Brown University
DATA 1010
Fall 2018: Practice Midterm II
Samuel S. Watson

Name:

You will have three hours to complete the exam. It consists of 24 written questions and one separate computational problem. Among the first 12 questions, you should only solve problems for standards for which you want to improve your score from the first exam. If you are completing the computational problem, you will hand in your answers to the written portion and then get out your laptop to submit a solution to the last question electronically.

For the written part of the exam, no calculators or other materials are allowed, except the Julia-Python-R reference sheet and the published exam reference sheet. For the computational part of the exam, you may use any internet technologies which do not involve active communication with another person.

You are responsible for explaining your answer to **every** *question. Your explanations do not have to be any longer than necessary to convince the reader that your answer is correct.*

I verify that I have read the instructions and will abide by the rules of the exam: _____

Problem 1 **[PMF]**

Suppose that X and Y are independent random variables whose probability mass functions m_X and m_Y are defined as follows:

$$m_X(1/3) = 1/2 \quad m_X(1) = 1/4 \quad m_X(2) = 1/4$$

$$m_Y(1/3) = 1/3 \quad m_Y(1) = 2/5 \quad m_Y(2) = 1/5 \quad m_Y(3) = 1/15$$

(a) How many points $(x, y) \in \mathbb{R}^2$ have the property that $m_{X,Y}(x, y) \neq 0$, where $m_{X,Y}$ is the joint PMF of X and Y?

(b) How many points $z \in \mathbb{R}$ have the property that $m_Z(z) \neq 0$, where m_Z is the PMF of $Z = X + Y$?

Put your answer in the box as an ordered pair (answer to (a), answer to (b)).

Solution

(a) The support of $m_{X,Y}$ contains $\boxed{12}$ points:

$$\{1/3, 1, 2\} \times \{1/3, 1, 2, 3\}$$

(b) The support of m_Z is

$$\left\{\frac{2}{3}, 2, 3, 4, 5, \frac{4}{3}, \frac{7}{3}, \frac{10}{3}\right\},$$

so it contains $\boxed{8}$ elements.

Final answer:

$(12, 8)$

Problem 2 **[PDF]**

Suppose that X is a random variable whose density is given by

$$f_X(x) = 2x\mathbf{1}_{\{0\le x\le 1\}}.$$

Which random variable has larger expected value, $\sqrt{X}$ or X^2?

Solution

We have

$$\mathbb{E}[\sqrt{X}] = \int_0^1 \sqrt{x}2x\,\mathrm{d}x = \frac{4}{5},$$

and

$$\mathbb{E}[X^2] = \int_0^1 x^2(2x)\,\mathrm{d}x = \frac{1}{2}.$$

Therefore, $\sqrt{X}$ has the larger expected value.

Alternatively, we could note that $x^2 < \sqrt{x}$ for all $x \in (0,1)$, and that implies $\mathbb{E}[X^2] < \mathbb{E}[\sqrt{X}]$ directly

Final answer:

$\sqrt{X}$

Problem 3 **[CONDPROB]**

Three cards are drawn without replacement from a well-shuffled standard deck. Find the conditional probability that the cards are all diamonds given that they are all red cards. (Note: 13 of the cards are diamonds, 26 of the cards are red, and all of the diamonds are red).

Solution

The probability that all of the cards are red is

$$\frac{1}{2}\cdot\frac{25}{51}\cdot\frac{24}{50},$$

and the probability that all of the cards are diamonds is

$$\frac{1}{4}\cdot\frac{12}{51}\cdot\frac{11}{50}.$$

Therefore, the conditional probability is

$$\frac{1}{2}\cdot\frac{12}{25}\cdot\frac{11}{24} = \frac{11}{100}.$$

Final answer:

$\frac{11}{100}$

Problem 4 [BAYES]

(a) Suppose that the conditional probability of an email (chosen uniformly at random from a large collection of emails) containing the phrase "additional income", given that the email is spam, is 14%. Suppose that the conditional probability of an email being spam, given that it contains the phrase "additional income", is 88%. Find the ratio of the probability that an email is spam to the probability that an email contains the phrase "additional income".

