Problem Set 9

1 (a)

$$f(x) = \sin \pi x \qquad f(0) = 0$$
$$f'(x) = \pi \cos \pi x \qquad f'(0) = \pi$$
$$f''(x) = -\pi^2 \sin \pi x \qquad f''(0) = 0$$
$$f'''(x) = -\pi^3 \cos \pi x \qquad f'''(0) = -\pi^3$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(2k+1)}(0) = (-1)^k \pi^{2k+1}$$
$$f^{(2k)}(0) = 0$$

So $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{(-1)\pi}{(2k+1)!} x^{2k+1}.$$

$$= \pi x - \frac{\pi^3}{3!}x^3 + \quad - \cdots$$

(b)

$$f(x) = 2^x$$
$$f'(x) = (\ln 2) 2^x$$
$$f''(x) = (\ln 2)^2 2^x$$
$$f^{(k)}(x) = (\ln 2)^k \, 2^x \Rightarrow f^{(k)}(0) = (\ln 2)^k.$$

So $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n.$$

* Note: this makes sense since $2^x = e^{x \ln 2}$.

2

$$f(x) = \frac{x}{1-2x} = x\left(1 + 2x + (2x)^2 + (2x)^3 + \cdots\right)$$
$$= x + 2x^2 + 4x^3 + 8x^4 + \cdots$$

This series converges if the common ratio $2x$ is strictly between -1 and 1, so if $-\frac{1}{2} < x < \frac{1}{2}$. Since this series is equal to f

over that interval, it must be equal to f's Taylor series. Therefore, the radius of convergence is $\frac{1}{2}\left(\frac{1}{2}-\left(-\frac{1}{2}\right)\right)=\boxed{\frac{1}{2}}$.

$\boxed{3}$ $f(c)=9/2$, $f'(c)=\frac{-1}{3}$, $f''(c)=-4$. So f is positive, decreasing, and concave down. This occurs only at $c=\boxed{7}$.

$\boxed{4}$ The Maclaurin series of f is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{(n+1)!}{n!}x^n$$
$$= \sum_{n=0}^{\infty}(n+1)x^n.$$

This series converges absolutely when it passes the ratio test:

$$\frac{|a_{n+1}|}{|a_n|}=\frac{(n+2)|x|^{n+1}}{(n+1)|x|^n} = \frac{n+2}{n+1}|x| \to |x|,$$

& this is less than 1 when x is between -1 and 1. So the radius of convergence is 1.

$\boxed{5}$ $f(x)=\sqrt{x}$ $\quad f(1)=1$

$f'(x)=\frac{1}{2\sqrt{x}}$ $\quad f'(1)=\frac{1}{2}$

$f''(x)=\frac{-1}{4x^{3/2}}$ $\quad f''(1)=\frac{-1}{4}$

$f'''(x)=\frac{3}{8x^{5/2}}$ $\quad f'''(1)=\frac{3}{8}$

So the 3rd order Maclaurin series is

$$P(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

then $P(101) = 1 + 50 - \frac{1}{8} \cdot 10{,}000 + \frac{1}{16} \cdot 1{,}000{,}000$

$$= 61{,}301.$$

Nowhere close! However,

$$P(1.01) = 1.0049875625,$$

which means $\sqrt{101} = \sqrt{100}\sqrt{\frac{101}{100}} = 10\sqrt{1.01}$
$\approx 10.049875625.$

Actually, $\sqrt{101} = 10.0498756211208\ldots$, so

thus our estimate is very accurate.