The Yoneda Lemma

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Abstract

Some notes on the Yoneda Lemma, starting with the notion of representable functors.

Covariant Representable Functors

We first define a *prototype* of a covariant representable functor out of a locally small category $\mathscr{A}$.

Definition. Let $\mathscr{A}$ be a locally small category, and fix $A \in \mathscr{A}$. Define the functor $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ by the following mapping on

1. **objects**: For $B \in \mathscr{A}$, define $H^A(B) = \mathscr{A}(A, B)$, the hom-set of arrows in $\mathscr{A}$ from A to B.
2. **morphisms**: For $g : B \to B'$, define $H^A(g) = \mathscr{A}(A, g) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$ by $p \mapsto g \circ p$ for all $p : A \to B$, sending morphisms from A to B to morphisms from A to B' by post-composition with g.

From this prototype of a representable functor, we can now define covariant representable functors:

Definition (Covariant Representable Functor)**.** Let $\mathscr{A}$ be a locally small category. A functor $X : \mathscr{A} \to \mathbf{Set}$ is representable if X is naturally isomorphic to H^A for some $A \in \mathscr{A}$. A representation of X is a choice of an object A along with a natural isomorphism from H^A to X.

Some examples of representable functors:

Example 1 (group G regarded as a one object category)**.** Regarding a group G as a one object category $\mathscr{G}$, with object $*$, $H^* = *(*, -) : \mathscr{G} \to \mathbf{Set}$ is a functor which maps

1. **objects**: There is only one object in $\mathscr{G}$, and $H^*(*)$ is the set of mappings from $*$ to $*$, otherwise known as the elements of the group G. So $H^*(*) \cong U(G)$ where $U : \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor that returns the underlying set of a group.

2. **morphisms**: Let $g \in G$, then $H^*(g)$ maps any element h of G to $g \circ h$, which, interpreted in the context of a group is just group mulitiplication on the left, i.e. gh.

Since $\mathscr{G}$ has only one object, there is only one representable functor on it (up to isomorphism), and the representable is the underlying set G acted on by left multiplication.

Example 2 ($H^1 : \mathbf{Set} \to \mathbf{Set}$)**.** Let 1 denote the set with just one element. The functor $1_{\mathbf{Set}}$ is represented by the functor $H^1 : \mathbf{Set} \to \mathbf{Set}$. To see this, let us first see what H^1 does and then show that it is naturally isomorphic to $1_{\mathbf{Set}}$.

Let $B \in \mathbf{Set}$, then $H^1(B) = \mathbf{Set}(1, B) = \{b : 1 \to B\}$ the set of functions from the singleton set to B that pick out an element of B. In particular, $H^1(B) \cong B$. Let $p : 1 \to B$ and $g : B \to B'$, then $H^1(g) = \mathbf{Set}(1, g) : \mathbf{Set}(1, B) \to \mathbf{Set}(1, B')$ sends functions p that pick out an element of B to functions $g \circ p$ that pick out an element of B'.

So we have the parallel functors $H^1, 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ and we know what they do. Now we construct a natural isomorphism $t : H^1 \Rightarrow 1_{\mathbf{Set}}$. We already have that $H^1(B) \cong B$, so the components of t are the isomorphisms $t_B : H^1(B) \to B$ (and of course $t_B^{-1} : B \to H^1(B)$). Now we need to check that the naturality square below commutes:

$$\begin{tikzcd}
H^1(B) \arrow[r, "H^1(g)"] \arrow[d, "t_B"'] & H^1(B') \arrow[d, "t_{B'}"] \\
B \arrow[r, "1_{\mathbf{Set}}(g) = g"'] & B'
\end{tikzcd}$$

where $g : B \to B'$.

