MASSACHUSETTS INSTITUTE OF TECHNOLOGY
INTERPHASE CALCULUS III MIDTERM I
INSTRUCTOR: SAMUEL S. WATSON

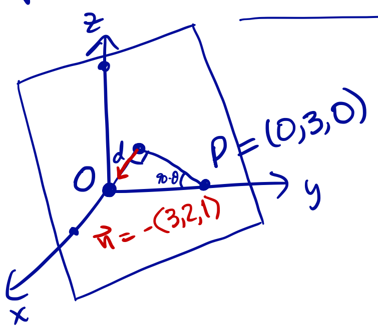
Solutions

Problem 1

Find the distance between the planes $3x + 2y + z = 0$ and $3x + 2y + z = 6$.

Solution

Since $\langle 3,2,1\rangle$ is parallel to $\langle 3,2,1\rangle$, the planes are parallel. Therefore, the distance to $3x+2y+z=6$ is the same for any point on $3x+2y+z=0$. Let's take $(0,0,0)$. Let θ be the angle between $\vec{n}=\langle 3,2,1\rangle$ and $\vec{PO}$, where $P=(0,3,0)$ is on $3x+2y+z=6$. Then



$$d = |\vec{PO}| \sin(90^\circ - \theta)$$
$$= |\vec{PO}| \cos\theta$$
$$= \frac{\vec{PO}\cdot\vec{n}}{|\vec{n}|}$$
$$= \frac{\langle 0,-3,0\rangle\cdot\langle -3,-2,-1\rangle}{\sqrt{3^2+2^2+1^2}}$$
$$= \frac{6}{\sqrt{14}}$$

Final answer:

$6/\sqrt{14}$

Problem 2

The position of a particle at time t is given by $\mathbf{r}(t) = \langle 2t, \frac{2}{t}, -t^2\rangle$. Find the positive time t when the velocity of the particle is perpendicular to its acceleration. You may express your answer as a radical.

Solution

We have

$$\vec{r}\,'(t) = \langle 2, -\tfrac{2}{t^2}, -2t\rangle$$
$$\vec{r}\,''(t) = \langle 0, \tfrac{4}{t^3}, -2\rangle,$$

so $0 = \vec{r}\,'(t)\cdot\vec{r}\,''(t) = \frac{-8}{t^5} + 4t$ when $t = \sqrt[6]{2}$.

Final answer:

$\sqrt[6]{2}$

Problem 3

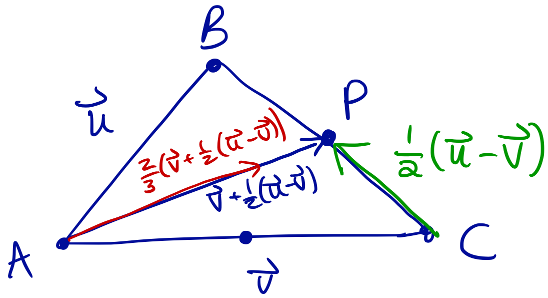
Suppose that ABC is a triangle. Define M to be the point which is two-thirds of the way from A to the midpoint of BC. Define N to be the point which is two-thirds of the way from B to the midpoint of AC.

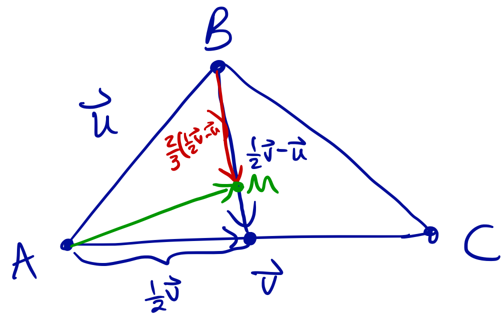
(a) Express the vector $\overrightarrow{AM}$ in terms of the vectors $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$.

(b) Express $\overrightarrow{AN}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

(c) What is the distance between M and N?

Solution





(a) $\overrightarrow{AM} = \frac{2}{3}\overrightarrow{AP} = \frac{2}{3}(\overrightarrow{AC}+\overrightarrow{CP}) = \frac{2}{3}(\vec{v}+\frac{1}{2}(\vec{u}-\vec{v})) = \frac{1}{3}\vec{u}+\frac{1}{3}\vec{v}.$

(b) $\overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{BN} = \vec{u} + \frac{2}{3}(\frac{1}{2}\vec{v}-\vec{u})$

$= \vec{u} + \frac{1}{3}\vec{v} - \frac{2}{3}\vec{u} = \frac{1}{3}(\vec{u}+\vec{v}).$

(c) $\overrightarrow{MN} = \overrightarrow{AM} - \overrightarrow{AN} = \vec{0}$, so $|\overrightarrow{MN}| = 0.$

Problem 4

Find a vector which is perpendicular to the line represented parametrically by $\langle 2-t, t, 4\rangle$ **and** the line represented parametrically by $\langle 1, t, 3t\rangle$.

