MATH 19 RECITATION
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1. Solve $f'(x) = (1 + e^{-x})(f(x) - 1)$.

$$\int \frac{f'(x)}{f(x)-1} = \int 1 + e^{-x}$$

$$\ln(f(x)-1) = x - e^{-x} + C$$

$$f(x) - 1 = Ce^{x-e^{-x}}$$

$$f(x) = 1 + Ce^{x-e^{-x}}.$$

2. Find a function f such that $(x^2 + 1)f'(x) = xf(x)$ and $f(0) = 1$.

$$\int \frac{f'(x)}{f(x)} = \int \frac{1}{2}\frac{2x}{x^2+1}$$

$$\ln f(x) = \frac{1}{2}\ln(x^2+1) + C$$

$$f(x) = Ce^{\ln(x^2+1)^{1/2}}$$

$$= C(x^2+1)^{1/2}.$$

$f(0) = 1 \Rightarrow C = 1$, so

$$f(x) = \sqrt{x^2+1}.$$

3. Derive and solve the differential equation governing the motion of an object falling in the presence of earth's gravitational field and an air resistance force proportional to the velocity of the object. Use k for this constant of proportionality, as well as m for the mass of the object and g for the acceleration due to gravity.

$$F = ma \Rightarrow$$

$$mg - ky' = my'' \Rightarrow$$

$$y'' + \frac{k}{m}y' = g.$$

This equation is nonhomogeneous, so we look for a solution with $y' = c$ for some constant c: $y'' = 0$, so

$$0 + \frac{k}{m}\cdot c = g \Rightarrow c = \frac{mg}{k}. \quad \text{So } y(t) = \frac{mgt}{k} \text{ is a particular solution.}$$

The general solution is $C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$ where λ_1, λ_2 are the roots of $\lambda^2 + \frac{k}{m}\lambda = 0$ namely $\{0, -k/m\}$. So

$$y(t) = C_1e^{0t} + C_2e^{-kt/m} + \frac{mgt}{k} = \boxed{C_1 + \frac{mgt}{k} + C_2e^{-kt/m}}.$$

4. In the context of the previous question, *terminal velocity* (the velocity at which the air resistance force matches the gravitational force) for a 60-kg human being is roughly 53 meters per second. Use this data and your answer from the previous question to determine how many seconds it takes after stepping out of a plane to reach a velocity of 52 meters per second.

$$y'(t) = \frac{mg}{k} - \frac{C_2k}{m}e^{-kt/m} \to \frac{mg}{k} \text{ as } t \to \infty.$$

So $\frac{mg}{k} \approx 53\frac{m}{s}$. Also, $y'(0) = 0$ since you're stepping out of the plane, so C_2 is such that

$$y'(t) = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m}.$$

This equals 52 when $e^{-kt/m}$ reaches $\frac{1}{53}$:

$$e^{-kt/m} = 1/53 \Rightarrow t = \frac{m\ln 53}{k} \approx \frac{53\ln 53}{g} \approx \boxed{21.5 \text{ seconds}}$$

b/c $\frac{mg}{k} \approx 53$