It is enough to consider an element $b : 1 \to B$ from $H^1(B)$. b picks out an element of B, which we will also identify by $b(1) = b$. Going down from $H^1(B)$, $t_B(b : 1 \to B)$ gives us this element b, which is then sent by g to some element of B', say b'. On the other hand, going right from $H^1(B)$, $H^1(g)(b : 1 \to B) = g \circ b$ which gives us the map $b' : 1 \to B'$ that picks out $g(b(1)) = b'$. Finally, $t_{B'}(b')$ is precisely this element b' of B'. So in fact the naturality square commutes.

Hence $1_{\mathbf{Set}}$ is represented by H^1.

Example 3 ($H^{\mathbf{1}}$: **Cat** $\to$ **Set**). Similarly to the example above, ob : **Cat** $\to$ **Set** which sends a small category to its underlying set of objects is represented by $H^{\mathbf{1}}$ where $\mathbf{1}$ is the one-object category. This is because when $\mathscr{B}$ is a category, $H^{\mathbf{1}}(\mathscr{B}) = \mathbf{Cat}(\mathbf{1}, \mathscr{B})$ which picks out objects of $\mathscr{B}$ and is isomorphic to ob$\mathscr{B}$. The proof that these functors are naturally isomorphic is essentially the same as the one for $1_{\mathbf{Set}}$ and H^1 in the example above.

Adjoints and Representables

Now we establish the claim that any set valued functor with a left adjoint is representable. We first prove the following lemma.

Lemma 1. *Let $\mathscr{A}, \mathscr{B}$ be locally small categories with functors $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A}$ such that $F \dashv G$. Fix $A \in \mathscr{A}$, then the functor*

$$\mathscr{A}(A, G(-)) : \mathscr{B} \to \textbf{\textit{Set}}$$

*that is, the composition $\mathscr{B} \xrightarrow{G} \mathscr{A} \xrightarrow{H^A}$ **Set** is representable.*

Proof. Because $F \dashv G$, we have the isomorphism $t^{-1}_{A,B} : \mathscr{A}(A, G(B)) \to \mathscr{B}(F(A), B)$ for each $B \in \mathscr{B}$. Now $H^{F(A)}$ maps B to $\mathscr{B}(F(A), B)$, and it maps a morphism $g : B \to B'$ to $\mathscr{B}(F(A), g) : \mathscr{B}(F(A), B) \to \mathscr{B}(F(A), B')$. So we suspect that the isomorphisms $t^{-1}_{A,B}$ gives rise to the natural isomorphism $t^{-1} : \mathscr{A}(A, G(-)) \Rightarrow H^{F(A)}$. We show that this is in fact the case.

We need to check naturality. Let $g : B \to B'$. We need to check that the following naturality square commutes:

$$\begin{tikzcd}
\mathscr{A}(A, G(B)) \arrow[r, "\mathscr{A}(A, Gg) = H^A(Gg)"] \arrow[d, "t^{-1}_{A,B}"'] & \mathscr{A}(A, G(B')) \arrow[d, "t^{-1}_{A,B'}"] \\
\mathscr{B}(F(A), B) \arrow[r, "H^{F(A)}(g)"'] & \mathscr{B}(F(A), B')
\end{tikzcd}$$

It suffices to consider a map $f : A \to G(B)$ from $\mathscr{A}(A, G(B))$ as it travels around the diagram. Going down from $\mathscr{A}(A, G(B))$, $t^{-1}_{A,B}(f : A \to G(B))$ is a map $t^{-1}_{A,B}(f) : F(A) \to B$, and then $H^{F(A)}(g)(t^{-1}_{A,B}) = g \circ t^{-1}_{A,B}$ which is a map from $F(A)$ to B'. On the other hand, going right from $\mathscr{A}(A, G(B))$, $H^A(Gg)(f : A \to G(B)) = Gg \circ f$ which is a map from A to $G(B')$ and then $t^{-1}_{A,B'}(Gg \circ f)$ is a map from $F(A)$ to B'. So the question is whether $t^{-1}_{A,B'}(Gg \circ f) = g \circ t^{-1}_{A,B}(f)$, but that is precisely what it means for t^{-1} to be natural with respect to B (See [Liu18]), so in fact $\mathscr{A}(A, G(-)) \cong H^{F(A)}$ and is therefore representable. $\square$

