9 Feb 2017

§1.6 continued (Applications)

Example

$$x_1\, C_3H_8 + x_2\, O_2 \longrightarrow x_3\, CO_2 + x_4\, H_2O$$

propane, oxygen, carbon dioxide, water

Find whole numbers x_1, x_2, x_3, x_4 that balance the above reaction.

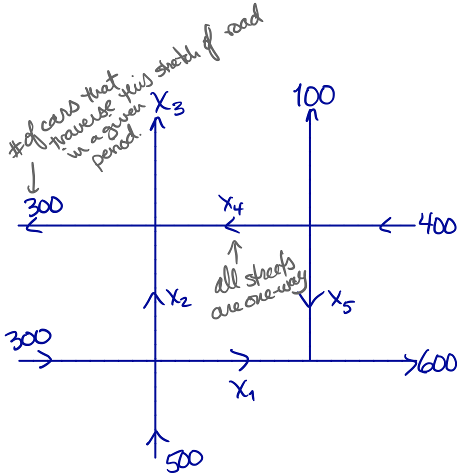
Solution

$$x_1\begin{pmatrix}3\\8\\0\end{pmatrix} + x_2\begin{pmatrix}0\\0\\2\end{pmatrix} + x_3\begin{pmatrix}-1\\0\\-2\end{pmatrix} + x_4\begin{pmatrix}0\\-2\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

steps skipped

$\Rightarrow \quad \vec{x} \in \mathrm{Span}\begin{pmatrix}1/4\\5/4\\3/4\\1\end{pmatrix}$. The least integer solution is $(1,5,3,4)$.

Example Describe the possible flows of traffic in the network shown.



Solution The constraints here are linear: traffic in = traffic out

$$\Rightarrow \quad x_2 + x_4 = x_3 + 300, \quad 400 = x_4 + x_5 + 100, \text{ etc}$$

[steps skipped] Ultimately, we get $\vec{x} = \begin{pmatrix}600-x_5\\200+x_5\\400\\500-x_5\\x_5\end{pmatrix}$. So if we knew one more flow value, we'd know them all.

§1.7 Linear dependence

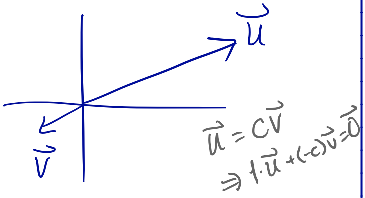
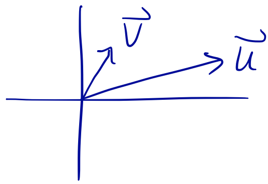
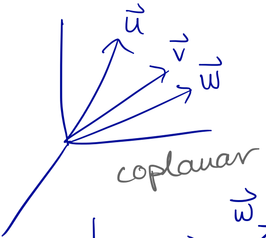
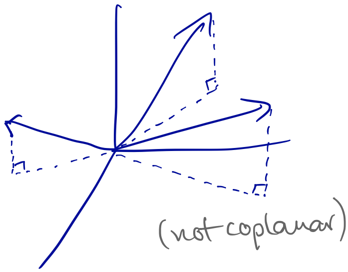
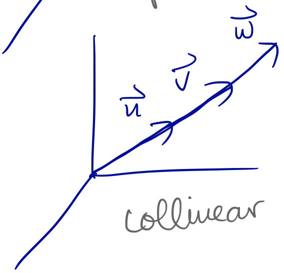
We have a term for the concept of whether a list of vectors is redundant in the sense that one or more vectors can be dropped without changing the span:

Definition $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is *linearly independent* if the only scalars $c_1, \ldots, c_n$ that make $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ are $c_1 = c_2 = \cdots = c_n = 0$.

