§4.1 Abstract Vector Spaces

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Many of the properties of vectors in $\mathbb{R}^n$ depend not so much on what these vectors are, but what you can do with them.

* like chess pieces: it doesn't matter how they look, but how they interact *

For example, we can consider a notion like linear independence:

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0} \quad \text{only if} \quad c_1 = \cdots = c_n = 0$$

for any collection of "things" $\vec{v}_1, \ldots, \vec{v}_n$ — regardless of what they look like — so long as we're told how to (i) multiply them by real numbers, (ii) add them, and (iii) tell which one is $\vec{0}$.

We also need assurance that these operations are sane, e.g., $0\vec{v} = \vec{0}$, and $\vec{0} + \vec{0} = \vec{0}$.

So you might wonder whether there is a short list of such properties that allows us to do all the stuff we have done with vectors in $\mathbb{R}^n$. There is!

> Note: the pattern of identifying the critical set of behaviors and properties of a system, and deliberately forgetting what the underlying objects look like, is extremely common in mathematics. It helps us reuse ideas we develop in one context by applying them to another.

Def$^{\underline{n}}$ A **vector space** is a set V together with two operations which tell us how to add elements of V and how to multiply elements of V by a real number that satisfies, for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $c, d \in \mathbb{R}$:

1. $\vec{u} + \vec{v} \in V$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
4. there is $\vec{0} \in V$ with $\vec{0} + \vec{u} = \vec{u}$
5. for all $\vec{u}$, there is $\vec{v}$ with $\vec{u} + \vec{v} = \vec{0}$ (we call $\vec{v}$ "$-\vec{u}$")
6. $c\vec{u} \in V$
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$.

Note that $0\vec{u} = \vec{0}$ wasn't in this list, and we wanted that! It's because this condition is a consequence of the other ones:

$$0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u}.$$

fact about real numbers

8

Now let $\vec{v}$ be the vector that adds to $0\vec{u}$ to give $\vec{0}$ (ensured by 4), & add $\vec{v}$ to both sides to give $\vec{0} = 0\vec{u}$, as desired. Some other consequences: $c\vec{0} = \vec{0}$ and $-\vec{u} = (-1)\vec{u}$

additive inverse of $\vec{u}$

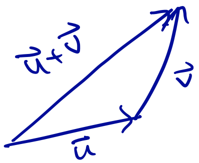
$\vec{u}$ times -1

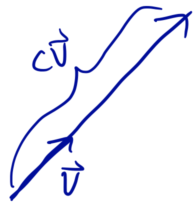
You can also show that the additive inverse of each vector is unique.

Examples of Vector Spaces

Example 1 $\mathbb{R}^n$, with usual addition & scalar multiplication. We know all about that.

Example 2 Let V be the set of all arrows in 3D space, with two considered equivalent if they have the same length and direction. Define operations via:





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as usual. Then V is a vector space: $\vec{0}$ is the zero-length arrow, $-\vec{u}$ is the reversal of $\vec{u}$, commutativity and associativity follow from geometry.

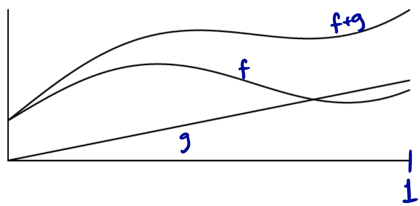
> This space is not-so-secretly "the same" as $\mathbb{R}^3$. It's elements look different (arrows vs. triples of real numbers), but if we identify $(x,y,z) \in \mathbb{R}^3$ with the arrow from $(0,0,0)$ to (x,y,z), then the vector space operations are the same in both pictures. Such spaces are said to be "isomorphic" — more on that later.

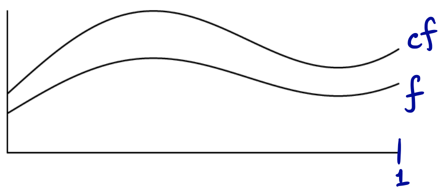
Example $\mathbb{P}_n$ = polynomials of degree at most n; usual + and $\cdot$.

Example "$\mathbb{R}^\infty$" $= \{(x_1, x_2, \ldots) : x_i \in \mathbb{R} \text{ for all } i\}$

This is like $\mathbb{R}^n$, but we have infinitely many slots to put real numbers in, instead of just n. We define addition & multiplication the same as for $\mathbb{R}^n$, & the vector space axioms work for the same reason.

Example The space $C([0,1])$ of continuous, real-valued functions on $[0,1]$, with usual addition of functions and multiplication of functions by a scalar:





The vector space properties are easy to check here too: continuous functions add to give continuous functions, $f+(g+h) = (f+g)+h$

works because it works for real numbers, etc.

Just as we had subspaces in $\mathbb{R}^n$, we can talk about subspaces in V, same idea:

Def$^{\underline{n}}$ A **subspace** H of a vector space V is a subset of V so that

(i) the zero vector of V is in H

(ii) $\vec{u}+\vec{v} \in H$ whenever $\vec{u}, \vec{v} \in H$

(iii) $c\vec{u} \in H$ whenever $c \in \mathbb{R}$, $\vec{u} \in H$.

