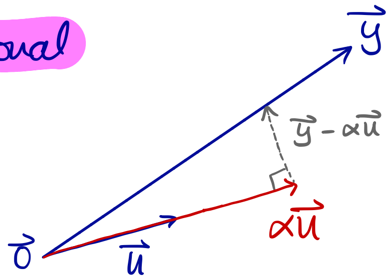
25 APR 2017

Orthogonal Projection

Given two vectors $\vec{u}$ and $\vec{y}$ in $\mathbb{R}^n$, how do we find the vector parallel to $\vec{u}$, let's call it $\alpha\vec{u}$, so that $\vec{u} \perp (\vec{y} - \alpha\vec{u})$?

This vector is called the **orthogonal projection** of $\vec{y}$ onto $\vec{u}$.



To solve for α, we can write down the perpendicularity requirement in equation form:

$$(\vec{y} - \alpha\vec{u}) \cdot \vec{u} = 0 \iff$$

$$\vec{y}\cdot\vec{u} - \alpha\vec{u}\cdot\vec{u} = 0 \iff$$

$$\alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

[we need to assume $\vec{u} \neq 0$ for orth. proj. to make sense]

So, we have $\boxed{\text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}}$

Example Find $\text{proj}_{\vec{u}}\vec{y}$ where $\vec{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

<u>Solution</u>

We calculate $\text{proj}_{\vec{u}}\vec{y} = \dfrac{\begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 4 \end{bmatrix}}$.

<u>Orthonormal lists</u>

An orthogonal list for which each vector has unit length is called **orthonormal**.

Example Normalize $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\div 1$ $\quad \div\sqrt{2}$ $\quad \div\sqrt{2}$

<u>Solution</u>

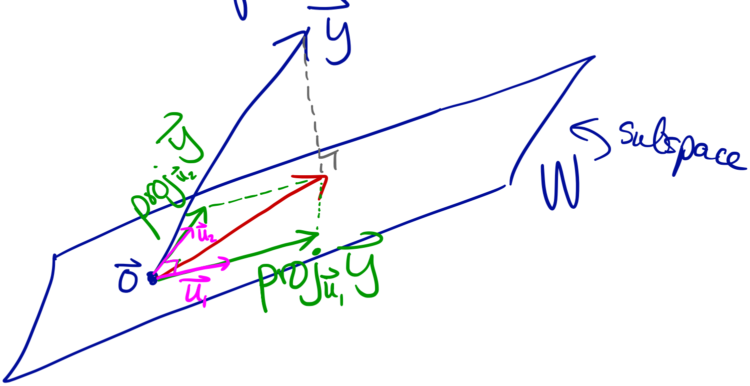
$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

Fact: A matrix has orthonormal columns iff $U^TU = I$

Proof: The $(i,j)^{\text{th}}$ entry of U^TU is equal to the i^{th} row of U^T (i.e., the i^{th} column of U) dotted with the j^{th} column of U. The $(i,j)^{\text{th}}$ entry of I is $\begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Orthogonal Projection, higher dimension

How to project $\vec{y}$ onto a plane or higher dimensional subspace of $\mathbb{R}^n$?



Geometrically, we might be tempted to project $\vec{y}$ onto two vectors that span W and add the results, since that makes sense in the picture (the green vectors add to give the red one). This is indeed the case:

Theorem If $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then $\hat{y} = \underbrace{\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p}_{\text{sum of projections}}$ is in W, and $\vec{y} - \hat{y}$ is orthogonal to W.

Proof We calculate

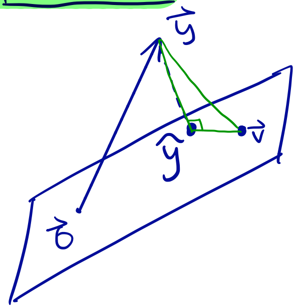
$$(\vec{y}-\hat{y})\cdot\vec{u}_1 = \vec{y}\cdot\vec{u}_1 - \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1\cdot\vec{u}_1 \underbrace{- 0 \cdots - 0}_{\text{rest of terms are zero}}$$ (by orthogonality)

$$= \vec{y}\cdot\vec{u}_1 - \vec{y}\cdot\vec{u}_1 = 0.$$

Similarly $(\vec{y}-\hat{y})\cdot\vec{u}_2 = \cdots = (\vec{y}-\hat{y})\cdot\vec{u}_p = 0.$

So $\vec{y}-\hat{y}$ is orthogonal to each vector in a basis of W, & therefore to every vector in W.

Theorem (Best Approximation)



If $W \subset \mathbb{R}^n$ is a subspace and $\hat{y} = \text{proj}_W \vec{y}$, then

$$|\vec{y}-\hat{y}| < |\vec{y}-\vec{v}|$$

for any $\vec{v} \in W$ other than $\hat{y}$.

Proof $$|\vec{y}-\hat{y}|^2 + |\hat{y}-\vec{v}|^2 = |\vec{y}-\vec{v}|^2$$ Pythagoras, green triangle

$$\Rightarrow |\vec{y}-\hat{y}| < |\vec{y}-\vec{v}|.$$ ∎