Now we are ready to prove the main claim of this subsection:

Theorem 1. *Any set-valued functor that has a left adjoint is representable.*

Proof. Let $G : \mathscr{A} \to \mathbf{Set}$ be a functor with left adjoint F. Let 1 denote the set with a single element. For all $A \in \mathscr{A}$, $G(A)$ is a set and by example 2 above, $H^1(G(A)) = \mathbf{Set}(1, G(A)) \cong G(A)$ naturally in A where H^1 is our functor from **Set** to **Set** in example 2 above. So $G \cong \mathbf{Set}(1, G(-))$ and by the lemma above, this means G is representable. $\square$

Covariant Embedding

Proposition 1. *Let $f : A' \to A$, then f induces a natural transformation $H^f : H^A \Rightarrow H^{A'}$ with components $H^f_B : H^A(B) \to H^{A'}(B)$ so that a map $p : A \to B$ in $H^A(B)$ gets mapped via precomposition with f to $p \circ f : A' \to B$.*

Proof. We need to show that this map H^f is indeed a natural transformation. Let $g : B \to B'$. This means checking that the naturality square:

$$\begin{tikzcd} H^A(B) \arrow[r, "H^A(g)"] \arrow[d, "H^f_B"'] & H^A(B') \arrow[d, "H^f_{B'}"] \\ H^{A'}(B) \arrow[r, "H^{A'}(g)"'] & H^{A'}(B') \end{tikzcd}$$

commutes.

It is sufficient to consider a map $p : A \to B$ in $H^A(B)$ as it travels around the square. Going down, we have first $H^f_B(p) = p \circ f$, and then to the right we have $H^{A'}(g)(p \circ f) = g \circ (p \circ f)$. Going to the right we have $H^A(g)(p) = g \circ p$, and then going down we have $H^f_{B'}(g \circ p) = (g \circ p) \circ f$. However, since morphism composition is associative, these two things are equal and so the square indeed commutes. $\square$

With this natural transformation between covariant representables, we have the following definition:

Definition. The *covariant embedding* is the functor $H^\bullet : \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$ defined by the following mapping on:

1. **objects:** For object A in $\mathscr{A}$, $H^\bullet(A) = H^A$.
2. **morphisms:** For morphism $f : A' \to A$, $H^\bullet(f) = H^f$.

We can do many of the same things with the dual case.

Contravariant Representable Functors

Again, we start with a definition of the *prototypical* contravariant representable functor:

Definition. Let $\mathscr{A}$ be a locally small category with object A. Define the functor $H_A : \mathscr{A}(-, A) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ by the following mapping on

1. **objects:** For object B in $\mathscr{A}$, define $H_A(B) = \mathscr{A}(B, A)$.
2. **morphisms:** For morphism $g : B' \to B$, define $H_A(g) = \mathscr{A}(g, A) : \mathscr{A}(B, A) \to \mathscr{A}(B', A)$ by $p : B \to A \mapsto p \circ g : B' \to B \to A$.

Similar to the covariant case, any functor $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ that is naturally isomorphic to a *prototypical* contravariant representable functor is called contravariantly representable.

Example 4. The functor $\mathscr{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ which maps sets to their powersets and maps functions $g : B' \to B$ to $\mathscr{P}(g) = g^{-1}U$ for all $U \in \mathscr{P}(B)$ where $g^{-1}U = \{x' \in B' | g(x') \in U\}$ is naturally isomorphic to $H_2 : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ where 2 denotes the set with two elements.

