MATH 19 RECITATION
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BROWN UNIVERSITY
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1. Determine the radius of convergence of the Taylor series for $\sqrt{x}$ centered at $x = 1$.

$f(x) = x^{1/2}$
$f'(x) = \frac{1}{2}x^{-1/2}$
$f''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2}$
$f'''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$
$f^{(4)}(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-7/2}$

So the Taylor series is $f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}f^{(3)}(1)(x-1)^3 + \cdots$

$= 1 + \frac{1}{2}(x-1) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x-1)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x-1)^3 + \cdots$

You can figure out this expression by looking at the pattern

The ratio of the absolute value of the $(n+1)$st term to the n^{th} term is

$$\frac{\frac{|x-1|^{n+1}}{(n+1)!}\left|\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)\left(-\frac{2n-1}{2}\right)\right|}{\frac{|x-1|^n}{n!}\left|\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)\right|} = \frac{|x-1|}{(n+1)}\left(\frac{2n-1}{2}\right)$$

This converges to $|x-1|$ as $n \to \infty$, so the series converges in $(0,2)$ and diverges outside $[0,2]$. So the radius of convergence is $\boxed{1}$.

2. Use Taylor series to find the exact value of $\sum_{n=0}^{\infty} \frac{1}{2^n n!} + \sum_{n=0}^{\infty} \frac{n}{2^n}$.

We have $\sum_{n=0}^{\infty} 2^{-n}\frac{1}{n!} = e^{1/2}$, since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Also, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

$$\Rightarrow \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

Substituting $x = 1/2$ gives $\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2 = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$.

So the sum is $\boxed{\sqrt{e} + 2}$

3. Use Taylor series to find the 2016th derivative of $f(x) = e^{x^5}$ evaluated at $x = 0$.

$$e^{x^5} = 1 + x^5 + \frac{x^{10}}{2!} + \frac{x^{15}}{3!} + \cdots,$$

so the only k for which the kth derivative of e^{x^5} at 0 are nonzero are the multiples of 5. So the 2016^{th} derivative is $\boxed{0}$.

4. Find

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots$$

We start with the geometric series formula

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$\int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots\,dx$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Sub $x = 1$ to get

$$\arctan 1 = \boxed{\frac{\pi}{4}} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$