$\Leftarrow$ "nonempty & closed under vector operations"

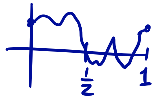
These properties ensure that H, with V's operations, is itself a vector space.

Example $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, because it isn't even a subset of $\mathbb{R}^3$.

However, $H = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$ is a subset of $\mathbb{R}^3$, and it's a subspace of $\mathbb{R}^3$ since it's closed under the vector operations.

H is basically the same as $\mathbb{R}^2$, but its elements look different ← H and $\mathbb{R}^2$ are "isomorphic", again, more details later

Example $H = \{ f \in C([0,1]) : f(\tfrac{1}{2}) = 0 \}$
is a subspace of $C([0,1])$.



why? $f, g \in H \Rightarrow (f+g)(\tfrac{1}{2}) = 0 + 0 = 0$, $cf(\tfrac{1}{2}) = c \cdot 0 = 0$.

Example If V is any vector space and $\{\vec{v}_1, \dots, \vec{v}_n\}$ are <u>any</u> vectors in V, then

← elements of a vector space are called vectors, no matter what they look like

$$H = \operatorname{Span}(\{\vec{v}_1, \dots, \vec{v}_n\}) = \{ c_1\vec{v}_1 + \dots + c_n\vec{v}_n : c_i \in \mathbb{R} \text{ for all } i \}$$

is a subspace of V.

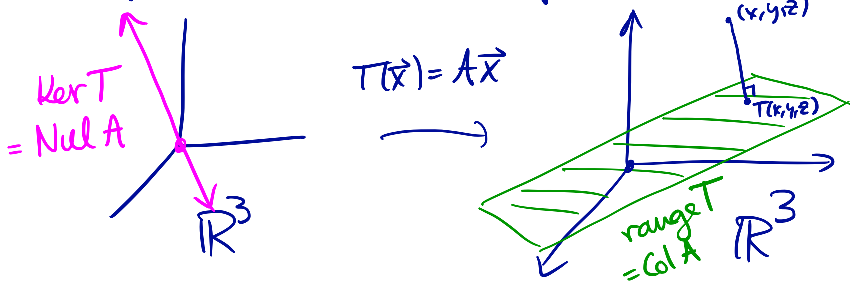
Why? if $\vec{u} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, $\vec{v} = d_1\vec{v}_1 + \dots + d_n\vec{v}_n$,

then $\vec{u} + \vec{v} = (c_1 + d_1)\vec{v}_1 + \dots + (c_n + d_n)\vec{v}_n$,

& same idea for $c\vec{u} \in H$.

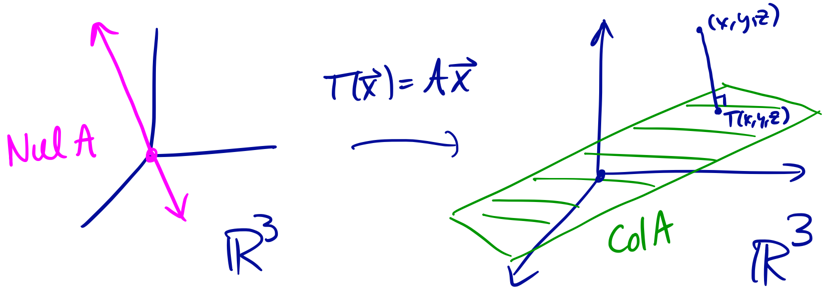
<u>Vocab</u>: If $H = \operatorname{Span}(\{\vec{v}_1, \dots, \vec{v}_n\})$, we say $\{v_1, \dots, v_n\}$ "spans" H. Think of it like the word "reach": we can say that a baby's reach is her tray, or we can say that the baby reaches the whole tray.

Null space and column space: a Review



Consider T which maps each $(x,y,z) \in \mathbb{R}^3$ to the nearest point on that green plane through $(0,0,0)$, then the range of T, a.k.a., the column space of T's matrix, is the green plane. The kernel of T, a.k.a the null space of T's matrix, is the set of all vectors that map to $\vec{0}$. These points form a line shown in pink above.

The following table summarizes some distinctions between null space & column space.



Nul A	Col A
Nul A = Ker T	Col A = range T
Nul $A \subset \mathbb{R}^n$	Col $A \subset \mathbb{R}^m$
vectors in Nul A are harder to find: solve $A\vec{x} = \vec{0}$	vectors in Col A are easy to find: columns of A
checking $\vec{x} \in$ Nul A is easy: $A\vec{x} \overset{?}{=} \vec{0}$	checking $\vec{y} \in$ Col A is harder: determine consistency of $A\vec{x} = \vec{y}$
Nul $A = \{\vec{0}\}$ iff A is injective iff A has a pivot in every column	Col $A = \mathbb{R}^m$ iff T is surjective iff A has a pivot in every row