Brown University
Problem Set 7
Instructor: Samuel S. Watson
Due: 3 November 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

In this problem, we will provide an explanation for the Lagrange multipliers formula which has an accompanying visualization different from the one provided in the book. Your explanations do not need to be rigorous; you may assume all curves are smooth and you can use facts which are merely visually apparent from your figures.

Suppose that f and g are differentiable functions of two variables and $c \in \mathbb{R}$. Suppose that the restriction of f to the c-level set of g has a local maximum of M at (a, b), and that $\nabla g(a, b) \neq \mathbf{0}$.

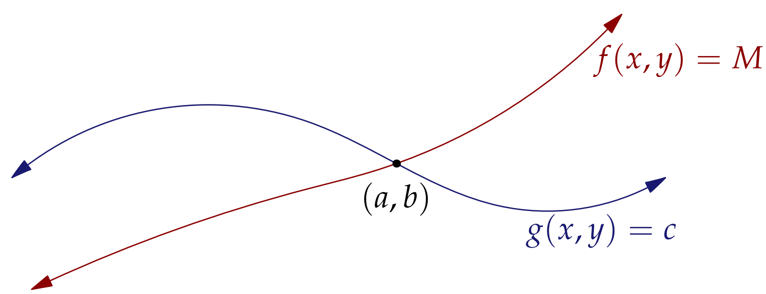
(a) Suppose that the M-level set of f crosses through the c-level set of g as shown. Why does this imply that $\nabla f(a, b) = 0$? Hint: suppose that $\nabla f(a, b) \neq \mathbf{0}$ and use the fact that f is larger on one side of $f(x, y) = M$ than the other to arrive at a contradiction.

(b) Use (a) to conclude that if $\nabla f(a, b) \neq \mathbf{0}$, then $\nabla f(a, b)$ and $\nabla g(a, b)$ point in the same or opposite directions. Draw a sketch of what the intersection of the level curves $\{(x, y) \in \mathbb{R}^2 : f(x, y) = M\}$ and $\{(x, y) \in \mathbb{R}^2 : g(x, y) = c\}$ would look like in this case.

(c) Use your figure from (b) to explain the following *duality* result: assuming $\nabla f(a, b) \neq \mathbf{0}$, if f restricted to the c-level set of g has a local maximum of M, then g restricted to the M-level set of f has a local minimum or maximum of c.

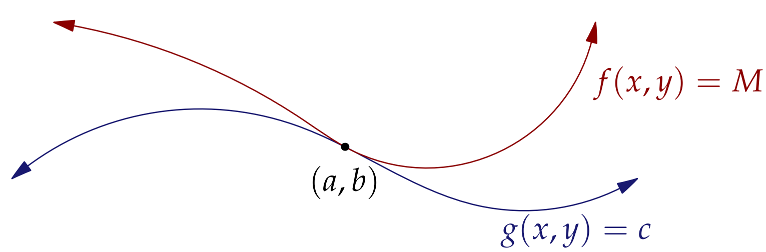
Solution

(a) If we assume $\nabla f(a, b) \neq \mathbf{0}$, then f increases in one direction orthogonal to ℓ and decreases in the other. Therefore, if the level curves cross as shown, then f increases as we move along the level curve of g in one direction and it decreases in the opposite direction. So we must have $\nabla f(a, b) = \mathbf{0}$.



(b) Since the two level curves cannot cross and must intersect (since they both include the point (a, b), they have to be tangent as shown in the second figure.

(c) Suppose that ∇g has a positive y component in the figure to the right (meaning that its direction is the same as that of ∇f). Then for any $c' < c$, the level curve $g = c'$ will be strictly lower (in this figure) than $g = c$ and will therefore no longer intersect the curve $f(x, y) = M$. Thus g has a minimum value of c in this case.



Likewise, if ∇g points opposite to ∇f, then g restricted to the level set $f = M$ has a maximum of c by the same idea.

See `http://web.csulb.edu/~saleem/Publications/FlipSide.pdf` for more details.

