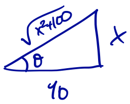
Solutions

MATH 19 PRACTICE MIDTERM I
FALL 2016
BROWN UNIVERSITY
SAMUEL S. WATSON

$\boxed{1}$ Find $\displaystyle\int_0^{\sqrt{44}} \left(\frac{x}{\sqrt{x^2+100}} - 2 \right) dx$.

We have:



$x = 10\tan\theta$

$dx = 10\sec^2\theta \, d\theta$

$$\int \frac{x}{\sqrt{x^2+100}} dx = \int \frac{10\tan\theta}{\sqrt{100\tan^2\theta+100}} (10\sec^2\theta \, d\theta)$$

$$= 10 \int \tan\theta \sec\theta \, d\theta$$

$$= 10\sec\theta$$

$$= \sqrt{x^2+100}.$$

So

$$\int_0^{\sqrt{44}} \left(\frac{x}{\sqrt{x^2+100}} - 2 \right) dx = \sqrt{x^2+100}\Big|_0^{\sqrt{44}} - 2x\Big|_0^{\sqrt{44}}$$

$$= 12 - 10 - 2\sqrt{44}$$

$$= 2 - 2\sqrt{44}.$$

2 Find any function $f(x)$ such that

$$\int_0^{2\pi} f(x)\cos 2x\,dx = \frac{1}{3}$$
$$\int_0^{2\pi} f(x)\sin 3x\,dx = 4$$
$$\int_0^{2\pi} f(x)(\sin x + \sin 2x)\,dx = 11.$$

We choose $f(x) = a\cos 2x + b\sin 3x + c\sin x$, where a, b, c are to be determined.

$$\int_0^{2\pi} f(x)\cos 2x\,dx = a\pi = \frac{1}{3},$$

because $\int_0^{2\pi} \sin 3x \cos 2x\,dx = 0 = \int_0^{2\pi} \sin x \cos 2x\,dx.$

Similarly, $\int_0^{2\pi} f(x)\sin 3x\,dx = b\pi = 4$

and $\int_0^{2\pi} f(x)(\sin x + \sin 2x)\,dx = c\pi = 11.$

So $\boxed{f(x) = \frac{\cos 2x}{3\pi} + \frac{4\sin 3x}{\pi} + \frac{11\sin x}{\pi}}$ works.

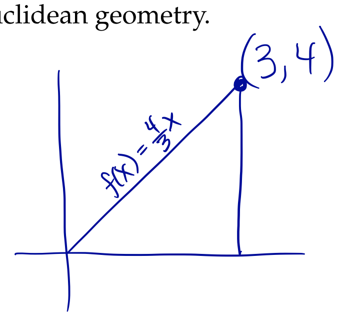
In fact $\frac{\cos 2x}{3\pi} + \frac{4\sin 3x}{\pi} + c\sin x + d\sin 2x$ works

whenever $c + d = \frac{11}{\pi}$.

$\boxed{3}$ Find the length of the line segment L from the origin to $(3,4)$ two ways:

a) by defining a function $f(x)$ whose graph over $[0,3]$ is L, and using the arclength integral formula applied to f, and

(b) using Euclidean geometry.



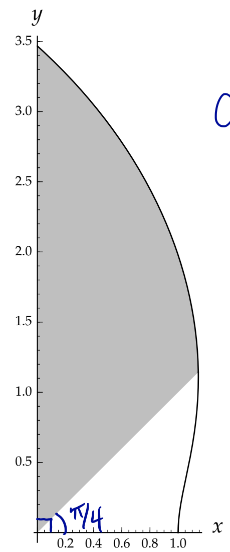
(a) $\int_0^3 \sqrt{1+f'(x)^2}\,dx = \int_0^3 \sqrt{1+\left(\frac{4}{3}\right)^2}\,dx$

$= \int_0^3 \sqrt{\frac{25}{9}}\,dx$

$= \left(\frac{5}{3}\right)(3) = \boxed{5}.$

(b) $\sqrt{3^2+4^2} = \boxed{5}$

4 The graph of the polar coordinate equation $r = 1 + \theta^2$ over $0 \le \theta \le \frac{\pi}{2}$ is shown below. Find the area of the portion of the region enclosed by this curve which lies above the line $y = x$. (In other words, find the area of the shaded region.)



$$\text{area} = \int_{\pi/4}^{\pi/2} \frac{1}{2}(1+\theta^2)^2 \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{2} + \theta^2 + \frac{\theta^4}{2} \, d\theta$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \left(\frac{\theta^3}{3} + \frac{\theta^5}{10}\right)\Bigg|_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{8} + \frac{1}{3}\left(\left(\frac{\pi}{2}\right)^3 - \left(\frac{\pi}{4}\right)^3\right) + \frac{1}{10}\left(\left(\frac{\pi}{2}\right)^5 - \left(\frac{\pi}{4}\right)^5\right)$$

$$= \frac{\pi}{8} + \frac{\pi^3}{3}\left(\frac{1}{8} - \frac{1}{64}\right) + \frac{\pi^5}{10}\left(\frac{1}{32} - \frac{1}{1024}\right)$$

$$= \frac{\pi}{8} + \frac{7\pi^3}{3 \cdot 64} + \frac{\pi^5}{10} \cdot \left(\frac{31}{1024}\right)$$

$$\begin{array}{r} 64 \\ \underline{3} \\ 192 \end{array}$$

$$= \boxed{\frac{\pi}{8} + \frac{7\pi^3}{192} + \frac{31\pi^5}{10240}}$$

$\boxed{5}$ Find all complex values of z satisfying the equation $(z-i)^3 = 8$. Express your answers in $a+bi$ form.

$(z-i)^3 = 8 \iff z-i$ is a cube root of 8.

the cube roots of 8 are of the form $r\,\text{cis}\,\theta$

where

$$(r\,\text{cis}\,\theta)^3 = 8 \iff$$

$$r^3\,\text{cis}\,3\theta = 2^3\,\text{cis}\,0 \iff$$

$$r = 2 \text{ and } 3\theta = 360^\circ k \text{ for } k \in \mathbb{Z}.$$

So we get $z-i = 2$ or $z - i = 2\,\text{cis}\,120^\circ = -1+\sqrt{3}\,i$

$\Rightarrow z = \boxed{2+i}$ $\qquad$ $\Rightarrow \boxed{z = -1+(\sqrt{3}+1)i}$

or $z-i = 2\,\text{cis}\,240^\circ$

$= -1-\sqrt{3}\,i$

$\Rightarrow \boxed{z = -1+(1-\sqrt{3})\,i}$

6 Find a function f which satisfies all of the following equations.

$$f''(x) - f(x) = 0$$
$$f''(x) - 3f'(x) + 2f(x) = 0$$
$$f(0) = 13.$$

$\boxed{\begin{array}{l} \lambda^2 - 1 = 0 \\ \Rightarrow \lambda = 1, -1 \end{array}}$ $f''(x) - f(x) = 0 \Rightarrow f(x) = Ae^x + Be^{-x}.$

$\boxed{\begin{array}{l} \lambda^2 - 3\lambda + 2 = 0 \\ (\lambda - 2)(\lambda - 1) = 0 \\ \lambda = 1, 2 \end{array}}$ $f''(x) - 3f'(x) + 2f(x) = 0 \Rightarrow f(x) = Ae^x + Be^{2x}$

The only way a function can fit both of these forms is if it is a multiple of e^x. So

$$f(x) = Ae^x.$$

then $f(0) = 13 \Rightarrow \boxed{f(x) = 13e^x}.$