BROWN UNIVERSITY
MATH 0350 MIDTERM
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Name:

Problem 1

In this problem, we will use vectors to show that an angle formed by connecting a point on a circle to two diametrically opposite points on the same circle is a right angle.

(a) Use vector addition/subtraction/scaling to label all three unlabeled vectors in the figure below in terms of **a** and/or **b**. Write your answers directly on the figure.

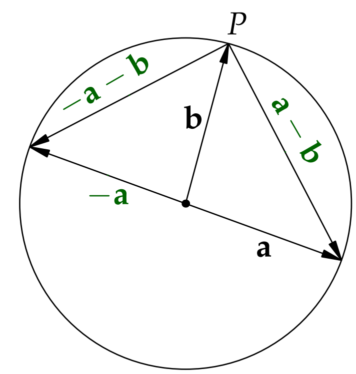
(b) Use part (a) to show that the dot product of the two vectors in the figure whose tails are at P is equal to zero.

Solution

We have

$$(-\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = -|\mathbf{a}|^2 - \mathbf{b}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 = \mathbf{0},$$

since **a** and **b** have the same length, as radii of the same circle.



Problem 2

The position of a particle at time t is given by $\mathbf{r}(t) = \langle 2t, \frac{2}{t}, -t^2\rangle$. Find the positive time t when the velocity of the particle is perpendicular to its acceleration. You may express your answer as a radical.

Solution

We have $\mathbf{r}'(t) = \langle 2, -2/t^2, -2t\rangle$ and $\mathbf{r}''(t) = \langle 0, 4/t^3, -2\rangle$. The dot product of these two vectors is $-8/t^5 + 4t$, which is equal to zero when $t = \sqrt[6]{2}$.

Final answer:

$$\sqrt[6]{2}$$

Problem 3

Consider the six points $(-1,1,0)$, $(1,-1,0)$, $(-1,0,1)$, $(1,0,-1)$, $(0,1,-1)$, and $(0,-1,1)$.

(a) Show that these six points are coplanar. (Hint: find the plane P passing through three of them, and then show that the other three also lie in this plane).

(b) Find the distance from the plane P to the line **segment** from $(4,3,1)$ to $(-2,-3,4)$. (Hint: think carefully about this one.)

Solution

(a) The vector from the first point to the second point is $\langle 2,-2,0\rangle$, and the vector from the first point to the third point is $\langle 0,-1,1\rangle$. The cross product of these two vectors is $\langle -2,-2,-2\rangle$. Therefore, the equation of the plane passing through the first three points is

$$\langle -2,-2,-2\rangle \cdot \langle x-(-1), y-1, z-0\rangle = 0,$$

which is equivalent to $x+y+z=0$. We see that the coordinates of the remaining three points sum to zero, so all six points are indeed coplanar.

(b) If we calculate

$$\langle -2,-2,-2\rangle \cdot \langle x-(-1), y-1, z-0\rangle$$

at the triple $(4,3,1)$, we get -14. Since the dot product of two vectors is equal to the product of the lengths of the vectors and the cosine of the angle between them, the sign of -14 implies that $(4,3,1)$ is on the side of the plane *opposite* to the direction in which $\langle -2,-2,-2\rangle$ points.

Evaluating the same expression at $(-2,-3,4)$, we get a dot product of 10. This means that $(-2,-3,4)$ is on the side of the plane where $\langle -2,-2,-2\rangle$ is pointing. Therefore, the two points are on opposite sides of the plane, and the line segment connecting them passes through the plane. Therefore, the desired distance is $\boxed{0}$.

Final answer:

0

Problem 4

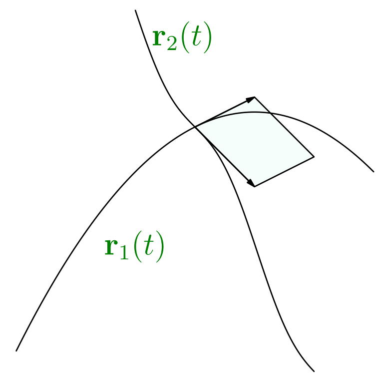
The figure below shows the two curves $\mathbf{r}_1(t) = \langle t, \frac{1}{2}t - \frac{1}{4}t^2 \rangle$ and $\mathbf{r}_2(t) = \langle t, -2t + \frac{1}{3}\sin 3t \rangle$, which intersect at the origin.

