Brown University
Problem Set 6
Instructor: Samuel S. Watson
Due: 27 October 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

Find the directions in which the directional derivative of $f(x,y) = ye^{-xy}$ at the point $(0,2)$ has value 1.

Solution

We use the formula for the directional derivative in the direction of a unit vector $\mathbf{u}$:

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

We have that $\nabla f(x,y) = \langle -y^2 e^{-xy}, e^{-xy}(1-xy) \rangle$, so $\nabla f(0,2) = \langle -4, 1 \rangle$. We let $\mathbf{u} = \langle a, b \rangle$, where $a^2 + b^2 = 1$. We use the formula above to give us a second equation involving a and b which allows us to solve for them explicitly:

$$D_{\mathbf{u}} f(0,2) = \nabla f(0,2) \cdot \langle a, b \rangle = \langle -4, 1 \rangle \cdot \langle a, b \rangle = -4a + b = 1.$$

With these two equations ($a^2 + b^2 = 1$ and $-4a + b = 1$) we solve for our directional vectors and get two solutions:

$$\mathbf{u}_1 = \langle 0, 1 \rangle, \quad \mathbf{u}_2 = \left\langle -\tfrac{8}{17}, -\tfrac{15}{17} \right\rangle$$

Problem 2

Find the derivative with respect to t of the function $g(t) = t^t$ by writing the function as $f(x(t), y(t))$ where $f(x,y) = x^y$ and $x(t) = t$ and $y(t) = t$.

Solution

Let $f(x(t), y(t)) = x^y$ where $x(t) = t$ and $y(t) = t$. We have that $\frac{\partial f}{\partial x} = yx^{y-1}$ and $\frac{\partial f}{\partial y} = x^y \ln x$. Since both derivatives of x and y with respect to t are 1, the chain rule implies that

$$g'(t) = t \cdot t^{t-1} + t^t \ln t = t^t(1 + \ln t).$$

Note that f is indeed differentiable, since it can be written as $f(x,y) = e^{y \ln x}$. Since y and $\ln x$ are differentiable, their product $y \ln x$ is differentiable, and then the composition with the differentiable exponential function is differentiable.

Problem 3

In this problem, we explore—in the context of the chain rule—what happens when we are careful with the errors associated with linear approximation, instead of just throwing them away and repeating the word "roughly".

For simplicity, suppose that $\mathbf{r}$ is a differentiable path satisfying $\mathbf{r}(0) = (0,0)$, and suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at the origin with $f(0,0) = 0$. We will show that $(f \circ \mathbf{r})'(0) = (\nabla f)(0,0) \cdot \mathbf{r}'(0)$.

(a) Suppose that L is the linearization of f at the origin. Consider the facts

$$f(x,y) = L(x,y) + R(x,y)|\langle x,y\rangle|, \tag{3.1}$$

where $R(x,y) \to 0$ as $(x,y) \to (0,0)$, and

$$\mathbf{r}(t) = t\,\mathbf{r}'(0) + t\,\mathbf{E}(t), \tag{3.2}$$

where $|\mathbf{E}(t)| \to 0$ as $t \to 0$. Fill in the blank: (3.1) follows from the fact that f is ________________, by definition. Equation (3.2) follows from the fact that $\mathbf{r}$ is ________________.

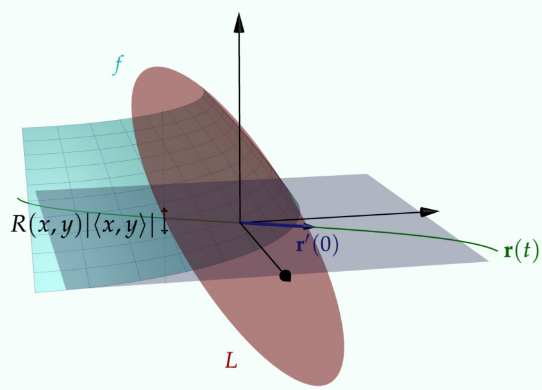
(b) Substitute one of the equations from (a) into the other to show that

