Brown University
Problem Set 1
Instructor: Samuel S. Watson
Due: 15 September 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

Without using determinants, show that the range the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (2x + y, -4x - 2y)$ is a line in $\mathbb{R}^2$. (Note: this requires that you identify the line, show that every point on that line is the image of some point (x, y), and show that every point (x, y) maps to that line.)

Solution

The range of f is the line $\ell = \{(s, t) \in \mathbb{R}^2 : t = -2s\}$. To see this, we need to show that (i) for every pair $(s, t) \in \ell$, there is some point $(x, y) \in \mathbb{R}^2$ that f maps to (s, t), and (ii), the image under f of every pair $(x, y) \in \mathbb{R}^2$ is in ℓ.

For (i), we note that $(s/2, 0)$ maps to (s, t) under f. For (ii), we note that $-2(2x + y) = -4x - 2y$, so if $(s, t) = f(x, y)$, then $t = -2s$.

Problem 2

Consider the line ℓ passing through $(1, -2, 0)$ and running parallel to the z-axis. Define $f(x, y, z)$ to be the distance from (x, y, z) to the line ℓ. Find a simple formula for f.

Solution

The squared distance from (x, y, z) to the point $(1, -2, t)$ is equal to

$$(x - 1)^2 + (y + 2)^2 + (z - t)^2.$$

This squared distance is minimized when $t = z$, so

$$f(x, y, z) = \sqrt{(x - 1)^2 + (y + 2)^2}.$$

Problem 3

Find an equation of the sphere with center $(-3, 2, 5)$ and radius 4. What is the intersection of this sphere with the yz-plane?

Solution

A point (x, y, z) is on the sphere if and only if the distance from (x, y, z) to $(-3, 2, 5)$ is equal to 4. This it true if and only if

$$(x+3)^2 + (y-2)^2 + (z-5)^2 = 16.$$

If a point is on this sphere and the y-z plane, then its x-coordinate is zero, which means that its y and z coordinates satsify $(y-2)^2 + (z-5)^2 = 16$. So the intersection is a circle in the y-z plane specified by the equation $(y-2)^2 + (z-5)^2 = 16$.

Problem 4

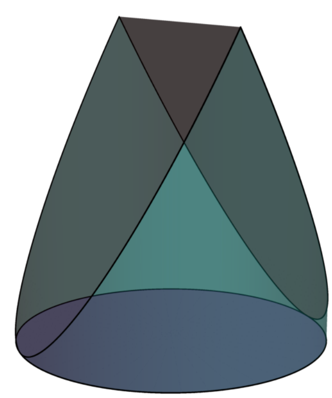
Sketch or describe a solid with the property that its shadows on the three coordinate planes are a circle, an isosceles triangle, and a square, respectively. (Note: the *shadow* of a solid S in $\mathbb{R}^3$ on the xy-plane is defined to be the set of all pairs $(x, y, 0)$ such that there is some $z \in \mathbb{R}$ so that $(x, y, z) \in S$. In other words, its the set of points you get when you smash the solid directly onto the xy-plane. And similarly for the other two coordinate planes.)

Solution

Let's start with a cylinder whose z-shadow is circular. Note that the cylinder already has a rectangular shadow x and y shadow, so this is a good start.

To fix the third shadow, we slice the solid along two planes which intersect in a line through the top face and meet the bottom face at a single point, as shown.

This solid meets the required description as long as the rectangular shadow is a square, which happens when the height of the shape matches its diameter.



Additional space