18.022 Recitation Handout (with solutions)
26 November 2014


1. Consider the polygon $P$ whose vertices are listed below, in counterclockwise order. Describe a simple algorithm for approximating the area of $P$.

| $x$ | $y$ |
| --- | --- |
| 1. | 0. |
| 0.990436 | 0.000510364 |
| 0.962169 | 0.00404386 |
| 0.916455 | 0.0134308 |
| 0.855307 | 0.031127 |
| 0.781385 | 0.059054 |
| 0.697852 | 0.0984698 |
| 0.608195 | 0.149879 |
| 0.516034 | 0.212985 |
| 0.42493 | 0.286693 |
| 0.338182 | 0.369151 |
| 0.258659 | 0.457849 |
| 0.188645 | 0.549739 |
| 0.129723 | 0.641405 |
| 0.0827009 | 0.729244 |
| 0.0475787 | 0.809659 |
| 0.0235696 | 0.879262 |
| 0.00916064 | 0.935053 |
| 0.00221855 | 0.974593 |
| 0.000130899 | 0.996135 |
| −0.0000248959 | 0.998721 |
| −0.00129456 | 0.982236 |
| −0.00665282 | 0.947413 |
| −0.0188134 | 0.895794 |
| −0.0400542 | 0.829645 |
| −0.0720676 | 0.751827 |
| −0.115845 | 0.665643 |
| −0.171601 | 0.574651 |
| −0.238745 | 0.482473 |
| −0.315899 | 0.392595 |
| −0.400957 | 0.30818 |
| −0.491197 | 0.231899 |
| −0.583415 | 0.165791 |
| −0.674106 | 0.111162 |
| −0.759645 | 0.0685255 |
| −0.836489 | 0.0375914 |
| −0.901371 | 0.0172975 |
| −0.951479 | 0.00589015 |
| −0.984611 | 0.00104315 |
| −0.999301 | 0.0000100651 |
| −0.994893 | −0.000198912 |
| −0.971584 | −0.00262612 |
| −0.930411 | −0.0101725 |
| −0.873191 | −0.0254117 |
| −0.802423 | −0.0504237 |
| −0.721151 | −0.0866563 |
| −0.632792 | −0.134824 |
| −0.540951 | −0.194848 |
| −0.449221 | −0.265846 |
| −0.360994 | −0.346165 |
| −0.279269 | −0.433459 |
| −0.206502 | −0.524811 |
| −0.144474 | −0.616889 |
| −0.0942056 | −0.706119 |
| −0.0559169 | −0.788884 |
| −0.0290287 | −0.86172 |
| −0.0122151 | −0.921505 |
| −0.00349795 | −0.965635 |
| −0.000378337 | −0.992164 |
| 0.000000440875 | −0.999913 |
| 0.000669899 | −0.988538 |
| 0.00464303 | −0.958545 |
| 0.0147221 | −0.911264 |
| 0.0333185 | −0.848777 |
| 0.0622956 | −0.773799 |
| 0.102843 | −0.689529 |
| 0.15539 | −0.599477 |
| 0.219565 | −0.507267 |
| 0.294196 | −0.41644 |
| 0.377366 | −0.330264 |
| 0.466505 | −0.251558 |
| 0.558525 | −0.182543 |
| 0.649984 | −0.124732 |
| 0.73727 | −0.0788538 |
| 0.816797 | −0.0448348 |
| 0.885207 | −0.0218148 |
| 0.939547 | −0.00821674 |
| 0.977439 | −0.00185515 |
| 0.997206 | −0.0000804643 |

*Solution.* By Green's theorem, the area of $P$ equals $\oint_{\partial P} x\, dy$. Since the vertices of $P$ are close to one another, not much is lost by approximating $x$ as a constant along each segment. One algorithm works as follows: for each vertex starting with the second, multiply the $x$ coordinate by the difference between the $y$-coordinate and the previous vertex's $y$-coordinate. Add up the total to find an approximation for the area of $P$. □


