Brown University
Math 0350, Honors Calculus
Final Exam
Instructor: Samuel S. Watson

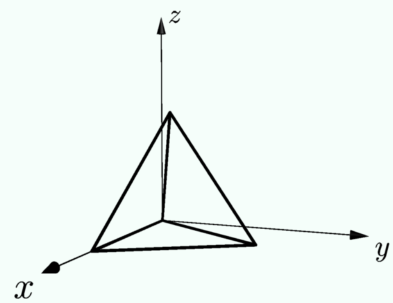
Name:

Problem 1

Consider the regular tetrahedron with vertices at $A = (0,0,0)$, $B = (1,0,0)$, $C = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, and $D = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3})$.

(a) Label the vertices on the figure shown.

(b) Find the angle between $\overrightarrow{AD}$ and the vector from A to the midpoint of $\overline{BC}$. You may express your answer in terms of radicals and inverse trig functions.



Solution

We label the figure as shown. Then

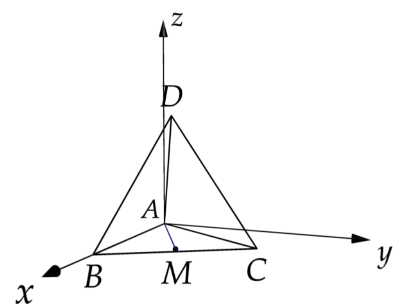
$$\overrightarrow{AM} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \left\langle \frac{3}{4}, \frac{\sqrt{3}}{4}, 0 \right\rangle,$$

so

$$\overrightarrow{AD} \cdot \overrightarrow{AM} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3} \right\rangle \cdot \left\langle \frac{3}{4}, \frac{\sqrt{3}}{4}, 0 \right\rangle = \frac{1}{2},$$

and $AM = \frac{\sqrt{3}}{2}$ and $AD = 1$. Thus

$$\theta = \cos^{-1}\left(\frac{1/2}{(\sqrt{3}/2)(1)}\right) = \cos^{-1}(1/\sqrt{3}).$$



Final answer:

$$\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Problem 2

Find the value of t such that the points $(4, 0, 2)$, $(12, 3, 0)$, and $(2 - 2t, -t, 4)$ are collinear.

Solution

The points are collinear if the vector

$$(12, 3, 0) - (4, 0, 2) = \langle 8, 3, -2 \rangle$$

from the first point to the second is parallel to the vector

$$\langle -2 - 2t, -t, 2 \rangle$$

from the first point to the third. Writing this relation as

$$\langle 8, 3, -2 \rangle = \lambda \langle -2 - 2t, -t, 2 \rangle$$

for some $\lambda \in \mathbb{R}$, we see from the third component that $\lambda = -1$. Thus $t = 3$.

Final answer:

$$3$$

Problem 3

Suppose that C_1 is a curve parametrized by $\mathbf{r}_1(t) = \langle t^2, t^3 \rangle$ as t ranges from 0 to 1 and C_2 is a curve parameterized by $\mathbf{r}_2(t) = \langle t^2, \frac{3}{4}t^4 \rangle$ as t ranges from 0 to 1. Which is longer: C_1 or C_2?

Note: this problem should be very light on calculation. Do not get into the weeds.

Solution

The length of C_1 is

$$\int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, \mathrm{d}t,$$

while the length of C_2 is

$$\int_0^1 \sqrt{(2t)^2 + (3t^3)^2}\, \mathrm{d}t.$$

Since $3t^3 < 3t^2$, the first integral is larger. Therefore, C_1 is the longer of the two curves.

Final answer:

$$C_1$$

Problem 4

How many of the following numbers c have the property that there exists a direction $\mathbf{u}$ such that

$$(D_{\mathbf{u}}f)(1,1) = c,$$

where $f(x,y) = x^2 + 3xy^2$?

$$8 \qquad -3 \qquad 7 \qquad -\sqrt{62} \qquad 0 \qquad 2 \qquad 13 \qquad 4 \qquad -1 \qquad \pi$$

Explain your reasoning.