Solution

Vectors parallel to these lines are $\langle -1,1,0\rangle$ & $\langle 0,1,3\rangle$, so:

$$\vec{n} = \langle -1,1,0\rangle \times \langle 0,1,3\rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = \langle 3,3,-1\rangle$$

$\vec{n}$ orthogonal to both.

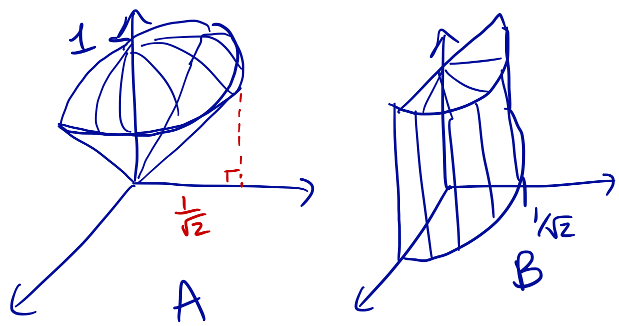
Final answer:

$\langle 3,3,-1\rangle$

Problem 5

Region A consists of all the points in 3D space satisfying the spherical coordinate in equalities $\rho \leq 1, 0 \leq \theta \leq \pi$, and $0 \leq \phi \leq \frac{\pi}{4}$. Region B consists of all the points in 3D space satisfying the cylindrical coordinate inequality $r \leq \frac{1}{\sqrt{2}}$, $0 \leq \theta \leq \pi$, and $0 \leq z \leq 1$. Without calculating any volumes, which region is larger? Sketch both regions carefully and explain how you can be sure one is larger than the other without calculating the volume of either.

Solution



Because the largest r value for region A is $\frac{1}{\sqrt{2}}$, region A is contained within B. So B's volume is larger

Final answer:

B

Problem 6

(a) Find $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$ and (b) explain a winning strategy for the limit game, to demonstrate that your answer to (a) is correct. (Note: for (b), this means saying how δ should be chosen in terms of ϵ to ensure that the function is within ϵ of its limit for all (x, y) within δ of $(0, 0)$.)

Solution

(a) $\frac{xy^3}{x^2+y^2} = \frac{r^4 \cos\theta \sin^3\theta}{r^2} = r^2 \cos^3\theta \sin\theta$ in polar,

& this is of the form (something going to 0)(something bounded)

so the limit is $\boxed{0}$.

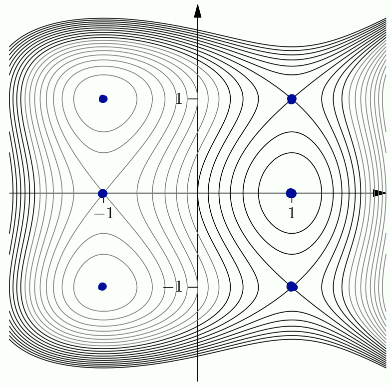
(b) Since the adversary's choice of r is constrained by $r < \delta$, choosing $\delta = \sqrt{\varepsilon}$ gives

$$\begin{aligned} |f(x,y)| &= |r^2 \cos\theta \sin^3\theta| \\ &\leq r^2 \quad \text{since } |\cos\theta \sin^3\theta| \leq 1 \\ &< (\sqrt{\varepsilon})^2 = \varepsilon \end{aligned}$$

for all (x,y) within δ of $(0,0)$.

Problem 7

(a) Find the critical points of $f(x,y) = -x^3 + 3x + y^4 - 2y^2$ and place dots at those locations in the figure below (which depicts some level curves of f).



(b) To find the maximum value of $f(x,y)$ for any point (x,y) satisfying $-2 \le x \le 2$ and $-2 \le y \le 2$, is it sufficient to look at the values of f at the critical points found in (a)? Explain why or why not.

Solution

(a) $\vec{\nabla} f = \langle -3x^2+3, 4y^3-4y \rangle = \vec{0}$

when $x \in \{\pm 1\}$ and $y \in \{-1,0,1\}$. So we have six critical points, shown above.

(b) No. Boundary critical points must also be taken into account.

Problem 8

Find a vector which is *tangent* to the hyperboloid $x^2 - y^2 + z^2 = 18$ at the point $(x,y,z) = (3,4,5)$. Hint: first find the vector normal to the hyperboloid at that point.

Solution

$\vec{\nabla}(x^2-y^2+z^2) = \langle 2x, -2y, 2z \rangle = 2\langle 3,-4,5 \rangle$, so $\langle 3,-4,5 \rangle$ is a vector normal to $x^2-y^2+z^2=18$ at $(3,4,5)$. So $\langle 4,-3,0 \rangle$, e.g., dots with $\langle 3,-4,5 \rangle$ to give 0 & so is tangent to the plane.

Final answer:

$\langle 4,-3,0 \rangle$

[or any $\vec{v} \neq \vec{0}$ with $\vec{v} \cdot \langle 3,-4,5 \rangle = 0$]

Additional space

Additional space