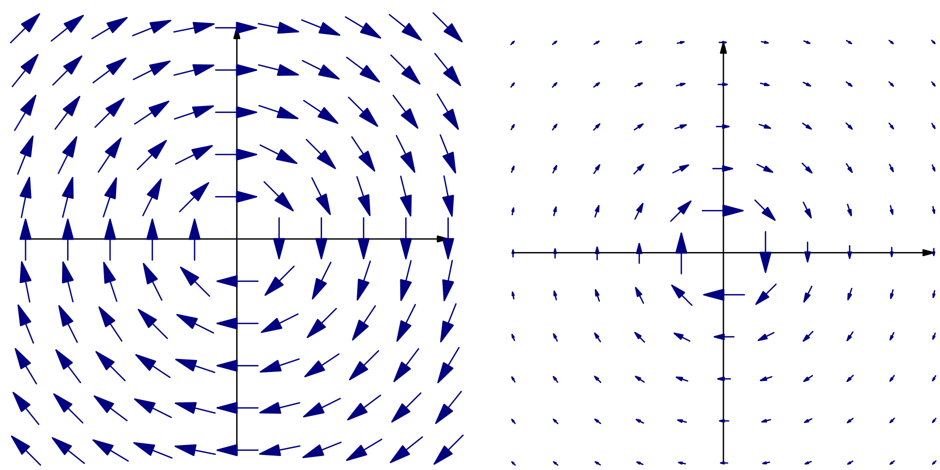
BROWN UNIVERSITY
PROBLEM SET 10
INSTRUCTOR: SAMUEL S. WATSON
DUE: 24 NOVEMBER 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

Sketch the vector fields $\mathbf{F}_1(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$ and $\mathbf{F}_2(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{x^2 + y^2}$.

Solution



Problem 2

Find $\int_C (x + 2y)\,dx + x^2\,dy$ where C is the concatenation of the line segment from $(0, 0)$ to $(2, 1)$ and the line segment from $(2, 1)$ to $(3, 0)$. (Note: the notation $(x + 2y)\,dx + x^2\,dy$ is another way of writing $\mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle x + 2y, x^2 \rangle$.)

Solution

The first segment can be parametrized as $\mathbf{r}(t) = \langle 2t, t \rangle$ as t ranges from 0 to 1, so the integral along that segment is

$$\int_0^1 \langle 2t + 2t, 4t^2 \rangle \cdot \langle 2, 1 \rangle\,dt = \int_0^1 8t + 4t^2\,dt = 4 + \frac{4}{3} = \frac{16}{3}.$$

The second segment can be parametrized as $\langle 2 + t, 1 - t \rangle$ as t ranges from 0 to 1, so the integral along that segment is

$$\int_0^1 \langle 2 + t + 2(1 - t), (2 + t)^2 \rangle \cdot \langle 1, -1 \rangle\,dt = \int_0^1 4 - t - (2 + t)^2 dt = -\frac{17}{6}.$$

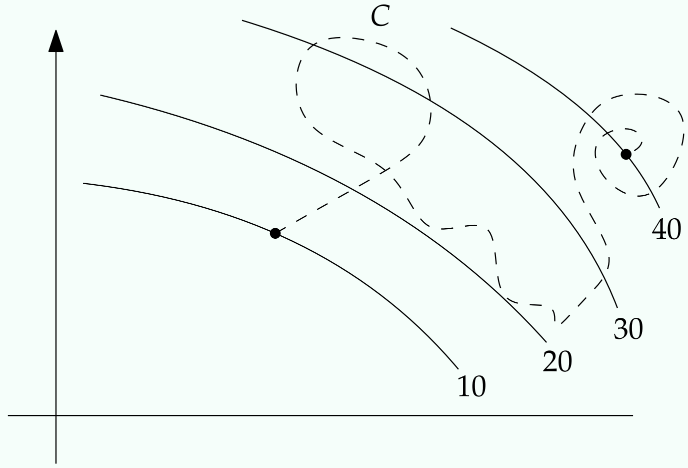
So the integral along the concatenation of the two segments is $\dfrac{16}{3} - \dfrac{17}{6} = \dfrac{5}{2}$.