Solution

We have $(D_{\mathbf{u}}f)(1,1) = \nabla f \cdot \mathbf{u} = \langle 2x + 3y^2, 6xy\rangle \cdot \mathbf{u} = \langle 5, 6\rangle \cdot \mathbf{u} = \sqrt{61}\cos\theta$, where θ is the angle between $\langle 5, 6\rangle$ and $\mathbf{u}$. Thus the possible values of the directional derivative are the real numbers between $-\sqrt{61}$ and $\sqrt{61}$. Exactly 7 of the given numbers fall in this range.

Final answer:

$$7$$

Problem 5

Suppose that P is a point selected uniformly at random from the set of points (x,y,z) satisfying $x^2 + y^2 + z^2 \le 1$. Find the expected value of the distance from P to the z-axis. Express your answer in terms of π.

Solution

The pdf of P is some constant k over the solid unit sphere, and we know that $\iiint_{\text{unit sphere}} k \,\mathrm{d}V = 1$, which implies that $k = \frac{3}{4\pi}$. Furthermore, the expected value of $g(x,y,z) = \sqrt{x^2+y^2}$ is

$$\begin{aligned}
\int_{\text{unit sphere}} g(x,y,z)\frac{3}{4\pi}\,\mathrm{d}V &= \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \overbrace{\rho\sin\phi}^{\sqrt{x^2+y^2}}\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{3}{4\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin^2\phi\,\mathrm{d}\phi\int_0^1\rho^3\,\mathrm{d}\rho \\
&= \left(\frac{3}{4\pi}\right)(2\pi)\left(\frac{\pi}{2}\right)\left(\frac{1}{4}\right) \\
&= \frac{3\pi}{16}
\end{aligned}$$

Final answer:

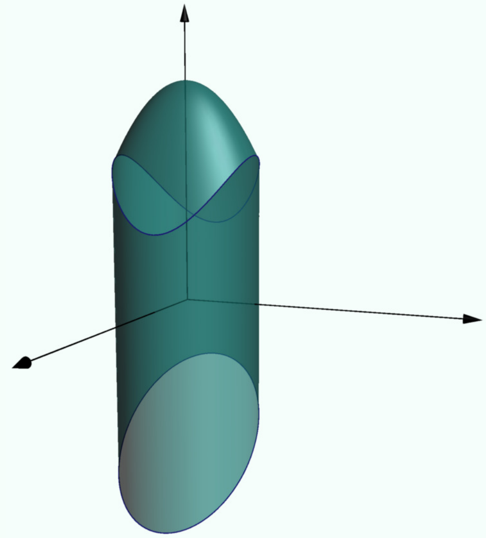
$$\frac{3\pi}{16}$$

Problem 6

Consider the solid R bounded by the cylinder $x^2 + y^2 = 4$, the surface $z = 6 - x^2 - \frac{1}{2}y^2$, and the plane $x + z = -4$. Find

$$\iiint_R 1\, dV.$$

(Hint: use a suitable coordinate system, and remember our trick for evaluating $\int_0^{2\pi} \cos^2\theta\, d\theta$ and $\int_0^{2\pi} \sin^2\theta\, d\theta$.)



