28 Feb

§2.8 Subspaces of $\mathbb{R}^n$

The solution set of $A\vec{x} = \vec{0}$ is a point, or a line, or a plane, etc., through the origin. These kinds of sets have a name, and we'll use a more direct definition:

Def$^{\underline{n}}$ A linear subspace of $\mathbb{R}^n$ is a set H for which

(i) $0 \in H$

(ii) $\vec{u} + \vec{v} \in H$ whenever $\vec{u} \in H$ and $\vec{v} \in H$,

(iii) $c\vec{u} \in H$ whenever $c \in \mathbb{R}$ and $u \in H$.

We say H is closed under scalar multiplication and vector addition.

Example Show that $\text{Span}(\{\vec{u}, \vec{v}\})$ is a linear subspace.

Solution $\vec{0} = 0\vec{u} + 0\vec{v}$, so $\vec{0} \in \text{Span}(\{\vec{u}, \vec{v}\})$.

$$\overbrace{(a_1\vec{u}+b_1\vec{v})}^{\text{arbitrary element of Span}(\{u,v\})}+\overbrace{(a_2\vec{u}+b_2\vec{v})}^{\text{...and another}}=\overbrace{(a_1+a_2)\vec{u}+(b_1+b_2)\vec{v}}^{\text{we can see this is in Span}(\{\vec{u},\vec{v}\})},$$

and $\quad c(a\vec{u}+b\vec{v}) = ca\vec{u}+cb\vec{v} \in \text{Span}(\{\vec{u},\vec{v}\})$. ∎

More generally, <u>the span of any list of vectors is a subspace</u>, including the empty list (which gives $\{\vec{0}\}$) and $\{\vec{e}_1,\dots,\vec{e}_n\}$ (which gives $\mathbb{R}^n$).

<u>**Def$^{\underline{n}}$**</u> The **column space** of a matrix A, "Col A", is the span of its columns

<u>**Def$^{\underline{n}}$**</u> The **null space** of a matrix A, "Nul A", is the solution set of $A\vec{x}=\vec{0}$.

<u>**Exercise**</u>: Show that the null space is a subspace.

Now we can generalize the notion of standard "basis" vectors $\vec{e}_1,\dots,\vec{e}_n$:

Def$^{\underline{n}}$ A basis of a linear subspace is a linearly independent list of vectors that span the space.

Example Find a basis for Nul A where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

<u>Solution</u> $[A \mid \vec{0}] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$, so

$$\text{Nul } A = \left\{ \begin{pmatrix} x_3 + x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} : x_3, x_4 \in \mathbb{R} \right\}$$

$$= \left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} : x_3, x_4 \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Since these two vectors are linearly independent, (we can see that by looking at their bottom two entries), they form a basis of Nul A.

Example Find a basis for Col A, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

<u>Solution</u> Taking all columns would give us a spanning list by definition, but it would not be linearly independent. We need to drop out columns until none of the ones remaining are linear combinations of the other ones left.

We can identify such a list using two principles:

(1) Row operations do not affect linear dependence relations among columns. For example:

$$\begin{array}{c} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \\ \vec{a} \;\; \vec{b} \;\; \vec{c} \end{array} \overset{\text{row 2} \leftarrow \text{row 2} - \text{row 1}}{\sim} \begin{array}{c} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{pmatrix} \\ \vec{a}' \;\; \vec{b}' \;\; \vec{c}' \end{array}$$

$\vec{c} = -\vec{a} + 2\vec{b}$ &

$\vec{c}' = -\vec{a}' + 2\vec{b}'$

② Non-pivot columns are linear combinations of pivot columns, and pivot columns are linearly independent.

For example:

$$\begin{array}{cccc} \vec{a} & \vec{b} & \vec{c} & \vec{d} \end{array}$$
$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

$\vec{b} = 2\vec{a}$ $\qquad$ $\vec{d} = 3\vec{a} - 2\vec{c}$

So we can take the pivot columns of A to get a basis for A.

$$A \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $\left\{ \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix} \right\}$ forms a basis for Col A.

§2.9 Dimension and Rank

A basis gives us <u>unique representation</u>:

Theorem If $\{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for H, then for every $\vec{v} \in H$, there is exactly one list $c_1, \dots, c_n$ of weights for which $\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$.

Why? If $\underbrace{c_1\vec{b}_1+\cdots+c_n\vec{b}_n}_{\text{two representations of } \vec{v}} = \underbrace{d_1\vec{b}_1+\cdots+d_n\vec{b}_n}_{\text{we want to show the weights are the same}}$, then

$$(c_1-d_1)\vec{b}_1+\cdots+(c_n-d_n)\vec{b}_n = 0.$$

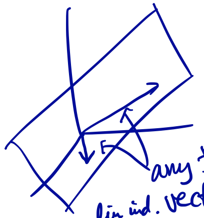
Since $\{\vec{b}_1,\dots,\vec{b}_n\}$ is linearly independent, $c_1-d_1=0$, $\dots$, $c_n-d_n=0$. So the c-list & d-list are the same. Thus there is at most one way to represent $\vec{v}$ as a lin. comb. of $\{\vec{b}_1,\dots,\vec{b}_n\}$. But since $\{\vec{b}_1,\dots,\vec{b}_n\}$ span H, there is also at least one way.

Theorem If $\{\vec{b}_1,\dots,\vec{b}_n\}$ and $\{\vec{a}_1,\dots,\vec{a}_p\}$ are both bases for H, then $n=p$.

Proof Suppose $p>n$. Define $B=[\vec{b}_1 \cdots \vec{b}_n]$ and note that since $\vec{a}_k\in H$, there exists c_k so that $B\vec{c}_k=\vec{a}_k$ for all k from 1 to p. Let $C=[\vec{c}_1 \cdots \vec{c}_p]$ and note that $C\vec{x}=\vec{0}$ has nontrivial solutions because C is wider than it is tall. Thus if we say $A:=[\vec{a}_1 \cdots \vec{a}_p]$, we get $BC = A$ by the definition of C. And if $C\vec{x} = \vec{0}$ has nontrivial solutions, then $BC\vec{x} = \vec{0}$ does too, so the columns of A are not linearly independent, a contradiction. So $p \le n$. By symmetry, $n \ge p$ too. So $p = n$. ∎

Def^{n} If H is a nonzero subspace, the dimension of H, **$\dim H$**, is the number of vectors in any basis of H.

<u>Note:</u> H has many bases, e.g. if $H \subset \mathbb{R}^3$ is a plane any two vectors in the plane will do, so long as neither is a multiple of the other.



<u>Def^{n}</u> The **rank** of a matrix is the dimension of its column space.

Theorem (Rank-Nullity)

$$\underbrace{\text{rank}\, A}_{\#\text{ pivot cols}} + \overbrace{\underbrace{\dim \text{Nul}\, A}_{\#\text{ non-pivot cols}}}^{\text{a.k.a. "nullity"}} = \#\text{ of columns of } A.$$