

Adjoint

Stephen Liu

June 12, 2018

Abstract

We should think about adjunctions as an interesting comparison of two categories that is somewhat more general and of a different nature than an equivalence of categories. Following [Lei14], we'll be looking at three different ways of understanding adjoint functors and showing that they are equivalent.

Hom-Set Definition

Definition (Adjoint Functors). Given a pair of functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$, we say F is left adjoint to G , and G right adjoint to F , written $F \dashv G$ if there is a natural isomorphism $t_{A,B} : \mathcal{B}(F(A), B) \rightarrow \mathcal{A}(A, G(B))$ for each A in \mathcal{A} and B in \mathcal{B} . An adjunction between F and G is a choice of natural isomorphism $t_{A,B}$.

So this means for each $g : F(A) \rightarrow B$, we have a map $t_{A,B}(g) : A \rightarrow G(B)$. We shall call this isomorphism the transpose of g (Leinster denotes this \bar{g}) and this process "transposing" g . Similarly, for each $f : A \rightarrow G(B)$, we have a map $t_{A,B}^{-1}(f) : F(A) \rightarrow B$.

Naturality

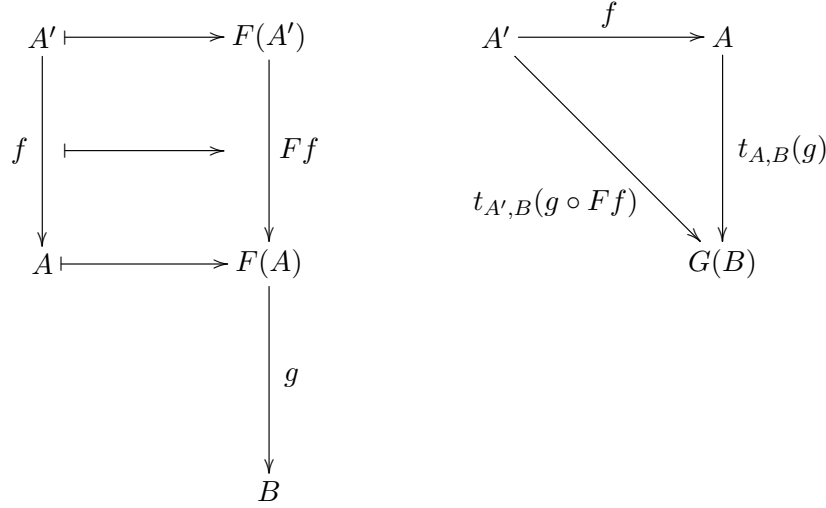
Let's take a closer look at what naturality means. In words it would mean that the transpose of a composition of two maps is equal to the composition of the transpose of the two maps. We have four options here:

1. naturality of t with respect to A
2. naturality of t^{-1} with respect to $F(A)$
3. naturality of t^{-1} with respect to B

4. and finally naturality of t with respect to $G(B)$.

Let's first take a look at *naturality of t with respect to A* :

We have the following data (left), and applying $t_{A',B}$ on $g \circ Ff$ and on them separately we get the commutative triangle on the right:

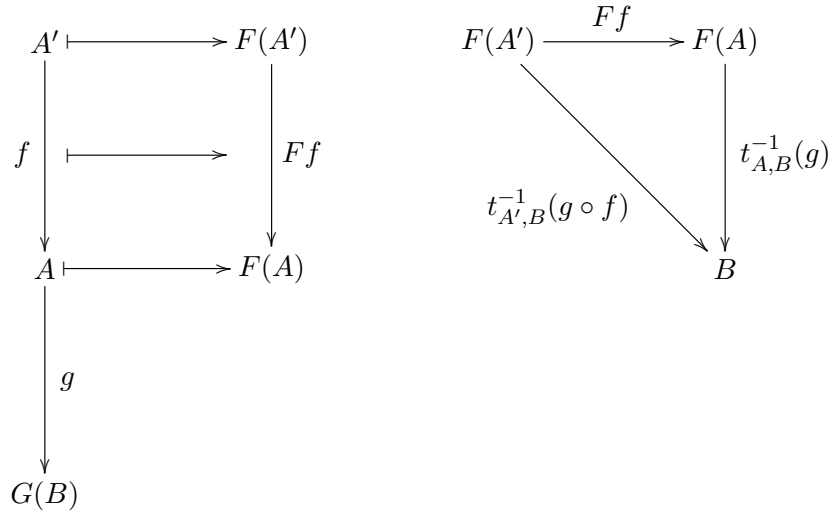


So $t_{A',B}(g \circ F(f)) = t_{A,B}(g) \circ f$ (here $t_{A',F(A)}(F(f)) = f$).

