Name: Key

MATH 19 PRACTICE FINAL
FALL 2016
BROWN UNIVERSITY
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$\boxed{\mathbf{1}}$ (10 points) Find $\displaystyle\int \cos^2 x + \cos^{10} x \sin x + xe^x \, dx$.

$$\cos^2 x = \frac{1+\cos 2x}{2} \implies \int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx$$

$$= \frac{x}{2} + \frac{\sin 2x}{4}$$

$$\int \cos^{10} x \sin x \, dx \overset{u=\cos x}{=} -\int u^{10} \, du$$

$$= \frac{-u^{11}}{11}$$

$$= \frac{-\cos^{11} x}{11}$$

$$\int xe^x \, dx = \int x (e^x)' \, dx$$

$$= xe^x - \int e^x \, dx$$

$$= xe^x - e^x$$

So $\quad \dfrac{x}{2} + \dfrac{\sin 2x}{4} - \dfrac{\cos^{11} x}{11} + xe^x - e^x + C$

2 (10 points) Consider the function

$$f(x) = \int_0^x \sqrt{4\cos^2 t - 1}\, dt.$$

Find the arclength of the graph of f over the interval $[0, \pi/2]$.

$$\int_0^{\pi/2} \sqrt{1 + f'(x)^2}\, dx = \int_0^{\pi/2} \sqrt{1 + (4\cos^2 x - 1)}\, dx$$

$$= \int_0^{\pi/2} 2\cos x\, dx$$

$$= 2\left[\sin x\right]_0^{\pi/2}$$

$$= 2$$

$\boxed{3}$ (10 points) Find the function f which satisfies $f(0) = 3$, $f'(0) = 6$, and

$$f''(x) + 2f(x) = 2f'(x) + e^x.$$

$$\Leftrightarrow f''(x) - 2f'(x) + 2f(x) = e^x$$

(i) solve homogeneous DE:

$$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{-(-2) \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

So $f(x) = Ae^x \cos x + Be^x \sin x$ solves $f'' - 2f' + 2f = 0$, for any A, B.

(ii) Particular solution: $f(x) = Ce^x \Rightarrow$

$$f''(x) - 2f'(x) + 2f(x) = Ce^x,$$

So $C = 1$.

(iii) Solve for constants:

$$f(x) = e^x + Ae^x \cos x + Be^x \sin x$$

$$f'(x) = e^x + Ae^x \cos x - Ae^x \sin x + Be^x \sin x + Be^x \cos x$$

So $f(0) = e^0 + Ae^0 \cos 0 + 0 = 3 \Rightarrow A = 2.$

then $f'(0) = 1 + A + B = 6 \Rightarrow B = 3$

So $f(x) = e^x + 2e^x \cos x + 3e^x \sin x.$

$\boxed{4}$ (10 points) Determine the convergence or divergence of each of the following series.

(a) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$\ln n > 1$ when $n \geq 3$,

so $\sum_{n=3}^{\infty} \frac{\ln n}{n} > \sum_{n=3}^{\infty} \frac{1}{n} = \infty$,

so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

by the comparison test.

(b) $1 + \frac{1}{1+2} + \frac{1}{1+2+4} + \frac{1}{1+2+4+8} + \cdots$

$= 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \cdots$

$= \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

Now $\sum_{n=10}^{\infty} \frac{1}{2^n-1} \leq \sum_{n=10}^{\infty} \frac{1}{2^n - \frac{1}{2}\cdot 2^n}$

$= \sum_{n=10}^{\infty} \frac{2}{2^n} < \infty$,

by geometric series. So the series converges, by the comparison test.

$\boxed{5}$ (10 points) Determine the convergence or divergence of each of the following series.

(a) $\sum_{n=1}^{\infty} \frac{n^n}{n^{n^2}}$

$$\sum_{n=2}^{\infty} n^{n-n^2} \le \sum_{n=2}^{\infty} n^{-2},$$ because $n-n^2 \le -2$ for all $n \geq 2$.

$< \infty$,

by the integral test, because $\int_1^{\infty} x^{-2}\,dx < \infty$. So $\sum \frac{n^n}{n^{n^2}}$ converges.

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{n^2+1+2^{-n}}{n^2-10}$

$\frac{n^2+1+2^{-n}}{n^2-10} \to 1$ as $n \to \infty$, since $2^{-n} \to 0$. So this series fails the n^{th} term test & therefore diverges.

$\boxed{6}$ (10 points) (a) Suppose f is a continuous function from $[0,\infty)$ to $\mathbb{R}$ and that $\int_0^\infty f(x)dx = 21$ and $\int_0^1 f(x)dx = 16$. Find

$$\lim_{b\to\infty} \int_1^b f(x)\,dx.$$

$$\lim_{b\to\infty}\int_1^b f(x)\,dx \xleftarrow{\text{by def}^n} = \int_1^\infty f(x)dx$$

$$= \int_0^\infty f(x)dx - \int_0^1 f(x)dx$$

$$= 21 - 16$$

$$= 5$$

(b) Suppose that the integrals $\displaystyle\int_0^1 x^p\,dx$ and $\displaystyle\int_1^\infty x^p\,dx$ are both improper and divergent. Find p.