(a) Which path is $\mathbf{r}_1$ and which is $\mathbf{r}_2$? Just label them in the figure.

(b) Find $\mathbf{r}_1'(0)$ and $\mathbf{r}_2'(0)$ (these are the two vectors shown).

(c) Use the cross product to find the area of the parallelogram which has $\mathbf{r}_1'(0)$ and $\mathbf{r}_2'(0)$ as two of its sides.

Solution



(b) We have $\mathbf{r}_1'(t) = \langle 1, \frac{1}{2} - \frac{1}{2}t \rangle$ and $\mathbf{r}_2'(t) = \langle 1, -2 + \cos 3t \rangle$. Substituting $t = 0$ gives $\langle 1, \frac{1}{2} \rangle$ and $\langle 1, -1 \rangle$.

(c) The cross product of $\langle 1, \frac{1}{2}, 0 \rangle$ and $\langle 1, -1, 0 \rangle$ is $\langle 0, 0, -5/2 \rangle$. Therefore, the area of the parallelogram is $\boxed{\frac{3}{2}}$.

Problem 5

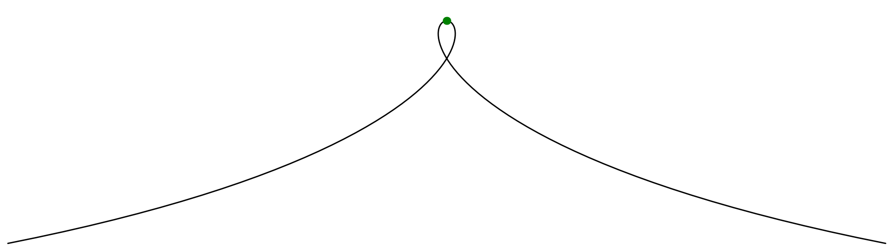
(a) Place a dot at the point of maximum curvature for the smooth path shown (no calculation!).

(b) The parametric equation for this path is $\mathbf{r}(t) = \langle 2t^3 - t, 5\cos(t) \rangle$. The path's curvature turns out to be

$$\kappa(t) = \frac{1}{\left(\left(6t^2-1\right)^2 + 25\sin^2 t\right)^2} \sqrt{25\left(6t^2-1\right)^2\left(6t^2\cos t - 12t\sin t - \cos t\right)^2 + \left(12t\left(\left(6t^2-1\right)^2 + 25\sin^2 t\right) + \left(-6t^2+1\right)\left(72t^3 - 12t + \frac{25}{2}\sin(2t)\right)\right)^2}.$$

Explain what steps you would take to arrive at the above formula for $\kappa(t)$. Just state the necessary formulas and name any intermediate quantities you would calculate; there is no need to do **any** computation.

Solution



We define the unit tangent vector $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ in terms of $\mathbf{r}$, and then $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$.

Problem 6

Each of the 40 tentacles in this piece of Chihuly-inspired graphic art is a random rotation of the path $\mathbf{r}(t) = \langle \sin t \cos t, \sin^2 t, t \rangle$ as t varies from 0 to 2. Find the **total** length of all the tentacles.

Hint: the integrand gets pretty tidy pretty fast. If the integration step is not trivial, go back and check your work.

(Credit to Oliver Knill for this problem concept)

Solution

We have

$$|\mathbf{r}'(t)|^2 = (-\sin^2 t + \cos^2 t)^2 + (2 \sin t \cos t)^2 + 1^2 =$$
$$\sin^4 t - 2\sin^2 t \cos^2 t + \cos^4 t + 4 \sin^2 t \cos^2 t = (\sin^2 t + \cos^2 t)^2 + 1 = 2,$$

so the length of one tentacle is $\int_0^2 \sqrt{2}\, dt = 2\sqrt{2}$. So the total length is $80\sqrt{2}$.

Final answer:

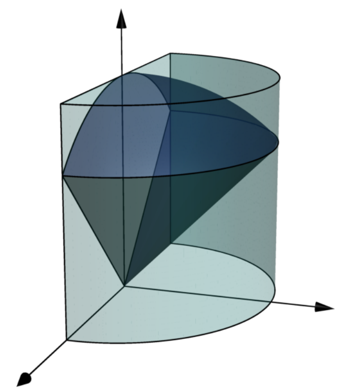
$$80\sqrt{2}$$

Problem 7

Region A consists of all the points in 3D space satisfying the spherical coordinate inequalities $\rho \leq 1, 0 \leq \theta \leq \pi$, and $0 \leq \phi \leq \frac{\pi}{4}$. Region B consists of all the points in 3D space satisfying the cylindrical coordinate inequalities $r \leq \frac{1}{\sqrt{2}}$, $0 \leq \theta \leq \pi$, and $0 \leq z \leq 1$. Without calculating any volumes, which region is larger? Sketch both regions carefully and explain how you can be sure one is larger than the other without calculating the volume of either.