Theorem The following are equivalent:

1. $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent
2. one of the vectors $\vec{v}_1, \ldots, \vec{v}_n$ can be written as a linear combination of the others.
3. one of the vectors may be dropped from the list without changing the span of the list.
4. $A\vec{x} = \vec{0}$ has a $\overbrace{\text{nontrivial}}^{=\text{"other than } \vec{x}=\vec{0}\text{"}}$ solution, where $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$ [i.e., the v_i's are columns]

Graphically:

Linearly dependent	Linearly independent
$\{\vec{u},\vec{v}\}$ 	$\{\vec{u},\vec{v}\}$ 
$\{\vec{u},\vec{v},\vec{w}\}$: 	$\{\vec{u},\vec{v},\vec{w}\}$ 
$\{\vec{u},\vec{v},\vec{w}\}$: 	

Example Show that $\left\{ \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 5 \end{pmatrix} \right\}$ is a linearly dependent list.

Solution: We have
$$\begin{pmatrix} 4 & -2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 1 & 2 & 5 & 0 \end{pmatrix} \to \begin{pmatrix} 4 & -2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & \frac{5}{2} & 5 & 0 \end{pmatrix} \to \begin{array}{cccc} & & \text{not a pivot column} & \\ & & \downarrow & \\ \end{array} \begin{pmatrix} 4 & -2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So x_3 is free, and every nonzero value of x_3 gives a nontrivial set of weights that $\overset{=\text{"makes equal to } \vec{0}\text{"}}{\text{annihilates}}$ the list. Like $x_3 = 1$,

$$x_2 = -\frac{6}{3}x_3 = -2, \quad x_1 = \frac{2}{4}x_2 = -1.$$

Explanation of theorem

① ⇒ ② Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent, so there exist $c_1, \ldots, c_n$ not all zero with

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$$

Let c_k be one of the nonzero scalars, let's assume $k=1$ for simplicity. Then

$$\frac{c_1\vec{V}_1}{c_1} = \frac{-c_2\vec{V}_2 - c_3\vec{V}_3 - \cdots - c_n\vec{V}_n}{c_1} \Rightarrow$$

$$\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 - \cdots - \frac{c_n}{c_1}\vec{v}_n.$$

So one of the vectors is a linear combination of the others.

② ⇒ ③ Suppose one vector is a linear combination of the others, like $v_1 = c_2v_2 + \cdots + c_nv_n$. Then any lin. com. of $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is also a lin. com. of $\{\vec{v}_2, \ldots, \vec{v}_n\}$ since we can sub.:

$$d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = d_1(c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) + d_2\vec{v}_2 + \cdots d_2\vec{v}_n = (d_1c_2 + c_2)\vec{v}_2 + \cdots + (d_1c_n + d_n)\vec{v}_n$$

(3) $\Rightarrow$ (1) If $\mathrm{Span}(\{\vec{v}_2, \dots, \vec{v}_n\}) = \mathrm{Span}(\{\vec{v}_1, \dots, \vec{v}_n\})$, then since $\vec{v}_1 \in \mathrm{Span}(\{\vec{v}_1, \dots, \vec{v}_n\})$, we have

$$\vec{v}_1 = c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

for some $c_2, \dots, c_n$. Therefore,

$$\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_n\vec{v}_n = 0$$

(this weight (1) is not zero!)

which means $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent.

(1) $\Leftrightarrow$ (4) Since $A\vec{x}$ is defined to be the linear combination of the columns of A with weights given by the entries of $\vec{x}$, the existence of a nontrivial solution of $A\vec{x}$ directly demonstrates linear dependence & vice versa.

Exercise Show that if $\vec{0}$ is in a list of vectors, then the list is linearly dependent.

__Solution__ $\underbrace{0\vec{v}_1}_{\vec{0}} + \cdots + \underbrace{1\vec{0}}_{\vec{0}} + \cdots + \underbrace{0\vec{v}_n}_{\vec{0}} = \vec{0}$.

Exercise Show that if $k > m$, then any list of k vectors is linearly dependent.

Solution We want to show $A\vec{x} = \vec{0}$ has a nontrivial solution, where A's columns are the vectors in our list. However, since there are $(k-m)$ more columns than rows, there are at least $k-m$ non-pivot columns. So there are infinitely many solutions to $A\vec{x} = \vec{0}$.

Geometric intuition: at most 1 vector fits in a line, at most two fit in the plane, 3 in space, etc., where 'fit' means that if you try to squeeze one more in, it will have to be in the span of the previous ones [assuming they weren't already linearly dependent.]