To show this, we first need to realize that a subset of a set B is just a boolean mapping from the set to 2 ($f(b) = 1$ means b is in that subset). So $H_2(B) = \mathbf{Set}(B, 2) \cong \mathscr{P}(B)$. Define $t_B : H_2(B) \to \mathscr{P}(B)$ with this isomorphism. We show that the t_B's are natural in B and therefore define the components of a natural transformation $t : H_2 \Rightarrow \mathscr{P}$. Let B, B' be sets, and let g be a map from B' to B. To do this, we show that the following naturality square commutes:

$$\begin{tikzcd}
H_2(B) \arrow[r, "H_2(g)"] \arrow[d, "t_B"'] & H_2(B') \arrow[d, "t_{B'}"] \\
\mathscr{P}(B) \arrow[r, "\mathscr{P}(g)"'] & \mathscr{P}(B')
\end{tikzcd}$$

Consider a map $u : B \to 2$ that picks out an associated subset $U \subseteq B$. $t_B(u)$ gives precisely this subset U, and $\mathscr{P}(g)(U)$ gives the subset U' of B' that maps along g into U. On the other hand, $H_2(g)(u)$ precomposes g with u to give the mapping $u \circ g : B' \to B \to 2$ which picks out the subset U' of B' that, when mapped along g gives back U. And $t'_B(u \circ g)$ gives this U'. So the diagram commutes.

Hence, $\mathscr{P}$ is represented by H_2.

Contravariant Embedding

Similar to the covariant case, the map $f : A \to A'$ induces a natural transformation $H_f : H_A \Rightarrow H_{A'}$ with components $H_f^B : H_A(B) \to H_{A'}(B)$ such that a map $p : B \to A \mapsto f \circ p : B \to A \to A'$.

This gives rise to the following contravariant embedding (which we will call the Yoneda Embedding)

Definition (Yoneda Embedding)**.** Let $\mathscr{A}$ be a locally small category. The Yoneda Embedding is the functor $H_\bullet : \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ defined by the following mapping on:

1. **objects:** For object A in $\mathscr{A}$, $H_\bullet(A) = H_A$.
2. **morphisms:** For morphism $f : A \to A'$, $H_\bullet(f) = H_f$.

Proposition 2. *$H_\bullet$ is injective on isomorphism classes of objects.*

Proof. Let A, A' be objects in $\mathscr{A}$ such that $H_A \cong H_{A'}$. We need to show $A \cong A'$, that is, find maps $f : A \to A'$, $g : A' \to A$ that are mutually inverse. Since $H_A \cong H_{A'}$ that means there are maps (natural transformations) α :

$H_A \Rightarrow H_{A'}$ and $\beta : H_{A'} \Rightarrow H_A$ such that $\alpha \circ \beta = 1_{H_{A'}}$ and $\beta \circ \alpha = 1_{H_A}$. We need to use α and β to construct the f and g we need.

First let's consider the components of α. Let B be an object in $\mathscr{A}$. Then α_B is a map from $H_A(B) \to H_{A'}(B)$. Letting $B = A$, we have $\alpha_A : H_A(A) \to H_{A'}(A)$. Now, we don't know what morphisms there are in $H_A(A)$, but we certainly know that 1_A exists. And applying α_A to 1_A, we get a map $\alpha_A(1_A) : A \to A'$.

Similarly, considering $\beta_{A'}$ applied to $(1_{A'})$, we get a map $\beta_{A'}(1_{A'}) : A' \to A$.

So now we have our maps between A and A'. We need to show that they are mutually inverse.

Naturality of α in B means that for every $p : B \to B'$ the following naturality square holds (note the contravariance):

$$\begin{tikzcd}
H_A(B) \arrow[d, "\alpha_B"'] & H_A(B') \arrow[l, "- \circ p"'] \arrow[d, "\alpha_{B'}"] \\
H_{A'}(B) & H_{A'}(B') \arrow[l, "- \circ p"]
\end{tikzcd}$$

In particular, if $x : B' \to A$ is a map $H_A(B')$, then $\alpha_B(x \circ p) = \alpha_{B'}(x) \circ p$.

