Math 520 Final Exam (alt.)
Spring 2017
Brown University
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Name: Solutions

Allowed materials are pen, pencil, and straightedge. You have three hours.

Problem 1 (8 points)

All of the 2×2 matrices with entries in $\{0, 1\}$ are shown below. How many of these matrices are row equivalent to the 2×2 identity matrix?

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution

We calculate determinants ↑

So 6 of them are invertible.

Note: a random $n \times n$ matrix with $\{0,1\}$ entries is extremely likely to be invertible when n is large, like even $n = 20$.

Final answer:

6

Problem 2 (5 points)

Find a vector $\mathbf{b}$ with the property that the equation $A\mathbf{x} = \mathbf{b}$ has no solutions, where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & -1 & -1 \\ -1 & 3 & 2 \end{bmatrix}$$

Show how you came up with your $\mathbf{b}$, and explain why $A\mathbf{x} = \mathbf{b}$ has no solutions for your $\mathbf{b}$.

Solution

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 \\ -1 & 3 & 2 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 \\ -3 & 9 & 6 & 3b_3 \end{array}\right] \sim \begin{bmatrix} 3 & -1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 \\ 0 & 8 & 8 & 3b_3+b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 \\ 0 & 0 & 0 & 3b_3+b_1+8b_2 \end{bmatrix}.$$

So we choose $\vec{b}$ so that $3b_3+b_1+8b_2 \neq 0$. like

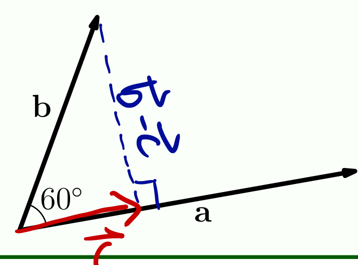
$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Final answer:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 3 (5 points)

Consider the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in the figure below. Denote by $\mathbf{c}$ the orthogonal projection of $\mathbf{b}$ onto (the line spanned by) $\mathbf{a}$. Find $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c})$.



Solution

By def^{n}, $\vec{a}$ is orthogonal to $\vec{b} - \vec{c}$. So

$$\vec{a} \cdot (\vec{b} - \vec{c}) = 0.$$

Final answer:

0

Problem 4 (7 points)

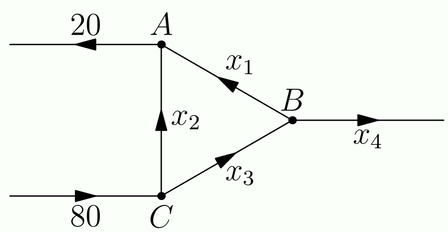
Suppose that the first, third, and seventh columns of a 3×7 matrix A are linearly independent. Show that the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is surjective.

Solution

the range of T is a dimension-3 subspace of $\mathbb{R}^3$, since $\{T(e_1), T(e_3), T(e_7)\}$ is linearly independent. Thus range $T = \mathbb{R}^3$, & so T is surjective.

Problem 5 (7 points)

Find the general solution for the traffic flow diagram below. Assuming all flows are nonnegative, what is the largest possible value for x_1?



Solution

We have

$$\begin{array}{ll} (A) & x_1 + x_2 = 20 \\ (B) & -x_1 + x_3 - x_4 = 0 \\ (C) & x_2 + x_3 = 80 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 20 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 80 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 80 \\ 0 & 1 & 1 & -1 & 20 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 80 \\ 0 & 0 & 0 & -1 & -60 \end{array}\right]. \quad \text{So } x_4 = 60$$

x_3 free, $x_2 = 80 - x_3$, $x_1 = 20 - (80 - x_3)$
$= x_3 - 60.$

Then $x_3 \leq 80 \Rightarrow x_1 \leq 80 - 60 = \boxed{20}$. ← & we can get 20 by letting $x_3 = 80$.

Problem 6 (6 points)

The matrix $U = \begin{bmatrix} \frac{3\sqrt{11}}{11} & \frac{13\sqrt{231}}{693} \\ 0 & \frac{\sqrt{231}}{21} \\ -\frac{\sqrt{11}}{11} & \frac{2\sqrt{231}}{99} \\ -\frac{\sqrt{11}}{11} & \frac{25\sqrt{231}}{693} \end{bmatrix}$ is orthogonal, which means that $U^T U = I$. Your friend says you to "if we take the transpose of both sides of this equation, we get $UU^T = I$. I believe that U and U^T are inverses of each other." *Both* of your friend's statements are incorrect. Explain why.

Solution

(i) $(U^TU)^T = U^T(U^T)^T = U^TU$, not UU^T.

