§4.7 Change of Basis

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It is often possible to choose a basis for a particular application which has nicer properties than other bases. [Foreshadowing: eigenvectors!]

In these situations, we want to be able to change coordinates so we can do linear algebra with respect to this new basis.

Example Suppose V is a vector space with bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$. If $\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$, $\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$, and $\vec{x} = 3\vec{b}_1 + \vec{b}_2$, write $\vec{x}$ as a linear comb. of $\vec{c}_1$ and $\vec{c}_2$.

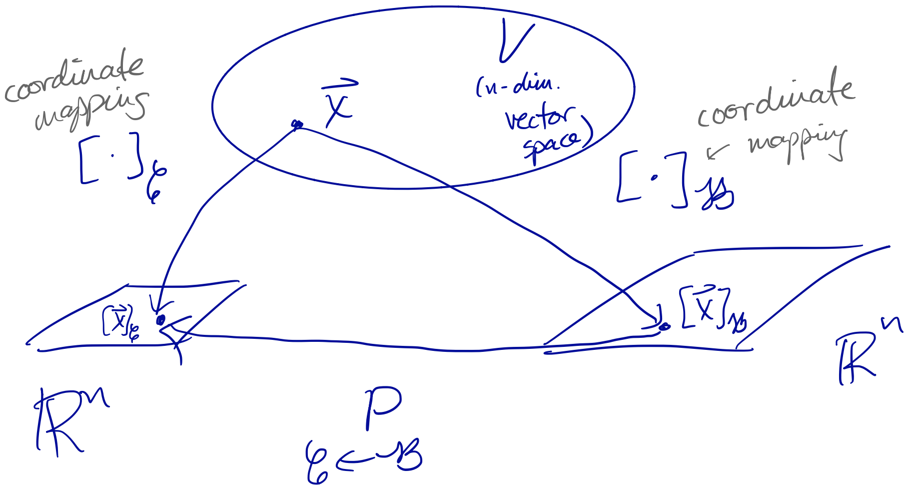
Solution: Substitute! $\vec{x} = 3(4\vec{c}_1 + \vec{c}_2) + (-6\vec{c}_1 + \vec{c}_2)$

$$= 6\vec{c}_1 + 4\vec{c}_2.$$ ∎

So, $[x]_\mathcal{C} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$. In matrix terms, what we did here is: $[x]_\mathcal{C} = \underbrace{\left[[\vec{b}_1]_\mathcal{C} \; [\vec{b}_2]_\mathcal{C} \right]}_{\text{"}\underset{C \leftarrow B}{P}\text{"}} [\vec{x}]_\mathcal{B}$, and this represents the general idea: find the coordinates of the old basis vectors w.r.t. the new ones, put the resulting coordinate vectors (which are now in $\mathbb{R}^n$) as column vectors in a matrix.

> ** it's easy to forget which way around old/new goes here. I like to remind myself by keeping a small example handy, like $\mathcal{B}=\{\vec{e}_1,\vec{e}_2\}$ in $\mathbb{R}^2$ and $\mathcal{C}=\{2\vec{e}_1, 2\vec{e}_2\}$ with $[\vec{x}]_{\mathcal{B}}=(3,3)$. Clearly $[\vec{x}]_{\mathcal{C}}=(\frac{3}{2},\frac{3}{2})$, so the change-of-basis matrix is $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ not $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **

The following diagram might help you visualize what's going on:



Now, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ maps $\mathcal{B}$ to $\mathcal{C}$ (that is: $\vec{b}_1$ to $\vec{c}_1$, etc.) which means $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ (since it is surjective and square) is invertible. So clearly, $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ maps $[x]_{\mathcal{C}}$ to $[\vec{x}]_{\mathcal{B}}$, for each $\vec{x}\in V$.

Example Let $\vec{b}_1=\begin{bmatrix}1\\-3\end{bmatrix}$, $\vec{b}_2=\begin{bmatrix}-2\\4\end{bmatrix}$, $\vec{c}_1=\begin{bmatrix}-7\\9\end{bmatrix}$, $\vec{c}_2=\begin{bmatrix}-5\\7\end{bmatrix}$.
Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

Solution We want to represent $\vec{c}_1$ & $\vec{c}_2$ in terms of $\vec{b}_1, \vec{b}_2$ to get $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ ("old in terms of new")
So the columns of of $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ solve $\vec{c}_1=[\vec{b}_1\ \vec{b}_2][?]$ and $\vec{c}_2=[\vec{b}_1\ \vec{b}_2][?]$. So $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}=[\vec{b}_1\ \vec{b}_2]^{-1}[\vec{c}_1\ \vec{c}_2]$.

We could invert $\begin{bmatrix}1&-2\\-3&4\end{bmatrix}$ by hand or row reduce $[\vec{b}_1\ \vec{b}_2\ \vec{c}_1\ \vec{c}_2]$. Either way, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}=\begin{bmatrix}5&3\\6&4\end{bmatrix}$.

For $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, we invert $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ to get $\begin{bmatrix}2&-3/2\\-3&5/2\end{bmatrix}$.