Final answer:

$$\frac{5}{2}$$

Problem 3

Some of the contour lines of a function $f(x, y)$ are shown in the figure below. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C is the curve shown dashed starting at the left point and ending at the right point.



Solution

By the fundamental theorem of vector calculus, the integral of ∇f along a curve is equal to the difference between the values of f at the starting and ending points. So we get

$$f(\text{end point}) - f(\text{starting point}) = 40 - 10 = 30.$$

Final answer:

$$30$$

Problem 4

Find $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$ where C is any path from $(1, 0)$ to $(2, 1)$. Make sure to explain why your answer is correct regardless of which path C you choose.

Solution

The vector field $\langle 2xe^{-y}, 2y - x^2e^{-y}\rangle$ is the gradient of $y^2 + x^2e^{-y}$. By the fundamental theorem of vector calculus,

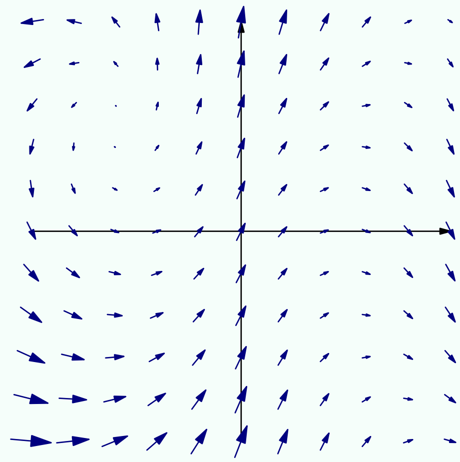
$$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy = y^2 + x^2e^{-y}\Big|_{(2,1)} - y^2 + x^2e^{-y}\Big|_{(0,0)} = 1 + \frac{4}{e}.$$

Final answer:

$$1 + \frac{4}{e}$$

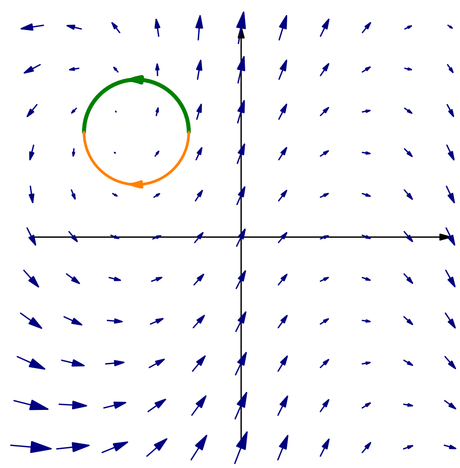
Problem 5

The vector field $\mathbf{F}$ plotted below is not conservative. Pick two points a and b and sketch two paths from a to b along which the line integrals of $\mathbf{F}$ are clearly different, and explain why the integrals are clearly different.



Solution

The integrals along the following two paths have opposite sign, since the integral along the green one is positive, and the integral along the orange one is negative.



Problem 6

Use Green's theorem to find the line integral of $\mathbf{F} = \langle \sqrt{x^2+1}, \arctan x \rangle$ along a counterclockwise traversal of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Solution

Green's theorem tells us the integral of $\mathbf{F} = \langle M, N \rangle$ around the given curve is equal to the double integral of $N_x - M_y = \frac{1}{1+x^2}$ over the triangle with the given vertices. We set this up as an iterated integral and get

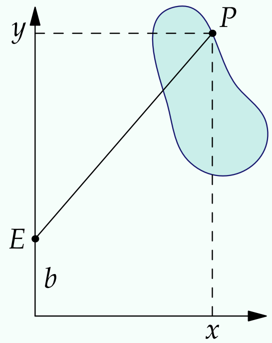
$$\int_0^1 \int_0^{1-x} \frac{1}{1+x^2}\, dy\, dx = \int_0^1 \frac{1-x}{1+x^2}\, dx = \frac{\pi}{4} - \frac{\log 2}{2}.$$

Final answer:

$$\frac{\pi}{4} - \frac{\log 2}{2}$$

Problem 7

A planimeter is a device used to calculate the area of a two-dimensional region. In this problem, we explore the mathematics behind how the planimeter works. (Thanks to Wikipedia for the picture and the description below).



