Brown University
Problem Set 3
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Due: 29 September 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

An astronaut is using a rope to move in space in such a way that her position at time t is given by $\mathbf{r}(t) = (2+t)\mathbf{i} + (2+\ln t)\mathbf{j} + \left(7 - \frac{4}{t^2+1}\right)\mathbf{k}$. The coordinates of the space station doorway are $(5, 4, 9)$. When should the astronaut let go of the rope so as to drift into the doorway?

Solution

We will let $\mathbf{q}$ be the vector pointing from the astronaut to the door at any given time. We thus have that $\mathbf{q}(t) = \left\langle 3-t, 2-\ln t, 2+\frac{4}{t^2+1} \right\rangle$.

Additionally, we have that the astronaut's velocity at any given time is $\mathbf{r}'(t) = \left\langle 1, \frac{1}{t}, \frac{8t}{(t^2+1)^2} \right\rangle$.

The astronaut should let go when these two vectors are parallel. We thus wish to find a t such that $\mathbf{q}(t) = \lambda \mathbf{r}'(t)$; in other words,

$$\left\langle 3-t, 2-\ln t, 2+\frac{4}{t^2+1} \right\rangle = \lambda \left\langle 1, \frac{1}{t}, \frac{8t}{(t^2+1)^2} \right\rangle,$$

for some positive value λ.

By inspection, we see that $t = 1$ and $\lambda = 2$ will solve this system of equations, and thus that the astronaut should let go at time $\boxed{t=1}$. To see that there are no other times that work, we can substitute the first equation into the second to get $2 - \ln t = \frac{3}{t} - 1$, which implies $\ln t + \frac{3}{t} = 3$. The function $\ln t + \frac{3}{t}$ is strictly decreasing on the interval $[0, 3]$ since its derivative is negative there, and this means it achieves the value 3 at most once there. The first equation combined with the positivity of λ implies that we may consider only values of t less than 3. (Note: there is another value of t, about 16.801, satisfying $\ln t + \frac{3}{t} = 3$.)

Problem 2

Find the curvature of the helix $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt \rangle$.

Solution

We have $\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b \rangle$, which means that

$$\mathbf{T} = \frac{\langle -a\sin t, a\cos t, b \rangle}{\sqrt{a^2+b^2}}.$$

Furthermore, by the fundamental theorem of calculus, if we define the arclength function $s(t) = \int_0^t |\mathbf{r}'(\tau)|\, d\tau$, we have

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{a^2+b^2}.$$

Therefore, the curvature is

$$\kappa(t) = \frac{|d\mathbf{T}/dt|}{|ds/dt|} = \frac{|\langle -a\cos t, -a\sin t, 0 \rangle| / \sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \boxed{\frac{a}{a^2+b^2}}.$$

This formula has the expected behavior as $b \to \infty$, because making b very large straightens out the helix. Also, if a is very large, then $\kappa(t) \approx 1/a$, which is the curvature of a circle of radius a.

Problem 3

Sketch the surface or solid described by the given equations and/or inequalities. If you feel your figure isn't sufficiently clear, feel free to supplement with a verbal description.

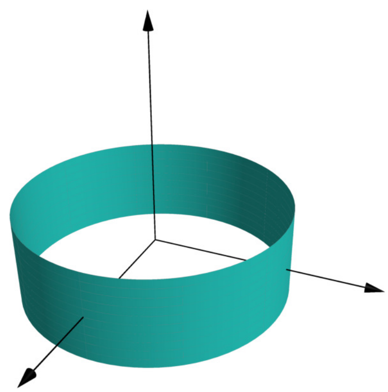
(a) $r = 3, \quad -1 \leq z \leq 1$

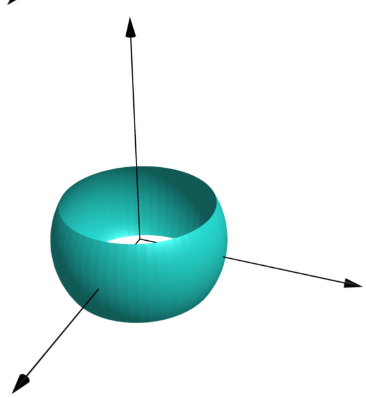
(b) $\rho = 2, \quad \pi/3 \leq \phi \leq 2\pi/3$

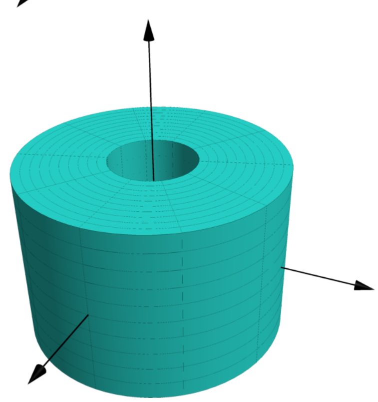
(c) $1 \leq r \leq 3, \quad -2 \leq z \leq 2$

Solution

The first two are surfaces, and the third is a solid:







Problem 4

Suppose that $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a curvy path with no straight portions. Given any positive integer n and real numbers $t_0, \ldots, t_n$ satisfying

$$a = t_0 < t_1 < \cdots < t_n = b,$$

explain briefly and in simple terms why

$$|\mathbf{r}(t_1) - \mathbf{r}(t_0)| + |\mathbf{r}(t_2) - \mathbf{r}(t_1)| + \cdots + |\mathbf{r}(t_n) - \mathbf{r}(t_{n-1})|$$

is less than the length of $\mathbf{r}$. Hint: draw a figure wherein the above expression has a natural geometric interpretation.

Solution

For each k from 1 to n, $|\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})|$ is the length of a straight line segment connecting where the path is at the two times t_k and t_{k-1}. Since **the shortest distance between two points is a straight line**, the actual arclength over this interval is greater than $|\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})|$. Summing all these inequalities over k, we find that the total arclength of the curve exceeds

$$|\mathbf{r}(t_1) - \mathbf{r}(t_0)| + |\mathbf{r}(t_2) - \mathbf{r}(t_1)| + \cdots + |\mathbf{r}(t_n) - \mathbf{r}(t_{n-1})|.$$

Problem 5

(a) The set of points satisfying $z = x^2$ and $y = 0$ is revolved around the z-axis. Write an equation for the surface generated in rectangular coordinates, and in cylindrical coordinates.

(b) The set of points satisfying $4x^2 + y^2 = 1$ and $z = 0$ is revolved around the y-axis. Find an equation in rectangular coordinates for the resulting ellipsoid.

Solution

(a) By definition of rotation, a point (x, y, z) is on the surface if and only if it lies on the curve $z = x^2$ when rotated into the xz-plane. Rotating the point (x, y, z) about the z-axis into the xz-plane changes its second coordinate to zero, and it leaves its third coordinate the same. It also preserves the distance the the z-axis, which means that the new x-coordinate must be $\sqrt{x^2 + y^2}$. So the rotated point is $(\sqrt{x^2 + y^2}, 0, z)$. Substituting into the equation for the parabola, we see that (x, y, z) is on the surface if and only if

$$z = \left(\sqrt{x^2 + y^2}\right)^2 \implies \boxed{z = x^2 + y^2}.$$

In cylindrical coordinates, this equation says $\boxed{z = r^2}$.

(b) By the same reasoning as for (a), we may replace x with $\sqrt{x^2 + z^2}$ to get the equation

$$\boxed{4x^2 + 4z^2 + y^2 = 1}.$$

Additional space