§1.1

26 Jan 2017

A **linear equation** can be written like

variables

$$a_1x_1 + \cdots + a_nx_n = b$$

coefficients

where $a_1, \ldots, a_n, b$ are some numbers and $x_1, \ldots, x_n$ are variables.

Linear	Nonlinear
$3x-2y-7=0$,	$x^2y = z$,
$\sqrt{7}\,x_1 = x_2$	$\dfrac{x_1}{x_2} = x_2$

A **system** of linear equations is one or more equations considered together, & we're interested in **solutions** of the system: values for all the variables that make all the equations true.

The set of all possible solutions is the **solution set** of the system, &

two systems are equivalent if they have the same solution set.

Example Show that

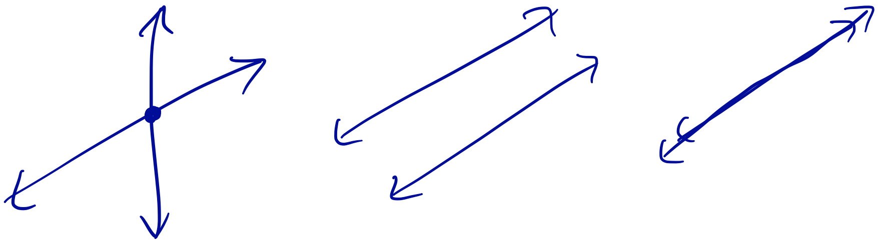
$$(i)\begin{cases} x+y=4 \\ x-y=2 \end{cases}, \quad (ii)\begin{cases} x+y=4 \\ 2x=6 \end{cases}$$

are equivalent.

Solution 1 If (x,y) is a solution to (i), then it also satisfies the sum of the two equations. So $2x=6$. Clearly also $x+y=4$, so (ii) is satisfied. Same idea for (x,y) a sol. to $(ii) \Rightarrow (x,y)$ a sol. to (i).

Solution 2 The solution set to (i) is $\{(3,1)\}$ and same for (ii). So the solution sets are equal.

Recall that two lines can intersect in 0, 1, or ∞ places:



1 solution   0 solutions   infinitely many solutions

So a 2-variable, 2-equation linear system has 0, 1, or ∞ solutions.

We make a matrix out of a system like so:

$$\begin{aligned} 3x - y &= 4 \\ x + 2y &= 0 \end{aligned} \longrightarrow \underbrace{\left(\begin{array}{c|c} \overbrace{\begin{matrix} 3 & -1 \\ 1 & 2 \end{matrix}}^{\text{matrix of coefficients}} & \begin{matrix} 4 \\ 0 \end{matrix} \end{array}\right)}_{\text{augmented matrix}}$$

We report matrix size as $m \times n$ where m is the number of rows & n the # of columns.

To solve a linear system, we may perform simplifying operations that leave the solution set the same:

$$\begin{cases} 3x - y = 4 \\ x + 2y = 0 \end{cases} \xrightarrow[\text{times } 2]{\text{row } 1} \begin{cases} 6x - 2y = 8 \\ x + 2y = 0 \end{cases}$$

$$\begin{cases} 3x - y = 4 \\ x + 2y = 0 \end{cases} \xrightarrow{\text{switch}} \begin{cases} x + 2y = 0 \\ 3x - y = 4 \end{cases}$$

$$\begin{cases} 3x - y = 4 \\ x + 2y = 0 \end{cases} \xrightarrow[-3 \cdot \text{row } 2]{\text{row } 1} \begin{cases} -7y = 4 \\ x + 2y = 0 \end{cases}$$

We will use these elementary row operations to work down the system where we can solve for one variable and substitute to get more, one at a time:

Example Solve ← i.e., find all solutions

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow[\text{row3} \to \text{row3} - 5\cdot\text{row1}]{} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\xrightarrow[\text{row3} \to \text{row3} - 5\cdot\text{row2}]{} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\xrightarrow{\text{scale}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

these zeros are what we're going for

Now $x_3 = -1$, and $x_2 - 4x_3 = 4$ so $x_2 = 0$, and $x_1 - 2x_2 + x_3 = 0 \Rightarrow x_1 = 1$. So $(1,0,-1)$ is the unique solution.

We say a system is **consistent** if it has a solution and **inconsistent** if not.

§1.2

What we were doing with the (augmented) matrix is called **row reducing**. We call the first nonzero entry in a row the **leading entry**. Our goal is to zero-out the entries below each leading entry & also sort the rows by which column their leading entry is in (& also put any zero rows at the bottom). Such a matrix is said to be in **echelon form**.

← nonzero ← can be anything

$$\begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We call the locations of leading entries in echelon form **pivot positions** & their columns **pivot columns**

Example Row reduce

$$\begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}$$

Solution

```
M = [0 3 -6 6; 3 -7 8 -5; 3 -9 12 -9]
```

$$\begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}$$

Step 1: switch to get nonzero entry at top of first column

```
M = rowswitch(M,1,3)
```

$$\begin{bmatrix} 3 & -9 & 12 & -9 \\ 3 & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \end{bmatrix}$$

-1

Step 2: add the first row times -1 to the 2nd row

```
M = rowadd(M,2,1,-1)
```

$$\begin{bmatrix} 3 & -9 & 12 & -9 \\ 0 & 2 & -4 & 4 \\ 0 & 3 & -6 & 6 \end{bmatrix}$$

Step 3: scale by $\frac{1}{2}$ (second row)

```
M = rowscale(M,2,1//2)
```

$$\begin{bmatrix} 3 & -9 & 12 & -9 \\ 0 & 1 & -2 & 2 \\ 0 & 3 & -6 & 6 \end{bmatrix}$$

Step 4: add -3 times the second row to the third

```
M = rowadd(M,3,2,-3)
```

$$\begin{bmatrix} 3 & -9 & 12 & -9 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This system is consistent because we could make up whatever value we want for x_3 and backsolve. The only thing that blocks consistency is a row like

nonzero

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \blacksquare \end{pmatrix}$$

since it corresponds to $0x_1 + \cdots + 0x_n = \blacksquare$, which is impossible.