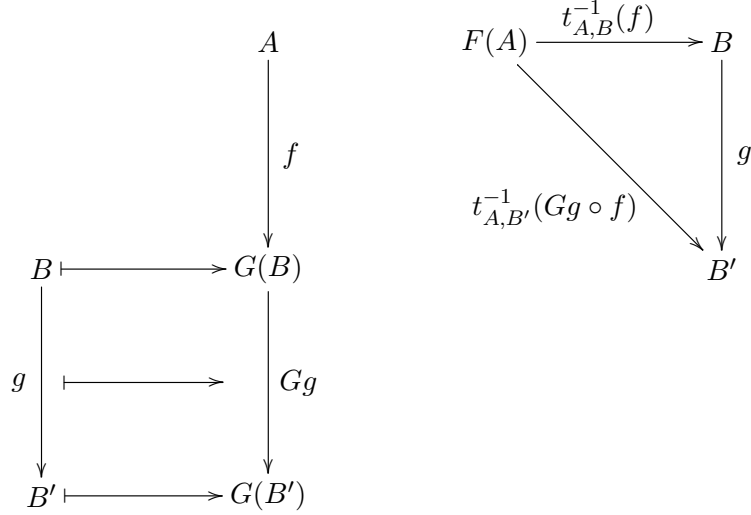
Similarly for 2, 3, and 4, we have the following data yielding the following commutative triangles:

naturality of t^{-1} with respect to $F(A)$:

We begin with the map $Ff : F(A') \rightarrow F(A)$, and taking the preimage, we get the following data and corresponding commutative triangle:



So $t_{A',B}^{-1}(g \circ f) = t_{A,B}^{-1}(g) \circ Ff$.
naturality of t^{-1} with respect to B :



So $t_{A,B'}^{-1}(Gg \circ f) = g \circ t_{A,B}^{-1}(f)$.
 Finally, *naturality of t with respect to $G(B)$:*

$$\begin{array}{ccc}
F(A) & & A \xrightarrow{t_{A,B}(f)} G(B) \\
\downarrow f & & \searrow t_{A,B'}(g \circ f) \\
B & \xrightarrow{\quad} & G(B) \\
\downarrow g & \xrightarrow{\quad} & \downarrow Gg \\
B' & \xrightarrow{\quad} & G(B')
\end{array}$$

So $t_{A,B'}(g \circ f) = Gg \circ t_{A,B}(f)$.

We call this understanding of adjoint functors the Hom-Set Definition because the important bit here is this isomorphism between the Hom-Sets of \mathcal{A} and \mathcal{B} .

There are a whole class of examples of adjoint functors that are the forgetful and free functors between algebraic theories. We'll be looking at one of these:

Example (Abelianization of Groups). There is an adjunction

$$\begin{array}{ccc}
& \mathbf{Ab} & \\
F \uparrow & \dashv & \downarrow U \\
& \mathbf{Grp} &
\end{array}$$

where U is the forgetful inclusion functor from the category of abelian groups to the category of groups, and F is the free functor from the category of groups to the category of abelian groups. For a group G in \mathbf{Grp} , $F(G)$ is the abelianization of the group G , or G/G' where G' is the commutator subgroup of G (see my writeup at [Liu18] for details). This abelianization gives rise to the universal property that for any group homomorphism ϕ out of G to an abelian group A , there is a unique $\bar{\phi} : G/G' \rightarrow A$ such that

$\phi = \bar{\phi} \circ \pi$ where π is the canonical quotient map from G to G/G' . This universal property is what allows us to specify what $t_{G,A} : \mathbf{Ab}(F(G), A) \rightarrow \mathbf{Grp}(G, U(A))$ should do: $t_{G,A}(\bar{\phi}) = \bar{\phi} \circ \pi = \phi$, and $t_{G,A}^{-1}(\phi) = \bar{\phi}$.

Units and Counits Definition

Definition (Unit and Counit of an Adjunction). Given $A \in \mathcal{A}$ and the identity map $1_{F(A)}$, $t_{A,F(A)}(1_{F(A)})$ defines the isomorphism $\eta_A : A \rightarrow GF(A)$. Similarly, given $B \in \mathcal{B}$ and the identity map $1_{G(B)}$, $t_{G(B),B}^{-1}(1_{G(B)})$ defines the isomorphism $\varepsilon_B : FG(B) \rightarrow B$. Together, η_A and ε_B define the natural transformations

$$\eta : 1_{\mathcal{A}} \rightarrow G \circ F, \quad \varepsilon : F \circ G \rightarrow 1_{\mathcal{B}}$$

called the unit and counit of the adjunction, respectively.

We have important triangle identities associated with the unit and counit.

Proposition 1 (Triangle Identities). *Given an adjunction $F \dashv G$ with unit η and counit ε , the triangles*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \varepsilon F \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow 1_G & \downarrow G\varepsilon \\ & & G \end{array}$$

commute.