Solution

Region A is the midnight blue region and region B is the light sea green region in the figure shown. Region B is clearly larger, because Region A is a subset of Region B.



To see this, we note that if a point has a ρ value which is at most 1 and a ϕ value which is at most $\pi/4$, the largest possible r value for that point occurs when $\rho = 1$ and $\phi = \pi/4$. The r value for such a point is $\rho \sin \phi = \frac{1}{\sqrt{2}}$. Similarly, the z value for a point in region A is bounded above by 1. Since the restrictions on θ are the same, we conclude that any point in Region A is indeed in Region B.

Problem 8

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with the property that, for all real numbers m, we have $\lim_{t \to 0} f(t, mt) = 6$. Answer the following questions and **use complete sentences** to clearly explain your reasoning.

(a) Based on this information, what can be determined about the value of $f(0,0)$? (In other words, for which values of $b \in \mathbb{R}$ does there exist a function satisfying the above properties and also $f(0,0) = b$?)

(b) If the limit of $f(x,y)$ as $(x,y) \to (0,0)$ exists and is equal to L, then what is the value of L?

(c) Suppose that f is continuous. Find $\lim_{t \to 0} f(t^2, \sin t)$.

(d) Suppose that f is continuous. Explain why $\lim_{t \to 0} f(t^2, \cos t)$ cannot be determined using the given information.

Solution

(a) Nothing can be said about $f(0,0)$. All the limiting statements $\lim_{t \to 0} f(t, mt) = 6$ pertain only to values of t which are nonzero, which means that they are all unaltered if the value of $f(0,0)$ is changed.

(b) The limiting value is equal to the common limiting value along all paths passing through the origin. Therefore, if the limit exists its value must be $\boxed{6}$.

(c) Because f is continuous, it is continuous at the origin. Because it is continuous at the origin, the limit of f exists at the origin. From (b), this means that the limit is 6. Since the path $\langle t^2, \sin t \rangle$ passes through the origin, the limit along that path must be equal to the limit of f at the origin, which is $\boxed{6}$.

(d) The path $\langle t^2, \cos t \rangle$ does not pass through the origin, so we don't have any information about the values of the function along that path.

Problem 9

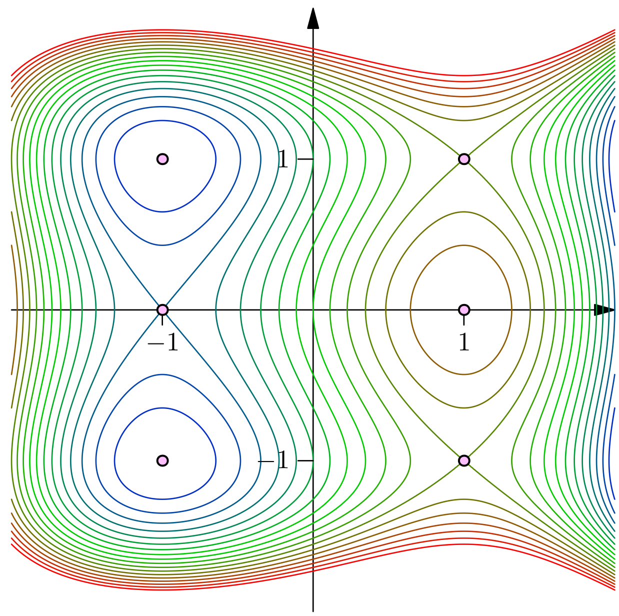
(a) Find the critical points of $f(x,y) = -x^3 + 3x + y^4 - 2y^2$ and place dots at those locations in the figure shown (which depicts some level curves of f).

(b) To find the maximum value of $f(x,y)$ for any point (x,y) satisfying $-2 \le x \le 2$ and $-2 \le y \le 2$, is it sufficient to find the maximum among the values of f at the critical points found in (a)? Explain why or why not.

(c) Apply the second derivative test to classify each of the critical points of f.

