PSet 7

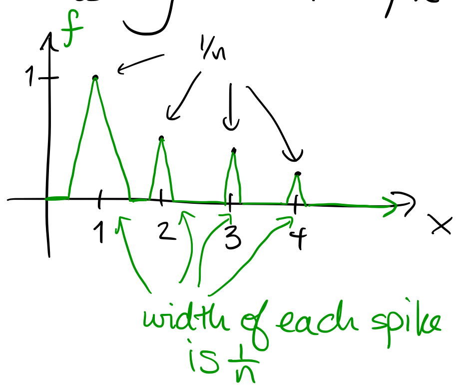
(1) Since $\sin 4n + \cos 3n$ is bounded between -2 and 2 and since $e^{-3n} \to 0$ as $n \to \infty$, the spring approaches a displacement of $p = \boxed{0}$ in the $n \to \infty$ limit. In other words, the spring comes to rest.

(2) By the n^{th} term test $a_n \to 0$ as $n \to \infty$, and by the definition of convergence of a series, $S_k \to 15$ as $k \to \infty$.

(3) The friend is correct to say the series passes the n^{th} term test (although they should say "... smaller & smaller, as close to 0 as desired" if they mean 'converges to 0'), but the sum nevertheless diverges, by the integral test: (Note $1/\sqrt{x}$ is positive & decreasing)

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left[x^{1/2}\Big|_1^b\right] = \infty.$$

④ (i) False. It must be assumed that f is decreasing. For example:



$$\int_0^\infty f(x)\,dx = \text{sum of triangle areas}$$

$$= \frac{1}{2}\left[\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots\right] < \infty$$

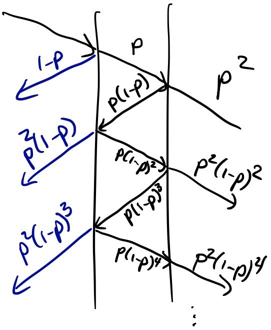
while $\sum f(n) = \sum \frac{1}{n} = \infty.$

(ii) False. $\frac{1}{n^2} < \frac{1}{n}$ and $\sum \frac{1}{n^2}$ converges,

but $\sum \frac{1}{n}$ diverges

(iii) False. If the common ratio is not between -1 and 1, the series diverges: $\sum 1 = \infty$

⑤



(a) The transmitted fractions are as labeled above, & their sum is

$$p^2 + p^2(1-p)^2 + p^2(1-p)^4 + \cdots = \frac{\text{first term}}{1-\text{common ratio}}$$

$$= \frac{p^2}{1-(1-p)^2} = \frac{p^2}{1-(1-2p+p^2)} = \frac{p^2}{2p-p^2} = \frac{p}{2-p}$$

(b) the reflected fraction is

$$(1-p) + \underbrace{p^2(1-p) + p^2(1-p)^3 + p^2(1-p)^5 + \cdots}_{\frac{p^2(1-p)}{1-(1-p)^2}}$$

$$= (1-p) + \frac{p^2(1-p)}{1-(1-p)^2} = 1-p+\frac{p^2(1-p)}{2p-p^2} = \frac{2(1-p)}{2-p}$$

(c) the part that stays inside forever is $1 - \left(\frac{2-2p}{2-p}\right) - \frac{p}{2-p} = 0.$

⑥ (a) $\sum \frac{1}{n \ln n}$ diverges because $\frac{1}{n\ln n}$ is decreasing and positive,

and $\int_2^\infty \frac{1}{x \ln x}dx = \lim_{b\to\infty}\left[\ln \ln b - \ln \ln 2\right] = \infty.$

(b) $\sum \frac{n}{\sqrt{n^2+4}}$ diverges by n^{th} term test: $\frac{n}{\sqrt{n^2+4}} \to 1 \neq 0.$

(c) $\sum ne^{-n}$ converges by the integral test

$$\int_0^\infty xe^{-x}dx = -(x+1)e^{-x}\Big|_0^\infty < \infty,$$ by L'Hospital's rule.