(b) We flip a weighted coin that has probability $\frac{3}{4}$ of turning up heads. If we get heads, we roll a six-sided die, and otherwise we roll an eight-sided die. Given that the die turns up 4, what is the conditional probability that the coin turned up heads?

Solution

(a) By the definition of conditional probability, we have

$$\frac{\mathbb{P}(A \mid B)}{\mathbb{P}(B \mid A)} = \frac{P(A \cap B)/\mathbb{P}(B)}{\mathbb{P}(B \cap A)/\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Therefore, the desired ratio is $88\%/14\% = 44/7$.

(b) We have

$$\mathbb{P}(\text{roll} = 4) = \frac{3}{4} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{8} = \frac{5}{32},$$

and the proportion of that probability mass which comes from the 'heads' branch of the tree is

$$\frac{\frac{3}{4} \cdot \frac{1}{6}}{\frac{5}{32}} = \frac{4}{5}.$$

Final answer:

$\frac{44}{7}, \frac{4}{5}$

Problem 5 [IND]

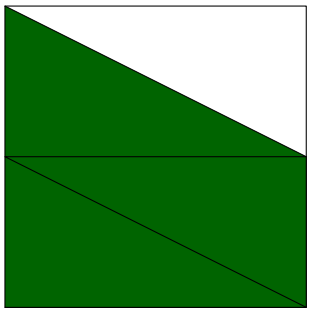
(a) Suppose that $X_1, \ldots, X_{10}$ are independent Bernoulli(p) random variables defined on a probability space Ω. What is the smallest possible cardinality of Ω?

(b) Suppose that U and V are independent random variables, each selected uniformly at random from $[0, 1]$. Find the probability of the event $\{\frac{1}{2}U + V \leq 1\}$.

Solution

(a) The range of the random vector $[X_1, \ldots, X_{10}]$ (thought of as a map from Ω to $\mathbb{R}^{10}$) has $2^{10} = 1024$ points in it. Therefore, Ω must have at least 1024 points. It can have exactly 1024 points; for example, we could take $\Omega = \{0, 1\} \times \{0, 1\} \times \cdots \times \{0, 1\}$ and $X(\omega) = \omega$.

(b) The joint distribution of (U, V) is the area measure on the unit square. The given event corresponds to the set of points shown shaded in the figure, which we can see from symmetry (using the extra lines drawn there) is $\frac{3}{4}$ of the square.



Problem 6 **[EXP]**

(a) Find the expected value of the sum of the sum and product of two independent die rolls.

(b) You roll a die, and if the result is prime you roll two more dice, and if it isn't prime you roll *three* more dice. Find the expected number of pips showing on the top faces of all of the dice rolled (so, either three dice or four dice).

Solution

(a) Let X and Y be the two die rolls. Then

$$\mathbb{E}[X + Y + XY] = \mathbb{E}[X] + \mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y] = 3.5 + 3.5 + 3.5^2 = 19.25.$$

(b) Let us adjust the experiment by rolling the fourth die anyway, If the first die roll isn't prime, we won't count the last one. Then the desired sum is

$$X_1 + X_2 + X_3 + YX_4,$$

where Y is the indicator of the event that X_1 is prime. Then by linearity of expectation,

$$\mathbb{E}[X_1 + X_2 + X_3 + YX_4] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[YX_4].$$

Since Y and X_4 are independent, this expression simplifies to

$$\mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[Y]\mathbb{E}[X_4] = \frac{7}{2} + \frac{7}{2} + \frac{7}{2} + \left(\frac{1}{2}\right)\left(\frac{7}{2}\right) = \frac{49}{4}.$$

Final answer:

$$\frac{49}{4}$$

Problem 7 **[COV]**

Suppose that X_1 and X_2 are independent and identically distributed.

