Name: Solutions

MATH 19 PRACTICE FINAL
FALL 2016
BROWN UNIVERSITY
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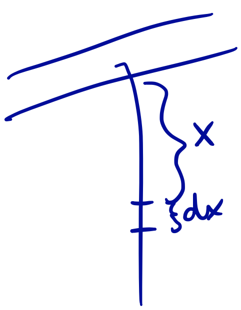
$\boxed{1}$ (10 points) Consider a 100-meter rope, 20 kg rope with one end secured to a bridge and the rest of the rope hanging straight down. How much work does it take to lift the rope onto the bridge? You may assume that $g = 10\,\mathrm{m/s^2}$.



Let x = the distance to the bridge.
Then the work to lift the portion between x and $x + dx$ is

$$W = \overset{\text{mass}}{m} g h$$

$$= \left(\frac{20\,\mathrm{kg}}{100\,\mathrm{m}} \cdot \underset{\text{meter}}{dx\,\mathrm{m}}\right)\left(10\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(x\,\mathrm{m})$$

$$= 2x\,dx\,\frac{\mathrm{kg\,m^2}}{\mathrm{s}^2}.$$

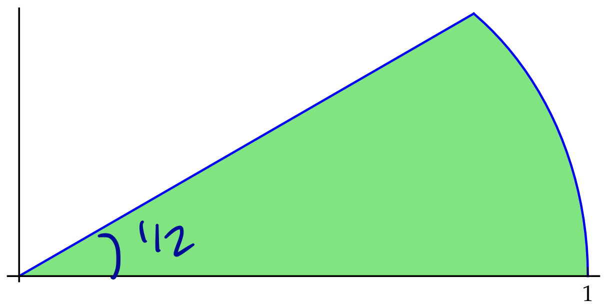
So the total work is $\displaystyle\int dW = \int_0^{100} 2\,x\,dx$

$$= x^2\Big|_0^{100}$$

$$= 10000 \text{ joules}$$

$\boxed{2}$ (10 points) Find the area of the shaded region below. All of the points on the boundary of the region lie on ~~the line $y = 0$ or the line $y = \frac{\sqrt{3}}{3}x$ or satisfy the polar coordinate equation $r^4 = 1 - \theta^2$.~~

$\theta = 0$, $\theta = \frac{1}{2}$, or $r^4 = 1 - \theta^2$



$$\text{area} = \frac{1}{2}\int_0^{1/2} \left[(1-\theta^2)^{1/4}\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{1/2} \sqrt{1-\theta^2}\, d\theta \qquad \begin{array}{l} \theta = \sin u \\ d\theta = \cos u\, du \end{array}$$

$$= \frac{1}{2}\int_0^{\pi/6} \sqrt{1-\sin^2 u}\, \cos u\, du$$

$$= \frac{1}{2}\int_0^{\pi/6} \cos^2 u\, du$$

$$= \frac{1}{2}\int_0^{\pi/6} \frac{1+\cos 2u}{2}\, du$$

$$= \frac{1}{2}\left[\frac{\pi}{12} + \frac{\sin 2u}{4}\Big|_0^{\pi/6}\right]$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{16}$$

$\boxed{3}$ Consider the differential equation

$$f''''(x) = f(x).$$

(a) (2 points) What is the order of this differential equation? How many free constants will the general solution have? Put your answers in the boxes below.

Order:

4

Number of free constants in general solution:

4

(b) (2 points) What is the characteristic polynomial of this differential equation?

$\lambda^4 = 1$

(c) (6 points) Find the general solution of this differential equation.

$\lambda^4 = 1 \Rightarrow \lambda^4 - 1 = 0$

$\Rightarrow (\lambda^2+1)(\lambda^2-1) = 0$

$\Rightarrow (\lambda+i)(\lambda-i)(\lambda+1)(\lambda-1) = 0$

So we get

$Ae^{x} + Be^{-x} + Ce^{-ix} + De^{ix}$

$= Ae^{x} + Be^{-x} + C\sin x + D\cos x.$

$\boxed{4}$ (a) (8 points) In the notes, we found the periodic solution of the DE

$$Q''(t) + Q'(t) + 4Q(t) = \sin t + \cos 2t$$

using complex Fourier series. Find the *general* solution of this differential equation (**not** using Fourier methods—for example, one of the things you should do is find a particular solution using the method of undetermined coefficients).