The pointer P at one end of the planimeter follows the contour C of the surface S to be measured. For the linear planimeter the movement of the "elbow" E is restricted to the y-axis. Connected to the arm PE is the measuring wheel with its axis of rotation parallel to PE. A movement of the arm PE can be decomposed into a movement perpendicular to PE, causing the measuring wheel to rotate, and a movement parallel to PE, causing the measuring wheel to skid, with no contribution to its reading.

Use Green's theorem to explain why the final reading on the measuring wheel is equal to the area of the surface S.

Hints: (i) define $b(x, y)$ to be the y-coordinate of the point E when the needle is at $P = (x, y)$. Then (ii) find the component of $\langle \Delta x, \Delta y \rangle$ which is perpendicular to $\overrightarrow{EP}$ and use your result to set up a line integral whose value equals to final reading on the meter. Then (iii) show that that the value of line integral is equal to the desired area.

Solution

Let the coordinates of P be x and y. Note that for each (x, y) close enough to the y-axis, there are two points on the y-axis whose distance from (x, y) is equal to the length $|EP|$ of the elbow. Define $b(x, y)$ to be the y-coordinate of the lower one. Note that as the needle makes a small move $\langle \Delta x, \Delta y \rangle$ along C, the measuring wheel only registers the component of $\langle \Delta x, \Delta y \rangle$ in the direction perpendicular to $\overrightarrow{EP} = \langle x, y - b \rangle$. The vector $\langle -(y - b), x \rangle$ is perpendicular to $\overrightarrow{EP}$, so the measuring wheel increments by

$$\langle \Delta x, \Delta y \rangle \cdot \langle -(y - b(x, y)), x \rangle$$

during this small step. Thus the final measurement on the planimeter is

$$\oint_C -(y - b(x, y))\, dx + x\, dy.$$

By Green's theorem, this is equal to

$$\iint_S \left(\frac{\partial x}{\partial x} + \frac{\partial (y - b(x, y))}{\partial y} \right) dx\, dy.$$

We claim that $\dfrac{\partial b(x, y)}{\partial y} = 1$. Indeed, if x is held constant while y is increased or decreased, $b(x, y)$ will change by the same amount that y changed (this is apparent geometrically; there is no need for a formula for b). Thus the partial derivative of b with respect to y is 1. Substituting into the above integral, we get $\iint_S dx\, dy = \text{area}(S)$.

Problem 8

Find the flow (from the inside to the outside) of $\mathbf{F}(x, y, z) = \langle xy, y, xz \rangle$ through the cube $[0, 1]^3$.

Solution

The normal vector for the top face is $\langle 0, 0, 1 \rangle$, so the flow through the top (we can substitute $z = 1$, since that is true for all points in that face) is

$$\int_0^1 \int_0^1 \langle xy, y, x \rangle \cdot \langle 0, 0, 1 \rangle \, dx \, dy = \frac{1}{2}.$$

The flow through the bottom face is

$$\int_0^1 \int_0^1 \langle xy, y, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dx \, dy = 0.$$

The flow through the $x = 1$ face is

$$\int_0^1 \int_0^1 \langle y, y, z \rangle \cdot \langle 1, 0, 0 \rangle \, dy \, dz = \frac{1}{2},$$

and through the $x = 0$ face:

$$\int_0^1 \int_0^1 \langle 0, y, 0 \rangle \cdot \langle -1, 0, 0 \rangle \, dy \, dz = 0.$$

Finally, the flow through the $y = 1$ and $y = 0$ faces are 1 and 0, respectively, so the net flow from inside to outside is $\frac{1}{2} + \frac{1}{2} + 1 = \boxed{2}$.

Final answer:

2