**Brown University**
**Problem Set 11**
**Instructor: Samuel S. Watson**
**Due: 6 December 2017**

*Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.*

## Problem 1

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{A}$, where

$$\mathbf{F} = \langle xy, y^2 + e^{xz^2}, \sin xy \rangle.$$

and $S$ is the boundary of the region $E$ bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

## Solution

We apply the divergence theorem to get

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{A} &= \iiint_E \nabla \cdot \mathbf{F} \, \mathrm{d}V \\
&= \iiint_E 3y \, \mathrm{d}V \\
&= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}x \\
&= \frac{184}{35}.
\end{aligned}$$

Final answer:

$$\frac{184}{35}$$

**Problem 2**

Suppose that $D$ is a region in $\mathbb{R}^3$ bounded by a piecewise smooth surface $S$. Suppose that $f$ is a differentiable function on $\mathbb{R}^3$, and define $\mathbf{n} : S \to \mathbb{R}^3$ to be the outward-pointing unit normal at each point of $S$. Show that

$$\iiint_D \nabla f \, \mathrm{d}V = \iint_S f\mathbf{n} \, \mathrm{d}A. \tag{2.1}$$

Hint: begin by applying the divergence theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is a constant vector.

Note: we define the integral of a vector-valued function to be the vector of integrals of its components.

**Solution**

Applying the divergence theorem to $\mathbf{F} = f\mathbf{i}$ gives

$$\iiint_D \partial_x f \, \mathrm{d}V = \iint_S f n_1 \, \mathrm{d}A,$$

where $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$. Similarly,

$$\iiint_D \partial_y f \, \mathrm{d}V = \iint_S f n_2 \, \mathrm{d}A,$$

and

$$\iiint_D \partial_z f \, \mathrm{d}V = \iint_S f n_3 \, \mathrm{d}A.$$

These three equations are the three components of (2.1).

**Problem 3**

Define $S$ to be the part of the sphere $x^2 + y^2 + z^2 = 4$ which lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy$-plane. Calculate the flow of $\nabla \times \mathbf{F}$ upward through $S$, where $\mathbf{F} = \langle xz, yz, xy \rangle$.

**Solution**

Stokes' theorem tells us that the desired flow is equal to the circulation of $\mathbf{F}$ around the boundary of $S$. The boundary of $S$ is the solution to the system $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. We can substitute the latter equation into the former to find that $z = \sqrt{3}$ and $x^2 + y^2 = 1$. Therefore, the boundary of $S$ can be parametrized as

$$\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle,$$

where $t$ ranges from 0 to $2\pi$. So

$$\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \sin t \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, \mathrm{d}t = 0.$$

Final answer:

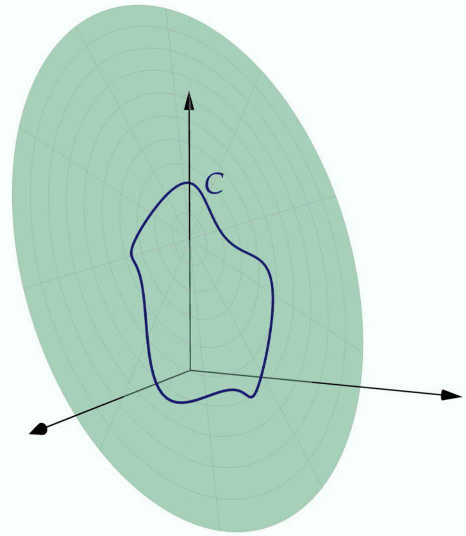
0

**Problem 4**

Let $C$ be a simple closed smooth curve in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,\mathrm{d}x - 2x\,\mathrm{d}y + 3y\,\mathrm{d}z$$

depends only on the area of the region enclosed by $C$ and not on its shape or location in the plane.




**Solution**

We apply Stokes' theorem to the surface $S$ in $x + y + z = 1$ enclosed by $C$ to obtain

$$\int_C z\,\mathrm{d}x - 2x\,\mathrm{d}y + 3y\,\mathrm{d}z = \iint_S \nabla \times \langle z, -2x, 3y\rangle \cdot \frac{\langle 1,1,1\rangle}{\sqrt{3}}\mathrm{d}A,$$

since $\langle 1,1,1\rangle$ is orthogonal to $S$ at every point. So we have

$$\int_C z\,\mathrm{d}x - 2x\,\mathrm{d}y + 3y\,\mathrm{d}z = \iint_S \langle 3,1,-2\rangle \cdot \frac{\langle 1,1,1\rangle}{\sqrt{3}}\,\mathrm{d}A = \frac{2}{\sqrt{3}}\,\mathrm{area}(S),$$

as desired.

Additional space