16 Mar 2017

Linear Transformations on Vector Spaces

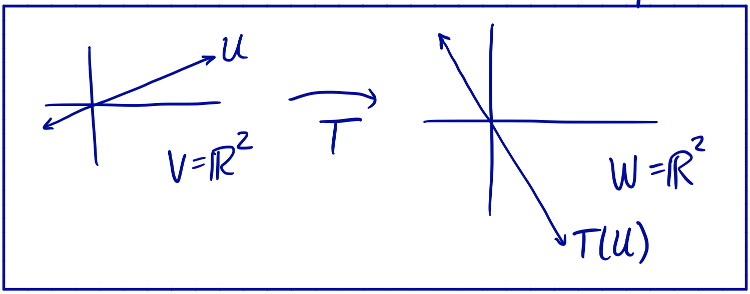
We can define the notion of a linear transformation from one vector space to another exactly as we did for maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let V, W be vector spaces:

Def$^{\underline{n}}$ A function $T: V \to W$ is a **linear transformation** if (i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$, and (ii) $T(c\vec{u}) = cT(\vec{u})$ for all $c \in \mathbb{R}$, $\vec{u} \in V$.

Example $T(\vec{x}) = A\vec{x}$, where A is a matrix

Example $T: \mathbb{P}_n \to \mathbb{P}_n$ defined by $T(p) =$ the derivative of p. We know T is linear from calculus, or it can be checked directly for polynomials, if you prefer. E.g., $T(1+2t+3t^4) = 2+12t^3$.

Theorem Suppose U is a subspace of V and $T: V \to W$ is linear. Then $T(U)$ is a subspace of W.



"Why?"

<u>Proof</u> We need to show that (i) $\vec{0} \in T(U)$, (ii) $T(U)$ is closed under vector addition, and (iii) $T(U)$ is closed under scalar multiplication.

the zero vector in V

We have $\vec{0} = T(\vec{0})$, and $\vec{0} \in U$ because U is

the zero vector in W

a subspace of V, so $\vec{0} \in T(U)$. If $\vec{x}, \vec{y} \in T(U)$, then there are $\vec{u}, \vec{v} \in U$ with $T(\vec{u}) = \vec{x}$ and $T(\vec{v}) = \vec{y}$, by definition. Then $\vec{u} + \vec{v} \in U$ because U is a subspace, and $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{x} + \vec{y}$. So $\vec{x} + \vec{y} \in T(U)$. Same idea for $c\vec{x} \in T(U)$.

§ 4.1 Linear Independence in abstract vector spaces

The definition of linear independence applies verbatim to an abstract vector spaces:

Def^n If V is a vector space and $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a list of vectors in V, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is **linearly independent** if

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0} \overset{\text{"implies"}}{\Rightarrow} c_1 = c_2 = \cdots = c_n = 0.$$

Example (i) The vectors $\{1, t, t^2, \dots, t^{10}\}$ in $C([0,1])$ are linearly <u>independent</u> because there is no linear combination of them that equals $\overset{\text{the zero function}}{\text{zero}}$, except the one with all zero weights. To see this, note that a $\overset{\text{not all zero weights}}{\text{nontrivial}}$ linear combination of them is a degree-n polynomial, where, $0 \le n \le 10$. Such a polynomial equals zero at 10 values of t <u>at most</u>! (ii) $\{1, \sin^2 t, \cos^2 t\}$ is linearly <u>dependent</u> since $1 = \sin^2 t + \cos^2 t \Rightarrow$

$$1 + \sin^2 t + \cos^2 t = \overset{\text{the zero function on } [0,1]}{0}.$$

Intuitively, if we build up a list of vectors one by one without ever inserting a redundant vector, the resulting list should be linearly independent:

Lemma (linear dependence lemma) $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent iff there is some j so that $\vec{v}_j \in \mathrm{Span}(\{\vec{v}_1, \dots, \vec{v}_{j-1}\})$

Proof If $\vec{v}_j \in \mathrm{Span}(\{\vec{v}_1, \dots, \vec{v}_{j-1}\})$ for some j, then $\vec{v}_j = c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1}$ for some $c_1, \dots, c_{j-1}$, and that means $\vec{0} = -\vec{v}_j + c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1}$. So the list is linearly independent.

