# Brown University
## Math 0350 Practice Midterm I
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Name:

### Problem 1

How many hemispheres $H$ have the property that (i) $H$ is a subset of the unit sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, and (ii) $H$ is the graph of some real-valued function $f$ defined on a subset of $\mathbb{R}^2$? Explain your answer.

### Solution

There are exactly two such hemispheres: the hemisphere of points above the $x$-$y$ plane and the hemisphere of points below it. Any other hemisphere has the property that there are vertical lines (that is, lines parallel to the $z$-axis) which intersect the graph at multiple points. Such a set cannot be the graph of a function, since a function has at most one output for every input.

Final answer:

$$2$$

### Problem 2

Suppose that $f(x, y) = (ax + by, cx + dy)$ has the property that $f(x, y)$ is the point obtained by rotating the point $(x, y)$ by 42 degrees counterclockwise about the origin. Find $bc - ad$.

### Solution

Note that $bc - ad$ is equal to $-1$ times the determinant of $f$. Furthermore, the determinant of $f$ is equal to 1, since rotations preserve area and orientation. Therefore, $bc - ad = -1$.

Final answer:

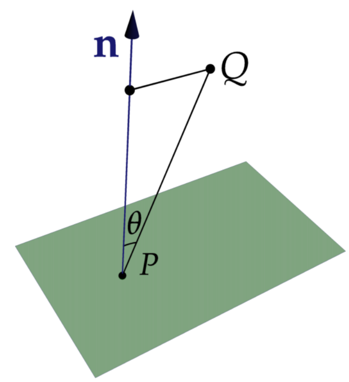
$$-1$$

**Problem 3**

Find the distance from the plane $3x + 2y + z = 6$ to the line passing through the point $(3, 4, 5)$ and parallel to the vector $\langle -2, 3, 0 \rangle$.

**Solution**

First, we should check that the line is parallel to the plane. The vector $\mathbf{n} := \langle 3, 2, 1 \rangle$ is normal to the plane, and this vector dotted with $\langle -2, 3, 0 \rangle$ gives zero, so the line is indeed parallel to the plane.




Since the line and plane are parallel, the distance between them is equal to the distance from $Q$ to the plane, where $Q$ is any point on the line. Let's take $Q = (3, 4, 5)$ for convenience, and let $P$ be a point on the plane, say $(1, 1, 1)$. Define $\theta$ to be the angle between $\langle 3, 2, 1 \rangle$ and the vector from $P$ to $Q$.

Forming the right triangle with $PQ$ has the hypotenuse and one leg along $\mathbf{n}$, as shown in the figure, we see that the desired distance is equal to $|\overrightarrow{PQ}||\cos\theta|$, which is equal to $|\overrightarrow{PQ} \cdot \mathbf{n}|/|\mathbf{n}|$. So we get a distance of

$$\frac{\langle 3, 2, 1 \rangle \cdot \langle 2, 3, 4 \rangle}{\sqrt{3^2 + 2^2 + 1^2}} = \frac{6 + 6 + 4}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

between the planes.

Final answer:

$$\frac{16}{\sqrt{14}}$$

**Problem 4**

The curve in $\mathbb{R}^3$ represented parametrically in $\mathbb{R}^3$ by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ is called the *twisted cubic*. Find a point on the twisted cubic at which the tangent line is parallel to the vector $\langle 4, 16, 48 \rangle$.

**Solution**

We have $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. This vector is parallel to $\langle 4, 16, 48 \rangle$ if and only if the equation

$$\langle 1, 2t, 3t^2 \rangle = c\langle 4, 16, 48 \rangle$$

has a solution $c \in \mathbb{R}$. The equality of first components tells us that $c = 1/4$, and from there we can use the second coordinate to find that $t = 2$. The third components are also equal with these values of $c$ and $t$. So the desired point is $\langle 2, 2^2, 2^3 \rangle = \langle 2, 4, 8 \rangle$.

Final answer:

$$\langle 2, 4, 8 \rangle$$

### Problem 5

The parallelogram law states that the sum of squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides of the parallelogram. Use vectors to prove the parallelogram law. (Hint: represent the two sides as **a** and **b**, and represent the two diagonals in terms of **a** and **b**.)

### Solution

Suppose that **a** and **b** are vectors with a common tail at one of the vertices of the parallelogram which extend to the two adjacent vertices of the parallelogram, respectively. Then the two diagonals of the parallelogram are $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$. Using linearity and the relationship between dot products and norms, we calculate

$$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})+(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= |\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b} \\ &= 2(|\mathbf{a}|^2+|\mathbf{b}|^2), \end{aligned}$$

as desired.

### Problem 6

Suppose that $f$ is a differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ with the property that $f_x = 1-4y\sin(2x)$ and $f_y = 2\cos 2x$.

