§1.4 Matrix-vector products

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It is useful to have a more direct way to say "think of this matrix as representing a system of equations," with the variables included explicitly. We do this by putting them in a column to the side:

$$\begin{aligned} 2x_1 - x_2 \quad &= 5 \\ 7x_1 \quad + x_3 &= 0 \end{aligned} \quad \longrightarrow \quad \begin{pmatrix} 2 & -1 & 0 \\ 7 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

matrix-vector product!

To facilitate this, we define the **product** of an $m \times n$ matrix and an $n \times 1$ vector to be the $m \times 1$ vector obtained as a linear combination of the columns of the matrix with weights given by the entries of the vector. So, for example,

$$\begin{pmatrix} 2 & -1 & 0 \\ 7 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} x_1 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_3$$

$$= \begin{pmatrix} 2x_1 - x_2 \\ 7x_1 + x_3 \end{pmatrix}.$$

We can also think of it like this:

$$\begin{pmatrix} 2 & -1 & 0 \\ 7 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + (-1)x_2 + 0x_3 \\ \end{pmatrix}$$

multiply in pairs & sum a.k.a. "dot"

$$\begin{pmatrix} 2 & -1 & 0 \\ 7 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \\ 7x_1 + 0x_2 + 1x_3 \end{pmatrix}$$

Note that for $m \times n$ times $k \times 1$ to work, we need $n = k$:

$$(m \times n)(n \times 1) = m \times 1$$

must match!

then get gobbled up

Example Calculate $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$.

<u>Solution</u>

$$\begin{pmatrix} 1(-1) + 2(0) + 3(2) + 4(0) \\ 5(-1) + 6(0) + 7(2) + 8(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

← column vectors of A

Theorem If $A = (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n)$ and $\vec{b} \in \mathbb{R}^m$

↖ $\vec{b}$ is an $m \times 1$ vector

then

$$A\vec{x} = \vec{b}$$ 1: matrix equation

is equivalent to $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$ 2: vector equation

and to the system with augmented matrix $(\vec{a}_1 \ldots \vec{a}_n\ \vec{b})$. 3: system of linear equations

Fact $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is in the span of A's columns.

Note: this follows directly from the theorem above.

Example Is $A\vec{x} = \vec{b}$ consistent for all $\vec{b} \in \mathbb{R}^3$, if $A = \begin{pmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{pmatrix}$?

Solution

We can row reduce

$$\begin{pmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{pmatrix}$$

by hand or using Julia [with the SymPy package, for handling variables].

```
In [1]: using SymPy
        using IntroLinearAlgebra

In [2]: @vars b_1 b_2 b_3 # declare b_1, b_2, b_3
                          # to be symbols

Out[2]: (b_1,b_2,b_3)

In [3]: M = [1 3 4 b_1; -4 2 -6 b_2; -3 -2 -7 b_3]
```

Out[3]: $\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$

```
In [4]: M = rowadd(M,2,1,4)
```

Out[4]: $\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$

```
In [5]: M = rowadd(M,3,1,3)
```

Out[5]: $\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix}$

```
In [6]: M = rowadd(M,3,2,-1//2)
```

Out[6]: $\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{b_2}{2} + b_3 \end{bmatrix}$

Since $b_1 - \frac{1}{2}b_2 + b_3$ is not always zero, the system is <u>not</u> always consistent. ∎

Note that the only way A could 'stop' what happened in this example is by having a pivot in each row. So:

Theorem The following are equivalent:

a matter of definition

(a) The columns of A span $\mathbb{R}^m$

(b) Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A

(c) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$

interesting! → (d) A has a pivot position in each row

Linearity

The matrix-vector product is <u>linear</u>, meaning that:

① $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$, for all $A, \vec{u}, \vec{v}$, and

② $A(c\vec{u}) = cA\vec{u}$ for all $A, \vec{u}$, and c.

Why? If $A = (\vec{a}_1 \cdots \vec{a}_n)$, then
$$\begin{aligned} A(\vec{u}+\vec{v}) &= \vec{a}_1(u_1+v_1) + \cdots \vec{a}_n(u_n+v_n) \\ &= (\vec{a}_1 u_1 + \vec{a}_1 v_1) + \cdots + (\vec{a}_n u_n + \vec{a}_n v_n) \\ &= A\vec{u} + A\vec{v}. \end{aligned}$$

Similarly for $A(c\vec{u}) = c(A\vec{u})$.

Exercise Suppose the columns of a 3×3 matrix A do not span $\mathbb{R}^3$. Show that $A\vec{x}=\vec{b}$ has either 0 solutions or infinitely many solutions.

Exercise Suppose that the columns of a 5×8 matrix A span $\mathbb{R}^5$. How many solutions does $A\vec{x}=\vec{b}$ have, for any $\vec{b}$?

Exercise What is the minimum number of vectors needed to span $\mathbb{R}^4$?

Exercise Find a column of

$$\begin{pmatrix} 2 & 0 & 4 & 7 \\ -1 & 3 & 2 & 4 \\ -6 & 5 & 1 & 1 \end{pmatrix}$$

that can be removed & still have the remaining columns span $\mathbb{R}^3$.

rows, columns

```
rref(M[:,[true,true,true,false]])
# colon means "take 'em all"
# array TTTF means "keep 1,2,3, drop 4"
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$