PSet6

1. We separate dependent and independent variables:

$$f'(x) = xe^{x^2 - \ln f(x)^2} \iff$$

$$f'(x) = \frac{xe^{x^2}}{f(x)^2} \iff$$

$$f(x)^2 f'(x) = xe^{x^2}.$$

Integrate both sides:

$$\frac{f(x)^3}{3} = \frac{1}{2}e^{x^2} + C$$

$$f(x) = \sqrt[3]{\frac{3}{2}e^{x^2} + 3C}.$$

Substituting $x=0$ gives $C = \frac{3}{2}$.

2. We multiply by $f(t)$ and integrate:

$$\int f(t)\, f'(t)\,dt = \int 1\,dt$$

$$\frac{1}{2}f(t)^2 = t + C$$

$$f(t) = \sqrt{2t+C}.$$

Now $f(1) = 3$ implies $3 = \sqrt{2(1)+C} \Rightarrow C = 7$. So

$$f(t) = \sqrt{2t+7}.$$

3. We substitute $f(t) = g'''(t)$, and we get

$$f'(t) = \sqrt{f(t)}.$$

So
$$\int \frac{f'(t)}{\sqrt{f(t)}}\,dt = \int dt \Rightarrow$$

$$2\sqrt{f(t)} = t + C \Rightarrow$$

$$f(t) = \left(\frac{t}{2} + c\right)^2$$

Substituting back, we get $g'''(t) = \left(\frac{t}{2} + c_1\right)^2$. Integrating gives $g''(t) = \frac{2}{3}\left(\frac{t}{2} + c_1\right)^3 + c_2$. Again:

$$g'(t) = \frac{2^2}{3 \cdot 4}\left(\frac{t}{2} + c_1\right)^4 + c_2 t + c_3. \text{ Again:}$$

$$g(t) = \frac{2^3}{3 \cdot 4 \cdot 5}\left(\frac{t}{2} + c_1\right)^5 + c_2 t^2 + c_3 t + c_4$$

$$= \frac{2}{15}\left(\frac{t}{2} + c_1\right)^5 + c_2 t^2 + c_3 t + c_4.$$

4. We have $T'(t) = -k(T(t) - T_a)$, from the description. We solve:

$$\int \frac{T'(t)}{T(t) - T_a}\,dt = \int -k\,dt \Rightarrow \ln(T - T_a) = -kt + C$$

$$\Rightarrow T = Ce^{-kt} + T_a$$

Substituting, we get

$$90 = T(10) = Ce^{-10k} + 70$$

so $Ce^{-10k} = 20$. Also $100 = T(0) = C + 70 \Rightarrow$

$C = 30$. So $30e^{-10k} = 20 \Rightarrow e^{-10k} = \frac{2}{3} \Rightarrow$

$-10k = \ln(2/3) \Rightarrow k = \frac{1}{10}\ln(3/2)$. So

$$T(t) = 70 + 30e^{-\frac{1}{10}\ln(\frac{3}{2})t}.$$

5. The solution of $f'(x) - f(x) = 0$ is $f(x) = Ce^x$. To find a particular solution, we try $A\sin 3x + B\cos 3x$:

$$\begin{array}{rl} f'(x) = & 3A\cos 3x - 3B\sin 3x \\ -\ (f(x) = & B\cos 3x + A\sin 3x) \\ \hline f'(x) - f(x) = & (3A - B)\cos 3x + (-3B - A)\sin 3x \end{array}$$

So we want $3A - B = 0$ and $-3B - A = 1$. So $A = \frac{-1}{10}$, $B = \frac{-3}{10}$.

The general solution is the homogeneous solution plus a particular solution: $\boxed{Ce^x - \frac{1}{10}\sin 3x - \frac{3}{10}\cos 3x}$

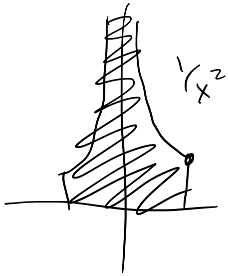
6. The characteristic equation is $\lambda^2+2\lambda+1=(\lambda+1)^2$, so the solution of the homogeneous equation is

$$f(x) = Ae^{-x} + Bxe^{-x}.$$

For the particular solution, we try ax^2+bx+c. Substituting and solving, we get

$$a=1, \quad b=-4, \quad c=6.$$

So: $f(x) = Ae^{-x} + Bxe^{-x} + x^2 - 4x + 6$ is the general solution.



