MATH 520 MIDTERM II
SPRING 2017
BROWN UNIVERSITY
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Name: Solutions

This is a pencil-and-paper-only exam. You have two hours.

Problem 1(a) (8 points)

Find $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Final answer:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 1(b) (8 points)

Suppose that A is a 3×3 matrix with the property that $A\mathbf{v}_1 = \mathbf{e}_1$, $A\mathbf{v}_2 = \mathbf{e}_2$, and $A\mathbf{v}_3 = \mathbf{e_2} + \mathbf{e}_3$, where $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$. Find A^{-1}. (Note: $\mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_3$ denote the standard basis vectors in $\mathbb{R}^3$.)

Solution

$A\vec{v}_3 = \vec{e}_3 - \vec{e}_2$

$\Rightarrow A(\vec{v}_3 - \vec{v}_2) = \vec{e}_3$,

So A^{-1} maps e_1 to $\vec{v}_1$

e_2 to $\vec{v}_2$

e_3 to $\vec{v}_3 - \vec{v}_2$. So $A^{-1} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 - \vec{v}_2]$

Final answer:

$$\begin{bmatrix} -4 & 0 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & -1 \end{bmatrix}$$

Problem 2(a) (8 points)

Suppose t is a real number and $A = \begin{bmatrix} t & 13 \\ 2 & t \end{bmatrix}$. Suppose further that the area of the image of any square S in $\mathbb{R}^2$ under the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is equal to 10 times the area of S. Find all four possible values of t (note: you will receive most of the credit for finding two of them).

Solution

$|\det A| = 10 \Rightarrow \det A = 10$ or -10.

$t^2 - 26 = 10 \Rightarrow t = \pm 6$

$t^2 - 26 = -10 \Rightarrow t = \pm 4.$

Final answer:

$\{-6, -4, 4, 6\}$

Problem 2(b) (8 points)

Show that $\det(A + B)$ is not always equal to $\det(A) + \det(B)$, where A and B are 2×2 matrices. Hint: just make up some examples for A and B; it will probably work.

Solution

2x2 identity

$\det(I_2 + I_2) = 4 \neq 2 = \det I_2 + \det I_2$

Final answer:

:)

Problem 3(a) (10 points)

The matrices $A = \begin{bmatrix} 1 & 4 & 5 & 8 & 2 \\ 0 & 1 & -2 & 3 & 5 \\ -2 & -7 & -12 & -13 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 13 & -4 & -18 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 2 & -4 & 6 & 10 \end{bmatrix}$ are row equivalent.

Find a basis for the row space of A and a basis for the column space of A.

Solution

$$\begin{bmatrix} 1 & 0 & 13 & -4 & -18 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 2 & -4 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 13 & -4 & -18 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the pivot rows of form a basis for the rowspace, & the pivot columns of $\underline{A}$ form a basis for col A.

Final answer:

$\{[1 \ 0 \ 13 \ -4 \ -18], [0 \ 1 \ -2 \ 3 \ 5]\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix} \right\}$

Problem 3(b) (5 points)

Find the rank and the nullity of A.

Solution

rank $A = 2$, so

nullity $A = 5-2 = 3$

Final answer:

rank $A = 2$

nullity $A = 3$

Problem 4(a) (8 points)

Consider the set $S = \{f \in C([0,1]) : f(x) \geq 0 \text{ for all } x \in [0,1]\}$ of continuous functions from $[0,1]$ to $\mathbb{R}$ which are nowhere negative. Show that S **is** closed under vector addition in $C([0,1])$ but is **not** closed under scalar multiplication. So is S a subspace?

Solution

If $f, g \in S$, then $(f+g)(x) = f(x) + g(x) \geq 0 + 0 = 0$, so $f+g \in S$. Thus S is closed under vector addition.

However, multiplying any nonzero function in S by a negative scalar yields a function no longer in S.

E.g. $f(x) = x^2$, $c = -1 \Rightarrow cf \notin S$.

So S is not a subspace of $C([0,1])$?

Problem 4(b) (8 points)

Consider the set S of polynomials in $\mathbb{P}_4$ which have no t^2 term. So, for example, $t - 4t^3$ is in the set, while $1 - 8t^2 + t^4$ is not. Is S a subspace of $\mathbb{P}^4$? Justify your answer.

