§6.1 Inner Products

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So far we have dealt only with notions pertaining to vector space structure (are two vectors parallel, is one vector in the span of some others, etc.). However, to really do geometry we need to get angle into the mix. Let's do it.

a.k.a. "dot product", $\vec{u}\cdot\vec{v}$

Def$^{\underline{n}}$ If $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$, the **inner product** of $\vec{u}$ and $\vec{v}$ is the scalar $\vec{u}^T\vec{v}$.

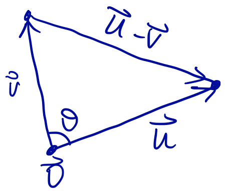
Proposition $\vec{u}\cdot\vec{u} = |\vec{u}|^2$

Proof $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \Rightarrow \vec{u}\cdot\vec{u} = u_1^2 + \cdots + u_n^2 = |\vec{u}|^2$. ∎

Will cut straight to the chase:

Proposition $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

Proof



Law of cosines: $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

Linearity: $(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v}$

solving for the two highlighted expressions shows us they're the same. ∎

Corollary $\vec{u}\cdot\vec{v}=0$ if & only if $\vec{u},\vec{v}$ are perpendicular.

Let's pause to reflect on this new development. In a vector space where we only have access to the vector space operations, we have no notion of perpendicularity. We require this additional structure, the inner product, for that.

Theorem The inner product satisfies

(*) $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ (commutative)

(*) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$ (distributive)

(*) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$ (associative with scalar product)

(*) $\vec{u}\cdot\vec{u}\geq 0$, with $\vec{u}\cdot\vec{u}=0$ iff $\vec{u}=0$. (positive)

Proof The first three follow from the corresponding matrix properties, since $\vec{u}\cdot\vec{v}=\vec{u}^T\vec{v}$. The last follows from $\vec{u}\cdot\vec{u}=|\vec{u}|^2$.

Unit Vectors

Example Find the vector whose length is 1 and which points in the same direction as $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

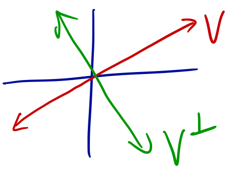
Solution If we scale $\vec{u}$ down by its length, then that new vector will have length 1:

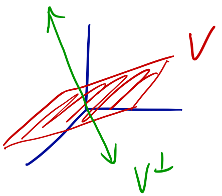
$$\frac{\vec{u}}{|\vec{u}|} = \frac{[1 \quad 2 \quad 3]^T}{\sqrt{1^2+2^2+3^2}} = \begin{bmatrix} \frac{1}{\sqrt{13}} \\ 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}.$$

We say $\frac{\vec{u}}{|\vec{u}|} = \begin{bmatrix} 1/\sqrt{13} \\ 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$ is the **unit vector** of $\vec{u}$.

Orthogonal Complements

If V is a subspace of $\mathbb{R}^n$, then we define the **orthogonal complement** of V to be the set of vectors in $\mathbb{R}^n$ perpendicular to every vector in V. We call this set $V^\perp$. E.g.:





Facts

perpendicular

(*) $\vec{x} \in V^\perp$ iff $\vec{x} \perp \vec{y}$ for all $\vec{y}$ in a spanning list V.

(*) $V^\perp$ is a subspace

Theorem Let A be an $m \times n$ matrix.

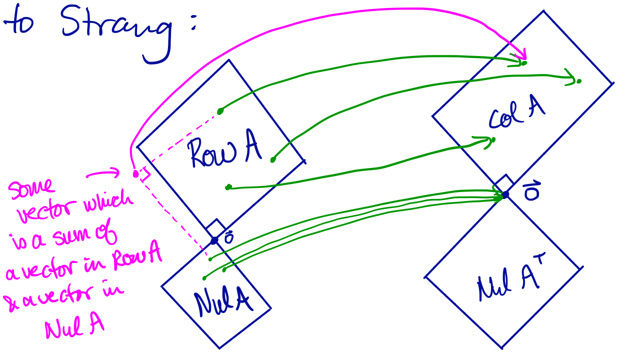
(i) $(\text{Row}\,A)^\perp = \text{Nul}\,A$, (ii) $(\text{Col}\,A)^\perp = \text{Nul}\,A^T$

Proof $\vec{x} \in \text{Nul}\,A \Rightarrow A\vec{x} = \vec{0} \Rightarrow \vec{x} \perp$ to each row of A, by the row-based description of the matrixvector product. So $\vec{x} \in (\text{Row}\,A)^\perp$. Conversely, if $\vec{x} \in (\text{Row}\,A)^\perp$ then clearly $A\vec{x} = \vec{0}$, so $x \in \text{Nul}\,A$. So $(\text{Row}\,A)^\perp = \text{Nul}\,A$.

Applying (i) to A^T gives (ii).

(the concept, <u>not</u> my drawing)

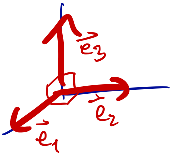
So we have the following beautiful schematic diagram, due to Strang:



§6.2 Orthogonal Lists

A list $\{\vec{v}_1, \dots, \vec{v}_n\}$ is said to be orthogonal if it's pairwise orthogonal: $\vec{v}_i \cdot \vec{v}_j = 0$ for every pair (i, j) with $i \neq j$.

Example $\{\vec{e}_1, \dots, \vec{e}_n\}$ in $\mathbb{R}^n$ is orthogonal.



Prop. An orthogonal list $\{\vec{v}_1, \dots, \vec{v}_n\}$ of nonzero vectors is lin. ind.

Proof

If $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$, then

$$\vec{v}_1 \cdot (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \vec{v}_1 \cdot \vec{0} \Rightarrow$$

$$c_1|\vec{v}_1|^2 + c_2 \cdot 0 + \cdots + c_n \cdot 0 = 0 \Rightarrow$$

$$c_1|\vec{v}_1|^2 = 0.$$

So $c_1 = 0$. Similarly, $c_2 = \cdots = c_n = 0$. So the list is linearly independent.

An orthogonal basis is what it says on the tin: a basis which is orthogonal.

What's great about them is how easy it is to find coordinates:

Proposition If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis of $V \subset \mathbb{R}^n$, then $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ implies $c_j = \dfrac{\vec{v} \cdot \vec{v}_j}{\vec{v}_j \cdot \vec{v}_j}$.

Proof $\vec{v} \cdot \vec{v}_j = (c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) \cdot \vec{v}_j = c_j \vec{v}_j \cdot \vec{v}_j$ + bunch of zeros.

so $c_j = \dfrac{\vec{v} \cdot \vec{v}_j}{\vec{v}_j \cdot \vec{v}_j}$.