Again, what we need is a specialization of this general phenomenon. Letting $x = 1_A$, which means making $B = A'$ and $B' = A$, our p becomes a map from $A' \to A$ and by naturality above we have $\alpha_{A'}(1_A \circ p) = \alpha_A(1_A) \circ p$. Now p is a map from $A' \to A$, so letting $p = \beta_{A'}(1_{A'})$ we have that $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = \alpha_{A'}(1_A \circ \beta_{A'}(1_{A'})) = \alpha_{A'}(\beta_{A'}(1_{A'}))$. But since $\alpha_{A'}$ and $\beta_{A'}$ are mutually inverse, this just means $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = 1_{A'}$, which is precisely what we wanted.

Similarly, we can show using the naturality of β and specializing to $1_{A'}$ that $\beta_{A'}(1_{A'}) \circ \alpha_A(1_A) = 1_A$.

Thus $A \cong A'$. $\square$

The Yoneda Lemma

Theorem 2 (The Yoneda Lemma)**.** *Let $\mathscr{A}$ be a locally small category. Then $[\mathscr{A}^{op}, \boldsymbol{Set}](H_A, X) \cong X(A)$ naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, \boldsymbol{Set}]$.*

Proof. First let's give a quick outline of the proof:

1. Produce a mapping $t_{A,X} : [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \to XA$.
2. Produce a mapping $t^{-1}_{A,X} : XA \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$.
3. Show that t, t^{-1} are mutually inverse.
4. Show that t is natural in A.
5. Show that t is natural in X.

We're not going to do these steps in order, instead we'll do these steps the way I did it when I came up with the proof, so you'll see my motivation for each successive step.

Let's begin with step 1. Let $\alpha : H_A \Rightarrow X$ be a natural transformation. Looking at α's components, $\alpha_B : H_A(B) \to XB$ sends a map $g : B \to A$ to the element of XB, $\alpha_B(g)$. $t_{A,X}$ needs to map α to one specific element of XA. Consider the special case where $B = A$. Then we have the component $\alpha_A : H_A(A) \to XA$. Since the only map we are certain exists in $H_A(A)$ is the identity map 1_A, this suggests we need to define $t_{A,X}(\alpha) = \alpha_A(1_A)$.

To check that this mapping is the correct one, let's first check that $t_{A,X}$ is natural in X. Let $F : X \Rightarrow X'$ be a natural transformation. $t_{A,X}$ natural in X means that the following naturality square holds:

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)} \arrow[r, "F \circ -"] \arrow[d, "t_{A,X}"'] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X')} \arrow[d, "t_{A,X'}"] \\
XA \arrow[r, "F_A"'] & X'A
\end{tikzcd}$$

Let $\alpha : H_A \Rightarrow X$ be a natural transformation. Then going down first and then to the right via F_A we have $F_A(t_{A,X}(\alpha)) = F_A(\alpha_A(1_A))$. On the other hand going right first via $F \circ -$ and then down we have $t_{A,X'}(F \circ \alpha) = (F \circ \alpha)_A(1_A)$. And by compositionality, $(F \circ \alpha)_A(1_A) = F_A \circ \alpha_A(1_A) = F_A(\alpha_A(1_A))$. So $t_{A,X}$ is indeed natural in X.

Now let us check that $t_{A,X}$ is natural in A. We won't succeed yet but this will give us a hint as to how we should define $t^{-1}_{A,X}$. Let f be a map from A to A'. We need to check that the following naturality square commutes:

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)} \arrow[d, "t_{A,X}"'] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_{A'}, X)} \arrow[l, "- \circ H_f"'] \arrow[d, "t_{A',X}"] \\
XA & XA' \arrow[l, "Xf"]
\end{tikzcd}$$

