Brown University
DATA 1010
Fall 2018: Practice Midterm III
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Name:

You will have three hours to complete the exam, which consists of 36 questions. Among the first 24 questions, you should only solve problems for standards for which you want to improve your medal from the second exam.

No calculators or other materials are allowed, except the provided reference sheets.

You are responsible for explaining your answer to **every** *question. Your explanations do not have to be any longer than necessary to convince the reader that your answer is correct.*

For questions with a final answer box, please write your answer as clearly as possible and strictly in accordance with the format specified in the problem statement. Do not write anything else in the answer box. Your answers will be grouped by Gradescope's AI, so following these instructions will make the grading process much smoother.

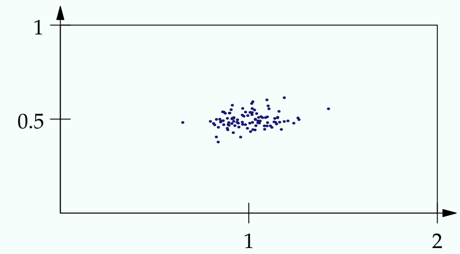
I verify that I have read the instructions and will abide by the rules of the exam: ____________________________

Problem 1 **[KDE]**

(a) Suppose that $f_\lambda(x, y)$ is the bandwidth-λ kernel density estimator associated with the set of samples shown in the figure (based on the tri-cube weight function).

Estimate the value of λ such that the points (x, y) for which $f_\lambda(x, y) = 0$ make up approximately half of the rectangle by area.

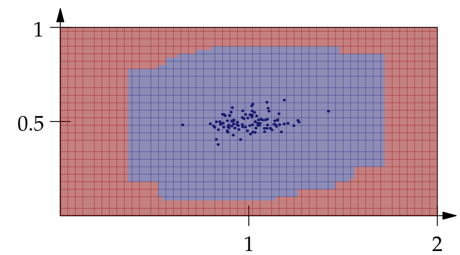
(b) Consider a set of points $\{(x_i, y_i)\}_{i=1}^n$ in $\mathbb{R}^2$ and a positive value of λ. Suppose that the vertical line $x = a$ passes through the three sample points (x_1, y_1), (x_2, y_2), and (x_3, y_3), and that no other sample points have an x value within λ of a. Find the value of $r_\lambda(a)$ (where r_λ is the Nadaraya-Watson estimator associated with the samples).



Solution

(a) If we choose a λ value of $\frac{1}{2}$, the union of the side-length-2λ squares centered at all the samples would fill up nearly the whole rectangle. I would estimate $\lambda \approx 0.3$ to fill up half the rectangle. (Checking it by computer reveals the answer to be about 0.302.)

(b) Integrating the kernel density estimator along a vertical line results in a weighted average of the y-values of the centers of the intersecting squares, with weight given by the kernel function evaluated at the horizontal distance to the square center. Since the vertical line passes through the center of each square, the weights are equal, so the result is $\frac{1}{3}(y_1 + y_2 + y_3)$.

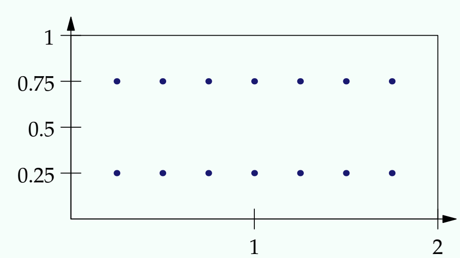


Final answer:

$\frac{1}{3}(y_1 + y_2 + y_3)$

Problem 2 **[LR]**

Find the residual sum of squares for the line of best fit for the samples shown.



Solution

The line which minimizes the residual sum of squares is $y = 0.5$, because for any function $r(x)$, the sum of squared residuals along each vertical line through a pair of sample points is $(0.75 - r(x))^2 + (r(x) - 0.25)^2$, which can be no smaller than its value when $r(x) = 0.5$.

Therefore, the minimum RSS is $14(0.25)^2 = \boxed{7/8}$.