Problem 2

Find the largest and smallest values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

Solution

We begin by finding critical points inside the disk by finding where $\nabla f = 0$. Then we will find boundary-critical points using the method of Lagrange multipliers. Lastly, we will test all critical points to find the maximum and minimum values of $f(x,y)$. We have

$$\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x = 0, \text{ and}$$

$$\frac{\partial f}{\partial y} = 4x = 0 \Rightarrow y = 0.$$

Therefore, the only critical point of f is the origin. As for boundary-critical points, we have

$$\nabla f = \langle 2x, 4y \rangle \text{ and } \nabla g = \langle 2x, 2y \rangle$$

so the Lagrange equations are

$$2x = \lambda 2x, \quad 4y = \lambda 2y, \quad x^2 + y^2 = 1.$$

This yields

$$\lambda = 1 \Rightarrow (x,y) = (1,0) \text{ or } (-1,0) \quad \text{ and } \quad \lambda = 2 \Rightarrow (x,y) = (0,1) \text{ or } (0,-1).$$

Testing the values of f at each critical or boundary-critical point, we get

$$f(0,0) = 0 \quad f(1,0) = 1 \quad f(0,1) = 2 \quad f(-1,0) = 1 \quad f(0,-1) = 2$$

Therefore, the minimum value of $f(x,y)$ within the bounded region is 0, and the maximum value is 2.

Problem 3

Use the method of Lagrange multipliers to find the minimum distance from the origin to the plane $3x + 2y + z = 6$.

Solution

We will minimize the squared distance

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constrant that $g(x,y,z) = 6$ where $g(x,y,z) = 3x + 2y + z$.

We have $\nabla g(x,y,z) = \langle 3,2,1 \rangle$ and $\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$. The Lagrange equations, therefore, are

$$3\lambda = 2x, \qquad 2\lambda = 2y, \qquad \lambda = 2z.$$

Using substitution into our original constraint equation ($g(x,y,z) = 6$), we get

$$x = \frac{9}{7}, \quad y = \frac{6}{7}, \quad z = \frac{3}{7}$$

The distance from the origin to this point is $\frac{\sqrt{126}}{7}$.

Final answer:

$$\frac{\sqrt{126}}{7}$$

Problem 4

Integrate $(x+y)^2$ over the triangular region with vertices at the origin, $(3,0)$, and $(0,4)$.

Note: the algebra is a little tedious; click here (`http://tinyurl.com/y9zubnaa`) for help with that.

Solution

We will begin by drawing this triangle in the x-y plane and see that it can be represented as a region between the x-axis, y-axis, and the line $y = 4 - \frac{4x}{3}$. We can set up the integral as follows: the least and greatest x values for any point in the triangle are 0 and 3, and for each fixed x between 0 and 3, the least and greatest values of y for a point in the figure having that x value are 0 and $4 - \frac{4x}{3}$:

$$\int_0^3 \int_0^{4-\frac{4x}{3}} (x+y)^2 \, \mathrm{d}y \, \mathrm{d}x = \int_0^3 \int_0^{4-\frac{4x}{3}} x^2 + 2xy + y^2 \, \mathrm{d}y \, \mathrm{d}x$$

Evaluating the integral with respct to y first, we get

$$\int_0^3 x^2y + xy^2 + \frac{y^3}{3}\bigg|_{y=0}^{y=4-\frac{4x}{3}} \mathrm{d}x = \int_0^3 x^2\left(4 - \frac{4x}{3}\right) + x\left(16 - \frac{32x}{3} + \frac{16x^2}{9}\right) + \frac{1}{3}\left(64 - \frac{192x}{3} - \frac{64x^2}{9} - \frac{64x^3}{27}\right) = 37.$$

Final answer:

