MATH 19 RECITATION
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BROWN UNIVERSITY
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Solutions

1. Find the Fourier series for $|\sin x|$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} |\sin x|\,dx$$
$$= \frac{2}{2\pi}\int_0^{\pi} \sin x\,dx$$
$$= \frac{1}{\pi}\left[-\cos\pi - (-\cos 0)\right]$$
$$= \frac{1}{\pi}\left[1+1\right]$$
$$= \frac{2}{\pi}$$

$$\boxed{\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha\\ \hline \tfrac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right) &= \sin\alpha\cos\beta.\end{aligned}}$$

$$\boxed{\begin{aligned}a_1 &= \frac{1}{2\pi}\left(\int_0^{\pi}\sin 2x - \int_{\pi}^{2\pi}\sin 2x\right)\\ &= \frac{1}{2\pi}\left[0 - 0\right] = 0\end{aligned}}$$

if $n=1$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos nx\,|\sin x|\,dx$$
$$= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\sin x\,dx + \int_{\pi}^{2\pi}\cos nx(-\sin x)\,dx\right]$$
$$= \frac{1}{\pi}\left[\frac{1}{2}\int_0^{\pi}\sin(n+1)x - \sin(1-n)x\,dx - \frac{1}{2}\int_{\pi}^{2\pi}\sin(n+1)x - \sin(1-n)x\,dx\right]$$
$$= \frac{1}{\pi}\left[\frac{1}{2}\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(1-n)x}{1-n}\right)_0^{\pi} - \frac{1}{2}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\right]_{\pi}^{2\pi}\right]$$

assuming $n>1$

these are both equal to $(-1)^{n+1}$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(1-n)\pi}{1-n} + \frac{\cos 0}{n+1} - \frac{\cos 0}{1-n}\right)\right.$$
$$\left. - \frac{1}{2}\left(\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(1-n)\pi}{1-n} + \frac{\cos 0}{n+1} + \frac{\cos 0}{1-n}\right)\right]$$
$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{2}\left(\frac{-1}{n+1} - \frac{1}{n-1}\right) + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) - \frac{1}{2}\left(\frac{-1}{n+1} + \frac{1}{n-1}\right)(-1)^{n+1} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)\right]$$
$$= \frac{1}{2\pi}\left[(-1)^{n+1}\left(\frac{-n+1-n-1}{n^2-1}\right) + \left(\frac{n-1+n+1}{n^2-1}\right) - \left(\frac{-n+1+n+1}{n^2-1}\right)(-1)^{n+1} - \left(\frac{n+1+n-1}{n^2-1}\right)\right]$$
$$= \frac{1}{2\pi}\left[\left(\frac{-2n}{n^2-1} - \frac{2}{n^2-1}\right)(-1)^{n+1} + \cancel{\left(\frac{2n}{n^2-1} - \frac{2n}{n^2-1}\right)}\right]$$
$$= (-1)^n\frac{2n}{\pi(n^2-1)}.$$

Also, $b_n = 0$ for all n, because f is even

So we get $a_0 + a_1\cos x + \cdots = \dfrac{2}{\pi} + \dfrac{4}{3\pi}\cos 3x - \dfrac{10}{24\pi}\cos 5x + \cdots$

2. Find the Fourier series for the function which is equal to x on $[0, \pi)$ and $2\pi - x$ on $[\pi, 2\pi]$.

We did this in class. We got

$$\frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x - \cdots$$

3. Differentiate the series you obtained in the previous exercise term-by-term and verify that the resulting series is the Fourier series of the derivative of f.

Differentiating term by term gives

$$\frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \cdots$$

which is the Fourier series for the square wave, which is the derivative of the triangle wave!