Final answer:

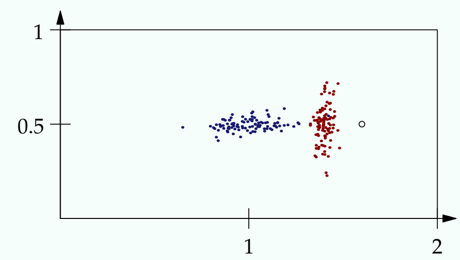
$\frac{7}{8}$

Problem 3 [QDA]

Select which of the two statements is correct (given that one of them is correct), and explain why the two classifiers behave differently.

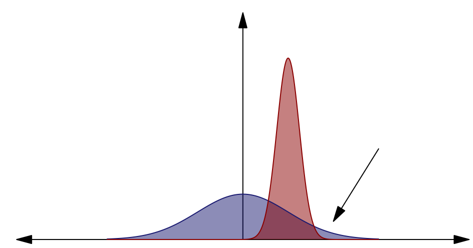
(a) The point marked with a hollow circle is classified as blue by a QDA classifier and red by a kernel density classifier.

(b) The point marked with a hollow circle is classified as red by a QDA classifier and blue by a kernel density classifier.



Solution

(a) is correct. The kernel density estimator will assign more red mass to the location of the hollow point, since there are many more nearby red points. However, QDA assumes that the distributions are multivariate Gaussian, which means that the horizontal component of blue distribution is a Gaussian with smaller mean but larger variance than the horizontal component of the red distribution. It can therefore be larger at the hollow point (as shown in the figure).



Final answer:

(a)

Problem 4 [STATLEARN]

Give an example which shows that a simple linear regression model can overfit the data. Give some ideas for how to mitigate the overfitting.

Solution

For simplicity, we'll consider linear regression with no bias term, though the same conclusions would apply. Suppose the $n \times 2$ feature matrix's columns measure the same quantity but with a different rounding rule, so that the columns are not actually identical. In geometric terms, those two columns point in nearly the same direction, but they nevertheless span a plane in $\mathbb{R}^n$.

Performing ordinary least squares linear regression will identify the vector in that plane which is closest to the vector of response values. However, this vector might very well be far from the lines spanned by the two columns. In that case, the regression optimization will leverage the difference between the two columns (which is noise) to obtain a vector which is much closer to the response vector than is meaningfully possible.

There are many possible solutions to this problem. We could penalize large regression coefficients in the loss functional, we could use PCA and throw out components corresponding to small singular values, or we could simply omit one of the two columns.

Problem 5 **[NPL]**

Show that the Bayes classifier is the classifier which minimizes the misclassification probability.

For simplicity, you may assume the context of a binary classification problem with a discrete sample space.

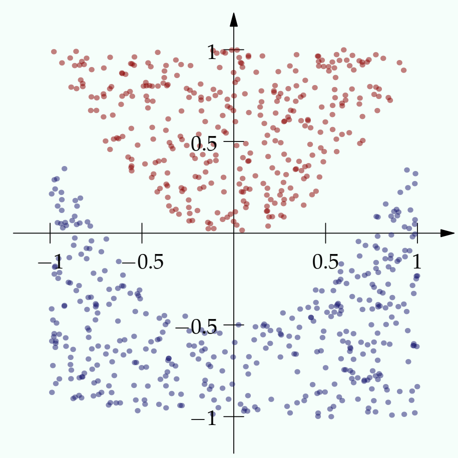
Solution

The misclassification probability is a sum of the probability masses at each un-predicted point in the space $\mathcal{X} \times \mathcal{Y}$. For example, if $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{-1, 1\}$, and `h.([1,2,3]) == [-1,+1,-1]`, then the misclassification probability would be the sum of the probability masses at $(1, +1)$, $(2, -1)$, and $(3, +1)$.

Since we can choose predictions for each $x \in \mathcal{X}$ independently, we can minimize the misclassification probability by classifying each x according to whether $(x, +1)$ or $(x, -1)$ has larger mass. In other words, we classify according to whether $p_{+1}f_{+1}(x)$ is larger than $p_{-1}f_{-1}(x)$. Since that is the rule for the Bayes classifier, this shows that the Bayes classification rule does minimize the misclassification probability.

Problem 6 **[SVM]**

Find a map from the plane to some other Euclidean space such that hard-margin SVM could, after applying the map, be used for classification problem shown in the figure.



Solution

We could map (x, y) *down* in dimension to $y - x^2 \in \mathbb{R}$. Then all of the red samples would lie to the right of the origin, and all of the blue samples would lie to the left.