$$Q''(t) + Q'(t) + 4Q(t) = 0$$

has char. eqn

$$\lambda^2 + \lambda + 4 = 0$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 4}}{2}$$

$$= \frac{-1}{2} \pm \frac{1}{2} i\sqrt{15}$$

so $Q(t) = Ae^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}}{2}t\right) + Be^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{15}}{2}t\right)$

To find a particular solution, we try

$$Q(t) = a\sin t + b\cos t + c\sin 2t + d\cos 2t.$$

We get

$$Q''(t) = -a\sin t - b\cos t - 4c\sin 2t - 4d\cos 2t$$

$$Q'(t) = a\cos t - b\sin t + 2c\cos 2t + 2d\sin 2t$$

$$+ \quad 4Q(t) = 4a\sin t + 4b\cos t + 4c\sin 2t + 4d\cos 2t$$

$$\sin t + \cos 2t = (3a-b)\sin t + (3b-a)\cos t + 2d\sin 2t + 2c\cos 2t$$

$3a - b = 1$
$3b - a = 0$
$a = \frac{3}{10}, b = \frac{1}{10}$

$2d = 0$
$2c = 1$
$\Rightarrow c = 1/2$

(b) (2 points) Using your answer to the previous question, explain why the solution $Q(t)$ of the differential equation approaches the periodic solution as $t \to \infty$, regardless of the values of the free constants.

So the full sol'n is

$$Ae^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right) + Be^{-1/2t}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

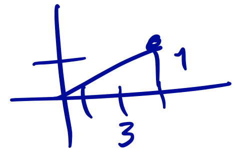
$$+ \frac{3}{10}\sin t - \frac{1}{10}\cos t + \frac{1}{2}\sin 2t$$

these two terms go to 0 as $t \to \infty$, so what's left is the periodic sol

$\boxed{5}$ In this problem, we will use complex numbers to find $\tan(\arctan(1/3) + \arctan(1/4))$. For (a), (b), and (c), please put your final answer in the box provided.

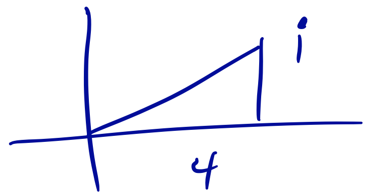
(a) (2 points) Consider $z = 3 + i$. Find the complex angle of z (in other words, if z is expressed in polar form $r \operatorname{cis} \theta$, find θ). You may express your answer using the arctan function.

$\boxed{\arctan(1/3)}$



(b) (2 points) Find a complex number w such that the complex angle of w is $\arctan(1/4)$. Hint: use (a) for inspiration.

$\boxed{4+i}$



(c) (2 points) Find zw. Express your answer in the usual $a + bi$ form.

$\boxed{11+7i}$

$$(3+i)(4+i) = 12 + i^2 + 4i + 3i$$
$$= 11 + 7i$$

(d) (4 points) Use (a) through (c) to find $\tan(\arctan(1/3) + \arctan(1/4))$.

we know $\text{angle}(zw) = \arctan(7/11)$

and $\text{angle}(zw) = \text{angle}(z) + \text{angle}(w)$

$= \arctan(\frac{1}{3}) + \arctan(\frac{1}{4})$.

So $\tan(\arctan(\frac{1}{3}) + \arctan(\frac{1}{4})) = 7/11$

$\boxed{6}$ Determine whether the following series converge or diverge. If the series converges, **find its sum**. For each example, clearly state which convergence test or tests you are using. Be sure to address the hypotheses of that test, if the test has any hypotheses.

(a) (5 points) $\sum_{n=1}^{\infty}(-1)^n\frac{\pi^{2n+1}}{3(2n+1)!9^n}$ $= -\frac{\pi^3}{3!\,3^3}+\frac{\pi^5}{5!\cdot 3^5}-\frac{\pi^7}{7!\cdot 3^7}+\cdots$

$\sin t = t - \frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots$

$$= \left(\frac{\pi}{3}-\frac{\pi^3}{3!\,3^3}+\frac{\pi^5}{5!\,3^5}-\cdots\right)-\frac{\pi}{3}$$

$$= \sin(\pi/3)-\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}-\frac{\pi}{3}$$

(b) (5 points) $\sum_{n=1}^{\infty}\frac{4}{n\ln n}$

$$\int_1^\infty \frac{1}{x\ln x}\,dx = \ln\ln x\Big|_1^\infty = \infty,$$

and $\frac{4}{n\ln n}$ is positive & decreasing to 0.

