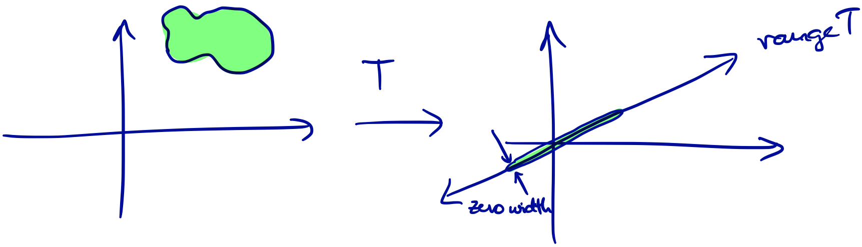
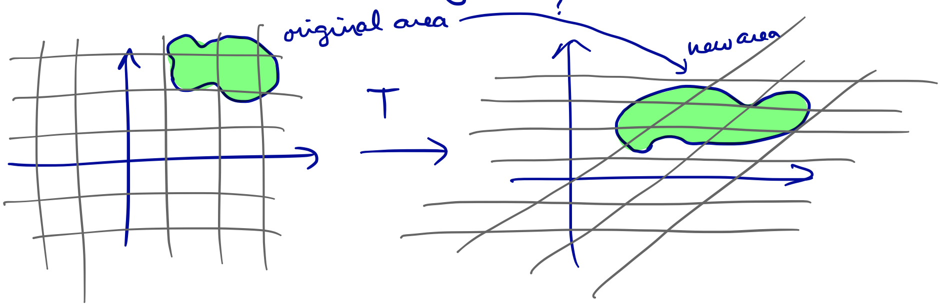
§ 3.1 Determinants

7 Mar 2017

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and not bijective, then it squishes any positive area region down to a <u>zero</u>-area region.



But what if T is bijective?



You can see that every small square has its area distorted by the same amount. Since any region can be approximated with small squares, this means that

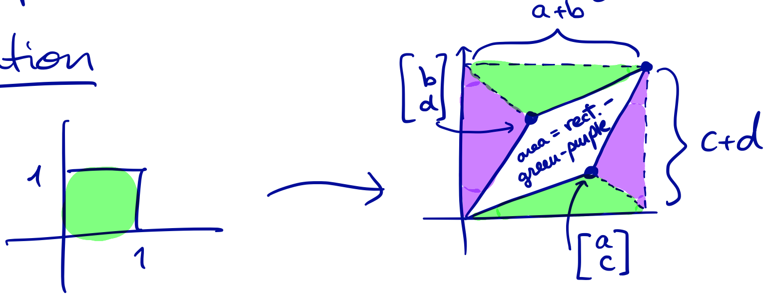
☆ all areas are distorted by the same factor. ☆

We call the factor by which T transforms areas the **determinant** of T, or $\det T$.

(warning: we will modify this defn slightly.)

Example Find $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ using the above defn.

Solution



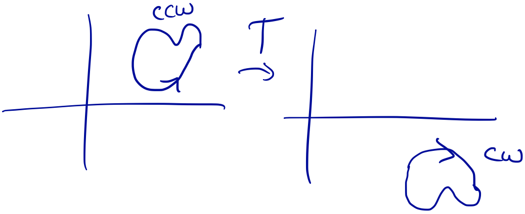
$$\text{area}(T(\text{unit square})) = (a+b)(c+d) - (c+d)b - (a+b)c$$

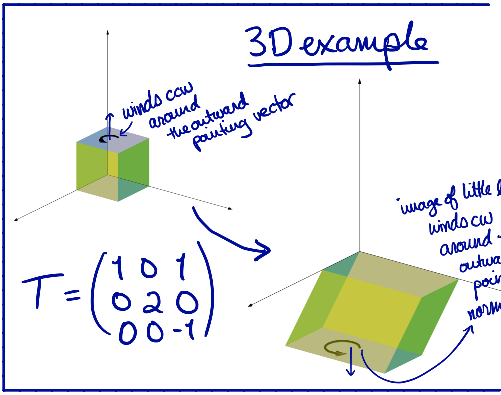
$$= ad - bc.$$

Example Find $\det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Solution $ad - bc = -1$. But that's negative! Note that

$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ is a reflection across the x-axis. So this negative sign is telling us that T <u>reverses orientations</u>:



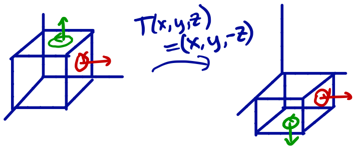


Since this sign gives us some useful info about T apart from its area distortion factor, we keep it.

Def^n (for real) The **determinant** $\det T$ of a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^n$ is $\underbrace{\text{volume}(T([0,1]^n))}$, times -1 if T reverses orientations.

= the factor by which T transforms volumes. Note: 2D vol = area, 1D vol = length

in higher dim. than 2, means that it reverses orientations of small pictures on the faces, w.r.t. "outward"



$\det A$ = factor by which A transforms oriented volumes

Cofactor Expansion of $\det A$

Here's a way to calculate $\det T$ from T's matrix.

① checkerboard sign: $\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$

② matrix entry

③ "minors"

$$\det\begin{pmatrix}1&\cancel{2}&\cancel{3}\\\cancel{4}&5&6\\\cancel{7}&8&10\end{pmatrix} = +1\det\begin{pmatrix}5&6\\8&10\end{pmatrix} - 2\det\begin{pmatrix}4&6\\7&10\end{pmatrix} + 3\det\begin{pmatrix}4&5\\7&8\end{pmatrix}$$

$$\begin{pmatrix}\cancel{1}&2&\cancel{3}\\4&\cancel{5}&6\\7&\cancel{8}&10\end{pmatrix} \nearrow \qquad \begin{pmatrix}\cancel{1}&\cancel{2}&3\\4&5&\cancel{6}\\7&8&\cancel{10}\end{pmatrix} \nearrow$$

$$= -3$$

the checkerboard sign times the determinant of the minor is called a cofactor, so this is called the 'cofactor expansion' of the determinant.

It doesn't matter the row/column we expand along.

It is TERRIBLE from a computational point of view

```
In [1]: function bad_det(A)
            if size(A,1) == 1
                return A[1,1]
            else
                return sum((-1)^(j+1)*A[1,j]*bad_det([A[2:end,1:j-1] A[2:end,j+1:end]]) for j=1:size(A,1))
            end
        end
```

checkerboard sign — appropiate minor — 1st row, jth column

```
Out[1]: bad_det (generic function with 1 method)
```

```
In [2]: A = [rand() for i=1:10,j=1:10];
```

← random 10×10 matrix

```
In [5]: @time det(A)
```

← built-in det

fast !!

```
0.000014 seconds (11 allocations: 1.297 KB)
Out[5]: 0.037052234870275196
```

```
In [6]: @time bad_det(A)
```

← the one we just made

slow !!

```
3.828347 seconds (97.16 M allocations: 3.666 GB, 13.72% gc time)
Out[6]: 0.03705223487027519
```

Example Cofactor-expand

$$\begin{vmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & -1 & 7 \\ 0 & 2 & 4 & 9 \\ 0 & 0 & 6 & 1 \end{vmatrix}.$$

← alternate notation for det

Solution Expand along first row:

$$0+0+0+(-2)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{vmatrix}$$

then along first column:

$$(-2)\left[1(2\cdot 6-0\cdot 4)+0+0\right]$$

$$=(-2)(1)(2)(6)=\boxed{-24}.$$