(ii) U is not square and thus has no inverse. UU^T and U^TU don't even have the same dimension.

Problem 7 (8 points)

An $n \times n$ matrix A is said to be a *square root* of an $n \times n$ matrix B if $A^2 = B$. Find a square root of $A = \begin{bmatrix} 9 & 10 \\ 0 & 4 \end{bmatrix}$.

Solution

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2+bc & ab+bd \\ \underbrace{ac+dc} & bc+d^2 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ 0 & 4 \end{bmatrix}$$

either $a=-d$ or $c=0$.

Try $c=0$:

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}^2 = \begin{bmatrix} a^2 & (a+d)b \\ 0 & d^2 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ 0 & 4 \end{bmatrix}$$

$\Rightarrow a=3, d=2, b=2$

Final answer:

$$\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$$

Note: $a=-d$ would have implied $9=4$, so that doesn't work.

Problem 8 (6 points)

Find $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$ $=: A^{-1}$. (Note: you don't need to write out every intermediate step; just the first couple and the final answer.)

Solution

$$[A \mid I] \sim \left[\begin{array}{cccccc|cccccc} 1&1&1&1&1&1&1&0&0&0&0&0 \\ 0&1&1&1&1&1&0&1&0&0&0&0 \\ 0&0&1&1&1&1&0&0&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0&0&1&-1 \\ 0&0&0&0&0&1&0&0&0&0&0&1 \end{array}\right] \quad \text{row5} \leftarrow \text{row5} - \text{row6}$$

$$\sim \left[\begin{array}{cccccc|cccccc} 1&1&1&1&1&1&1&0&0&0&0&0 \\ 0&1&1&1&1&1&0&1&0&0&0&0 \\ 0&0&1&1&1&1&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0&1&-1&0 \\ 0&0&0&0&1&0&0&0&0&0&1&-1 \\ 0&0&0&0&0&1&0&0&0&0&0&1 \end{array}\right] \quad \text{row4} \leftarrow \text{row4} - \text{row5}$$

$\vdots$

etc.

Final answer:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 9(a) (4 points)

Consider the polynomial function

$$D(a,b,c,d,e,f,g,h,i) = aei - afh - bdi + bfg + cdh - ceg.$$

Find $D(3,4,5,6,8,10,114,561,-111)$. (Hints: yes, this is a linear algebra question, and no, you don't have to do much calculation).

Solution

$$D(3,4,5,6,8,10,114,561,-111) = \begin{vmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \\ 114 & 561 & -111 \end{vmatrix}$$

$$= 0,$$

since rows 1 & 2 are linearly dependent.

Final answer:

0

Problem 9(b) (3 points)

Suppose that A is an $n \times n$ matrix. Use determinants to show that $A^T A$ is invertible if and only if A is invertible. (Note: be sure to argue both directions: if $A^T A$ is invertible then A is invertible, and if A is invertible then $A^T A$ is invertible.)

Solution

$$\det(A^T A) = \det(A^T)\det A = (\det A)^2,$$

so $A^T A$ invertible $\Rightarrow (\det A)^2 \neq 0 \Rightarrow \det A \neq 0 \Rightarrow A$ invertible,

& A invertible $\Rightarrow \det A \neq 0 \Rightarrow (\det A)^2 \neq 0 \Rightarrow A^T A$ invertible.

Problem 10(a) (6 points)

Consider the set $S = \{f \in C([0,1]) : f(x) \notin [17,20] \text{ for all } x \in [0,1]\}$. This is the set of continuous functions from $[0,1]$ to $\mathbb{R}$ which do not realize any values between 17 and 20. Show that S is not a subspace of $C([0,1])$.

Solution

$f(x) = x$ on $[0,1]$ is in S, but

$100f \notin S$ because $(100f)(0.2) = 20$.

Problem 10(b) (6 points)

Define S' to be the set of functions f in $C([0,1])$ with the property that there exists a number K for which

$$f(y) - f(x) \leq K(y-x)$$

for all $0 \leq x < y \leq 1$. Show that (i) S' contains the zero function and (ii) $S'$is closed under function addition. (Note: two functions f and g might have different K's, which you might want to call K_f and K_g to keep them separate; your goal for (ii) will be to find a new K that works for $f+g$).

Solution

(i) $0(y) - 0(x) = 0 \leq K(y-x)$, for any $K \geq 0$.