So the series diverges

$\boxed{7}$ (Same directions as previous question)

(a) (5 points) $\displaystyle\sum_{n=0}^{\infty} \frac{\ln(2)+3^{n+1}}{7^n}$

$$\sum_{n=0}^{\infty} \frac{\ln 2}{7^n} = \ln 2 \sum_{n=0}^{\infty} \frac{1}{7^n}$$
$$= \ln 2\left(\frac{1}{1-1/7}\right)$$
$$= \frac{7\ln 2}{6}$$

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^n} = 3\sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^n$$
$$= 3\left(\frac{1}{1-\frac{3}{7}}\right)$$
$$= \frac{21}{4}.$$

So $\dfrac{7\ln 2}{6} + \dfrac{21}{4}$

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\arctan(\log(n))$

$\ln n \to \infty$ as $n \to \infty$.

So $\arctan \ln n \to \pi/2$

as $n \to \infty$.

So this series fails the n^{th} term test &

thus diverges.

$\boxed{8}$ Consider the function $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$, defined for all $-1 < x < 1$.

(a) (2 points) Find $f^{(61)}(0)$. You may use factorial expressions in your answer.

$$f^{(61)}(0) = 61! \cdot (\text{coeff. of } x^{61})$$
$$= 62 \cdot 61! = 62!$$

(b) (3 points) Find $\int f(x)\,dx$.

$$\int f(x)\,dx = x + x^2 + x^3 + x^4 + \cdots$$
$$= \frac{x}{1-x}$$

(c) (5 points) Find $f\left(\frac{3}{4}\right)$.

$$f(x) = \left(\frac{x}{1-x}\right)' = \frac{(1-x)\cdot 1 - x(-1)}{(1-x)^2}$$
$$= \left(\frac{1}{1-x}\right)^2$$

$$f\left(\tfrac{3}{4}\right) = \frac{1}{\left(1-\frac{3}{4}\right)^2} = 16$$

$\boxed{9}$ (7 points) (a) Find the real Fourier series of the 2π-periodic function $f(x)$ defined by $f(x) = 0$ for $0 \le x < \pi$ and $f(x) = 1$ for $\pi \le x < 2\pi$. Express your answer in ellipsis notation.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_\pi^{2\pi} \cos nx\,dx$$

$$= \frac{1}{\pi} \sin nx\Big|_\pi^{2\pi}$$

$$= \frac{1}{\pi}(0 - \sin n\pi) = 0$$

$$b_n = \frac{1}{\pi}\int_\pi^{2\pi} \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2\pi n + \cos \pi n}{n}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

So $$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2} - \frac{2}{\pi}\sin x - \frac{2}{3\pi}\sin 3x - \frac{2}{5\pi}\sin 5x - \cdots$$

(b) (3 points) For which values of x does the Fourier series you found in part (a) *not* converge to $f(x)$? Be sure to address the hypotheses of the Fourier convergence test.

The series converges to $\frac{1}{2}$ at $x = k\pi$ for any integer k, & to the value of the function elsewhere.

We know this because f is continuous except for finitely many jumps, and it has finitely many extrema over each period.

$\boxed{10}$ (a) (3 points) Discuss some similarities and differences between Taylor series and Fourier series. All sincere answers with at least three sentences will receive full credit.

Fourier series approximate periodic functions, and it's OK if the function we're approximating has discontinuities. Taylor series require the approximated function to be infinitely differentiable, but it does not have to be periodic. Fourier series use sines and cosines to approximate, while Taylor series use powers of x.

(b) (7 points) Consider a spring system which is driven by a 2π-periodic driving function $F(t)$ and responds by moving according to the solution $S(t)$ of the differential equation

$$S''(t) + S'(t) + 4S(t) = F(t).$$

($S(t)$ describes the position of end of the spring at time t.) Now suppose that the spring system is hooked up to a circuit in such a way that the spring's motion applies a voltage to the circuit. So the charge in the system satisfies the DE

$$Q''(t) + Q'(t) + Q(t) = S(t).$$

Suppose that the complex Fourier coefficients of F, S, and Q are $(f_n)_{n=-\infty}^{\infty}$, $(s_n)_{n=-\infty}^{\infty}$, and $(q_n)_{n=-\infty}^{\infty}$, respectively. Write an equation which expresses q_n in terms of f_n.

$$s_n = \frac{1}{(in)^2 + in + 4} f_n, \quad \text{and}$$

$$q_n = \frac{1}{(in)^2 + in + 1} s_n, \quad \text{so}$$

$$q_n = \frac{1}{(in)^2 + in + 4} \cdot \frac{1}{(in)^2 + in + 1} f_n$$

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