(a) Find the covariance of $X_1 + X_2$ and $X_1 - X_2$.

(b) Show that if X_1 and X_2 are normal random variables, then $X_1 + X_2$ and $X_1 - X_2$ are independent. Hint: use your knowledge of the multivariate normal distribution density.

Solution

(a) We have $\mathbb{E}[(X_1 + X_2)(X_1 - X_2)] = \mathbb{E}[X_1^2] - \mathbb{E}[X_2^2]$, and we have $\mathbb{E}[X_1 + X_2]\mathbb{E}[X_1 - X_2] = \mathbb{E}[X_1]^2 - \mathbb{E}[X_2]^2$. Subtracting, we find that the covariance of $X_1 + X_2$ and $X_1 - X_2$ is $\mathrm{Var}(X_1) - \mathrm{Var}(X_2)$, which is zero since X_1 and X_2 have the same distribution and hence also the same variance. We didn't even need the independence hypothesis!

(b) If X_1 and X_2 are independent normal random variables with the same mean $\boldsymbol{\mu}$ and variance σ^2, then the distribution of

$$\begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is multivariate Gaussian since it's an affine transformation of a vector of independent standard normals. The covariance is $2\sigma^2 I$, and the mean is $\boldsymbol{\mu} = [2\mu, 0]$, so the density is

$$\mathbf{y} \mapsto \frac{1}{\sqrt{(2\pi)^2(2\sigma^2)^2}} e^{-\frac{1}{2(2\sigma^2)}(\mathbf{y}-\boldsymbol{\mu})'(\mathbf{y}-\boldsymbol{\mu})}.$$

Therefore, the density can be written as

$$\frac{1}{\sqrt{2\pi}(\sqrt{2\sigma^2})} e^{-(y_1-2\mu)^2/(2\cdot 2\sigma^2)} \frac{1}{\sqrt{2\pi}(\sqrt{2\sigma^2})} e^{-y_2^2/(2\cdot 2\sigma^2)},$$

which is the product of the density of Y_1 and the density of Y_2. Therefore, the two random variables are independent.

Problem 8 **[CONDEXP]**

(a) Suppose that, for all $x \in \mathbb{R}$, the conditional distribution of Y given $X = x$ is exponential with parameter $\lambda = 2|x| + 1$. Find $\mathbb{E}[Y \mid X]$.

(b) What is the strongest conclusion that can be drawn about the distribution of X, based on the information in (a)?

Solution

(a) The conditional expectation is the expectation calculated with respect to the conditional measure. Therefore, the conditional expectation given $X = x$ is the mean of the exponential distribution with parameter $2|x| + 1$, which is $\frac{1}{2|x|+1}$. Upper-casing x gives $\frac{1}{2|X|+1}$.

(b) The only conclusion that can be drawn is that X has either probability mass or probability density at every point on the number line (since otherwise we couldn't make sense of the conditional distribution of Y there). Besides that, it can have any distribution whatsoever, since we can generate X from any distribute we like and then generate Y from the exponential distribution with parameter $2|X| + 1$. The resulting pair (X, Y) will satisfy the conditions of the problem and have the chosen marginal distribution on X.

Final answer:

$$\frac{1}{2|X|+1}$$

Problem 9 **[COMDISTD]**

Suppose that $S = X_1 + \ldots + X_n$, where the X_i's are independent $\text{Ber}(p)$ random variables.

(a) The distribution of S is a named probability measure. Which one is it, and what are the parameters?

(b) Find the probability mass function for the conditional distribution of S given $\{X_1 = 1\}$.

(c) You collect some data over a few years, and you find that the number of near-doorings you experience per month on your bicycle commute is approximately Poisson distributed. Give an explanation for why the Poisson distribution might be expected to emerge in this context.

Solution

(a) The distribution of S is a Binomial distribution with parameters n and p.