Solution

S is a subspace:

☑ 0 has no t^2 term

☑ the sum of two polynomials with no t^2 term also has no t^2 term:

$$a + bt + ct^3 + dt^4 + \tilde{a} + \tilde{b}t + \tilde{c}t^3 + \tilde{d}t^4$$
$$= a + \tilde{a} + (b + \tilde{b})t + (c + \tilde{c})t^3 + (d + \tilde{d})t^4$$

☑ A constant times a polynomial with no t^2 term also has no t^2 term: $k(a + bt + ct^3 + dt^4) = ka + kbt + kct^3 + kdt^4$.

Problem 5 (10 points)

Consider the linear transformation $T : \mathbb{P}_7 \to \mathbb{P}_7$, where $T(p) = p'$. In other words, T acts by taking the derivative. So, for example, $T(-3t^4 + t^2 - t) = -12t^3 + 2t - 1$.

Find the range and the kernel of T. Feel free to describe these sets using either a verbal description or math notation, as you prefer (as long as they are clearly specified). Explain your reasoning.

Solution

* range $T = \mathbb{P}_6$:
  - ☑ range $T \subset \mathbb{P}_6$, because any poly. of deg. 7 or less differentiates to a poly of deg 6 or less.
  - ☑ $\mathbb{P}_6 \subset$ range T, because if $p \in \mathbb{P}_6$, then $\int_0^t p(u)\,du$ is in $\mathbb{P}_7$ and differentiates to p.

* kernel $T = \mathbb{P}_0$:
  - ☑ kernel $T \subset \mathbb{P}_0$, because if $p' = 0$, then p is a constant polynomial.
  - ☐ $\mathbb{P}_0 \subset$ kernel T, because a constant polynomial differentiates to zero.

Problem 6(a) (6 points)

Explain why $\det A$ is an integer whenever A is a square matrix with integer entries. Hint: for partial credit, work out $\det A$ for some 3×3 matrix with integer entries. How could you have known the answer would be an integer before you did all the calculations?

Solution

$\det A$, where A is $n \times n$, is a polynomial (of degree n) in the entries of A, with coefficients in $\{\pm 1\}$. Since $\mathbb{Z}$ (the set of integers) is closed under multiplication, this implies $\det A \in \mathbb{Z}$.

Problem 6(b) (9 points)

Use Cramer's rule to show that if A is an invertible $n \times n$ matrix with integer entries and $\mathbf{b}$ is an $n \times 1$ vector with integer entries, then the unique solution $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{b}$ is a vector whose entries are rational numbers (that is, simplified fractions with integer numerator and denominator) **whose denominators evenly divide** $\det A$. (So, for example, if $\det A = 8$, then the denominators of the entries of $\mathbf{x}$ are necessarily in the set $\{1, 2, 4, 8\}$).

Solution

Cramer's rule says $x_i = \dfrac{\det([\vec{a}_1 \cdots \vec{a}_{i-1}\ \vec{b}\ \vec{a}_{i+1} \cdots \vec{a}_n])}{\det A}$.

By 6(a), num. & denom. here are integers. So the denom. of x_i is either $\det A$, if the above fraction is simplified, or some factor thereof if the fraction happens to reduce.

Problem 7 (12 points)

Suppose that W is a ten-dimensional vector space and V is a subspace of W whose dimension is 6. Show that there is a four-dimensional subspace U of W with the property that $U \cap V = \{\mathbf{0}\}$ (in other words, so that U and V have no vectors in common except the zero vector). Explain your reasoning precisely. Hint: begin by considering some basis of V.

Solution

Let $\{\vec{b}_1, \dots, \vec{b}_6\}$ be a basis of U, & extend it to a basis $\{\vec{b}_1, \dots, \vec{b}_{10}\}$ of W.

We define $U = \operatorname{span}(\{\vec{b}_7, \vec{b}_8, \vec{b}_9, \vec{b}_{10}\})$. Then U is four-dim, because $\{\vec{b}_7, \vec{b}_8, \vec{b}_9, \vec{b}_{10}\}$ are lin. ind. by virtue of being some of the vectors in a basis and span U by definition of U.

Also, $U \cap V = \{\vec{0}\}$, because if $\vec{w} \in U \cap V$, then $\vec{w} = c_1\vec{b}_1 + \dots + c_6\vec{b}_6$ and $\vec{w} = c_7\vec{b}_7 + \dots + c_{10}\vec{b}_{10}$ for some $c_1, \dots, c_{10}$. So

$$\vec{0} = c_1\vec{b}_1 + \dots + c_6\vec{b}_6 - c_7\vec{b}_7 - \dots - c_{10}\vec{b}_{10},$$

which implies $c_1 = \dots = c_{10} = 0$. So $\vec{w} = \vec{0}$.

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