Let $\alpha' : H_{A'} \Rightarrow X$ be a natural transformation. Then going down first and then left via Xf we have $Xf(t_{A',X}(\alpha')) = Xf(\alpha'_{A'}(1_{A'}))$. On the other hand, going to the left first via $-\circ H_f$ and then down, we have $t_{A,X}(\alpha' \circ H_f) = (\alpha'_A \circ H_f^A)(1_A) = \alpha'_A(f)$ (Because $H_f^A(1_A) = f \circ 1_A) = f$. So we need to have the equality

$$Xf(\alpha'_{A'}(1_{A'})) = \alpha'_A(f)$$

Which we can't prove just yet. However, this does give us a hint as to how we should define $t^{-1}_{A,X}$. Notice that $\alpha'_{A'}(1'_A)$ is an element of XA' and that for every $g : B \to A' \in H_{A'}(B)$, $g \mapsto Xg(\alpha'_{A'}(1'_A))$ defines a mapping from $H_{A'}(B)$ to XB, which is beginning to look like the components of a natural transformation $H_{A'} \Rightarrow X$.

With that let us define our mapping $t^{-1}_{A,X} : XA \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$. Let x be an element of XA. We define $t^{-1}_{A,X}(x) = (X(f : B \to A)(x))_{B \in \mathscr{A}}$. So for each B, we have a family of morphisms $H_A(B) \to XB$ given by $f \mapsto Xf(x)$. Now we need to check that this family of morphisms does indeed form a natural transformation $H_A \Rightarrow X$.

Let $k : B \to B'$. We need to check that the following naturality square commutes:

$$\begin{tikzcd}
H_A(B) \arrow[d, "X(- : B \to A)(x)"'] & H_A(B') \arrow[l, "- \circ k"'] \arrow[d, "X(-' : B' \to A)(x)"] \\
XB & XB' \arrow[l, "Xk"]
\end{tikzcd}$$

Let $p : B' \to A$. Then going left and down we first have $p \circ k : B \to A$ which then becomes $X(p \circ k)(x)$. On the other hand, going down first and then left we have $Xk(Xp(x))$. These expressions are equal by the (co)functoriality of X. So $t^{-1}_{A,X}(x)$ does indeed yield a natural transformation.

Now we need to check that $t_{A,X}$ and $t^{-1}_{A,X}$ are in fact mutually inverse. Let $x \in XA$. Then $t_{A,X}(t^{-1}_{A,X}(x)) = t_{A,X}(X(-)(x)) = (X(-)(x))_A(1_A) = X1_A(x) = 1_{XA}(x) = x$. So $t_{A,X} \circ t^{-1}_{A,X}$ does indeed equal 1_{XA}. The other direction is a little bit harder:

Let α be a natural transformation from $H_A \Rightarrow X$. $t^{-1}_{A,X}(t_{A,X}(\alpha)) = t^{-1}_{A,X}(\alpha_A(1_A)) = X(-)(\alpha_A(1_A))$. Let $f : B \to A$. For $X(-)(\alpha_A(1_A)) = \alpha$, we need to have $Xf(\alpha_A(1_A)) = \alpha_B(f)$ for every f. The only tool we have at our disposal is naturality, either naturality of $X(-)(x)$ which we just established, or the naturality of α. When I first went about trying to prove this I thought we ought to use the naturality of $X(-)(x)$, however it turns out that the correct thing to do is use the naturality of α. (Thank you [Lei14]!) Let's write out the naturality square for α with f:

$$\begin{tikzcd}
H_A(B) \arrow[d, "\alpha_B"'] & H_A(A) \arrow[l, "- \circ f"'] \arrow[d, "\alpha_A"] \\
XB & XA \arrow[l, "Xf"]
\end{tikzcd}$$

Considering the image of 1_A as it moves around the square, we have $\alpha_B(1_A \circ f) = \alpha_B(f) = Xf(\alpha_A(1_A))$, which is exactly what we needed.

So t and t^{-1} are mutually inverse, and in fact this last step is also what we needed to prove that $t_{A,X}$ is natural in A, so we are done. □

References

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[Liu18] Stephen Liu. Adjoints, 2018. https://ssyl55.github.io/files/adjoints.pdf.