BROWN UNIVERSITY
PROBLEM SET 2
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DUE: 22 SEPTEMBER 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

If a and b are scalars and $\mathbf{u}$ is a vector in $\mathbb{R}^3$, then $(ab)\mathbf{u} = a(b\mathbf{u})$. Explain the difference between the meaning of $(ab)\mathbf{u}$ and the meaning of $a(b\mathbf{u})$ and use coordinates to show that the two sides are in fact equal.

Solution

$(ab)\vec{u}$ means first multiply the real numbers a and b and then multiply the resulting product by the vector $\vec{u}$

$a(b\vec{u})$ means first multiply b by the vector $\vec{u}$ and then multiply the result by a.

$$
\begin{aligned}
(ab)\vec{u} &= \langle (ab)u_1, (ab)u_2, (ab)u_3 \rangle \\
&= \langle a(bu_1), a(bu_2), a(bu_3) \rangle \\
&= a\langle bu_1, bu_2, bu_3 \rangle \\
&= a(b\langle u_1, u_2, u_3 \rangle) = a(b\vec{u}).
\end{aligned}
$$

associative property of real multiplication

Problem 2

Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors such that $\mathbf{u} \cdot \mathbf{v} = 3$, $|\mathbf{u}| = 4$, $|\mathbf{w}| = 2$, and the cosine of the angle between $\mathbf{u}$ and $\mathbf{w}$ is $\frac{3}{4}$. Show that $\mathbf{u}$ is perpendicular to $-2\mathbf{v} + \mathbf{w}$.

Solution

We have $\vec{u} \cdot \vec{w} = |\vec{u}||\vec{w}|\cos\theta = (4)(2)(\frac{3}{4}) = 6.$

then

$$
\begin{aligned}
\vec{u} \cdot (-2\vec{v} + \vec{w}) &= -2\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \\
&= -2(3) + 6 = 0.
\end{aligned}
$$

Thus $\vec{u}$ is perpendicular to $-2\vec{v} + \vec{w}$

Problem 3

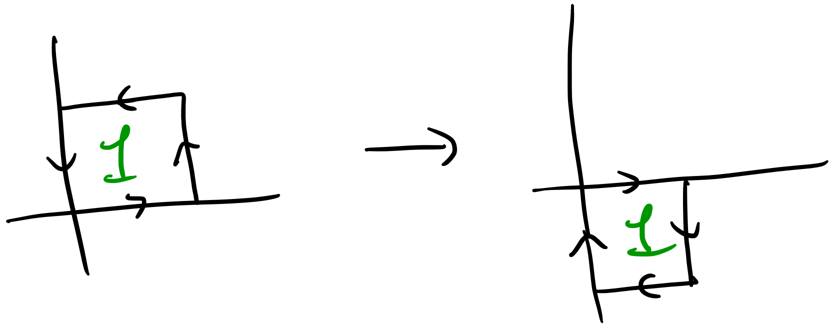
Find the determinant of each of the following matrices, and draw the image of the unit square under the corresponding linear transformations to see that value of the determinant you computed makes sense.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

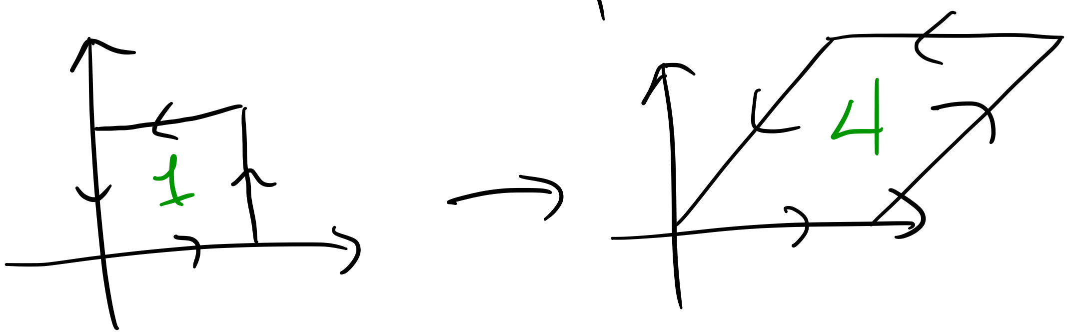
(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