Alternatively, we could apply the map $(x, y) \mapsto (x, y, y - x^2)$, which retains the original information in the data but also permits a separating plane (any plane of the form $z = c$, where c is between $-\frac{1}{2}$ and 0).

Final answer:

$(x, y) \mapsto$
$(x, y, y - x^2)$

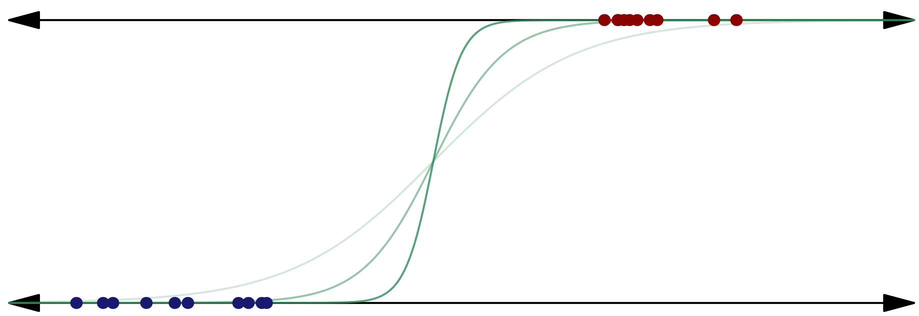
Problem 7 **[LOGIST]**

Consider a binary classification problem for which there exists a hyperplane separating the classes. What goes wrong if you try to apply logistic regression?

Solution

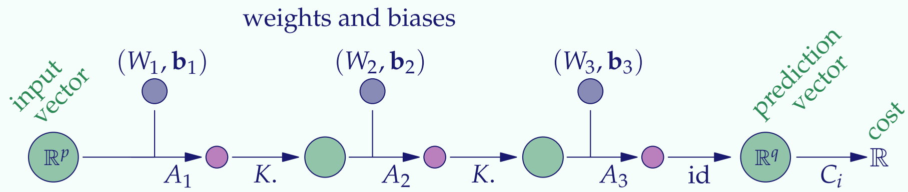
The problem is that the logistic regression optimization problem is unbounded. Given any logistic function $\mathbf{x} \mapsto \sigma(\boldsymbol{\beta} \cdot \mathbf{x} + \alpha)$ whose decision boundary separates the two classes, we can improve it by merely scaling up $\boldsymbol{\beta}$ and α. This will preserve the decision boundary while enhancing the confidence of all predictions (which are already correct). Therefore, the loss function can always be decreased, and no minimum exists.

The figure below shows a 1D classification problem with separable classes. Higher opacity corresponds to scaled-up α and $\boldsymbol{\beta}$ values.



Problem 8 **[NN]**

Suppose that weights and biases have been chosen for the neural network shown, and that a vector has been forward propagated through the network. Suppose that the vectors recorded at the purple nodes are $[1, -4, 2]$, $[6, 3]$, and $[9, 7, -4, -1, 5]$.



(a) What is the architecture of this neural net?

(b) What vector is recorded at the second green node (the one between A_1 and A_2)?

(c) Now suppose that we are in the midst of the backpropagation process, and we have just determined that the derivative of the cost with respect to the vector in the second purple node is equal to $[-3, -4]'$. Calculate the derivative of the cost with respect to the matrix W_2.

Solution

(a) The architecture is $[p, 3, 2, 5]$, since those are the dimensions of the Euclidean spaces in the green nodes in the diagram.

(b) We apply $K.$ to $[1, -4, 2]$, and we get $[1, 0, 2]$.

(c) We learned that this derivative is equal to the outer product of the *gradient* matrix stored in the following purple node and the original vector stored in the previous green node. So we get that the derivative of cost with respect to W_2 is

$$\begin{bmatrix} -3 \\ -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -6 \\ -4 & 0 & -8 \end{bmatrix}$$

Problem 9 [DR]

The first graph below shows the dot product of each of the first 5000 (de-meaned) vectors in the MNIST training set with the first principal component and the second principal component. The remaining graphs are similar, but using different pairs of principal components. One uses the second and third, one uses the second and tenth, one uses the 80th and 81st, and one uses the 80th and 120th. Figure out which is which.