7 (a) Looking at the graph of $\frac{1}{x^2}$, we see that the area must be computed as

$$\lim_{b\to 0^-} \int_{-1}^{b} x^{-2}\,dx + \lim_{a\to 0^+} \int_{a}^{1} x^{-2}\,dx$$

$$= \lim_{b\to 0^-}\left(\frac{1}{b}+1\right) + \lim_{a\to 0^+}\left(\frac{1}{a}-1\right)$$

$$= \infty + \infty$$

$$= \infty$$

(b) $\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\left[-\cos x\right]_0^b = \lim_{b\to\infty}(\cos b - 1).$

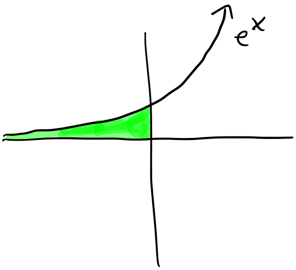
Since $\cos b$ oscillates between -1 and 1 as $b \to \infty$, this limit does not exist.

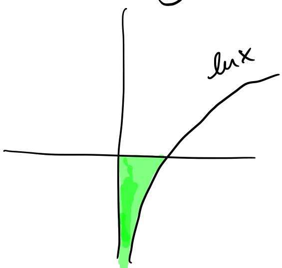
8. $$\int_{-\infty}^{0} e^x\,dx = \lim_{a\to-\infty}\left(e^x\Big|_a^0\right) = \lim_{a\to-\infty} 1-e^a = 1.$$

$$\lim_{a\to 0}\int_a^1 \ln x\,dx = \lim_{a\to 0}\left(x\ln x - x\right)\Big|_a^1$$
$$= \lim_{a\to 0}\left[-1-(a\ln a - a)\right]$$
$$= -1,$$

since $\lim_{a\to 0} a\ln a = \lim_{a\to 0}\frac{\ln a}{1/a} = \lim_{a\to 0}\frac{1/a}{-1/a^2} = 0.$

Also the areas of these two regions are equal because they are related by a reflection.





(the graphs of e^x and $\ln x$ are related by a reflection because e^x and $\ln x$ are inverses)

9. $$\int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 0-(-1) = 1$$

$$\int_0^\infty x e^{-x}\,dx = \int_0^\infty x\left(-e^{-x}\right)'\,dx = -xe^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})\,dx$$
$$= 0 + 1 = 1.$$

$$\int_0^\infty x^2 e^{-x}\,dx = \int_0^\infty x^2(-e^{-x})'\,dx$$

$$= -x^2e^{-x}\Big|_0^\infty - \int_0^\infty 2x(-e^{-x})\,dx$$

$$= 0 + 2\int_0^\infty x e^{-x}\,dx$$

$$= 2.$$

$$\int_0^\infty x^3 e^{-x}\,dx = -x^3e^{-x}\Big|_0^\infty + \int_0^\infty 3x^2e^{-x}\,dx$$

$$= 0 + 3\cdot 2 = 6.$$

$$\int_0^\infty x^4 e^{-x}\,dx = -x^4e^{-x}\Big|_0^\infty + \int_0^\infty 4x^3e^{-x}\,dx$$

$$= 0 + 4\cdot 6 = 24.$$

More generally, $\int_0^\infty x^n e^{-x}\,dx = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$= n!$ ($\leftarrow$ 'factorial')

10.

(a) $$\int_1^\infty x^p\,dx = \begin{cases} \dfrac{x^{p+1}}{p+1}\Big|_1^\infty & \text{if } p \neq -1 \\ \ln x\Big|_1^\infty & \text{if } p = -1 \end{cases}$$

$$= \begin{cases} \dfrac{-1}{p+1} & \text{if } p < -1 \\ \infty & \text{if } p > -1 \\ \infty & \text{if } p = -1 \end{cases}$$

(b) $$\int_0^1 x^p\,dx = \begin{cases} \left.\dfrac{x^{p+1}}{p+1}\right|_0^1 & \text{if } p \neq -1 \\ \left.\ln x\right|_0^1 & \text{if } p = -1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{p+1} & \text{if } p > -1 \\ \infty & \text{if } p < -1 \\ \infty & \text{if } p = -1 \end{cases}$$

(c) Therefore, $\int_0^\infty x^p\,dx = \int_0^1 x^p\,dx + \int_1^\infty x^p\,dx$ does not converge for any value of p (since one of the integrals diverges if $p \neq -1$ or both diverge if $p=-1$).