Solution

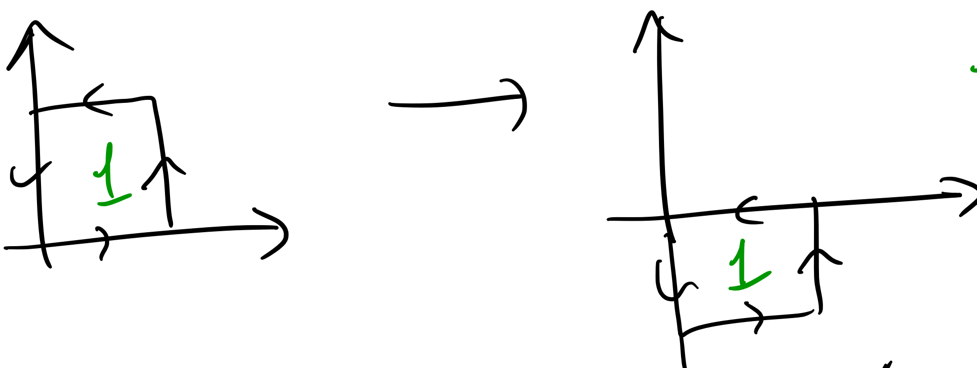
determinants: $-1, 4, 1, 0$



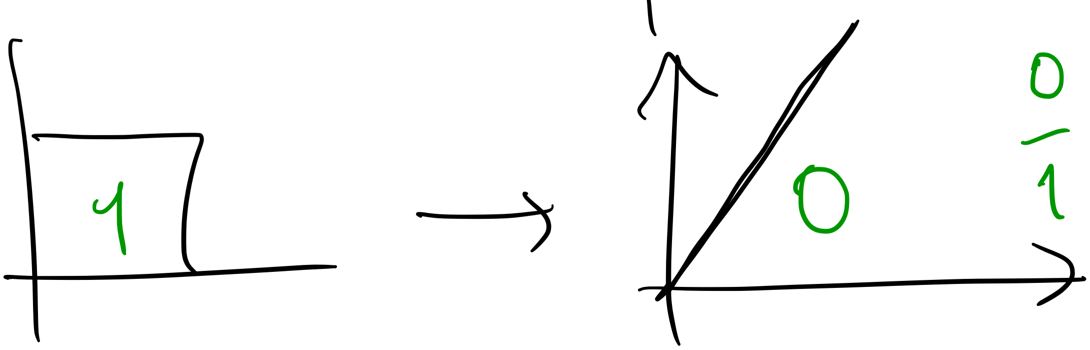
$\frac{1}{1} \stackrel{\checkmark}{=} |-1|$



$\frac{4}{1} \stackrel{\checkmark}{=} |4|$



$\frac{1}{1} \stackrel{\checkmark}{=} |1|$



$\frac{0}{1} \stackrel{\checkmark}{=} |0|$

Problem 4

Find two vectors **u** and **v** which are both perpendicular to $\langle -1, 4, 3\rangle$ and are perpendicular to each other.

Solution

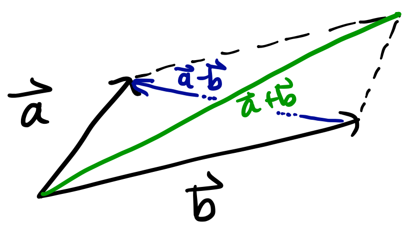
Let $\vec{u} = \langle 4,1,0\rangle$, so that $\vec{u} \cdot \langle -1,4,3\rangle = 0$. Then let $\vec{v}$ be $\langle -1,4,3\rangle \times \langle 4,1,0\rangle = \langle -3,12,-17\rangle$.

Note: many possible answers.

Problem 5

Use dot products to show that the diagonals of a parallelogram have the same length if and only if the parallelogram is a rectangle. (Hints: let **a** and **b** be vectors along two sides of the parallelogram, and express vectors running along the diagonals in terms of **a** and **b**.)

Solution



Suppose the diagonals have the same length, so $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$.