Conversely, if $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$ for some $c_1, \dots, c_n$ not all zero, then we can define j so c_j is the last nonzero coefficient. Then

$$\vec{v}_j = -c_j^{-1}(c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1}) \in \mathrm{Span}(\{\vec{v}_1, \dots, \vec{v}_{j-1}\}),$$

as desired.

#1 **Proposition** If $\{\vec{v}_1, \dots, \vec{v}_n\}$ spans V (a vector space), then $\{\vec{w}_1, \dots, \vec{w}_{n+1}\}$ is linearly dependent, for any $\vec{w}_1, \dots, \vec{w}_{n+1} \in V$.

#2 **Proposition** If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent vectors and $\vec{v}_1, \dots, \vec{v}_n \in V$, then $\{\vec{w}_1, \dots, \vec{w}_{n-1}\}$ does not span V, for any $\vec{w}_1, \dots, \vec{w}_{n-1} \in V$.

Proofs For #1, we begin with the list

$$\{\vec{w}_1, \vec{v}_1, \dots, \vec{v}_n\}$$

which spans V since it includes all the v's.

It's also linearly dependent because $\vec{w} \in \text{Span}(\{v_1, \dots, v_n\})$.

So we can find some vector from this list which is in the span of the vectors before it (by the linear dependence lemma), and removing such a vector does not change the span of the list [because you can replicate the removed vector using other ones. like if you have chili powder, cumin, paprika & oregano, you don't need taco seasoning]. Doing this, tacking $\vec{w}_2$ on the front and repeating, we will eventually either find some $\vec{w}$ that is a linear combination of some other $\vec{w}$'s (if a $\vec{w}$ is ever removed), or we'll arrive at $\{\vec{w}_n, \vec{w}_{n-1}, \dots, \vec{w}_1\}$. In that case, $\vec{w}_{n+1} \in \text{Span}(\{\vec{w}_1, \dots, \vec{w}_n\}) = V$ (so $\{\vec{w}_1, \dots, \vec{w}_{n+1}\}$ is l.d.).

Now #2 follows from #1, because it's the contrapositive of #1. Said another way,

$\underbrace{\text{contrapositive}}$: the contrapositive of $A \Rightarrow B$ is (not B) $\Rightarrow$ (not A), & is logically equivalent

#1 says "any spanning list is at least as long as any linearly independent list", and that's what #2 says too.

Def$^{\underline{n}}$ A **basis** of a vector space V is a linearly independent spanning list

Examples

(·) $\mathbb{P}_3$ has $\{1, t, t^2, t^3\}$ as a basis.

(·) $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is the "standard" basis of $\mathbb{R}^n$

(•) <u>any</u> 3 linearly independent vectors form a basis of $\mathbb{R}^3$.

(•) $C([0,1])$ and $\mathbb{R}^\infty$ do not have any finite spanning lists. So they do not have finite bases. ← "BAYSS-eez"

<u>Proof</u> $(1,0,\dots),(0,1,0,\dots),\dots$ in $\mathbb{R}^\infty$

$\{1,t,t^2,t^3,\dots\}$ in $C([0,1])$ ← infinite linear independent lists

(•) Any linearly independent list is a basis of the subspace generated by it (that is, the span of the list).

<u>Fact</u> Any two bases of V contain the same number of vectors.

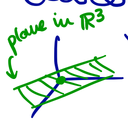
<u>Proof</u> If B_1 and B_2 are bases with m and n vectors, respectively, then $m \geq n$ because B_1 spans and B_2 is lin. ind. Similarly, $m \leq n$. So $m = n$.

Def^{n} The **dimension** of a vector space is the number of vectors in a basis of the vector space, if there is a finite basis. Otherwise, it's ∞.

Examples

(*) $\dim \mathbb{P}_n = n+1$, since $\{1, t, t^2, \ldots, t^n\}$ is a basis of $\mathbb{P}_n$

(*) $\dim C([0,1]) = \infty$, since $C([0,1])$ has no finite basis

(*) $\dim\Big($ 

$\Big) = 2$, because two linearly ind. vectors in the plane forms a basis for it.