Brown University
Problem Set 8
Instructor: Samuel S. Watson
Due: 10 November 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

Find the volume of the region W that represents the intersection of the solid cylinder $x^2 + y^2 \leq 1$ and the solid ellipsoid $2(x^2 + y^2) + z^2 \leq 10$.

Solution

Expressed in cylindrical coordinates, the inequalities become

$$r^2 \leq 1 \quad \text{and} \quad 2r^2 + z^2 \leq 10 \Rightarrow z^2 \leq 10 - 2r^2.$$

The latter inequality is equivalent to $-\sqrt{10-2r^2} \leq z \leq \sqrt{10-2r^2}$. The least and greatest values for θ for any point in the region are 0 and 2π, and for each θ, the least and greatest possible r values are 0 and 1 for any point in the region having that θ value. Finally, for each r and θ, the least and greatest z values (for the upper half of the solid) are $-\sqrt{10-2r^2}$ and $\sqrt{10-2r^2}$. So we get

$$\begin{aligned}\int_0^{2\pi}\int_0^1\int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta &= 4\pi\int_0^1\int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r\,\mathrm{d}z\,\mathrm{d}r \\ &= 4\pi\int_0^1 r\sqrt{10-2r^2}\,\mathrm{d}r \\ &= 4\pi\left(-\frac{1}{6}\right)(10-2r^2)^{\frac{3}{2}}\Big|_0^1 \\ &= \frac{2\pi}{3}[8^{\frac{3}{2}} - 10^{\frac{3}{2}}] \approx 18.84.\end{aligned}$$

Final answer:

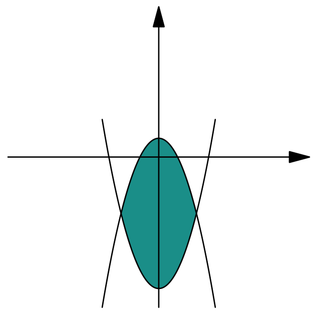
$\frac{2\pi}{3}[16\sqrt{2} - 10\sqrt{10}]$

Problem 2

Consider the region R between the parabolas $y = 1 - x^2$ and $y = x^2 - 7$. Find $\iint_R xy \,\mathrm{d}A$.

Solution

We can find the bounds on x by finding the intersection of the two curves.



We see that the curves intersect at $x = -2$ and $x = 2$. So we get

$$\int_{-2}^{2}\int_{x^2-7}^{1-x^2} xy \,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^{2} \frac{xy^2}{2}\bigg|_{x^2-7}^{1-x^2} \mathrm{d}x = \int_{-2}^{2} \frac{x}{2}[(1-2x^2+x^4)-(x^4-14x^2+49)]\,\mathrm{d}x$$

$$= \int_{-2}^{2} 6x^3 - 24x \,\mathrm{d}x = \frac{3x^4}{2} - 12x^2\bigg|_{-2}^{2} = 0$$

Final answer:

$$0$$

Problem 3

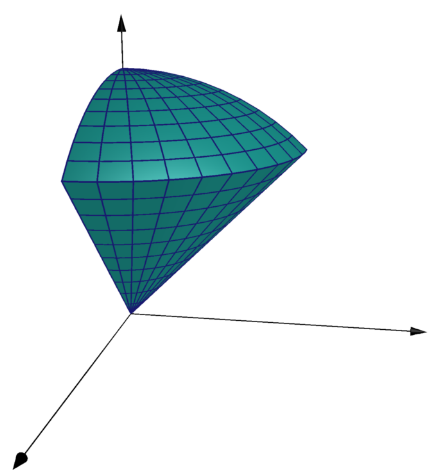
Evaluate

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2 \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

by rewriting the integral in spherical coordinates.

Solution

To set up this integral, we write the integral as a 3D integral over this region:



We then write this integral in spherical coordinates, including the volume differential $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$:

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{18}} \rho^4 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta &= \frac{\pi}{2}\int_0^{\pi/4}\int_0^{\sqrt{18}} \rho^4 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi \\
&= \frac{\pi}{2}\int_0^{\pi/4} \frac{1}{5}\rho^5 \sin\phi \bigg|_0^{\sqrt{18}} \mathrm{d}\phi \\
&= \frac{\pi}{2}\frac{\sqrt{18}^5}{5}\int_0^{\pi/4} \sin\phi \,\mathrm{d}\phi \\
&= \frac{\pi}{2}\frac{\sqrt{18}^5}{5}(-\cos\phi)\bigg|_0^{\pi/4} \\
&= \frac{\pi}{2}\frac{\sqrt{18}^5}{5}\left(1 - \frac{\sqrt{2}}{2}\right) \\
&= \frac{486\pi}{5}\left(\sqrt{2}-1\right) \approx 126.5
\end{aligned}$$

