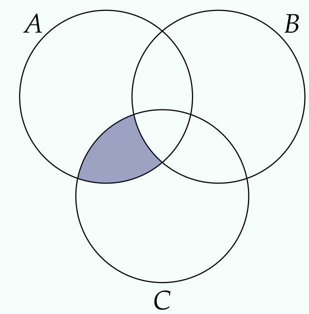
DATA 1010
PROBLEM SET 4
DUE 05 OCTOBER 2018 AT 11 PM

Problem 1



(a) Use set operations to express the shaded region in the Venn diagram in terms of A, B, and C.

(b) Suppose that $|A| = 5$, $|\Omega| = 10$, and $A \subset \Omega$. How many sets $B \subset \Omega$ satisfy the property that $A \cap B = \emptyset$?

(c) Find the least possible cardinality of a set which can be written as a disjoint union of 10 sets of pairwise unequal cardinalities.

Solution

(a) The shaded region contains exactly the points which are in A and C but not B. In other words, $(A \cap C) \setminus B$.

(b) There are $2^5 = 32$ such sets, since A satisfies the given conditions if and only if it is a subset of $\Omega \setminus B$.

(c) A set of 45 elements, such as $\{1, 2, 3, \ldots, 45\}$, can be written as such a disjoint union:

$$\{1, 2, 3, \ldots, 45\} = \{\} \cup \{1\} \cup \{2, 3\} \cup \cdots \cup \{37, 38, \ldots, 45\}$$

Problem 2

Write (from scratch) a function `binary` which interprets a string of 0's and 1's as the binary representation of an integer and returns the value of that integer.

```
binary("001001") == 9
binary("1001001011110101010") == 300970
```

Solution

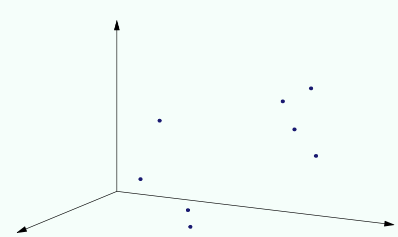
We can do this with an array comprehension:

```
bit(s) = s == '1' ? 1 : 0
binary(s) = sum(bit(s[k])*2^(length(s)-k) for k=1:length(s))
```

Problem 3

The columns of the following matrix are plotted in the figure to the right.

```
1.25  0.75  0.25  1.25  1.5  1.5   0.75  0.75
2.25  1.0   1.75  1.25  1.0  2.25  1.75  0.75
1.25  0.0   0.5   0.25  0.5  1.0   1.0   0.75
```



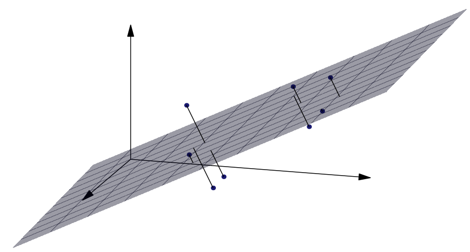
Identify the plane passing through the origin whose sum of squared distances to these points is as small as possible.

Solution

The plane through the origin which best approximates a point cloud in the sum-of-squares sense is the span of the first two columns of U, where U is the matrix of left singular vectors in the SVD of the matrix A whose columns contain the points.

Therefore, we calculate the SVD and find that the plane is equal to the span of $[-0.506, -0.785, -0.358]$ and $[0.861, -0.436, -0.26]$. We can write an equation for this plane using the components of the third column of U as coefficients of x, y, and z, since that column is orthogonal to the first two:

$$0.048x - 0.44y + 0.897z = 0.$$



Problem 4

Show that every representable `Float64` between 2^{-1021} and 2^{1022} can be doubled or halved with no roundoff error.

Solution

An arbitrary `Float64` in that range can be written as $2^e(1 + k/2^{52})$, where e is an integer between -1021 and 1021, and k is an integer between 0 and $2^{52} - 1$. Doubling this number yields $2^{e+1}(1 + k/2^{52})$, which is also a representable `Float64`. Likewise, $2^{e-1}(1 + k/2^{52})$ is also a representable `Float64`.

Problem 5

Recall that if A is an $m \times n$ matrix with rank n and $\mathbf{b} \in \mathbb{R}^m$, then the vector $\mathbf{x}$ which minimizes $|A\mathbf{x} - \mathbf{b}|^2$ is

$$\mathbf{x} = (A'A)^{-1}A'\mathbf{b}. \tag{5.1}$$

Also, recall that if U is a matrix with orthonormal columns, then the vector in the span of the columns of U is closest to $\mathbf{b}$ is given by $UU'\mathbf{b}$. Show that the formula $UU'\mathbf{b}$ can be obtained using (5.1).

Solution

Substituting U into (5.1), we get

$$\mathbf{x} = (U'U)^{-1}U'\mathbf{b} = I^{-1}U'\mathbf{b} = U'\mathbf{b}.$$

Therefore, the vector whose distance to $\mathbf{b}$ is minimal is $U\mathbf{x} = UU'\mathbf{b}$, as desired.

Problem 6

Suppose that A is an invertible, symmetric matrix. Which of the following may we conclude? Select all that apply, and explain your reasoning.