Proof. We prove the equivalent statement that the triangles

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\eta_A)} & FGF(A) \\ & \searrow 1_{F(A)} & \downarrow \varepsilon_{F(A)} \\ & & F(A) \end{array} \qquad \begin{array}{ccc} G(B) & \xrightarrow{\eta_{G(B)}} & GFG(B) \\ & \searrow 1_{G(B)} & \downarrow G(\varepsilon_B) \\ & & G(B) \end{array}$$

commute for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

For the triangle on the left, we use *naturality of t^{-1} with respect to $F(A)$* that we explained above where we replace f with η_A and g with $1_{GF(A)}$. So we have the following data giving rise to the commutative triangle on the right:

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & F(A) \\
 \eta_A \downarrow & \xrightarrow{\quad} & \downarrow F(\eta_A) \\
 GF(A) & \xrightarrow{\quad} & FGF(A) \\
 \downarrow 1_{GF(A)} & & \\
 GF(A) & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(A) & \xrightarrow{F(\eta_A)} & FGF(A) \\
 & \searrow & \downarrow t_{GF(A),F(A)}^{-1}(1_{GF(A)}) \\
 & & F(A)
 \end{array}$$

Now by definition $t_{GF(A),F(A)}^{-1}(1_{GF(A)}) = \varepsilon_{F(A)}$, and $t_{A,F(A)}^{-1}(1_{GF(A)} \circ \eta_A) = t_{A,F(A)}^{-1}(\eta_A)$ and by definition, $t_{A,F(A)}(1_{F(A)}) = \eta_A$, so $t_{A,F(A)}^{-1}(\eta_A) = 1_{F(A)}$. So from the triangle we get $1_{F(A)} = \varepsilon_{F(A)} \circ F(\eta_A)$, proving the commutative triangle.

Similarly, for the triangle on the right, we use *naturality of t with respect to $G(B)$* that we explained above where we replace f with $1_{FG(B)}$ and g with ε_B . So we from the resulting commutative triangle we have

$$t_{G(B),B}(\varepsilon_B \circ 1_{FG(B)}) = G(\varepsilon_B) \circ t_{G(B),FG(B)}(1_{FG(B)}).$$

And again, by definition $t_{G(B),FG(B)}(1_{FG(B)}) = \eta_{G(B)}$ and $t_{G(B),B}(\varepsilon_B \circ 1_{FG(B)}) = t_{G(B),B}(\varepsilon_B) = 1_{G(B)}$, so $1_{G(B)} = G\varepsilon_B \circ \eta_{G(B)}$, proving the identity. \square

It turns out the unit and counit determine the whole adjunction.

Proposition 2. *Given an adjunction $t_{A,B} : \mathcal{B}(F(A), B) \rightarrow \mathcal{A}(A, G(B))$ for any $g : F(A) \rightarrow B$, $t_{A,B}(g) = Gg \circ \eta_A$, and for any $f : A \rightarrow G(B)$, $t_{A,B}^{-1}(f) = \varepsilon_B \circ Ff$.*

Proof. For any $g : F(A) \rightarrow B$, by naturality, $t_{A,B}(g) = t_{A,B}(g \circ 1_{F(A)})$ which by naturality of t with respect to $GF(A)$, is equal to $Gg \circ \eta_A$. Similarly, for any $f : A \rightarrow G(B)$, $t_{A,B}^{-1}(f) = t_{A,B}^{-1}(1_{G(B)} \circ f)$ which by naturality of t^{-1} with respect to $FG(B)$, is equal to $\varepsilon_B \circ Ff$. \square

Using this fact, we can equivalently define adjunctions by specifying pairs of units and counits.

Theorem 1. *Given functors $F : \mathcal{A} \rightarrow \mathcal{B}$, $G : \mathcal{B} \rightarrow \mathcal{A}$, there is a bijection between adjunctions $F \dashv G$ and pairs of units and counits (η, ε) that satisfy the triangle identities.*

Proof. We have already shown that given an adjunction t , we can define natural transformations η, ε that satisfy the triangle identities. Now, we just need to show that given unit and counit η, ε , we can uniquely define a natural isomorphism $t_{A,B} : \mathcal{B}(F(A), B) \rightarrow \mathcal{A}(A, G(B))$ for all $A \in \mathcal{A}$, for all $B \in \mathcal{B}$.

Given $g : F(A) \rightarrow B$, define $t_{A,B}(g) = Gg \circ \eta_A$, and given $f : A \rightarrow G(B)$, define $t_{A,B}^{-1}(f) = \varepsilon_B \circ Ff$. We need to show that t and t^{-1} are well-defined, mutually inverse, natural, and that η, ε are in fact their unit and counit.

Well-Defined Let $g : F(A) \rightarrow B$, $h : F(A) \rightarrow B$ with $g = h$. Since G is well-defined, $t_{A,B}(g) = Gg \circ \eta_A = Gh \circ \eta_A = t_{A,B}(h)$, so t is well-defined. Similarly for t^{-1} .

Isomorphism

\square

References

- [Lei14] Tom Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.
- [Liu18] Stephen Liu. Abelianization of groups, 2018. ssyl55.github.io/files/abelianization.pdf.