31 Jan 2016

Review (§1.2)

Recall our use of matrices to solve linear systems:

$$\begin{cases} 3x_1 - 2x_2 + x_3 = 2 \\ x_1 \qquad\quad + 6x_3 = 19 \\ \qquad x_2 + x_3 = 5 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 1 & 0 & 6 & 19 \\ 0 & 1 & 1 & 5 \end{array}\right)$$

we then row-reduce the matrix to get it to row echelon form (where leading entries are sorted by column & have only zeros below them). This system is equivalent & may be solved by back-substitution.

$$\xrightarrow[\cdots]{\text{row}2 \to \text{row}2 - \frac{1}{3}\cdot\text{row}1} \begin{pmatrix} 3 & -2 & 1 & 2 \\ 0 & 2/3 & 17/3 & \frac{55}{3} \\ 0 & 1 & 1 & 5 \end{pmatrix} \xrightarrow{\text{row}2 \to 3\cdot\text{row}2} \begin{pmatrix} 3 & -2 & 1 & 2 \\ 0 & 2 & 17 & 55 \\ 0 & 1 & 1 & 5 \end{pmatrix}$$

$$\xrightarrow{\text{switch } 2\&3} \begin{pmatrix} 3 & -2 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 17 & 55 \end{pmatrix} \xrightarrow[\text{row3}-2\cdot\text{row2}]{\text{row3}\to} \begin{pmatrix} 3 & -2 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 15 & 45 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{15}\text{row3}]{\text{row3}\to} \begin{pmatrix} 3 & -2 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Now we can back-substitute: $x_3 = 3$,

$x_2 = 5 - x_3 = 2$, $x_1 = \frac{1}{3}(2x_2 - x_3 + 2) = 1$

So $(1,2,3)$ is the unique solution

Reduced row echelon form

Alternatively, we could keep going & create more zeros:

$$\xrightarrow[\text{row2}-\text{row3}\ \cdots]{\text{row2}\to} \begin{pmatrix} 3 & -2 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow[\text{row1}+2\text{row2}]{\text{row1}\to} \begin{pmatrix} 3 & 0 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{\text{row1}\to\text{row1}-\text{row3}} \begin{pmatrix} 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow[\frac{1}{3}\text{row1}]{\text{row1}\to} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

This is called **reduced row echelon form**: all leading entries have zeros above them as well, and they're all 1.

Examples: $\begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Theorem Every matrix may be row reduced to a matrix in reduced row echelon form, and the RREF of a matrix is unique.

Solving a system in RREF

Let's backsolve this system:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 6 & 4 \\ 0 & 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We can let x_4 and x_5 be whatever, and then x_3 would have to be $x_3 = x_4 - 2x_5 + 3$.

We can also let x_2 be whatever, & then

x_1 would have to be $4-2x_2-3x_4-6x_5$.

Thus we may write the solution as

$$\left\{\left(4-2x_2-3x_4-6x_5,\ x_2,\ 3+x_4-2x_5,\ x_4,\ x_5\right) : x_2, x_4, x_5 \in \mathbb{R}\right\}$$

"is an element of"

"the set of real numbers"

means these 3 variables can take any real value

pivot columns

The variables x_1 and x_3 are called **basic** variables, and x_2, x_4, x_5 are called **free** variables.

other columns.

Theorem A consistent linear system has one solution if there are no free variables (i.e., every column is a pivot column) and infinitely many otherwise

Recall: consistent $\Longleftrightarrow$ no row like $(0\ 0\ \cdots\ 0\ \blacksquare)$

§1.3 Vectors

A matrix with one column is a column vector, or just 'vector.' The fundamental vector operations are addition:

$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$$

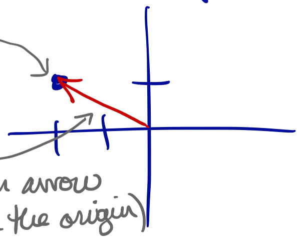
and scalar multiplication

a real number is called a scalar in this context

$$4 \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 28 \\ -4 \end{pmatrix}.$$