(b) Given that $X_1 = 1$, the sum of the remaining random variables is a Binomial with parameters $n - 1$ and p. Therefore, the conditional distribution of S_n given X_1 is one plus a $\text{Bin}(n-1, p)$:

$$m(k) = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}.$$

(c) Your probability of getting doored on a particular block is low, but you traverse many blocks on your commute. Therefore, the number of doorings is a binomial random variable with large n and small p (small enough that np is still modest; otherwise you'd have stopped commuting by bike). The Poisson approximation says that such a distribution is approximately Poisson with parameter $\lambda = np$.

Problem 10 **[COMDISTC]**

(a) Find the probability density function of the distribution of $\sqrt{X}$, where X is an exponential random variable with parameter λ.

(b) Find $\mathbb{P}(Z = 0.5)$, where Z is a standard normal random variable.

Solution

(a) We calculate $\mathbb{P}(\sqrt{X} > t) = \mathbb{P}(X > t^2) = e^{-\lambda t^2}$, which implies that the density function of $\sqrt{X}$ is

$$\frac{d}{dt}\mathbb{P}(X^2 \leq t) = -\frac{d}{dt}\mathbb{P}(X^2 > t) = 2\lambda t e^{-\lambda t^2}.$$

(b) The probability that a normal random variable equals any particular value is 0.

Problem 11 **[RVINEQ]**

(a) Suppose $k > 0$. Explain why $\mathbb{P}(|X - \mu| > k\sigma) < 1/k^2$, if μ and σ are the mean and standard deviation of X, respectively.

(b) Use Chebyshev's inequality to find an interval centered at 3.5 which contains X with probability 99%, where X is the average of 10,000 independent fair die rolls. (Note: the variance of a fair die roll is $\frac{35}{12}$.) Feel free to leave your answer in unsimplified form.

Solution

(a) This inequality is true since $\mathbf{1}_{\{|x-\mu|>k\sigma\}} < (x-\mu)^2/(k\sigma)^2$ for all $x \in \mathbb{R}$, and we can substitute $X(\omega)$ for x and take the expectation to get

$$\mathbb{P}(|x - \mu| > k\sigma) < \mathbb{E}[(x-\mu)^2]/(k^2\sigma^2) = 1/k^2.$$

(b) The inequality $\mathbb{P}(|x - \mu| > k\sigma) < 1/k^2$ tells us that $\mathbb{P}(|x - \mu| > 10\sigma) < 1/100$, which means that the average of the samples is between $\frac{7}{2} - 10 \cdot \frac{1}{100}\sqrt{35/12}$ and $\frac{7}{2} + 10 \cdot \frac{1}{100}\sqrt{35/12}$ with probability 99%.

Problem 12 **[CLT]**

The **chi-squared distribution** with parameter n is the distribution of the sum of the squares of n independent standard normal random variables.

Let S_k be the sum of k independent chi-squared random variables with parameter 8. Find the limit as $k \to \infty$ of

$$\mathbb{P}(8k \leq S_k \leq 8.01k).$$

Solution

The mean of the chi-squared distribution is

$$\mathbb{E}[Z_1^2 + \cdots + Z_8^2],$$

where Z_i's are independent standard normals. Applying linearity and using the fact that $\mathbb{E}[Z_i^2] = \operatorname{Var} Z_i = 1$, we find that the mean of the chi-squared distribution is 8. The variance of the chi-squared distribution is not as straightforward to calculate explicitly; let's call it σ^2.

The sum S_k has mean $8k$ and variance $k\sigma^2$. Therefore, its typical values are close to $8k$, with fluctuations on the order of $\sigma\sqrt{k}$. Since $0.01k$ is much larger than $\sigma\sqrt{k}$ when k is large (and since the normal distribution is symmetric), approximately $\boxed{\frac{1}{2}}$ of the mass is between $8k$ and $8k + 0.01k$.

Final answer:

$$\frac{1}{2}$$