Squaring both sides gives $|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2 \Rightarrow$

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) \Rightarrow$$

$$|\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \Rightarrow$$

$$4\vec{a}\cdot\vec{b} = 0 \Rightarrow$$

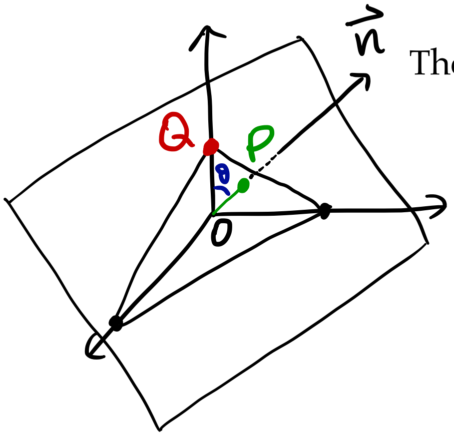
$\vec{a}, \vec{b}$ are perpendicular.

Conversely, if $\vec{a}$ and $\vec{b}$ are perpendicular, then the same steps in reverse show that $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$.

Problem 6

Find the distance from the origin to the plane $x + 2y + 3z = 6$.

Solution



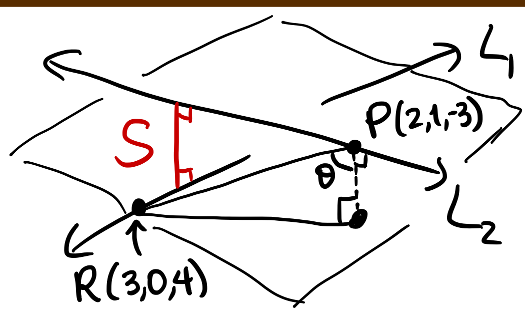
The vector of coefficients $\vec{n} = \langle 1,2,3 \rangle$ is normal to the plane. The point $Q = (0,0,2)$ is on the plane. We can see in the figure that the distance OP is equal to $|\overrightarrow{OQ}|\cos\theta$, &

$$|\overrightarrow{OQ}|\cos\theta = \frac{\overrightarrow{OQ}\cdot\vec{n}}{|\vec{n}|} = \frac{6}{\sqrt{1^2+2^2+3^2}} = \boxed{\frac{6}{\sqrt{14}}}$$

Problem 7

The line L_1 is described by the parametric equation $(x(t), y(t), z(t)) = (3 + 2t, t, 4)$, and the line L_2 passes through the points $P(2, 1, -3)$ and $Q(0, 8, 4)$. These two lines are *skew*, meaning that they do not intersect and are not parallel. Find the shortest possible distance between a point on L_1 and a point on L_2.

Solution



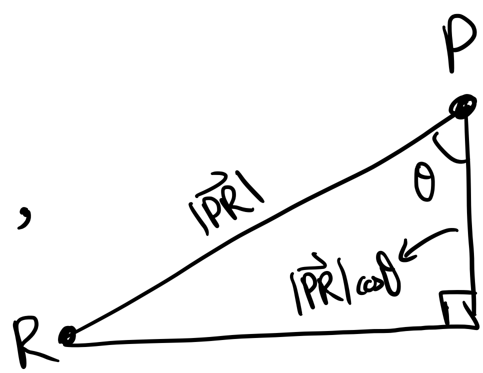
Geometric intuition suggests that the segment S connecting the closest pair of points on the two lines is perpendicular to both lines, as shown. Indeed, if either angle were not 90 degrees, then the segment could be shortened by moving one point or the other to make the angle closer to 90 degrees.

Therefore, we can cross vectors running parallel to each line to get the direction of S:

$$\vec{n} := \langle 2,1,0 \rangle \times \langle -2,7,7 \rangle = \langle 7,-14,16 \rangle.$$

The component of $\overrightarrow{PR}$ in the direction of $\vec{n}$ is

$$|\overrightarrow{PR}|\cos\theta = \frac{\overrightarrow{PR}\cdot\vec{n}}{|\vec{n}|} = \frac{133}{\sqrt{501}},$$



and this is the distance between the two planes.

Additional space