MATH 520 PRACTICE MIDTERM II
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BROWN UNIVERSITY
SAMUEL S. WATSON

Solutions

This is a pencil-and-paper-only exam. You have two hours.

Problem 1(a)

Solve the matrix equation

$$AB\mathbf{x} + \mathbf{b} = 2AB\mathbf{x}$$

for $\mathbf{x}$, where A and B are invertible $n \times n$ matrices and $\mathbf{b}$ is an $n \times 1$ vector. Your final answer should be in terms of A, B, and $\mathbf{b}$ and should not contain parentheses.

Solution

$$\begin{aligned} AB\vec{x} + \vec{b} &= 2AB\vec{x} \\ -AB\vec{x} \quad &\quad -AB\vec{x} \\ \hline \vec{b} &= AB\vec{x} \\ A^{-1}\vec{b} &= B\vec{x} \\ B^{-1}A^{-1}\vec{b} &= \vec{x} \end{aligned}$$

Final answer:

$$\vec{x} = B^{-1}A^{-1}\vec{b}$$

Problem 1(b)

Show by substitution that the matrix $C = B^{-1}A$ satisfies the matrix equation $B^2CA^{-1} = B$.

Solution

$$\begin{aligned} B^2(B^{-1}A)A^{-1} &= BBB^{-1}AA^{-1} \\ &= BII \\ &\overset{\checkmark}{=} B \end{aligned}$$

Problem 2

The matrices

$$\begin{bmatrix} -2 & 4 & 6 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -3 & 1 & -4 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -2 & 0 & 1 & -15 & -6 \\ 0 & 1 & 0 & -1 & 13 & 5 \\ 0 & 0 & 1 & \frac{1}{2} & -5 & -2 \end{bmatrix}$$

are row equivalent. Find

$$\begin{bmatrix} -2 & 4 & 6 \\ 1 & 0 & 2 \\ -3 & 1 & -4 \end{bmatrix}^{-1}.$$

Solution

We just need to do one more row operation to get this part to be I. So:

$$\left[\begin{array}{ccc|c} -2 & 4 & 6 & \\ 1 & 0 & 2 & I \\ -3 & 1 & -4 & \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 11 & 4 \\ 0 & 1 & 0 & -1 & 13 & 5 \\ 0 & 0 & 1 & \frac{1}{2} & -5 & -2 \end{array}\right]$$

Final answer:

$$\begin{bmatrix} -1 & 11 & 4 \\ -1 & 13 & 5 \\ \frac{1}{2} & -5 & -2 \end{bmatrix}$$

Problem 3(a)

The set $\mathcal{M}_{2\times 2}$ of 2×2 matrices with real entries, equipped with matrix addition and scalar multiplication, is a vector space. The *trace* $T(A)$ of a 2×2 matrix A is defined to be the sum of its diagonal entries. In other words, the trace of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $a+d$. Show that T is a linear transformation from $\mathcal{M}_{2\times 2}$ to $\mathbb{R}^1$.

Solution

We check

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = T\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right)$$

$$= a+e+d+h$$

$$\overset{\checkmark}{=} T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$$

and $$T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = ka+kd$$

$$\overset{\checkmark}{=} kT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Problem 3(b)

Find the rank and the nullity of T.

Solution

The range of T is all of $\mathbb{R}$, because for any $x \in \mathbb{R}$, $T\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = x$. So the rank is 1.

The null space of T is the set $\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

$= \operatorname{Span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}\right)$, & so is 3-dimensional.

[or: $\operatorname{nullity} T + \operatorname{rank} T = \dim M_{2\times 2}$
$= 4$
$\Rightarrow \operatorname{nullity} T = 3$]

Final answer:

rank $T = 1$

nullity $T = 3$

Problem 4(a)

For which values of t is the following matrix invertible? Hint: this problem requires almost no computation; inspect the matrix carefully.

$$\begin{bmatrix} 2 & -3 & 5 & 1 & 5 & -2 \\ 1 & 1 & -5 & -3 & 0 & -5 \\ 2 & -3 & 5 & t^2 & 5 & -2 \\ -4 & -3 & -2 & 4 & -2 & -1 \\ 5 & -5 & 3 & -4 & 0 & -4 \\ -2 & -5 & 1 & 3 & -3 & 5 \end{bmatrix}$$

Solution

The first and third rows are equal if $t^2 = 1$, so the det is zero when $t \in \{-1, 1\}$. Furthermore, the det is a quadratic polynomial in t, since the t^2 entry appears at most once in each term of the expansion of the determinant. So there can be at most two values of t that make $\det A = 0$.