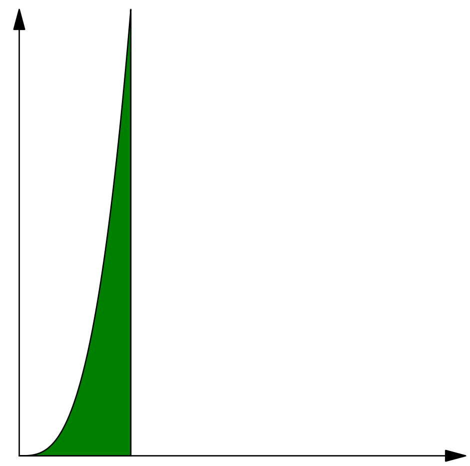
37

Problem 5

Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, \mathrm{d}x\, \mathrm{d}y$ by reversing the order of integration.

Solution

Let's begin by drawing the region, as shown to the right.



We can see that this is the region under the graph of $y = x^3$ from $x = 0$ to $x = 2$. Thus we integrate as x ranges from 0 to 2 and (for each fixed value of x) as y ranges from 0 to x^3. We get

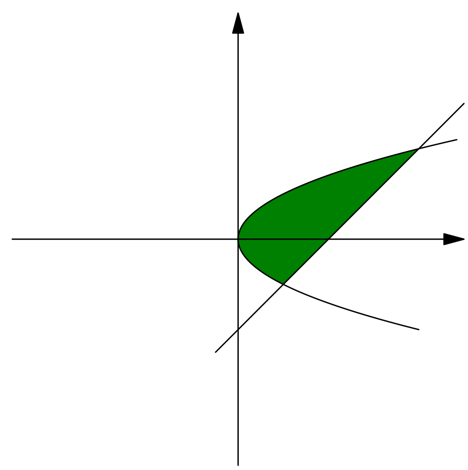
$$\begin{aligned}\int_0^2 \int_0^{x^3} e^{x^4}\, \mathrm{d}y\, \mathrm{d}x &= \int_0^2 \int_0^{x^3} ye^{x^4}\Big|_{y=0}^{y=x^3} \mathrm{d}x \\ &= \int_0^2 x^3 e^{x^4}\, \mathrm{d}x = \frac{e^{x^4}}{4}\Big|_0^2 = \frac{e^{16}-1}{4} \approx 2221527.\end{aligned}$$

Problem 6

Set up iterated integrals for both orders of integration for the integral of $f(x,y) = y$ over the region D bounded by $y = x - 2$, $x = y^2$. Then evaluate the double integral using the easier order and explain why it's easier.

Solution

We can begin by drawing the region D and finding the points of intersection of the two curves. By solving the system of equations, we get that the curves intersect at $(1, -1)$ and $(4, 2)$.



The order $\mathrm{d}y\, \mathrm{d}x$ is tricky because we would have two different expressions for the lower bound, depending on the value of x. In the figure, notice that when $0 \le x \le 1$, y is bounded between $y = -\sqrt{x}$ and $y = \sqrt{x}$, but for $1 \le x \le 4$, y is bounded between $y = -\sqrt{x}$ and $y = x - 2$. Therefore, the set of integrals for which we would integrate over y first would look like this:

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\, \mathrm{d}y\, \mathrm{d}x + \int_1^4 \int_{x-2}^{\sqrt{x}} y\, \mathrm{d}y\, \mathrm{d}x.$$

Integrating over x on the inside is simpler, because the lower bounds of the intervals of intersection of the region and each horizontal line are described by the same expression, and similarly for the upper bound. If we do it this way, we have only one integral:

$$\int_{-1}^2 \int_{y^2}^{y+2} y\, \mathrm{d}x\, \mathrm{d}y$$

Evaluating, we get:

$$\int_{-1}^2 \int_{y^2}^{y+2} y\, \mathrm{d}x\, \mathrm{d}y = \int_{-1}^2 xy\Big|_{y^2}^{y+2} \mathrm{d}y = \int_{-1}^2 y^2 + 2y - y^3\, \mathrm{d}y = \frac{y^3}{3} + y^2 - \frac{y^4}{4}\Big|_{-1}^2 = \boxed{\frac{9}{4}}.$$

Additional space