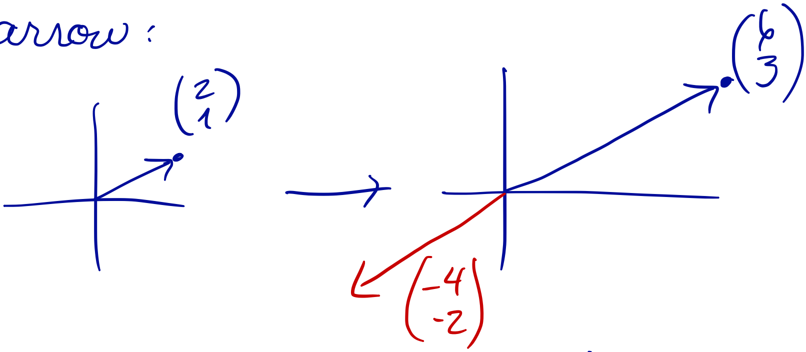
We associate an $n \times 1$ vector with a point in n-dimensional space:

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$



(or as an arrow from the origin)

then scalar multiplication scales (& flips, if the scalar is negative) the length of the arrow:



And vector addition works like this:

Note: we have $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$, $\vec{u}+(-\vec{u}) = \vec{0}$, (vector of all zeros)
& a bunch of other properties you'd expect.

Linear Combinations

Recipe analogy: $\vec{v}$'s are ingredients
c's are the amounts needed to obtain $\vec{w}$

If $$\vec{w} = c_1\vec{v_1} + c_2 v_2 + \cdots c_n \vec{v_n}$$

we say that $\vec{w}$ is a **linear combination** of $\{\vec{v}_1, \ldots, \vec{v_n}\}$ with **weights** $c_1, c_2, \ldots, c_n$.

Example Determine whether $\vec{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$ is a linear combination of $\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$ and $\vec{a}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$.

Solution: We seek a solution of

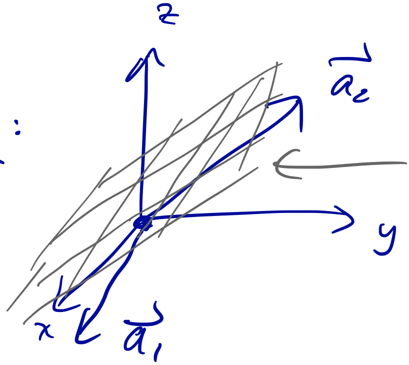
$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$$

i.e.

$$\begin{cases} x_1 + x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -5x_1 + 6x_2 = -3 \end{cases}$$

this is just the system w/ columns $\vec{a}_1, \vec{a}_2, \vec{b}$!! We can solve it [steps omitted] to get $x_1 = 3$ and $x_2 = 2$.

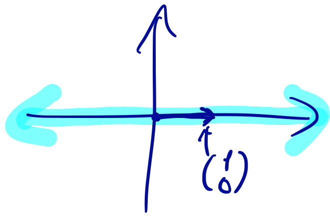
Graphically:



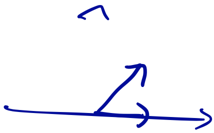
we figured out that $\vec{b}$ is in the plane containing $\vec{a}_1, \vec{a}_2 \,\&\, \vec{0}$

the set of all linear combinations of a collection of vectors is called its **span**. So:

$$\mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}\right) =$$

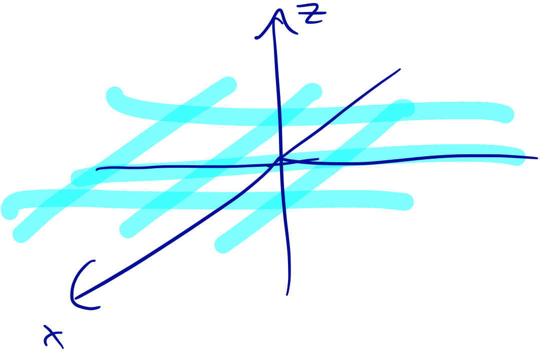


$$\mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}\right) =$$



every vector in $\mathbb{R}^2$ is a linear combination of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$

$$\mathrm{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right) =$$



← (xy plane)