18.022 Recitation Handout 8 December 2014

Let $D \subset \mathbb{R}^2$ be the region enclosed by the curve $r = g(\theta)$, for some C^1 , non-negative $g \colon \mathbb{R} \to \mathbb{R}$ such that $g(x + 2\pi) = g(x)$ for all $x \in \mathbb{R}$.

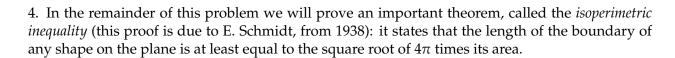
1. Calculate the length of ∂D , the boundary of D. Express your answer as an integral involving g and its first derivative.

2. Let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D . Calculate the length of ∂D again, but this time express the answer as an integral involving the derivatives of x and y.

3. Calculate the length of ∂D for the case that $g(\theta) = 1 - \cos \theta$.



Let C be the unit circle. Explain why

$$\operatorname{area}(D) + \pi = \oint_{\partial D} (0, x) \cdot d\mathbf{s} + \oint_{C} (-y, 0) \cdot d\mathbf{s}. \tag{1}$$

5. Assume henceforth that $g(x) \le 1$ and $g(0) = g(\pi) = 1$. Show that

$$(x(\theta), w(\theta)) = \begin{cases} (g(\theta)\cos\theta, \sqrt{1 - g(\theta)^2\cos^2\theta} & 0 \le \theta \le \pi \\ (g(\theta)\cos\theta, -\sqrt{1 - g(\theta)^2\cos^2\theta} & \pi \le \theta \le 2\pi \end{cases}$$
 (2)

is a parametrization of a unit circle.

6. Let $(x(\theta), w(\theta))$ be the parametrization of the unit circle C from (2). Again let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D . Using (1), show that

$$\operatorname{area}(D) + \pi = \oint_0^{2\pi} (x(\theta), -w(\theta)) \cdot (y'(\theta), x'(\theta)) \, d\theta.$$

7. Explain why it follows from the previous question that

$$\operatorname{area}(D) + \pi \le \oint_0^{2\pi} \sqrt{(x(\theta)^2 + w(\theta)^2) \cdot (x'(\theta)^2 + y'(\theta)^2)} \, \mathrm{d}\theta.$$

8. Explain why

$$area(D) + \pi \le length(\partial D)$$
.

9. Recall the AMGM inequality: for a, b > 0 it holds that $\sqrt{ab} \le (a + b)/2$. Use this to show that $\sqrt{4\pi \cdot \operatorname{area}(D)} \le \operatorname{length}(\partial D)$.

For which shape are these two quantities equal?