Final answer:

$$\mathbb{R} \setminus \{-1, 1\}$$

$(= \{x \in \mathbb{R} : x = 1 \text{ and } x \neq -1\})$

Problem 4(b)

Show that if A is a square matrix then $\det(A^T A) \geq 0$.

Solution

$$\det(A^T A) = \det A^T \det A$$
$$= (\det A)^2 \geqslant 0.$$

Problem 5

Show that $S = \{f \in C([0,1]) : f(0)f(1) \leq 0\}$ is not a linear subspace of $C([0,1])$. (In words: S is the set which contains every continuous function f from $[0,1]$ to $\mathbb{R}$ with the property that its values at 0 and at 1 have a nonpositive product.)

Solution

Take $f \in C([0,1])$ so that $f(0)=3$, $f(1)=-4$

& $g \in C([0,1])$ so that $g(0)=-4$, $g(1)=3$

Then

$$(f+g)(0)\,(f+g)(1) = (3-4)(-4+3)$$
$$= (-1)(-1)$$
$$= 1.$$

So $f, g \in S$, but $f+g \notin S$. Thus

S is not a subspace.

Problem 6

Consider the vector space $\mathbb{P}_3$ of polynomials of degree 3 or less, and consider the basis

$$\mathcal{B} = \{1, \quad 1+t, \quad 1+t+t^2, \quad 1+t+t^2+t^3\}$$

of $\mathbb{P}_3$. Find the coordinates of $-1+t^2-3t^3$ with respect to $\mathcal{B}$.

Solution

$$\begin{aligned}
-1+t^2-3t^3 = &-3(t^3+t^2+t+1) \\
&+4(t^2+t+1) \\
&-1(t+1) \\
&-1(1)
\end{aligned}$$

so $[-1+t^2-3t^3]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -3 \end{bmatrix}$

Final answer:

$$\begin{bmatrix} -1 \\ -1 \\ 4 \\ -3 \end{bmatrix}$$

Problem 7(a)

Suppose that W is a ten-dimensional vector space. Suppose that U and V are subspaces of W, and that $\dim U = 8$ and $\dim V = 4$. Show that $U \cap V$ is a subspace of W.

Solution

Suppose $v \in U \cap V$ and $w \in U \cap V$. Then $v \in U$ and $v \in V$ and $w \in U$ and $w \in V$. So $v + w \in U$, since U is a subspace. and $v + w \in V$ since V is a subspace. So, $v + w \in U \cap V$.

Similarly, if $v \in U \cap V$ and $c \in \mathbb{R}$, then $v \in U$ and $v \in V$ so $cv \in U$ and $cv \in V$, so $cv \in U \cap V$.

Lastly, $\vec{0} \in U$ and $\vec{0} \in V$, so $\vec{0} \in U \cap V$.

So $U \cap V$ is a subspace.

Problem 7(b)

Show that $2 \leq \dim U \cap V \leq 4$.

Solution

Since $U \cap V \subset V$, any basis of V spans $U \cap V$. Therefore, $\dim(U \cap V) \leq 4$.

Conversely, suppose $k = \dim(U \cap V)$ and $\{w_1, \ldots, w_k\}$ is a basis for $U \cap V$. Extend $\{w_1, \ldots, w_k\}$ to a basis $\{w_1, \ldots, w_8\}$ for U and a basis $\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_4\}$ for V. Then the combined list $\{w_1, \ldots w_8, v_{k+1}, \ldots v_4\}$ has $12-k$ vectors in it, and they are all in the 10-dim. space W. Also, they are linearly independent, since $\{w_1, \ldots, w_8\}$ is lin. ind., and none of $v_{k+1}, \ldots, v_4$ is in the span of the preceding vectors in the list since none of them are in U [for if they were, they'd be in $U \cap V$ and thus $\{w_1, \ldots, w_k, v_{k+1}, \ldots v_4\}$ would have been lin. dep.]. So, $12 - k \leq 10 \Rightarrow k \geq 2$.