Math 520 Midterm II
Spring 2017
Brown University
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Name: Solutions

This is a pencil-and-paper-only exam. You have two hours.

Problem 1(a) (8 points)

Find $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$$

Final answer:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

Problem 1(b) (8 points)

Suppose $A^{-1}\mathbf{v}_1 = \mathbf{e}_1$, $A^{-1}\mathbf{v}_2 = \mathbf{e}_2$, and $A^{-1}\mathbf{v}_3 = \mathbf{e}_3$, where $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$. Find A.

(Note: $\mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_3$ denote the standard basis vectors in $\mathbb{R}^3$.)

Solution

We know $A\mathbf{e}_i = \vec{v}_i$ for $i = 1, 2, 3$,

so $A = \begin{bmatrix} -4 & 0 & 3 \\ 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$

Final answer:

$$\begin{bmatrix} -4 & 0 & 3 \\ 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

Problem 2(a) (8 points)

Consider the linear transformation $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x \\ 4y \end{bmatrix}$. Define T to be the 135-degree counterclockwise rotation in $\mathbb{R}^2$. Find the determinant of the composition $S \circ T$. Explain your reasoning.

Solution

$$\det S \circ T = \underbrace{(\det S)}\underbrace{(\det T)}$$

$$= \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} \cdot 1 \leftarrow \text{because rotations preserve area and orientation}$$

$$= 12$$

Problem 2(b) (8 points)

Show that $\det(kA) = k^n \det A$, if k is a real number and A is an $n \times n$ matrix.

Solution

$$\det(kA) = \begin{vmatrix} ka_{1,1} & \cdots & ka_{1,n} \\ \vdots & \ddots & \vdots \\ ka_{n,1} & \cdots & ka_{n,n} \end{vmatrix} = k^n a_{1,1} \cdots a_{n,n} - k^n a_{1,2} a_{2,1} \cdots a_{n,n}$$

$$+ \cdots (n! - 2 \text{ other terms})$$

$$= k^n (a_{1,1} \cdots a_{n,n} - a_{1,2} a_{2,1} \cdots a_{n,n} + \cdots)$$

$$= k^n \det A;$$

the point is that each term in the rook expansion is multiplied by k^n, so the whole sum is multiplied by k^n.

Final answer:

:)

Problem 3(a) (10 points)

The matrices $A = \begin{bmatrix} 1 & 4 & 5 & 8 & 2 \\ 0 & 1 & -2 & 3 & 5 \\ -2 & -7 & -12 & -13 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 13 & -4 & -18 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 2 & -4 & 6 & 10 \end{bmatrix}$ are row equivalent.

Find a basis for the row space of A and a basis for the column space of A.

Solution

[See original midterm]

Final answer:

Problem 3(b) (5 points)

Find the rank and the nullity of A.

Solution

Final answer:

Problem 4(a) (8 points)

Consider the set $S = \{f(x) = 0$ for **some** $x \in [0,1]\}$ of continuous functions from $[0,1]$ to $\mathbb{R}$ which are zero somewhere. For example, the function $\left(x - \frac{1}{2}\right)^3$ is in S since it vanishes at $x = \frac{1}{2}$, but the function $1 + x^2$ is not in S.

Show that S **is** closed under scalar multiplication in $C([0,1])$ but is **not** closed under vector addition. So is S a subspace of $C([0,1])$?

Solution

If $f \in S$, then $f(x) = 0$ for some x. Then for the same x, we have $(cf)(x) = c(f(x)) = c \cdot 0 = 0$. So $cf \in S$.

Note $x \in S$ and $1-x \in S$ but $x + 1 - x = 1 \notin S$. So S is not closed under vector addition & thus is not a subspace!

Problem 4(b) (8 points)

Consider the subset S of $\mathbb{P}_4$ consisting of all polynomials whose cubic and quadratic terms have the same coefficient. For example, $-1 + 3t^2 + 3t^3 + t^4$ is in this set, while $-1 + 2t^2 + 3t^3 + t^4$ is not. Is this set a subspace of $\mathbb{P}_4$? Explain your reasoning.

