Math 0350 Practice Final
Fall 2017
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Problem 1

Verify that if $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors, the vector $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

Solution

The cosine of the angle between $\mathbf{a}$ and $\mathbf{c}$ is

$$\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|} = \frac{|\mathbf{a}|\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}||\mathbf{a}|^2}{|\mathbf{a}||\mathbf{c}|},$$

while the angle between $\mathbf{b}$ and $\mathbf{c}$ is

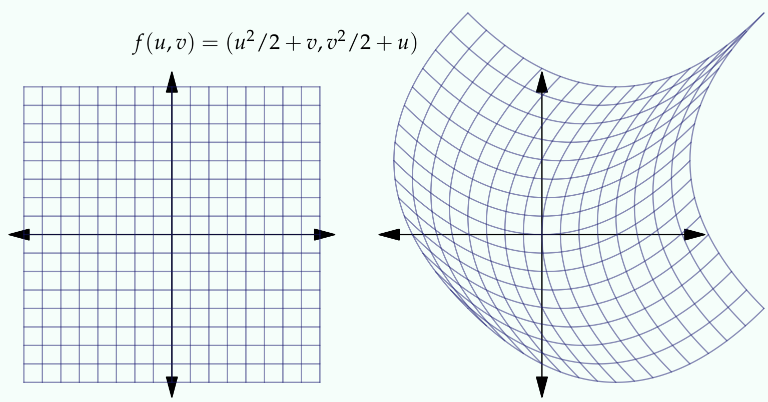
$$\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = \frac{|\mathbf{a}||\mathbf{b}|^2 + |\mathbf{b}|\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}||\mathbf{c}|}.$$

Canceling the factor of $|\mathbf{a}|$ in the first expression and $|\mathbf{b}|$ in the second, we see that these two expressions are equal. Therefore, the angle between $\mathbf{a}$ and $\mathbf{c}$ is congruent to the angle between $\mathbf{b}$ and $\mathbf{c}$. Furthermore, since $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, it is in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. Therefore, it bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

Problem 2

Consider the transformation shown, which maps the square $[-1,1]^2$ to an awesome manta-ray-looking region. (a) Where are the two points in the domain of f where the Jacobian of f is equal to 0? Where is the Jacobian of f at a maximum?

(b) Does the transformation map its domain onto its image in an orientation-preserving way or an orientation-reversing way?



Solution

(a) The Jacobian of f is $uv - 1$, so the Jacobian goes to 0 at the top right and bottom left of the square. You can see these points in the image of f at the head and tail of the manta ray. The Jacobian is at a maximum at the opposite corners, where $|uv - 1| = 2$. You can see these points at the wing-tips of the manta ray, where the areas of the image patches are at a maximum.

(b) The map is orientation-reversing, since its Jacobian is negative everywhere. You can also see this by considering the images of the four corners of the domain.

Problem 3

Suppose that $f(x, y) = xy - x$. Find the set of real numbers c such that there exists a differentiable path $\mathbf{r}$ satisfying $\mathbf{r}(0) = \mathbf{0}$ and $(f \circ \mathbf{r})'(0) = c$.

Solution

By the chain rule, we have
$$(f \circ \mathbf{r})'(0) = \nabla f(\mathbf{0}) \cdot \mathbf{r}'(0).$$
Since $\nabla f(\mathbf{0}) = \langle -1, 0 \rangle \neq \mathbf{0}$, the expression $\nabla f(\mathbf{0}) \cdot \mathbf{r}'(0)$ can be made equal to any desired value by choosing a path $\mathbf{r}$ which has an appropriate derivative at 0. For example, we can take $\mathbf{r}(t) = \langle -ct, 0 \rangle$. Therefore, the set of possible c values is $\boxed{\mathbb{R}}$.

Problem 4

Find the set of all points on the plane $x + y + z = 1$ which are equidistant from the points $(4, 2, 2)$, and $(3, 5, 1)$.

Solution

We are looking for points $(x, y, z) \in \mathbb{R}^3$ which satisfy both $x + y + z = 1$ and
$$(x-4)^2 + (y-2)^2 + (z-2)^2 = (x-3)^2 + (y-5)^2 + (z-1)^2 \iff -2x + 6y - 2z = 11.$$
This is the intersection of two planes, and it is a line since the two planes are clearly not parallel. Therefore, the desired set of points is some line ℓ.

To find a parametric equation for ℓ, there are at least a couple approaches: let z be our parameter, and solve the resulting system for x and y in terms of z. We get
$$x = -z - \frac{5}{8}, \quad y = \frac{13}{8}, \quad z = z.$$
So $\ell = \{(-t - \frac{5}{8}, \frac{13}{8}, t) : t \in \mathbb{R}\}$.