2. (Fun/Challenge problem) A planimeter is a device used to calculate the area of a two-dimensional region. In this problem, we explore the mathematics behind how the planimeter works. (Thanks to Wikipedia for the figures and the description below).
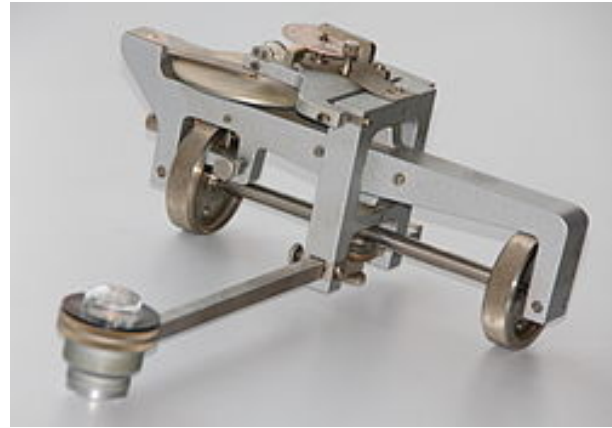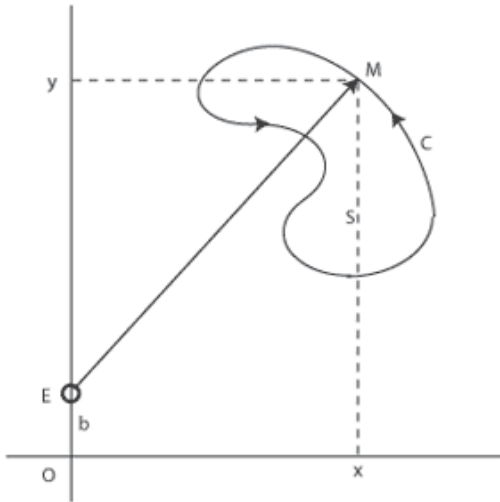
The pointer $M$ at one end of the planimeter follows the contour $C$ of the surface $S$ to be measured. For the linear planimeter the movement of the "elbow" $E$ is restricted to the $y$-axis. Connected to the arm $ME$ is the measuring wheel with its axis of rotation parallel to $ME$. A movement of the arm $ME$ can be decomposed into a movement perpendicular to $ME$, causing the measuring wheel to rotate, and a movement parallel to $ME$, causing the measuring wheel to skid, with no contribution to its reading.

Use Green's theorem to explain why the final reading on the measuring wheel is equal to the area of the surface $S$.

*Solution.* Let the coordinates of $M$ be $x$ and $y$. Note that for each $(x, y)$ close enough to the $y$-axis, there are two points on the $y$-axis whose distance from $(x, y)$ is equal to the length $|EM|$ of the elbow. Define $b(x, y)$ to be the $y$-coordinate of the lower one. Note that as we move the needle around $C$, the measuring wheel only registers the component of $(dx, dy)$ in the direction perpendicular to $\overrightarrow{EM} = (x, y - b)$. To rotate a vector 90 degrees, we swap the coordinates and negate the first. In other words, the final measurement is

$$\oint_C -(y - b(x, y)) \, dx + x \, dy.$$

By Green's theorem, this is equal to

$$\iint_S \left( \frac{\partial x}{\partial x} + \frac{\partial (y - b(x, y))}{\partial y} \right) dx \, dy.$$

We claim that $\frac{\partial b(x,y)}{\partial y} = 1$. Indeed, if $x$ is held constant while $y$ is increased or decreased, $b(x, y)$ will change by the same amount that $y$ changed (this is apparent geometrically; there is no need for a formula for $b$). Thus the partial derivative of $b$ with respect to $y$ is 1. Substituting into the above integral, we get $\iint_S dx \, dy = \text{area}(S)$. □