Solution

The first one shows the first two principal components, as stated in the problem. The key idea for identifying the remaining four is that the principal components are arranged in decreasing order of variation. This means that the sum of squared lengths of the vectors obtained by projecting the points onto the first principal component is larger than for the second principal component, which in turn is larger than for the third, and so on.

So, the middle and last figures show $(2, 3)$ and $(2, 10)$, which we can distinguish by the amount of vertical variation. The remaining two show $(80, 120)$ and $(80, 81)$, respectively, which we can again distinguish by the amount of vertical variation.

Problem 10 **[R]**

(a) Write an R function which computes the ReLU activation function of a vector:
`all(ReLU(c(-2,4,-5,0,1)) == c(0,4,0,0,1))`

(b) Write a line of R code with the same effect as the Julia line `[x > 0 ? 2x : -x for x in v]`, where `v` is a vector. Hint: use `ifelse` and `sapply`.

(c) What is difference between `A * A` in Julia and `A * A` in R (where `A` is a matrix)? What is Julia's equivalent of R's `A * A`, and what is R's equivalent of Julia's `A * A`?

Solution

(a) We make a boolean vector indicating where the positive entries are and then multiply that by `v`. Since `TRUE` is treated arithmetically as `1` and `FALSE` is treated as `0`, this has the desired effect.

```
ReLU <- function(v) {v * (v > 0)}
```

(b) We make an anonymous function which evaluates the given ternary conditional and use `sapply`:

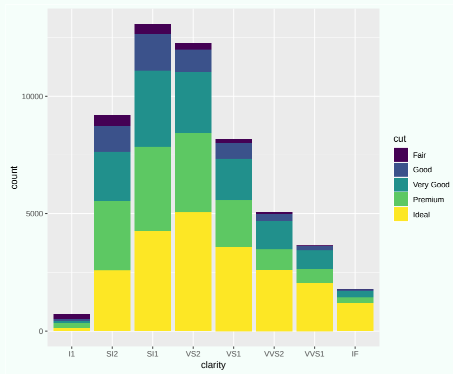
```
sapply(v, function(x) {ifelse(x > 0,2*x,-x)})
```

(c) In Julia, the asterisk operator performs matrix multiplication, while in R it performs pointwise multiplication. To obtain matrix multiplication in R we use `%*%`, and to obtain pointwise multiplication in Julia we use `.*`.

Problem 11 [GGPLOT]

Write code to generate the figures shown. The name of the data frame is `diamonds`.

Note that no style customizations have been applied. If you get the geoms and aesthetic mappings correct, your code will produce the desired figures.



Solution

The geom is `geom_bar`, with an aesthetic mapping of clarity to x, and fill color to cut.

```
library(tidyverse)
ggplot(diamonds) +
  geom_bar(aes(x = clarity, fill = cut))
```

For the second figure, we're using the point geom, and color is mapped to `clarity` (as well as `carat` to x and `price` to y).

```
ggplot(diamonds) +
  geom_point(aes(x = carat, y = price, color = clarity))
```

Problem 12 **[DPLYR]**

The first few records in the data frame `diamonds` are shown below.

	carat	cut	color	clarity	depth	table	price	x	y	z
1	0.23	Ideal	E	SI2	61.50	55.00	326	3.95	3.98	2.43
2	0.21	Premium	E	SI1	59.80	61.00	326	3.89	3.84	2.31
3	0.23	Good	E	VS1	56.90	65.00	327	4.05	4.07	2.31
4	0.29	Premium	I	VS2	62.40	58.00	334	4.20	4.23	2.63
5	0.31	Good	J	SI2	63.30	58.00	335	4.34	4.35	2.75
6	0.24	Very Good	J	VVS2	62.80	57.00	336	3.94	3.96	2.48

(a) Write code which returns a data frame containing only the ideal cut diamonds which are at least one carat.

(b) Write code which returns a data frame whose rows correspond to color values and whose columns show the maximum price and average "volume" (where "volume" is defined to be xyz).

Solution

(a) We apply a filter with two conditions:

```
library(tidyverse)
diamonds %>% filter(cut == 'Ideal', carat >= 1)
```

(b) We group by color and `summarise`:

```
diamonds %>%
  group_by(color) %>%
  summarise(vol=mean(x*y*z,na.rm=TRUE),max.price=max(price,na.rm=TRUE))
```