Alternatively, we can find a point on ℓ by setting z to any particular value and solving for x and y. For example, setting $z = 0$ gives $(x, y) = \left(-\frac{5}{8}, \frac{13}{8}\right)$. Then the vectors $\langle 1, 1, 1 \rangle$ and $\langle -2, 6, -2 \rangle$ which are normal to the planes are both perpendicular to ℓ (since ℓ is contained in both planes). It follows that we can find the direction of ℓ by calculating the cross product
$$\langle 1, 1, 1 \rangle \times \langle -2, 6, -2 \rangle = \langle -8, 0, 8 \rangle.$$
Thus we arrive at the parametric representation $\ell = \{(-8t - \frac{5}{8}, \frac{13}{8}, 8t) : t \in \mathbb{R}\}$.

Problem 5

Find the center of mass of the parabolic lamina $0 \leq y \leq 1 - x^2$.

Solution

The x-coordinate of the center of mass is 0, by symmetry.

The y-coordinate is equal to the integral of y of the lamina divided by the total mass of the lamina:

$$\frac{\int_{-1}^{1}\int_{0}^{1-x^2} y \; dy\, dx}{\int_{-1}^{1}\int_{0}^{1-x^2} 1 \; dy\, dx} = \frac{8/15}{4/3} = \frac{2}{5}.$$

Final answer:

$$\frac{2}{5}$$

Problem 6

Find the moment of inertia about the z-axis of the tetrahedron with vertices at the origin, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$. Assume that the mass density of the tetrahedron is $\rho(x, y, z) = 1$.

Solution

The moment of inertia is

$$\int_0^1 \int_0^x \int_0^y (x^2 + y^2)\, dz\, dy\, dx = \frac{3}{20}.$$

Final answer:

$$\frac{3}{20}$$

Problem 7

Find the critical point of $f(x,y) = xy + 2x - \ln(x^2y)$ in the first quadrant, and determine whether f has a local maximum, a local minimum, or a saddle point there.

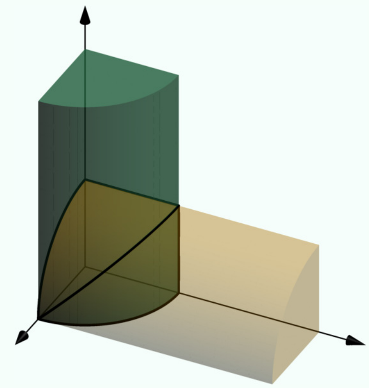
Solution

We have $f_x = y - 2/x + 2$ and $f_y = x - 1/y$. Solving $f_y = 0$ for x and substituting into the equation $f_x = 0$ gives $(x,y) = \boxed{(1/2,2)}$.

We have $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2/x^2 & 1 \\ 1 & 1/y^2 \end{vmatrix} = \frac{2}{(xy)^2} - 1 = 1$, which means that f has either a max or a min at $(1/2,2)$. Since the diagonal terms are positive, f has a $\boxed{\text{min}}$.

Problem 8

Find the volume of the region in the first octant common to the cylinders $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$, as shown.



Solution

We set up the integral in Cartesian coordinates, with the integration order $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$. We get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 1 - x^2\,\mathrm{d}x = \frac{2}{3}.$$

Final answer:

$$\frac{2}{3}$$

Problem 9

Find the flow of $\mathbf{F} = \langle 0, 0, x^2 + y^2 \rangle$ outward through the portion of the surface surface $z^2 + 1 = x^2 + y^2$ between the planes $z = 0$ and $z = \sqrt{3}$.

Solution

We can parametrize the surface using the cylindrical coordinates r and θ. The point on the surface with coordinates r and θ is $\mathbf{r}(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{r^2 - 1})$, where θ ranges over $[0, 2\pi]$ and r ranges from 1 to 2. Then $\mathbf{r}_r \times \mathbf{r}_\theta = \left\langle -\frac{r^2\cos\theta}{\sqrt{r^2-1}}, -\frac{r^2\sin\theta}{\sqrt{r^2-1}}, r \right\rangle$. Since the z-component of this vector field, namely r, is positive, the normal vectors $\left\langle -\frac{r^2\cos\theta}{\sqrt{r^2-1}}, -\frac{r^2\sin\theta}{\sqrt{r^2-1}}, r \right\rangle$ are all pointing upward and therefore (given the shape of the surface) inward. So the desired orientation is represented by the negation of this vector field, namely $\left\langle \frac{r^2\cos\theta}{\sqrt{r^2-1}}, \frac{r^2\sin\theta}{\sqrt{r^2-1}}, -r \right\rangle$.