Final answer:

$$\frac{486\pi}{5}\left(\sqrt{2}-1\right)$$

Problem 4

Find the mass of the cylinder bounded by the surfaces $z = 0, z = 1$, and $x^2 + y^2 = 1$ whose density at (x, y, z) is given by $\rho(x, y, z) = z\sqrt{x^2 + y^2}$.

Solution

We can describe the solid by its cylindrical coordinate inequalities:

$$0 \leq z \leq 1, \qquad 0 \leq r \leq 1, \qquad 0 \leq \theta \leq 2\pi \qquad \text{and} \quad \rho(r, \theta, z) = z\, r$$

Then the mass of the solid is equal to the integral of the density, so the mass is

$$\int_0^1 \int_0^1 \int_0^{2\pi} zr \overbrace{r \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z}^{\mathrm{d}A} = \int_0^1 \int_0^1 \int_0^{2\pi} zr^2 \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z.$$

Evaluating the integral yields

$$\int_0^1 \int_0^1 \int_0^{2\pi} zr^2 \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z = 2\pi \int_0^1 \int_0^1 zr^2 \, \mathrm{d}r \, \mathrm{d}z = 2\pi \int_0^1 \frac{1}{3} r^3 \Big|_0^1 z \, \mathrm{d}z = 2\pi \int_0^1 \frac{1}{3} z \, \mathrm{d}z = 2\pi \frac{1}{6} z^2 \Big|_0^1 = 2\pi \frac{1}{6} = \frac{\pi}{3}$$

Final answer:

$$\frac{\pi}{3}$$

Problem 5

Find the region E in $\mathbb{R}^3$ for which

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2)\,\mathrm{d}V$$

is as large as possible.

Solution

If S is a small region in $\mathbb{R}^3$ on which $1 - x^2 - 2y^2 - 3z^2 > 0$, then including S increases the value of the given integral. Conversely, if $1 - x^2 - 2y^2 - 3z^2 < 0$ at all points in S, then including S decreases the value of the integral. Therefore, we should include in E only those points where $1 - x^2 - 2y^2 - 3z^2 \geq 0$, which is to say, the ellipsoid $x^2 + 2y^2 + 3z^2 \leq 1$.

Problem 6

(a) Explain why the integral of the function $f(x,y,z) = \frac{1}{x+y+z+1}$ over the cube $[0,1] \times [0,1] \times [0,1]$ is equal to $\lim_{n\to\infty} S_n$, where

$$S_n = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{f(i/n, j/n, k/n)}{n^3}.$$

(b) At `sagecell.sagemath.org`, use the code [click here]

```
n = 20
var("x","y","z","i","j","k")
f(x,y,z) = 1/(x+y+z+1)
assume(0<x<1);assume(0<y<1);assume(0<z<1)
I = integrate(integrate(integrate(f(x,y,z),x,0,1),y,0,1),z,0,1)
S = sum(sum(sum(f(i/n,j/n,k/n)/n^3,k,1,n),j,1,n),i,1,n)
(I,S,N(I),N(S))
```

which gives the exact values of the integral and S_{20} followed by their decimal representations, to show that S_{20} is close to the value of the integral. Increase n (just change the first line of code to assign a different value to n) by multiples of 5 to **find the least value of** n such that n is a multiple of 5 and S_n differs from I by less than 0.01.

Solution

(a) S_n is a Riemann sum approximating f with cube sizes shrinking to zero as $n \to \infty$. So if f is integrable over the unit cube—which it is since f is continuous—S_n will converge to it as $n \to \infty$.

(b) Running the above code for increasing values of n starting with $n = 20$, we can see that $n = 30$ is the first one for which the error is less than 0.01.

Programming note: you can use a loop to automate the process of check through various values of n. For example:

```
while abs(N(S-I)) > 0.01:
    n += 5
    S = sum(sum(sum(f(i/n,j/n,k/n)/n^3,k,1,n),j,1,n),i,1,n)
print(n)
```

Final answer:

30