(a) A is orthogonal.

(b) A^2 is symmetric.

(c) A^{-1} is symmetric.

(d) $A - 3I$ is symmetric.

(e) A is orthogonally diagonalizable.

(f) The singular values of A are all positive.

Solution

(a) Any invertible symmetric matrix with positive entries is not orthogonal, since vectors with positive entries cannot have a dot product of zero. For example, $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ is symmetric and invertible but not orthogonal.

(b) We check that $(A^2)' = (AA)' = A'A' = A^2$, so the square of a symmetric matrix is symmetric.

(c) Taking the transpose on both sides of $AA^{-1} = I$, we get $(A^{-1})'A' = I'$, which implies $(A^{-1})'A = I$. Since A^{-1} is the only matrix whose product with A is the identity, this implies that $(A^{-1})' = A^{-1}$. Therefore, the inverse of a symmetric invertible matrix is symmetric.

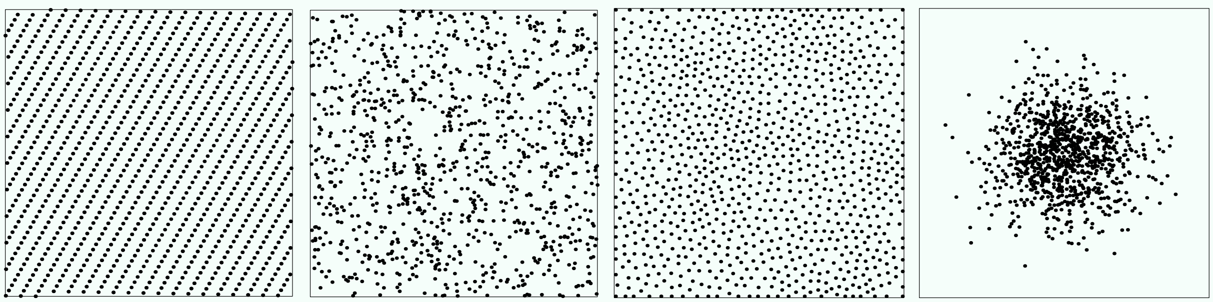
(d) Subtracting a diagonal matrix does not change the off-diagonal entries, so $A - 3I$ is still diagonal.

(e) Yes, this is the content of the spectral theorem.

(f) The number of positive singular values is equal to the rank of the matrix. Since A has an inverse, its rank is equal to its dimension, and it follows that all of the singular values of A are positive.

Problem 7

One of the following three pictures was obtained by generating 2000 independent random numbers uniformly distributed in $[0, 1]$, arranging them into blocks of 2, and plotting the resulting 1000 ordered pairs in the plane. Which is it?



Hint: `using Plots; scatter(rand(1000),rand(1000);aspect_ratio=1)`

Solution

The second one features points which are independently and uniformly distributed in the square. The points in the first figure lie on a relatively small number of lines, and the points in the third figure are too spread out. The points in the fourth figure are not uniformly distributed.

Problem 8

Show, using only the three basic properties of a probability space, that $\mathbb{P}(E \cup F \cup G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G)$ for any events E, F, and G which are pairwise disjoint.

Solution

We can apply the additivity property twice, since $E \cup F$ is disjoint from G:

$$\mathbb{P}(E \cup F \cup G) = \mathbb{P}((E \cup F) \cup G) = \mathbb{P}(E \cup F) + \mathbb{P}(G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G).$$

Problem 9

Write a Julia program to find the exact value of the probability that in eight flips of a fair coin, no two consecutive flips turn up heads.

Two hints: (i) in class we wrote a recursive function to generate all possible outcomes for a similar random experiment, and (ii) if you store your flip sequences as strings, you can use the function `occursin` to check for the substring `HH`.

Solution

```
"""
Returns an array of all of the length-n
strings with characters in {H,T}
"""
function flipsequences(n)
    if n == 1
        ["H","T"]
    else
        [c*s for s in flipsequences(n-1) for c in ("H","T")]
    end
end
# we count up the number of sequences containing HH
sum([occursin("HH",s) for s in flipsequences(8)])
```

Problem 10

The 52 cards in a standard deck (13 spades, 13 clubs, 13 diamonds, 13 hearts), are dealt into four hands of 13 cards each. What is the probability that one of the hands contains all of the hearts?

Solution

The number of ways to deal the four hands is $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$, since there are $\binom{52}{13}$ ways of dealing the first hand, then $\binom{39}{13}$ ways of dealing the second hand given any particular dealing of the first hand, and so on.

The number of dealings that have all of the hearts in the first hand is $1\binom{39}{13}\binom{26}{13}\binom{13}{13}$, since there is one way to put all of the hearts in the first hand, then $\binom{39}{13}$ ways to put 13 of the remaining cards in the second hand, and so on. The number of ways to deal all of the hearts to the second hand is the same, and similarly for the third and fourth hands.

So the ratio of the number of dealings where someone gets all of the hearts to the total number of dealings is

$$\frac{4}{\binom{52}{13}} = 6.3 \times 10^{-12}.$$