$\int_0^1 x^p dx$ diverges if $p \le -1$, since

$$\int x^p dx = \begin{cases} \frac{x^{p+1}}{p+1} & p \neq -1 \\ \ln x & p = 1. \end{cases}$$

$\int_1^\infty x^p dx$ diverges if $p \geq -1$. So

$$p = -1.$$

$\boxed{7}$ (10 points) (a) Find the Fourier series of any 2π-periodic function $f(x)$ which is equal to 1 for all x strictly between $-\pi/2$ and $\pi/2$ and 0 for all x strictly between $\pi/2$ and $3\pi/2$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{2\pi}(\pi) = \frac{1}{2}.$$

For $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\,dx = \left.\frac{\sin nx}{n\pi}\right|_{-\pi/2}^{\pi/2}$$

$$= \frac{\sin(n\pi/2)}{n\pi} - \frac{\sin(-n\pi/2)}{n\pi}$$

$$= \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi} = \begin{cases} 0 & n \text{ even} \\ 2/n\pi & n = 4k+1 \\ -2/n\pi & n = 4k+3 \end{cases}$$

the b_n's are zero b/c f is even.

So the F.S. is $\frac{1}{2} + \frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x + \frac{2}{5\pi}\cos 5x - \frac{2}{7\pi}\cos 7x + \cdots$

(b) Suppose f is one of the functions described in part (a), and suppose that the Fourier series for f converges to $f(x)$ for all x. Calculate $f(0)$, $f(\pi/2)$, and $f(\pi)$.

$f(0) = 1$ and $f(\pi) = 0$ by def$^{\underline{n}}$.

$$f(\pi/2) = \frac{1}{2}\left(f\left(\tfrac{\pi}{2}+\right) + f\left(\tfrac{\pi}{2}-\right)\right)$$

$$= \frac{1}{2}(1+0) = \frac{1}{2}$$

$\boxed{8}$ (10 points) Consider a physical system which responds to a periodic stimulus $V(t)$ by behaving according to the periodic solution Q of the differential equation

$$Q''(t) + 6Q'(t) + 13Q(t) = V(t).$$

Express $V(t)$ as a real Fourier series given that

$$Q(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2+1)} e^{int}.$$

Note: You may assume that Q is twice differentiable (it is).

$$Q'(t) = \sum_{n=-\infty}^{\infty} \frac{in}{\pi(n^2+1)} e^{int}$$

$$Q''(t) = \sum_{n=-\infty}^{\infty} \frac{(in)^2}{\pi(n^2+1)} e^{int}$$

so $$Q''(t) + 6Q'(t) + 13Q(t) = \sum_{n=-\infty}^{\infty} \frac{(in)^2 + 6in + 13}{\pi(n^2+1)} e^{int}$$

$$= V(t).$$

So the F.S. for $V(t)$ has

$$a_n = 2\operatorname{Re} c_n = \frac{2(13-n^2)}{\pi(n^2+1)},$$

$$b_n = -2\operatorname{Im} c_n = \frac{12n}{\pi(n^2+1)}$$

$\boxed{9}$ (10 points) Consider the power series $\sum_{n=0}^{\infty} \frac{n!}{n^n}(x-3)^n$.

(a) Find the radius of convergence of this power series.

$$\lim_{n\to\infty} \frac{|x-3|^{n+1}\cancel{(n+1)!}^{\,n+1}}{(n+1)^{n+1}} \cdot \frac{n^{n+1}}{n\,|x-3|^n\cancel{n!}} = |x-3| \lim_{n\to\infty} \frac{n+1}{\left(\frac{n+1}{n}\right)^n n}$$

$$= |x-3| \lim_{n\to\infty}\left(\frac{n+1}{n}\right) \lim_{n\to\infty}\left(\frac{1}{(1+\frac{1}{n})^n}\right)$$

$$= |x-3| \cdot 1 \cdot \frac{1}{e}$$

$$= \frac{|x-3|}{e}$$

then $-1 < \frac{|x-3|}{e} < 1$ if $3-e < x < 3+e$. If $x < 3-e$ or $x > 3+e$ then the series diverges by the ratio test.

So the radius is e

(b) Let us define $f(x) = \sum_{n=0}^{\infty} \frac{n!}{n^n}(x-3)^n$ for all x such that the infinite series on the right-hand side converges. Calculate $f^{(4)}(3)$.

the coeff. of $(x-3)^4$ is $\frac{f^{(4)}(3)}{4!} = \frac{f^{(4)}(3)}{24}$. So

$$f^{(4)}(3) = 24 \cdot \frac{4!}{4^4}$$

$$= \frac{\cancel{8}\cdot 3 \cdot \cancel{8} \cdot 3}{\cancel{16}^{2} \cdot \cancel{16}^{2}}$$

$$= \frac{9}{4}$$

$\boxed{10}$ (10 points) Show that for all real numbers x, we have

$$1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \frac{x^8}{2^4 \cdot 4!} - \cdots \neq 0.$$

(b) the LHS is $e^{-x^2/2}$ because

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

and $e^{-x^2/2} = 0$ has no solution x.

(a) $e^z = e^{x+iy} = e^x e^{iy}$

nonzero because $e^x > 0$ for all real x

nonzero because $e^{iy} = \mathrm{cis}(y)$ is on the unit circle

the product of two nonzero complex numbers is nonzero because

$$|zw| = |z||w| = \text{nonzero real} \times \text{nonzero real}$$

$$= \text{nonzero}$$