BROWN UNIVERSITY
PROBLEM SET 5
INSTRUCTOR: SAMUEL S. WATSON
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Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

(a) Find the quadratic Maclaurin polynomial for the function $f(x, y) = e^{3x+y}$ by calculating all the relevant partial derivatives.

(b) Find the quadratic Maclaurin polynomial for e^t and substitute $t = 3x + y$.

Solution

(a) The relevant partial derivatives of f at the origin are

$$\begin{aligned} f(0,0) &= 1 \\ f_x(0,0) &= 3 \\ f_y(0,0) &= 1 \\ f_{xx}(0,0) &= 9 \\ f_{yy}(0,0) &= 1 \\ f_{xy}(0,0) &= 3. \end{aligned}$$

So the quadratic Maclaurin polynomial for f is

$$1 + 3x + y + \tfrac{9}{2}x^2 + 3xy + \tfrac{1}{2}y^2.$$

(b) The Maclaurin polynomial for e^t is $1 + t + t^2/2$. If we substitute $t = 3x + y$, we get $1 + 3x + y + (3x + y)^2/2 = 1 + 3x + y + \frac{9}{2}x^2 + 3xy + \frac{1}{2}y^2$, the same as in (a).

Problem 2

Consider the function $f(x,y) = \frac{e^{xy}}{e(1+x^2)}$.

(a) Use a quadratic Taylor polynomial centered at $(1,1)$ to approximate $f(0.99, 0.98)$. Compare your answer to Example 4.3.3 in the book.

(b) Use a quadratic Taylor polynomial centered at $(0,0)$ to approximate $f(0.99, 0.98)$.

(c) The following code can be copy-pasted at `sagecell.sagemath.org` (or click here) to calculate the degree-50 Maclaurin polynomial of f and evaluate it at $(0.99, 0.98)$.

```
var("x y") # declares x and y to be symbolic variables
f(x,y) = exp(x*y-1)/(1+x^2) # defines f
taylor(f(x,y),(x,0),(y,0),50).subs(x=0.99,y=0.98).n()
```

How good is this estimate compared to the ones in (a) above and in Example 4.3.3 in the text?

(d) Repeat (c) but with the function $g(x,y) = \frac{e^{xy}}{e(9+x^2)}$. Does the degree-50 Maclaurin polynomial approximate the value of $g(0.99, 0.98)$ well?

Solution

(a) The Taylor polynomial centered at $(1,1)$ is $\frac{1}{2}y + \frac{1}{2}(x-1)(y-1) + \frac{1}{4}(y-1)^2$. Substituting $(0.99, 0.98)$ gives exactly 0.4902. The linear approximation yielded 0.49, and the actual value is $0.4901972\ldots$, so the quadratic approximation is much more accurate.

(b) We get an approximation of $(x^2 + xy + 1)/e = 0.364237$, which is not nearly as good as the approximations centered at $(1,1)$.

(c) We get $0.4492\ldots$, which is also not very good.

(d) We run the code

```
var('x y') # declares x and y to be symbolic variables
f(x,y) = exp(x*y-1)/(9+x^2) # defines f
taylor(f(x,y),(x,0),(y,0),50).subs(x=0.99,y=0.98).n(), f(0.99,0.98)
```

which returns

```
(0.0972575066447653, 0.0972575066447653)
```

Remarkably, this estimate is so good that all the digits displayed by the system are the same.

Here's an amazing fact: the Maclaurin polynomial of the function $1/(1+x^2)$ only approximates it well up to a certain distance from the origin. This critical distance is determined by the least distance from the origin to a *complex* root of the polynomial $1 + x^2$ in the denominator. This is one sense in which the study of complex numbers can be relevant even when looking at real-valued functions. This idea is explored further in my course notes for Math 19 [link].

Problem 3

Consider the function $f(x,y) = \frac{1}{xy}$. Show that f has no maximum or minimum value on the open unit square $S = (0,1) \times (0,1)$. In other words, show that for any $(x,y) \in S$, there exists $(x',y') \in S$ with $f(x',y') > f(x,y)$ (and similarly for the minimum).

Solution

Given any $(x,y) \in S$, we can choose x' to be any number strictly between 0 and x, and we can set $y' = y$. Then $f(x',y') = 1/(x'y') > 1/(xy) = f(x,y)$. Thus f has no maximum value on S.

Similarly, if $(x,y) \in S$, then we can choose x' to be strictly between x and 1 and set $y' = y$. Then $f(x',y') = 1/(x'y') < 1/(xy) = f(x,y)$. Thus f has no minimum value on S.

This example does not contradict the extreme value theorem because the set S is not **closed**.

Problem 4

Let D be the closed unit disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Come up with a function $f : D \to \mathbb{R}$ with the property that f does not have a maximum value on D. Explain why your function does indeed have this property.

Solution

There are many such functions, but let us set $f(x,y) = 1/(1 + x^2 + y^2)$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then from any $(x,y) \neq (0,0)$, we can increase the value of f by moving a bit closer to the origin, and $f(0,0)$ is actually the function's *minimum* value.

This function does not contradict the extreme value theorem because the function is **discontinuous**.

Problem 5

Find the maximum and minimum values of $f(x,y) = x^4 + y^4 - 4xy$ on the rectangle $[0,3] \times [0,2]$.

Solution

The system of equations $\partial_x f = \partial_y f = 0$ tells us that $y = x^3$ and $x = y^3$. Substituting, we find $x = x^9$, which can be rearranged and factored (by repeated applications of difference of squares) to get

$$x(x-1)(x+1)(x^2+1)(x^4+1) = 0.$$

The only value of x in the interior of the rectangle which satisfies this equation is $x = 1$. Substituting into $y = x^3$, we find the critical point $(1,1)$ in the interior of the rectangle. The value of the function this point is -2.

Along the bottom edge of the rectangle, the value of the function at $(x,0)$ is x^4 which ranges monotonically from 0 to 81. Along the top edge, we have $f(x,2) = x^4 - 8x + 16$. This has a critical point at $x = \sqrt[3]{2}$.

Along the left edge, the function is equal to y^4, which has no interior critical points. And finally, along the right edge, we have $f(3,y) = y^4 - 12y + 81$, which has a critical point at $\sqrt[3]{3}$.

So altogether, checking critical points and corners gives

(x,y)	$f(x,y)$
$(1,1)$	-2
$(0,0)$	0
$(3,0)$	81
$(0,2)$	16
$(3,2)$	73
$(3,\sqrt[3]{3})$	$81 - 9\sqrt[3]{3} \approx 68.02$
$(\sqrt[3]{2},2)$	$16 - 6\sqrt[3]{2} \approx 8.44$

So the absolute maximum is $\boxed{81}$ and the absolute minimum is $\boxed{-2}$.

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