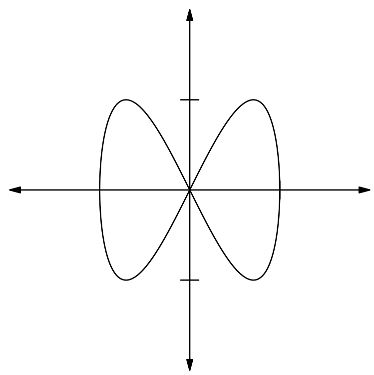
18.022 Recitation Handout (with solutions)
15 October 2014

1. Sketch the image of the path $\mathbf{x}(t) = (\cos t, \sin 2t)$.

Solution. As t goes from 0 to $\pi/2$, the x-coordinate of $\mathbf{x}$ varies from 1 to 0, while the y-coordinate varies from 0 to 1 and back to 0. Superimposing these two "pen" movements (like an etch-a-sketch), we get a bump going from $(1,0)$ to $(0,0)$, as shown in the first quadrant below. Letting t continue to increase produces the three more copies of this shape in the other three quadrants.



2. Find the arclength of the graph of $f(x) = \frac{2}{3}(x-1)^{3/2}$ between the points $(1,0)$ and $(4, 2\sqrt{3})$.

Solution. We calculate $\int_1^4 \sqrt{1+f'(x)^2} = \int_1^4 \sqrt{1+(\sqrt{x-1})^2}\,dx = \boxed{14/3}$.

3. Consider the function $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\mathbf{F}(x,y,z) = (3x/y, 2x+e^z)$.

(a) Find $D\mathbf{F}$.

Solution. The total derivative is the matrix of partial derivatives:

$$\begin{pmatrix} 3/y & -3x/y^2 & 0 \\ 2 & 0 & e^z \end{pmatrix}$$

(b) Show that there exists an open set $U \subset \mathbb{R}$ containing 1 and a function $\mathbf{f}: U \to \mathbb{R}^2$ such that for all $x \in U$, the equations $\mathbf{F}(x,y,z) = \mathbf{F}(1,-2,0)$ have a unique solution $(y,z) = \mathbf{f}(x)$. Show that $\mathbf{f}$ is C^1.

Solution. The implicit function theorem ensures that we can solve (abstractly) for (y,z) in terms of x if the matrix of partial derivatives corresponding to the y and z columns has nonvanishing determinant. In this case, that means

$$\det\begin{pmatrix} -3x/y^2 & 0 \\ 0 & e^z \end{pmatrix} = \det\begin{pmatrix} -3/4 & 0 \\ 0 & 1 \end{pmatrix} = -3/4 \neq 0,$$

so the implicit function theorem does apply and gives us the desired function $\mathbf{f}$. The theorem also tells us that $\mathbf{f}$ is C^1.

(c) Find $D\mathbf{f}(1)$.

Solution. Since $\mathbf{F}(x, \mathbf{f}(x)) = \mathbf{c}$ for some constant $\mathbf{c}$, we can differentiate both sides to get $D\mathbf{f} = -A^{-1}B$, where the matrix A is the one obtained by taking the y and z columns of $D\mathbf{F}$ and B is the matrix obtained by considering the remaining columns. We get

$$D\mathbf{f}(1) = \begin{pmatrix} -3/4 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3/2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ -2 \end{pmatrix}}$$

4. (from 3.2.15 in *Colley*) Determine the moving frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, the curvature, the torsion, and the arc length parameter $s(t)$ for the curve

$$\mathbf{x} = \left(5, \frac{1}{3}(t+1)^{3/2}, \frac{1}{3}(1-t)^{3/2}\right), \qquad -1 < t < 1.$$

Solution. We have

$$\begin{aligned}
\mathbf{T} &= \mathbf{x}'(t)/\|\mathbf{x}'(t)\| = \left(1, \frac{1}{2}\sqrt{t+1}, -\frac{1}{2}\sqrt{1-t}\right)/\sqrt{3/2}, \\
\mathbf{N} &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|} = 2\sqrt{2}\left(0, \frac{1}{2}\sqrt{1-t}, \frac{1}{4}\sqrt{t+1}\right), \\
\mathbf{B} &= \mathbf{T} \times \mathbf{N} = (1, -\sqrt{t+1}, \sqrt{1-t})/\sqrt{3}.
\end{aligned}$$

The curvature and torsion are $\dfrac{\sqrt{2}}{6(1-t^2)}$ and $\dfrac{1}{3\sqrt{1-t^2}}$, respectively. The arclength parameter is $s(t) = \int_0^t \|\mathbf{x}'(\tau)\|\, d\tau = t/\sqrt{2}$.