MATH 520 PRACTICE MIDTERM I
SPRING 2017
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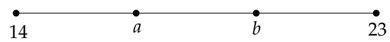
Key

This is a pen-and-paper-only exam. You have two hours.

1 Suppose that that value associated with each interior node in the figure below is equal to the average of the two adjacent nodes.



(a) Write two linear equations that must be satisfied by a and b.

$$\frac{14+b}{2} = a \qquad \frac{a+23}{2} = b$$

(b) Write this system of equations in augmented matrix form and row reduce it to solve for a and b.

$$\begin{aligned} 2a - b &= 14 \\ -a + 2b &= 23 \end{aligned} \rightarrow \begin{pmatrix} 2 & -1 & 14 \\ -1 & 2 & 23 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & -23 \\ 2 & -1 & 14 \end{pmatrix}$$

$$\begin{array}{r} 46 \\ -14 \\ \hline 32 \end{array} \qquad \sim \begin{pmatrix} 1 & -2 & -23 \\ 0 & 3 & 60 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & -23 \\ 0 & 1 & 20 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 17 \\ 0 & 1 & 20 \end{pmatrix}$$

So $\boxed{a = 17 \text{ and } b = 20.}$

2 For each of the following statements, indicate whether it is true or false. If it is false, give a counterexample demonstrating that it is false.

F (a) Given a matrix A, there is only one matrix in row echelon form which is row equivalent to A.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

both row echelon form,
& row equivalent

T (b) Given a matrix A, there is only one matrix in reduced row echelon form which is row equivalent to A.

F (c) If the column in an augmented matrix corresponding to the variable x_3 is not a pivot column, then there are necessarily infinitely many solutions to the corresponding system of equations.

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \leftarrow \text{no sol.}$$

not a pivot column

F (d) Every linear map from $\mathbb{R}^{10}$ to $\mathbb{R}^5$ is surjective but not injective.

$T(\vec{x}) = \vec{0}$ is neither surjective nor injective

T (e) The transpose of A times B is equal to the transpose of B times the transpose of A.

$\boxed{3}$ Determine whether the columns of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

are linearly independent.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can see that the third column is not a pivot column, so $A\vec{x} = \vec{0}$ will have nontrivial solutions. Thus the columns of A are <u>linearly dependent</u>.

$\boxed{4}$ Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation which maps $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Find the matrix which represents T.

If $T(\vec{x}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a+2b \\ c+2d \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -a+b \\ -c+d \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

so $\begin{cases} a+2b=0 \\ -a+b=3 \end{cases} \Rightarrow 3b=3 \Rightarrow b=1, \ a=-2$

$\begin{cases} c+2d=-1 \\ -c+d=0 \end{cases} \Rightarrow 3d=-1, \text{ so } d=-\frac{1}{3}, \ c=-\frac{1}{3}.$

so $\boxed{\begin{pmatrix} -2 & 1 \\ -1/3 & -1/3 \end{pmatrix}}$

$\boxed{5}$ Suppose that running for one hour burns 450 calories and costs \$1 (worth of wear-and-tear on your shoes). Suppose that cycling for one hour burns 350 calories and costs \$3 (worth of bicycle maintenance cost).

(a) Write down a vector expression that represents the calories burned and cost of running r hours and cycling c hours.

$$r\begin{pmatrix} 450 \\ 1 \end{pmatrix} + c\begin{pmatrix} 350 \\ 3 \end{pmatrix}$$

(b) Write down a vector equation satisfied by the numbers r and c such that running r hours and cycling c hours burns 16000 calories and costs 80 dollars.

$$r\begin{pmatrix} 450 \\ 1 \end{pmatrix} + c\begin{pmatrix} 350 \\ 3 \end{pmatrix} = \begin{pmatrix} 16000 \\ 80 \end{pmatrix}$$

(c) Rewrite the vector equation from (b) as a matrix equation (that is, a matrix of the form $A\mathbf{x} = \mathbf{b}$).

$$\begin{pmatrix} 450 & 350 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} r \\ c \end{pmatrix} = \begin{pmatrix} 16000 \\ 80 \end{pmatrix}$$

$\boxed{6}$ Find a matrix equation of the form $A\mathbf{x} = \mathbf{b}$ whose solution set is equal to

$$\left\{ \begin{bmatrix} 2-2x_3 \\ -4x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\}.$$

We reverse the procedure used to infer the solution from the rref:

$$\boxed{\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 4 & 0 \end{bmatrix}}$$

$\boxed{7}$ (a) What does it mean to say that a matrix transformation is *one-to-one* (injective)? What does it mean to say the matrix transformation is *onto* (surjective)? Any correct definitions are acceptable.

$T(\vec{x}) = A\vec{x}$ is surjective if for every $\vec{b} \in \mathbb{R}^m$, there is an $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{b}$

$T(\vec{x}) = A\vec{x}$ is injective if $T(\vec{x}) = T(\vec{y})$ occurs only when $\vec{x} = \vec{y}$.

(b) Show that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation which is both one-to-one and onto, then $m = n$.

'one-to-one' requires a pivot in every column of T's matrix,

& 'onto' requires a pivot in every row.

So # pivots = # rows and

\# pivots = # columns,

So # rows = # columns.

$\boxed{8}$ Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent list of vectors in $\mathbb{R}^{50}$, then the list $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ of vectors in $\mathbb{R}^{50}$ is also linearly dependent.

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent,

there exist c_1, c_2, c_3 not all zero with

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

Thus $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + 0\vec{v}_4 + 0\vec{v}_5 + 0\vec{v}_6 = \vec{0}$,

& $\{c_1, c_2, c_3, 0, 0, 0\}$ are not all zero. So

$\{\vec{v}_1, \ldots, \vec{v}_6\}$ is not linearly independent.