# Adjoints

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**Abstract**

We should think about adjunctions as an interesting comparison of two categories that is somewhat more general and of a different nature than an equivalence of categories. Following [Lei14], we'll be looking at three different ways of understanding adjoint functors and showing that they are equivalent.

## Hom-Set Definition

**Definition** (Adjoint Functors)**.** Given a pair of functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$, we say $F$ is left adjoint to $G$, and $G$ right adjoint to $F$, written $F \dashv G$ if there is a natural isomorphism $t_{A,B} : \mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$ for each $A$ in $\mathscr{A}$ and $B$ in $\mathscr{B}$. An adjunction between $F$ and $G$ is a choice of natural isomorphism $t_{A,B}$.

So this means for each $g : F(A) \to B$, we have a map $t_{A,B}(g) : A \to G(B)$. We shall call this isomorphism the transpose of $g$ (Leinster denotes this $\overline{g}$) and this process "transposing" $g$. Naturality here means that the transpose of a composition of two maps is equal to the composition of the transpose of the two maps. In symbols, we have the following composition of maps $F(A) \xrightarrow{Ff} F(A') \xrightarrow{g} B$ and applying $t$ we get the following commutative triangle:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "t_{A,B}(g \circ F(f))"'] & A' \arrow[d, "t_{A',B}(g)"] \\
 & G(B)
\end{tikzcd}$$

So $t_{A,B}(g \circ F(f)) = t_{A',B}(g) \circ f$ (here $t_{A,A'}(F(f)) = f$). Also note this diagram is naturality in $A$, we have a similar triangle for naturality in $B$ and probably involving $t^{-1}$.

We call this understanding of adjoint functors the Hom-Set Definition because the important bit here is this isomorphism between the Hom-Sets of $\mathscr{A}$ and $\mathscr{B}$.

There are a whole class of examples of adjoint functors that are the forgetful and free functors between algebraic theories. We'll be looking at one of these:

**Example** (Abelianization of Groups)**.** There is an adjunction

$$\begin{tikzcd} \mathbf{Ab} \arrow[d, shift left=3ex, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Grp} \arrow[u, shift left=3ex, "F"] \end{tikzcd}$$

where $U$ is the forgetful inclusion functor from the category of abelian groups to the category of groups, and $F$ is the free functor from the category of groups to the category of abelian groups. For a group $G$ in $\mathbf{Grp}$, $F(G)$ is the abelianization of the group $G$, or $G/G'$ where $G'$ is the commutator subgroup of $G$ (see my writeup at [Liu18] for details). This abelianization gives rise to the universal property that for any group homomorphism $\phi$ out of $G$ to an abelian group $A$, there is a unique $\overline{\phi} : G/G' \to A$ such that $\phi = \overline{\phi} \circ \pi$ where $\pi$ is the canonical quotient map from $G$ to $G/G'$. This universal property is what allows us to specify what $t_{G,A} : \mathbf{Ab}(F(G), A) \to \mathbf{Grp}(G, U(A))$ should do: $t_{G,A}(\overline{\phi}) = \overline{\phi} \circ \pi = \phi$, and $t^{-1}_{G,A}(\phi) = \overline{\phi}$.

### Units and Counits Definition

## References

[Lei14] Tom Leinster. *Basic Category Theory.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.

[Liu18] Stephen Liu. Abelianization of groups, 2018. ssyl55.github.io/files/abelianization.pdf.