Using the parametric formula for the vector surface integral, we find that

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \int_0^{2\pi} \int_1^2 \left\langle 0, 0, r^2 \right\rangle \cdot \left\langle \frac{r^2\cos\theta}{\sqrt{r^2-1}}, \frac{r^2\sin\theta}{\sqrt{r^2-1}}, -r \right\rangle \mathrm{d}r\, \mathrm{d}\theta = -\frac{15\pi}{2}$$

Final answer:

$$-\frac{15\pi}{2}$$

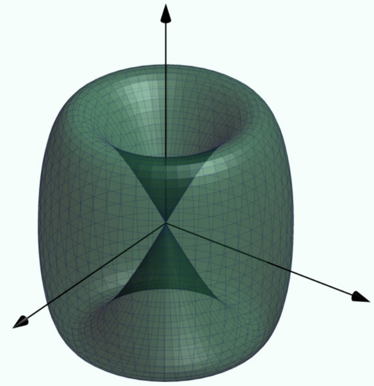
Problem 10

Consider the surface parametrized by

$$\mathbf{r}(u, v) = (\cos u \cos v, \cos u \sin v, \sin 2u).$$

as u ranges over $[-\pi/2, \pi/2]$ and v ranges over $[0, 2\pi]$. Use the divergence theorem to find the volume enclosed by the surface.

Hint: you'll have occasion to make use of the identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.



Solution

The divergence theorem implies that the volume enclosed by the surface is equal to the flow of $\mathbf{F}$ through the surface for any vector field $\mathbf{F}$ with divergence 1. We find that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2\cos u \cos 2u \cos v, -2\sin v \cos u \cos 2u, -\sin u \cos u \rangle .$$

Spotting that the third component is the simplest one, we take $\mathbf{F} = \langle 0, 0, z \rangle$, so that $\operatorname{div} \mathbf{F} = 1$. With the orientation corresponding to $\mathbf{r}_u \times \mathbf{r}_v$, we get a flow through the surface of

$$\begin{aligned}
\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \sin 2u \cdot -\sin u \cos u \,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \sin 2u \cdot -\frac{1}{2}\sin 2u \,\mathrm{d}u\,\mathrm{d}v \\
&= -\frac{1}{2}\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \frac{1-\cos 4u}{2} \,\mathrm{d}u\,\mathrm{d}v \\
&= -\frac{1}{2}(2\pi)\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{2}.
\end{aligned}$$

Since this value is negative, we can conclude *a posteriori* that the orientation we used was inward rather than outward. We could also have figured this out by checking the direction of $\mathbf{r}_u \times \mathbf{r}_v$ at some particular point, like $(u, v) = (0, 0)$. In any case, we find that the volume enclosed by the surface is $\pi^2/2$.

Final answer:

$$\frac{\pi^2}{2}$$

Problem 11

Suppose that S_1 is the set of points on the sphere $x^2+y^2+z^2=1$ which are not inside the sphere $x^2+y^2+(z+1)^2=1$, and suppose that S_2 is the set of points on the sphere $x^2+y^2+(z+1)^2=1$ which are not inside the sphere $x^2+y^2+z^2=1$. We may interpret S_1 and S_2 as surfaces carrying an orientation from the inside to the outside. Find

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{A} \quad \text{and} \quad \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{A},$$

where $\mathbf{F} = \langle yz, x, e^{xyz} \rangle$.

Solution

We use Stokes' theorem to say that the first integral is equal to

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the circle where the two spheres intersect. This integral may be evaluated by parametrizing C as $\langle \frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t, -\frac{1}{2} \rangle$ where t ranges from 0 to 2π. The counterclockwise orientation is correct, because that is the one for which the surface is on the left. Thus the first integral equals

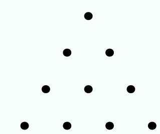
$$\int_0^{2\pi} \langle -\tfrac{\sqrt{3}}{4}\sin t, \tfrac{\sqrt{3}}{2}\cos t, [\text{something}] \rangle \cdot \langle -\tfrac{\sqrt{3}}{2}\sin t, \tfrac{\sqrt{3}}{2}\cos t, 0 \rangle \, dt = \tfrac{3}{4}\int_0^{2\pi} \tfrac{1}{2}\sin^2 t + \cos^2 t \, dt = \boxed{\frac{9\pi}{8}},$$

where [something] indicates an expression whose value is immaterial because of the 0 in the third component of $\mathbf{r}'(t)$.

The second integral equals $\boxed{-\frac{9\pi}{8}}$, since Stokes' theorem tells us that this surface integral is equal to the same line integral as above, but with the opposite orientation.

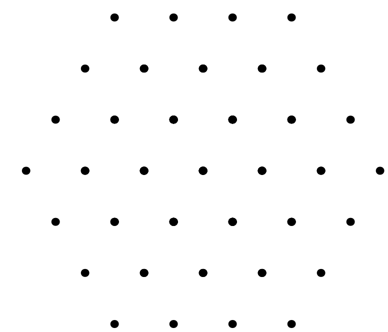
BONUS

How many distinct vectors can be formed by selecting a tail and a head from the grid of points shown?



Solution

The set of vectors that can be formed using points from the given grid, if positioned so that their tails are all the origin, have heads spanning the hexagonal grid of ponts shown to the right. There are 37 points in this grid.



Final answer:

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