Solution

We have

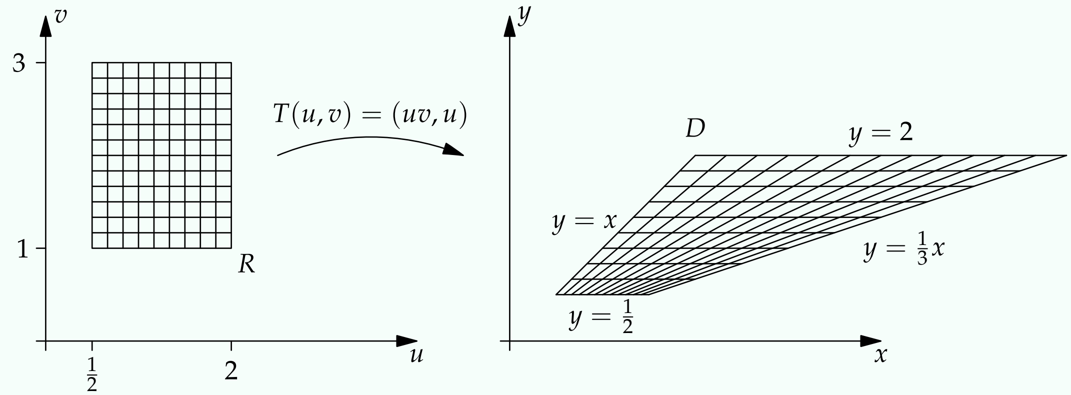
$$\begin{aligned}
\iiint_R 1\, \mathrm{d}V &= \int_0^{2\pi} \int_0^2 \int_{-4-r\cos\theta}^{6-r^2\cos^2\theta-\frac{1}{2}r^2\sin^2\theta} 1r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \int_0^{2\pi} \left[5r^2 - \frac{1}{4}r^4\cos^2\theta - \frac{r^4}{8}\sin^2\theta + \frac{1}{3}r^3\cos\theta\right]_0^2 \mathrm{d}\theta \\
&= \int_0^{2\pi} 20 - 4\cos^2\theta - 2\sin^2\theta + \frac{8}{3}\cos\theta\, \mathrm{d}\theta \\
&= 40\pi - 4\pi - 2\pi = \boxed{34\pi}.
\end{aligned}$$

Final answer:

34π

Problem 7

Consider the trapezoid D shown below. This trapezoid is the image of a rectangle under the transformation $(x, y) = (uv, u)$, as shown.



(a) Based on the figure, along which side of R do you expect the Jacobian of this transformation to be largest?

(b) Calculate the Jacobian of T. Is the result actually bigger along the side you predicted in (a) than along the other sides?

(c) Use this coordinate transformation in the change of variables formula to calculate $\iint_D xy\,dx\,dy$. Express your answer as a mixed number.

Solution

(a) Whichever side maps to the top has the largest Jacobian, since those patches are biggest. Matching up vertices, we see that this is the right edge.

(b) The Jacobian is

$$\left|\det \begin{bmatrix} v & u \\ 1 & 0 \end{bmatrix}\right| = |-u| = u,$$

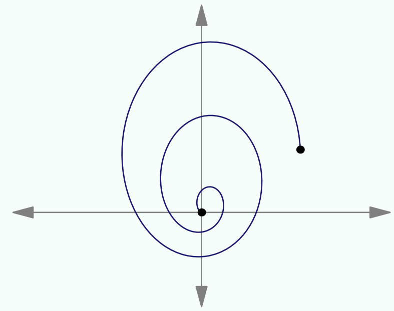
which is indeed largest along the right edge.

(c) We have

$$\int_1^3 \int_{1/2}^2 uv \cdot u \cdot u \, \mathrm{d}u \, \mathrm{d}v = \int_1^3 v \, \mathrm{d}v \int_{1/2}^2 u^3 \, \mathrm{d}u = \boxed{15\frac{15}{16}}.$$

Problem 8

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle y\cos(xy), y^3 + x\cos(xy)\rangle$ and C is the spiral curve shown, which starts at $(\pi/2, 1)$ and ends at the origin.



