y_4 - \alpha - \beta x_4)^2,$$

where $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\} = \{(1, 4), (2, 5), (3, 5), (4, 7)\}$. Sketch these ordered pairs in the plane and interpret your findings graphically.

Solution

We are looking to minimize

$$|A\vec{x} - \vec{b}|^2$$



where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 5 \\ 7 \end{bmatrix}$. This happens when $A^T(A\vec{x} - \vec{b}) = \vec{0}$, i.e.,

$$\vec{x} = (A^TA)^{-1}A^T\vec{b}$$

$$= \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 21 \\ 57 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix}\begin{pmatrix} 21 \\ 57 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 60 \\ 18 \end{pmatrix} = \begin{pmatrix} 3 \\ 0.9 \end{pmatrix}$$

The graphical interpretation is that the line $3+1.9x$ minimizes the sum of squared error & thus does the best job of approximating the four given data points.