Solution

Yes, because 0 has the same t^3 and t^2 coeffs (both 0), & $a + bt + ct^2 + ct^3 + dt^4 + \tilde{a} + \tilde{b}t + \tilde{c}t^2 + \tilde{c}t^3 + \tilde{d}t^4$

$$= (a+\tilde{a}) + (b+\tilde{b})t + \underbrace{(c+\tilde{c})}_{\text{equal}}t^2 + \underbrace{(c+\tilde{c})}_{\text{equal}}t^3 + (d+\tilde{d})t^4,$$

so $p, q \in S \Rightarrow p+q \in S$.

Also $k(a + bt + ct^2 + ct^3 + dt^4) = ka + kbt + \underbrace{kc}_{\text{equal}}t^2 + \underbrace{kc}_{\text{equal}}t^3 + kdt^4 \in S$.

Problem 5 (10 points)

Consider the linear transformation $T : \mathbb{P}_7 \to \mathbb{P}_7$, where $T(p) = p''$. In other words, T acts by taking the second derivative. So, for example, $T(-3t^4 + t^2 - t) = -36t^2 + 2$.

Find the range and the kernel of T. Feel free to describe these sets using either a verbal description or math notation, as you prefer (as long as they are clearly specified). Explain your reasoning.

Solution

the range of T is $\mathbb{P}_5$ since $p'' \in \mathbb{P}_5$ whenever $p \in \mathbb{P}_7$, and for any $q \in \mathbb{P}_5$, $\int\int q \in \mathbb{P}_7$ and satisfies $\left(\int\int q\right)'' = p$.

The kernel of T is $\mathbb{P}_1$, since $p'' = 0$ iff $p(t) = a + bt$ for some $a, b \in \mathbb{R}$.

Problem 6(a) (6 points)

Show that $\{1+t^2, 1+t+2t^4\}$ is a basis of the span of $\{1+t^2, 1+t+2t^4\}$ (here $1+t^2$ and $1+t+2t^4$ are polynomials in the vector space of all polynomials, with the usual notions of addition and scalar multipliation). Find the coordinates of $1+4t-3t^2+8t^4$ with respect to the basis.

Solution

$\{1+t^2, 1+t+2t^4\}$ is a basis of its span because it's linearly independent (since $a(1+t^2)+b(1+t+2t^4)=0 \Rightarrow a=b=0$).

the coordinates are $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ since $(2t^4)(4)=8t^4$ and $(-3)(t^2)=-3t^2$.

Problem 6(b) (9 points)

Use Cramer's rule to solve

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

for x and y in terms of a, b, c, d, e, f, assuming that $ad - bc \neq 0$.

Solution

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

Problem 7 (12 points)

Suppose that U and V are four-dimensional subspaces of a 10-dimensional vector space W. (a) Show that $U \cup V$ is not necessarily a subspace of W. (b) Show that the span of $U \cup V$, which *is* a subspace of W, is not equal to W.

Solution

$U = \operatorname{Span}(\{e_1, e_2, e_3, e_4\})$

$V = \operatorname{Span}(\{e_5, e_6, e_7, e_8\})$

then $U \cup V = \{\vec{x} \in \mathbb{R}^{10} : \text{either } x_5 = x_6 = \cdots = x_{10} = 0 \text{ or } x_1 = \cdots = x_4 = 0 = x_9 = x_{10}\}$

which is clearly not a subspace: $(1,1,1,1,0,0,0,0,0,0) + (0\,0\,0\,0\,1\,1\,1\,1\,0\,0) \notin U \cup V$

The span of $U \cup V$ is spanned by $\{u_1, \ldots, u_4, v_1, \ldots, v_4\} = L$ where $\{u_1, \ldots, u_4\}$ is a basis for U & $\{v_1, \ldots, v_4\}$ is a basis for V. To see this, note that $w \in \operatorname{Span}(U \cup V)$ implies $w = u_1 + \cdots + u_m + v_1 + \cdots + v_m$ for some vectors $u_1, \ldots, u_m, v_1, \ldots, v_m$ with each u_k in U & each v_k in V. Since each u_k is in the span of L & so is each v_k, $w \in \operatorname{Span} L$ too. But a 10 D space cannot have a spanning list of length less than 10, so $\operatorname{Span} U \cup V \neq W$.

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