Also $f(x) = xe^{-x}$ is eventually decreasing because

$$f'(x) = \underbrace{(1-x)}_{\text{negative, when } x>1}\underbrace{e^{-x}}_{\text{positive, always}}$$

(d) $\sum \frac{1}{n^{1.001}}$ converges by the integral test: $\frac{1}{n^{1.001}}$ is positive & decreasing, and $\int_1^\infty x^{-1.001}\,dx = \lim_{b\to\infty}\left[\frac{-b^{-0.001}+1}{1000}\right]$

$< \infty$.

(7) (a) $\sum \frac{n}{n^2-1} \geq \sum \frac{n}{n^2} = \sum \frac{1}{n} = \infty$, so <u>diverges</u>.

(b) $\sum \frac{1}{n^2+n+e^{2n}} \leq \sum \frac{1}{e^{2n}}$, which is a <u>convergent</u> geometric series

(c) $\sum \frac{1}{3^n-2^n} = \sum \frac{1}{(\frac{3}{2})^n(2^n-1)} \leq \sum \frac{1}{(\frac{3}{2})^n} < \infty$, because $\sum (\frac{2}{3})^n$ is a convergent geometric series

(d) $\sum \frac{\sin(1/n)}{n^2} \leq \sum \frac{1}{n^2} < \infty$, so <u>converges</u>.

(8)

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

$$\begin{array}{rllll}
1 & \frac{1}{2} + & \frac{1}{4} + & \frac{1}{8} + & \frac{1}{16} + \cdots \\
+\frac{1}{2} & & +\frac{1}{4} + & \frac{1}{8} + & \frac{1}{16} + \cdots \\
+\frac{1}{4} & & & +\frac{1}{8} + & \frac{1}{16} + \cdots \\
+\cdots & & & \frac{1}{8} + & \frac{1}{16} + \cdots \\
=\boxed{2} & & & &
\end{array}$$

So $\sum_{k=1}^{\infty} \frac{k}{2^k} = \boxed{2}$

(9) We calculate $\sin\theta + \cdots + \sin n\theta = \mathrm{Im}\left[e^{i\theta} + e^{2i\theta} + \cdots + e^{in\theta}\right]$

$$= \mathrm{Im}\left[\frac{e^{i(n+1)\theta} - e^{i\theta}}{e^{i\theta} - 1}\right]$$

$$= \mathrm{Im}\left[\frac{e^{-i\theta/2}\left(e^{i(n+1)\theta} - e^{i\theta}\right)}{e^{-i\theta/2}\left(e^{i\theta} - 1\right)}\right]$$

$$= \mathrm{Im}\left[\frac{e^{i(n+\frac{1}{2})\theta} - e^{i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right]$$

$$= \mathrm{Im}\left[\frac{\cos(n+\frac{1}{2})\theta + i\sin(n+\frac{1}{2})\theta - \cos(-\theta/2) - i\sin(-\theta/2)}{\cos(\theta/2) + i\sin(\theta/2) - \left[\cos(-\theta/2) + i\sin(-\theta/2)\right]}\right]$$

$$= \mathrm{Im}\left[\frac{\cos(n+\frac{1}{2})\theta + i\sin(n+\frac{1}{2})\theta - \cos(\theta/2) + i\sin(\theta/2)}{2i\sin(\theta/2)}\right]$$

$$= \boxed{\frac{\cos(\theta/2) - \cos(n+\frac{1}{2})\theta}{2\sin(\theta/2)}}$$

(10) It's false. Set $a_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{if } n \text{ is composite.} \end{cases}$ Then $(a_n)_{n=2}^{\infty}$ does not converge, because it goes back & forth between 0 & 1 forever. But for any prime p,

$$B_p = 1, 0, 0, 0, 0, \ldots$$

which goes to 0.