Solution

The function $f(x, y) = \sin(xy) + \frac{y^4}{4}$ has $\mathbf{F}$ as its gradient. Therefore, the fundamental theorem of vector calculus implies that the desired integral is

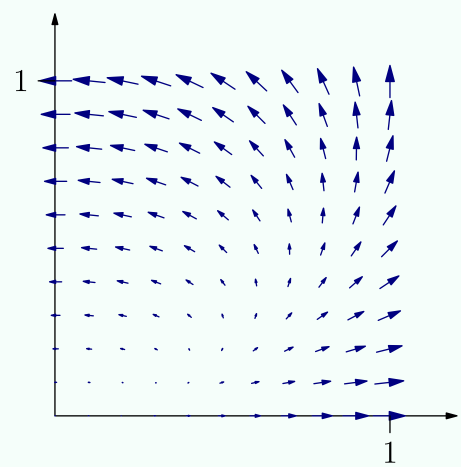
$$f(0,0) - f\left(\frac{\pi}{2}, 1\right) = 0 - \left(1 + \frac{1}{4}\right) = -\frac{5}{4}.$$

Final answer:

$$-\frac{5}{4}$$

Problem 9

(a) Consider the vector field $\mathbf{F} = \langle x^2 - y, xy\rangle$. Explain how to infer the sign of $\nabla \cdot \mathbf{F}$ at the point $(0.5, 0.5)$ based only on the vector plot of $\mathbf{F}$ shown. Then confirm your answer by calculating $(\nabla \cdot \mathbf{F})(0.5, 0.5)$ exactly using the above expression for $\mathbf{F}$.



(b) Suppose now that $\mathbf{F}$ is a 3D vector field with the property that

$$\mathbf{F}(x, y, 0) = \langle x^2 - y, xy, 0\rangle$$

for all x and y. Explain how to infer the sign of the third component of $\nabla \times \mathbf{F}$ at the point $(0.5, 0.5, 0)$ based only on the vector plot of $\mathbf{F}$ shown. Then confirm your answer by calculating the third component of $(\nabla \times \mathbf{F})(0.5, 0.5, 0)$ exactly using the above expression for $\mathbf{F}$.

Solution

(a) The arrow pointing away from $(0.5, 0.5)$ is longer that the one pointing toward it. Therefore, the vector plot leads us to believe that the divergence at $(0.5, 0.5)$ is positive.

We calculate $\nabla \cdot \mathbf{F} = 2x + x = 3x = \frac{3}{2} > 0$ at $(0.5, 0.5)$, which comports with our graphical conclusion.

(b) A small paddle wheel situated at $(0.5, 0.5, 0)$ with its axle in the z direction would rotate counterclockwise viewed from above, because the vectors pushing on the top right part of the paddle wheel are longer than the vectors pushing on the bottom left. Therefore, we would expect the third component of $\nabla \times \mathbf{F}$ to be positive at $(0.5, 0.5, 0)$.

If we calculate the curl of $\mathbf{F}$, we get

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \partial_x(xy) - \partial_y(x^2 - y) = y + 1 = \frac{3}{2},$$

which is indeed positive.

Problem 10

Define the surface S to be the (whole) boundary of the solid cone $r \leq z \leq 1$. Find the flow outward through S of the vector field $\mathbf{F} = \langle y, x, z^2 \rangle$.

Solution

We apply the divergence theorem to find that the given flow is equal to $\iiint_R 2z \, \mathrm{d}V$, where R is the cone described. We can integrate over this cone in cylindrical coordinates to get

$$\int_0^{2\pi} \int_0^1 \int_r^1 2zr \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{2\pi} \int_0^1 r(1 - r^2) \, \mathrm{d}r \, \mathrm{d}\theta = \left(\frac{1}{4}\right)(2\pi) = \frac{\pi}{2}.$$

Final answer:

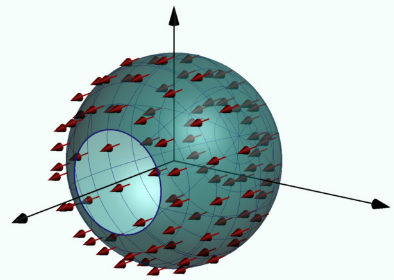
$$\frac{\pi}{2}$$

Problem 11

Consider the surface S of points where $x^2 + y^2 + z^2 = 1$ and $x \le \frac{\sqrt{3}}{2}$. Suppose that $\mathbf{G} = \langle 2, 0, 0 \rangle$, which is equal to the curl of $\langle 0, -z, y \rangle$.

