MATH 19 RECITATION
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BROWN UNIVERSITY
INSTRUCTOR: SAMUEL S. WATSON

1. By calculating derivatives, find the fourth-order Taylor polynomial for $f(x) = xe^x$ centered at $x = 0$.

$f'(x) = xe^x + e^x$

$f''(x) = xe^x + e^x + e^x$

$f'''(x) = xe^x + 3e^x$

$f^{(4)}(x) = xe^x + 4e^x$

So $f^{(k)}(0) = k$. therefore, the Taylor Series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{(k-1)!}$$

2. Find the Taylor series representation of $f(x) = \frac{1}{1-x}$ centered at $x = 0$. Multiply the resulting infinite series by $1 - x$ (meaning distribute and collect terms); what do you get?

$f(x) = \frac{1}{1-x}$

$f'(x) = \frac{1}{(1-x)^2}$

$f''(x) = \frac{2}{(1-x)^3}$

$\vdots$

$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$ So the Taylor Series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} x^k.$$

Note: $(1-x)(1+x+x^2+x^3+\cdots)$

$$\begin{aligned} = {} & 1 + x + x^2 + x^3 + \cdots \\ & \quad - x - x^2 - x^3 - \cdots \\ \hline = {} & 1, \end{aligned}$$

which makes sense because $(1-x) \cdot \frac{1}{1-x} = 1$.

3. Determine the radius of convergence of each of the following series.

(a) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

$$\lim_{n\to\infty} \frac{(n+1)!\,|x|^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!\,|x|^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}|x|\,n^n}{\cancel{(n+1)}(n+1)^n}$$

$$= \frac{|x|}{e}.$$

this is less than 1 when $-e < x < e$,

So the radius of convergence is e.

(b) $\sum_{n=1}^{\infty} (-7)^n x^n$.

$$\lim_{n\to\infty} \frac{|(-7)^{n+1}|\,|x|^{n+1}}{|(-7)^n|\,|x|^n} = \lim_{n\to\infty} |7x|,$$

which is less than 1 when

$$|x| < \boxed{1/7}$$

4. Find the nth order Taylor approximations of $\sin x$, $\cos x$, and e^x. You may express your answer either in summation notation or using an ellipsis.

Substitute $x = i\theta$ in the Taylor approximation for e^x, add the Taylor approximation for $\cos x$ to i times the Taylor approximation for $\sin x$. Comment on how your answer relates to Euler's formula.

$\sin x$: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\cos x$: $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

e^{ix} : $1 + \frac{ix}{1} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$

$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$

$=$ taylor series for $\cos x$

$+ i$ (taylor serie for $\sin x$)