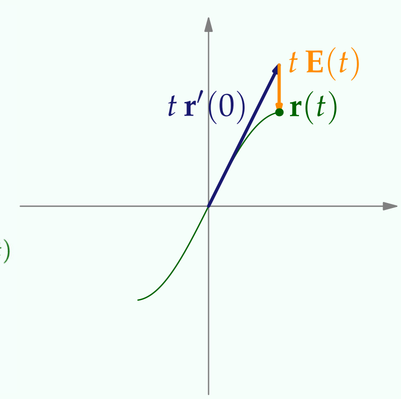
$$\frac{f(\mathbf{r}(h))}{h} = L(\mathbf{r}'(0)) + L(\mathbf{E}(h)) \pm R(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h))|\mathbf{r}'(0) + \mathbf{E}(h)|. \tag{3.3}$$

You will want to make use of the fact that L is linear in the sense that $L(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha L(\mathbf{v}) + \beta L(\mathbf{w})$ for all scalars α and β and vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$.

(c) For each of the terms on the right-hand side of (3.3) except the first, explain why it converges to zero as $h \to 0$.

(d) Take the limit of both sides of (3.3) as $h \to 0$ and thereby establish the chain rule.





Solution

Substituting (3.2) into (3.1), we get

$$\begin{aligned}\frac{f(\mathbf{r}(h))}{h} &= \frac{f(h\,\mathbf{r}'(0) + \mathbf{E}(h))}{h} \\ &= \frac{L(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h)) + R(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h))h|\mathbf{r}'(0) + \mathbf{E}(h)|}{h}.\end{aligned}$$

Since $f(0,0) = 0$, the linearization L is of the form $ax + by$ for some constants a and b (to wit, the partial derivatives of f at the origin). It follows that $L(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha L(\mathbf{v}) + \beta L(\mathbf{v})$ for all $\alpha, \beta \in \mathbb{R}$ and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$. Applying this rule, we get

$$\begin{aligned}\frac{f(\mathbf{r}(h))}{h} &= \frac{hL(\mathbf{r}'(0)) + hL(\mathbf{E}(h))}{h} + \frac{R(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h))h|\mathbf{r}'(0) + \mathbf{E}(h)|}{h} \\ &= L(\mathbf{r}'(0)) + L(\mathbf{E}(h)) + R(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h))|\mathbf{r}'(0) + \mathbf{E}(h)|.\end{aligned}$$

Taking $h \to 0$ on both sides, we see that $L(\mathbf{E}(h)) \to 0$ as $h \to 0$, since $\mathbf{E}(h) \to \mathbf{0}$ as $h \to 0$ and L is continuous and equal to 0 at the origin. The factor $R(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h))$ converges to zero since the argument $h\,\mathbf{r}'(0) + h\,\mathbf{E}(h)$ converges to zero and the limit of R exists and is equal to zero at the origin.

The last factor $|\mathbf{r}'(0) + \mathbf{E}(h)|$ converges to $|\mathbf{r}'(0)|$ since $\mathbf{E}(h) \to \mathbf{0}$. Therefore, the whole term $R(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h))|\mathbf{r}'(0) + \mathbf{E}(h)|$ converges to zero.

All together, we see that $\lim_{h\to 0} \frac{f(\mathbf{r}(h))}{h} = L(\mathbf{r}'(0))$. Since $L(x,y) = f_x(0,0)x + f_y(0,0)y$, we have

$$\lim_{h\to 0} \frac{f(\mathbf{r}(h))}{h} = L(\mathbf{r}'(0)) = f_x(0,0)r_1'(0) + f_y(0,0)r_2'(0),$$

where $\mathbf{r} = \langle r_1, r_2 \rangle$, as desired.

Additional space