(a) Determine from the figure what the sign of the flow of $\mathbf{G}$ through S, from the inside to the outside, should be. Explain your reasoning.

(b) Find the flow of $\mathbf{G}$ through S (from the inside to the outside).



Solution

(a) The net flow of the region through the sphere is from the outside to the inside, since the contributions of all the outward pointing arrows are canceled by corresponding inward-pointing errors on the opposite side of the sphere. Some of the contributions of inward-pointing vectors (in the back, where x is at its most negative) are not canceled. Therefore, the answer to (a) should be negative.

(b) Stokes' theorem implies that the desired flow is equal to the line integral of $\langle 0, -z, y \rangle$ along the boundary of the surface. This boundary is the circle of radius $\sqrt{1 - (\frac{\sqrt{3}}{2})^2} = \frac{1}{2}$ centered at $\left(\frac{\sqrt{3}}{2}, 0, 0\right)$ and perpendicular to $\langle 1, 0, 0 \rangle$. The boundary orientation corresponding to the inside-to-outside surface orientation is clockwise as viewed from the positive x-axis.

So we can use the parametrization $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\cos t, -\frac{1}{2}\sin t\right)$ for this loop and calculate

$$\int_0^{2\pi} \left\langle 0, \frac{1}{2}\sin t, \frac{1}{2}\cos t \right\rangle \cdot \left\langle 0, -\frac{1}{2}\sin t, -\frac{1}{2}\cos t \right\rangle \mathrm{d}t = -\frac{\pi}{2}.$$

Alternative solution for (b): Stokes' theorem allows us to write the desired flow as the flow (in the direction of decreasing x) of $\mathbf{G}$ through the disk of radius $\frac{1}{2}$ centered along the x-axis at the point $\left(\frac{\sqrt{3}}{2}, 0, 0\right)$. This flow is equal to $-(2)(\pi/4) = -\pi/2$. No integration!

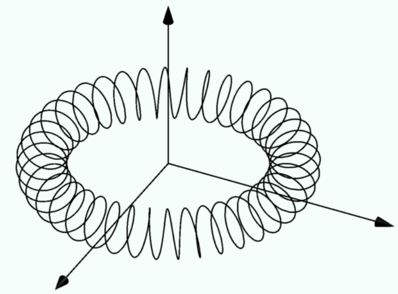
Final answer:

$$-\frac{\pi}{2}$$

BONUS (0 points)

The business end of a frother looks like the coil in the figure shown. Find a parameterized curve which looks as similar as possible to the one shown.

Note: the coil winds around a donut-shaped solid with the following property: the intersection of the solid with any level surface of the coordinate function θ is a disk of radius 0.2 which is centered at some point on the unit circle in the xy-plane.



Solution

We define a curve $\mathbf{r}_1(t) = \langle \cos t, \sin t, 0\rangle$ which traces out the unit circle passing through the middle of the angular cross sections. We also define a curve $\mathbf{r}_2$ which winds around the sphere of radius 0.2 along a longitude line $\theta(t) = t$. The point whose ϕ value is kt and whose θ value is given by t is $\langle \sin kt \cos t, \sin kt \sin t, \cos kt\rangle$. We choose the value $k = 40$ based on the number of times the coil winds around in a full revolution.

Altogether, we have

$$\mathbf{r}_1(t) + \mathbf{r}_2(t) = \langle \cos t + 0.2 \cos t \sin 40t, \sin t + 0.2 \sin t \sin 40t, \cos 40t\rangle.$$

Final answer:

$$\langle \cos t + 0.2 \cos t \sin 40t, \sin t + 0.2 \sin t \sin 40t, \cos 40t\rangle$$

Additional space

Additional space

Additional space