§5.1 continued

11 Apr 2017

We have seen that it's pretty cool to have a basis for $\mathbb{R}^n$ consisting of eigenvectors of $A:\mathbb{R}^n \to \mathbb{R}^n$, because then A acts very simply on vectors represented in that basis:

$$A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \underbrace{\lambda_1 c_1 \vec{v}_1}^{\text{here } \vec{v}_i \text{ has eigenvalue } \lambda_i,\ \text{for } i=1,\dots,n} + \cdots + \lambda_n c_n \vec{v}_n.$$

It turns out that we <u>automatically</u> get a basis of eigenvectors if there happen to be n different eigenvalues:

<u>**Theorem**</u> If $\{\vec{v}_1,\dots,\vec{v}_r\}$ are eigenvectors corresponding to distinct eigenvalues $\{\lambda_1,\dots,\lambda_r\}$, then $\{\vec{v}_1,\dots,\vec{v}_r\}$ is a linearly independent list.

Proof Assume the list is lin. dep. Apply the linear dependence lemma to extract some p, the <u>least possible one</u>, and $c_1,\dots,c_p$ so that

$$\vec{v}_{p+1} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p.$$

Apply A to both sides to get

$$\lambda_{p+1}\vec{v}_{p+1} = c_1\lambda_1\vec{v}_1 + \cdots + c_p\lambda_p\vec{v}_p.$$

Now substituting $\vec{v}_{p+1} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$, we get

$$c_1(\lambda_1-\lambda_{p+1})\vec{v}_1 + c_2(\lambda_2-\lambda_{p+1})\vec{v}_2 + \cdots + c_p(\lambda_p-\lambda_{p+1})\vec{v}_p = \vec{0}$$

Now all the eigenvalue differences are nonzero, so all the c's are zero. But that contradicts the minimality assumption for p, so we're done. ∎

§ 5.2 Finding eigenvalues

We have seen that finding eigenvectors once we know the corresponding eigenvalue is a matter of solving the linear homogeneous equation $(A-\lambda I)\vec{v}=\vec{0}$. But how to find λ? Well, we know $(A-\lambda I)\vec{v}=\vec{0}$ has nonzero solutions only if $A-\lambda I$ is non-invertible. So we solve $\det(A-\lambda I)=0$ to find the eigenvalues. Let's do one we've seen already.

Example Find the eigenvalues of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Solution We solve

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda) - 6\cdot 5$$

$$= \lambda^2 - 3\lambda + 2 - 30$$

$$= \lambda^2 - 3\lambda - 28$$

$$= (\lambda - 7)(\lambda + 4).$$

So $\boxed{\lambda = 7}$ and $\boxed{\lambda = -4}$ are the only two eigenvalues. ∎

We can see that if A is an $n \times n$ matrix, then $\det(A - \lambda I)$ will be an n-degree polynomial in λ. That's because the only λ^n term comes from $(-\lambda)^n$ in the diagonal rook arrangement. $\det(A-\lambda I)$ is called the **characteristic polynomial** of A, and $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Theorem An $n\times n$ matrix has at most n eigenvalues.

Proof $\det(A-\lambda I)=0$ has at most n solutions, by the fundamental theorem of algebra. ∎

Let's go back & see how the golden ratio shows up with our Fibonacci matrix:

Example Find the eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution

$$0=\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$= -\lambda(1-\lambda)-1$$

$$= \lambda^2-\lambda-1$$

when $\lambda = \dfrac{-(-1)\pm\sqrt{(-1)^2-4(-1)(1)}}{2} = \dfrac{1\pm\sqrt{5}}{2}$. ∎