(a) Explain why this is not enough information to approximate $f(0.1, 0.02)$.

(b) Approximate $f(0.1, 0.02) - f(0,0)$ using the linear approximation of $f$.

### Solution

(a) No information is given about $f(0,0)$. In fact, for any function with the given partial derivatives, we can add any constant we wish to obtain another function with the same partial derivatives. Therefore, $f(0.1, 0.02)$ could be any number whatsoever.

(b) We have $f(0.1,0.02) - f(0,0) \approx f_x(0,0)(0.1) + f_y(0,0)(0.02) = (1)(0.1) + (2)(0.02) = \boxed{0.14}$

### Problem 7

Your friend slices her spherical orange into 8 congruent pieces using three cuts. Write a system of inequalities in spherical coordinates whose graph is shaped like one of her pieces.

### Solution

There are various possible answers here, but one would be (for any positive number $R$, which represents the radius of the orange):

$$\rho \leq R, \quad 0 \leq \theta \leq \pi/2, \quad \phi \leq \pi/2.$$

### Problem 8

(a) Determine whether each statement is true or false, and explain your answer.

(i) If every "$z =$ constant" slice of a surface is a circle, then the surface must be a sphere.

(ii) The graph of $z = 2x^2 - 2y^2$ is shaped like a saddle.

(iii) If every "$x =$ constant" slice of a quadric surface is a parabola, then every "$y =$ constant" slice is a parabola.

(b) Sketch the graph of $z(z-1)(z+1) = x^2 + y^2$.

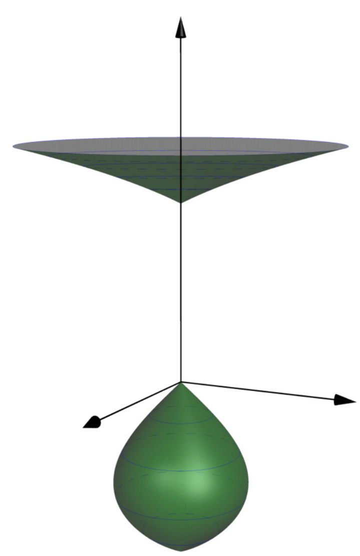
### Solution

(a) (i) This is false. Consider $z = x^2 + y^2$, which is a paraboloid.

(ii) This is true; $z = 2x^2 - 2y^2$ is shaped like a saddle because it has one down-turned parabolic slice (where $x = 0$) and an another up-turned parabolic slice (where $y = 0$).

(iii) This is false; consider $z + y^2 = x$, whose intersection with every level set of the $x$ function is a parabola and whose intersection with every level set of the $y$ function is a plane.

(b) The intersection of the graph with the plane $z = c$ is a circle of radius $\sqrt{c}$ if $c \geq 0$ and is empty if $c < 0$. The polynomial $z(z-1)(z+1)$ is non-negative if and only if $z \geq 1$ or $-1 \leq z \leq 0$. So the graph looks like the figure shown.



**Problem 9**

Determine each of the following limits or explain why it doesn't exist.

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x^2+y^4}$ (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^2+y^2}$

**Solution**

(a) The function $\frac{xy}{x^2+y^2}$ is obtained from the coordinate functions $x$ and $y$ by the continuous operations of multiplication, addition, and reciprocation. Therefore, it is continuous everywhere the denominator is nonzero. This implies that the limit exists and is equal to the value of the function at $(1,1)$, which is $1/2$.

(b) The restriction of the function to $x = 0$ is $f(0,y) = 1/y^2$, which does not converge as $y \to 0$. Therefore, the limit of the function does not exist at the origin.

(c) The value of the function at $(r\cos\theta, r\sin\theta)$ is equal to $r^3\cos^3\theta\sin^2\theta$. Since $|r^3\cos^3\theta\sin^2\theta| \le r^3$, we can win the limit game by choosing $\delta = \epsilon^{1/3}$.

## Problem 10

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with the property that $f\left(\frac{1}{n}, \frac{1}{\sqrt{n}}\right) = 0$ for all positive integers $n$. Explain in rigorous terms why it cannot be true that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

## Solution

The adversary can win the limit game by choosing $\epsilon = 1/2$. Then no matter which $\delta$ is chosen, the adversary can choose some integer $n$ large enough that $\sqrt{1/n^2 + 1/n} < \delta$ and select $(x,y) = (1/n, 1/\sqrt{n})$. This point wins the game, because $f(x,y) = 0$, which is not within $\epsilon$ of $L = 1$.

Alternatively, the minimum value of $f$ on the ball of radius $r$ centered at the origin satisfies $m(r) \leq 0$ for all $r$, since $f$ takes on the value 0 for some pairs $(x,y)$ in the ball of radius $r$. Since $m(r) \leq 0$ for all $r$, it is not possible that $m(r)$ converges to 1.