MATH 19 PRACTICE MIDTERM II
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Solutions

1 Find the general solution of the differential equation

$$f'(x) + f(x) = xe^x.$$

the general solution of

$$f'(x) + f(x) = 0$$

is $f(x) = Ce^{-x}$, since the characteristic polynomial is $\lambda + 1 = 0$.

For a particular solution, we try

$$f(x) = Axe^x + Be^x.$$

We get

$$\begin{aligned} f'(x) &= Axe^x + Ae^x + Be^x \\ + \quad f(x) &= Axe^x + Be^x \\ \hline f'(x) + f(x) &= 2Axe^x + (2B + A)e^x \end{aligned}$$

So we want $2A = 1$, $2B + A = 0$. So $A = 1/2$,

$B = -\frac{A}{2} = -\frac{1}{4}$. So the general soln is

$$f(x) = Ce^{-x} + \frac{1}{2}xe^x - \frac{1}{4}e^x,$$

where C is an arbitrary constant.

2 Consider the sequence $(a_n)_{n=1}^{\infty}$ for which $a_0 = 1$ and for all $n > 0$, then nth term is obtained from the previous one by adding $1/n$ to it. So, for example, the first few terms are

$$\begin{aligned} a_0 &= 1 \\ a_1 &= a_0 + \tfrac{1}{1} = 2 \\ a_2 &= a_1 + \tfrac{1}{2} = \tfrac{5}{2} \\ a_3 &= a_2 + \tfrac{1}{3} = \tfrac{17}{6} \\ &\vdots \end{aligned}$$

Determine whether the sequence $(a_n)_{n=1}^{\infty}$ converges.

Note that $a_n = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k}.$$

therefore a_n is 1 plus the n^{th} partial sum of the harmonic series. Since the harmonic series diverges by the integral test, a_n $\boxed{\text{diverges}}$ to ∞.

$\boxed{3}$ (a) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, using the comparison test.

(a) Find the exact value of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ by calculating its Nth partial sum and taking a limit of the resulting expression as $N \to \infty$. Hint: check that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, and then use that identity to write out the first few partial sums, looking for cancellation.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

comparison test

integral test: $\frac{1}{x^2} \downarrow 0$

and $\displaystyle\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2}$

$\displaystyle = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$ converges

(b) $\displaystyle\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\displaystyle \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$

$\displaystyle = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{N}} - \frac{1}{N+1}$

$\displaystyle = 1 - \frac{1}{N+1}.$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = \underline{1}.$

$\boxed{4}$ Determine the convergence or divergence of each of the following series

(a) $\frac{1}{1}+\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{1\cdot 3\cdot 5\cdot 7\cdot 9}+\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}+\cdots$

$\times 2/3 \qquad \times 3/5 \qquad \times 4/7 \qquad \times 5/9 \qquad 6/11$

Each time we multiply by $\frac{k}{2k-1}$ to get from the $(k-1)$st term to the k^{th}. Since

$$\lim_{k\to\infty}\left(\frac{k}{2k-1}\right) = \frac{1}{2} < 1$$

The ~~ratio~~ test implies that the series $\boxed{\text{converges}}$

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi)}{\sqrt{n}}$

Note $$(-1)^n\cos(n\pi) = \begin{cases}(-1)(-1)=1 & \text{if } n \text{ is odd}\\ (1)(1)=1 & \text{if } n \text{ is even}\end{cases} = 1.$$

So $$\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi)}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \infty$$

by the integral test, $\frac{1}{\sqrt{x}}\downarrow 0$ and

$$\int_1^{\infty}\frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} 2\sqrt{x}\Big|_1^b$$
$$= \lim_{b\to\infty}\left[2\sqrt{b}-2\right] = \infty.$$

So it $\boxed{\text{diverges}}$

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

$$f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{when } x > 1.$$

So, f is decreasing on $[2,\infty)$. Also $\lim_{n\to\infty} f(n) = 0$, & $f(n) > 0$.

So $\sum (-1)^n \frac{n}{n^2+1}$ $\boxed{\text{converges}}$ by the alternating series test.

(d) $\sum_{n=1}^{\infty} \frac{e^n (n!)^2}{(2n)!}$

Ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{n+1}((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{e^n (n!)^2}$$

$$= \lim_{n\to\infty} \frac{e(n+1)^2}{(2n+2)(2n+1)}$$

$$= \frac{e}{4} < 1,$$

So by the ratio test, this series $\boxed{\text{converges}}$.

$\boxed{5}$ (a) Suppose that p is a real number. Find the fourth-order Taylor polynomial of $f(x) = (1+x)^p$ centered at $x = 0$. Express your answer in terms of p.

(b) Use your answer to part (a) to find the fourth-order Taylor polynomial of $g(x) = \frac{1}{\sqrt{1-x^2}}$ centered at $x = 0$. (Hint: first find the Taylor series for h where $h(y) = 1/\sqrt{1-y}$ and then substitute $y = x^2$.)

(a)
$$f(x) = (1+x)^p$$
$$f'(x) = p(1+x)^{p-1}$$
$$f''(x) = p(p-1)(1+x)^{p-2}$$
$$f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$$
$$f^{(4)}(x) = p(p-1)(p-2)(p-3)(1+x)^{p-4}$$

So
$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$$
$$= \boxed{1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \frac{p(p-1)(p-2)(p-3)}{24}x^4}$$

(b) $h(y) = (1-y)^{-1/2}$, so the Taylor polynomial for h is:
$$P_2(y) = 1 - \frac{1}{2}(-y) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-y)^2$$
$$= 1 + \frac{y}{2} + \frac{3}{8}y^2$$

So the Taylor polynomial for g is, up to 4^{th} order:
$$\boxed{1 + \frac{x^2}{2} + \frac{3}{8}x^4}$$

$\boxed{6}$ Suppose that you get to put plus or minus signs between the following terms *however you wish*:

$$\frac{1}{3} \qquad \frac{1}{9} \qquad \frac{1}{27} \qquad \frac{1}{81} \cdots$$

So if you put all plus signs, you'd get $\frac{1}{3} + \frac{1}{9} + \cdots$. If you put all minus signs, then you'd get $-\frac{1}{3} - \frac{1}{9} - \cdots$. (In addition to these two, there are many, many other ways you could fill in the signs).

(a) Show that the resulting series is absolutely convergent, regardless of your choice of signs.

(b) Show that it is not possible to fill in the signs in such a way that the sum of the resulting series is 0.

(a) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} 3^{-n} < \infty$ because the series is geometric with common ratio $1/3$. The signs are irrelevant to absolute convergence

(b) If the first sign is $+$, then the smallest possible sum would be achieved with all rest negatives:

$$\frac{1}{3} - \frac{1}{9} - \frac{1}{27} - \cdots = \frac{1}{3} - \left(\frac{1}{9} + \frac{1}{27} + \cdots\right)$$
$$= \frac{1}{3} - \left(\frac{1/9}{1 - 1/3}\right)$$
$$= \frac{1}{3} - \frac{1}{9} \cdot \frac{3}{2}$$
$$= \frac{1}{6}.$$

Similarly, if the first term is negative, the sum may be no larger than $-1/6$. So actually, no value in $(-\frac{1}{6}, \frac{1}{6})$ may be realized.