Solution

(a) The partial derivatives of f are $-3x^2 + 3$ and $4y^3 - 4y$, respectively. Setting these quantities equal to zero, we find $x \in \{-1, 1\}$ and $y \in \{-1, 0, 1\}$. Therefore, there are six critical points as shown in the figure.



(b) Some continuous functions on a closed and bounded domain realize their maxima on the boundary of the domain. Therefore, to find the maximum and minimum of the function, we must consider both the critical points on the inside of the domain *and* the values of the function on the boundary.

(c) We have $f_{xx}(x,y) = -6x$, $f_{yy}(x,y) = 4(3y^2 - 1)$, and $f_{xy}(x,y) = 0$. Therefore, $D = -24x(3y^2 - 1)$. So we get:

a	b	$f_{xx}(a,b)$	$D(a,b)$	classification
-1	-1	6	48	local min
1	-1	-6	-48	saddle point
-1	0	6	-24	saddle point
1	0	-6	24	local max
-1	1	6	48	local min
1	1	-6	-48	saddle point,

since the second derivative test says that an ordered pair is a saddle point if $D < 0$ and a local min or max, respectively, if $f_{xx} > 0$ and $D > 0$ or $f_{xx} < 0$ and $D > 0$.

Problem 10

Suppose that $2x + 3y + 2z = 9$ is the equation of the plane tangent to the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(1, 1, 2)$.

(a) Rewrite the equation of the tangent plane in the form $z = a(x-1) + b(y-1) + c$, where a and b are numbers.

(b) Use (a) to find $(\partial_x f)(1, 1)$ and $(\partial_y f)(1, 1)$.

(c) Find $L(0.99, 1.02)$, where L is the linear approximation of f at the point $(1, 1)$.

Solution

(a) We get $z = \frac{9}{2} - x - \frac{1}{3}y$. To get it in the desired form, we can write

$$\frac{9}{2} - x - \frac{3}{2}y = \frac{9}{2} - (x-1) - 1 - \frac{3}{2}(y-1) - \frac{3}{2} = 2 - (x-1) - \frac{3}{2}(y-1).$$

(b) The partial derivatives of f at a point are the coefficients of x and y in the linearization of f at the point, so in this case they are -1 and $-\frac{3}{2}$, respectively.

(c) We substitute into the answer for (a) to get $2 - (-0.01) - (1.5)(0.02) = \boxed{1.98}$.

BONUS 1 (0 points)

Find the third-order Maclaurin polynomial of

$$x^3y^3 + 6x^3y^2 - 8x^3y - 9x^3 - 4x^2y^3 - 10x^2y^2 + x^2y + 6x^2 + 8xy^3 - 4xy^2 - 2xy + 8x + 4y^3 - 9y^2 - 11$$

Solution

We can just drop the terms of order higher than 3:

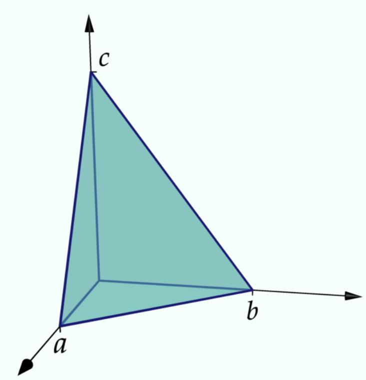
$$\boxed{-9x^3 + x^2y + 6x^2 + 8xy^3 - 4xy^2 - 2xy + 8x + 4y^3 - 9y^2 - 11}$$

This polynomial matches all of the original function's derivatives of order up to 3.

BONUS 2 (0 points)

Prove the 3D Pythagorean theorem, for tetrahedrons with a trirectangular vertex (that is, a vertex where all three incident faces have a right angle): *the sum of the squares of the areas of the three smallest faces is equal to the square of the area of the largest face.*

You may work a tetrahedron with vertices at the origin, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, as shown.



Solution

The sum of the squares of the areas of the three smallest faces is $\frac{1}{4}(a^2c^2 + a^2b^2 + b^2c^2)$. The "hypotenuse" face has the vectors $\langle a, -b, 0\rangle$ and $\langle -a, 0, c\rangle$ as two of its sides. The cross product of these vectors is $-\langle bc, ac, ab\rangle$. So the sqaured area of the largest face is the square of one-half of norm of this vector, which is equal to $\frac{1}{4}(a^2c^2 + a^2b^2 + b^2c^2)$.

Additional space

Additional space