(ii) $f(y) - f(x) \leq K_f(y-x)$

$+ \quad g(y) - g(x) \leq K_g(y-x)$

$(f+g)(y) - (f+g)(x) \leq (K_f + K_g)(y-x)$,

So $f+g \in S'$ with a K value of $K_f + K_g$

Problem 11 (7 points)

Let us say that the *deficiency* of an eigenvalue of a matrix A is the difference between (a) its multiplicity as a root of the characteristic polynomial of A and (b) the dimension of its eigenspace. Find the eigenvalue with the **greatest** deficiency for the following matrix. Explain your reasoning clearly.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

e'vals: 2, 4, 1

2: $A-2I$ has <u>one</u> non-pivot column,
so $\text{def}(2) = 3-1 = 2$.

4: $A-4I$ has <u>4</u> non-pivot columns,
so $\text{def}(4) = 5-4 = 1$.

1: $A-I$ has 2 nonpivot columns,
so $\text{def}(1) = 2-2 = 0$.

Final answer:

2

Problem 12 (6 points)

Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}\}$ is a linearly independent list of vectors in $\mathbb{R}^{20}$ with the following properties:

1. each of the last seven vectors is in the orthogonal complement of the span of the first three, and
2. the last seven vectors form an orthogonal list.

Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{10}\}$ be the orthogonal list obtained by applying the Gram-Schmidt procedure to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}\}$. Show that $\mathbf{b}_k = \mathbf{v}_k$ for all $k = 1, 4, 5, 6, 7, 8, 9, 10$.

Solution

$b_1 = v_1$ because that's how GS always starts.

Now

$$b_4 = v_4 - \text{proj}_{b_1} v_4 - \text{proj}_{b_2} v_4 - \text{proj}_{b_3} v_4$$

$$= v_4 - 0 - 0 - 0$$

$$= v_4,$$

since $\text{span}(\{b_1, b_2, b_3\}) = \text{span}(\{v_1, v_2, v_3\})$ and v_4 is orthogonal to that span. Then

$$b_5 = v_4 - \text{proj}_{b_1} v_4 - \text{proj}_{b_2} v_4 - \text{proj}_{b_3} v_4 - \text{proj}_{b_4} v_5$$

$$= v_4 - \underbrace{0 - 0 - 0}_{\text{same reason}} - 0$$

b/c $\{v_4, \ldots, v_{10}\}$ is an orthogonal list

& same for $b_6, \ldots, b_{10}$.

Problem 13(a) (5 points)

Find an orthogonal matrix (i.e., a matrix with orthonormal columns) whose column space is equal to the column space of $\begin{bmatrix} 1 & 6 \\ -3 & 2 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 6 - 6 = 0, \text{ so:}$$

$$\begin{bmatrix} 1/\sqrt{10} & 6/\sqrt{40} \\ -3/\sqrt{10} & 2/\sqrt{40} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

Problem 13(b) (5 points)

Find matrices P and C such that C is a pure rotation-scaling matrix $\left(\text{that is, a matrix of the form } \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right)$ and

$$PCP^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & 6 \end{bmatrix}.$$

Hint: $\frac{9}{2} + \frac{\sqrt{7}i}{2}$ is an eigenvalue of A, with corresponding eigenvector $\begin{bmatrix} -\frac{3}{8} + \frac{\sqrt{7}i}{8} \\ 1 \end{bmatrix}$.

Solution

$$P = \begin{bmatrix} \operatorname{Re} v & \operatorname{Im} v \end{bmatrix} = \begin{bmatrix} -3/8 & \frac{\sqrt{7}}{8} \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{bmatrix} = \begin{bmatrix} 9/2 & \frac{\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{2} & 9/2 \end{bmatrix}.$$

Problem 14 (6 points)

Suppose that A is an $m \times n$ matrix and $\mathbf{b}$ is an $m \times 1$ vector. Show that, although the equation $A\mathbf{x} = \mathbf{b}$ might have no solutions, the equation $A^T A\mathbf{x} = A^T\mathbf{b}$ always has at least one solution. (Hint: rearrange terms, and think about orthogonal projection).

Solution

$$A^T A\vec{x} = A^T\vec{b} \iff A^T(A\vec{x} - \vec{b}) = \vec{0}$$

$\iff A\vec{x} - \vec{b}$ is orthogonal to $\operatorname{Col} A$.

By def$^{\underline{n}}$, $\operatorname{proj}_{\operatorname{Col} A}\vec{b} \in \operatorname{Col} A$, which means there exists $\vec{x}$ so that $A\vec{x} = \operatorname{proj}_{\operatorname{Col} A}\vec{b}$.

But for that $\vec{x}$, we will indeed have that $A\vec{x} - \vec{b} = \operatorname{proj}_{\operatorname{Col} A}\vec{b} - \vec{b}$ is orthogonal to $\operatorname{Col} A$, because $(\operatorname{proj}_W \vec{y}) - \vec{y} \perp W$ for any $\vec{y}$, W.

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