The Yoneda Lemma

Stephen Liu

June 18, 2018

Abstract

Some notes on the Yoneda Lemma, starting with the notion of representable functors.

Covariant Representable Functors

We first define a *prototype* of a covariant representable functor out of a locally small category \mathscr{A} .

Definition. Let \mathscr{A} be a locally small category, and fix $A \in \mathscr{A}$. Define the functor $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ by the following mapping on

- 1. **objects**: For $B \in \mathcal{A}$, define $H^A(B) = \mathcal{A}(A, B)$, the hom-set of arrows in \mathcal{A} from A to B.
- 2. **morphisms**: For $g: B \to B'$, define $H^A(g) = \mathscr{A}(A,g): \mathscr{A}(A,B) \to \mathscr{A}(A,B')$ by $p \mapsto g \circ p$ for all $p: A \to B$, sending morphisms from A to B to morphisms from A to B' by post-composition with g.

From this prototype of a representable functor, we can now define covariant representable functors:

Definition (Covariant Representable Functor). Let \mathscr{A} be a locally small category. A functor $X: \mathscr{A} \to \mathbf{Set}$ is representable if X is naturally isomorphic to H^A for some $A \in \mathscr{A}$. A representation of X is a choice of an object A along with a natural isomorphism from H^A to X.

Some examples of representable functors:

Example 1 (group G regarded as a one object category). Regarding a group G as a one object category \mathscr{G} , with object $*, H^* = *(*, -) : \mathscr{G} \to \mathbf{Set}$ is a functor which maps

- 1. **objects**: There is only one object in \mathscr{G} , and $H^*(*)$ is the set of mappings from * to *, otherwise known as the elements of the group G. So $H^*(*) \cong U(G)$ where $U : \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor that returns the underlying set of a group.
- 2. **morphisms**: Let $g \in G$, then $H^*(g)$ maps any element h of G to $g \circ h$, which, interpreted in the context of a group is just group mulitiplication on the left, i.e. gh.

Since \mathscr{G} has only one object, there is only one representable functor on it (up to isomorphism), and the representable is the underlying set G acted on by left multiplication.

Example 2 ($H^1: \mathbf{Set} \to \mathbf{Set}$). Let 1 denote the set with just one element. The functor $1_{\mathbf{Set}}$ is represented by the functor $H^1: \mathbf{Set} \to \mathbf{Set}$. To see this, let us first see what H^1 does and then show that it is naturally isomorphic to $1_{\mathbf{Set}}$.

Let $B \in \mathbf{Set}$, then $H^1(B) = \mathbf{Set}(1, B) = \{b : 1 \to B\}$ the set of functions from the singleton set to B that pick out an element of B. In particular, $H^1(B) \cong B$. Let $p : 1 \to B$ and $g : B \to B'$, then $H^1(g) = \mathbf{Set}(1, g) : \mathbf{Set}(1, B) \to \mathbf{Set}(1, B')$ sends functions p that pick out an element of B to functions $g \circ p$ that pick out an element of B'.

So we have the parallel functors $H^1, 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ and we know what they do. Now we construct a natural isomorphism $t : H^1 \Rightarrow 1_{\mathbf{Set}}$. We already have that $H^1(B) \cong B$, so the components of t are the isomorphisms $t_B : H^1(B) \to B$ (and of course $t_B^{-1} : B \to H^1(B)$). Now we need to check that the naturality square below commutes:

$$H^{1}(B) \xrightarrow{H^{1}(g)} H^{1}(B')$$

$$t_{B} \downarrow \qquad \qquad \downarrow t_{B'}$$

$$B \xrightarrow{1_{\mathbf{Set}}(g) = g} B'$$

where $g: B \to B'$.

It is enough to consider an element $b: 1 \to B$ from $H^1(B)$. b picks out an element of B, which we will also identify by b(1) = b. Going down from $H^1(B)$, $t_B(b: 1 \to B)$ gives us this element b, which is then sent by g to

some element of B', say b'. On the other hand, going right from $H^1(B)$, $H^1(g)(b:1\to B)=g\circ b$ which gives us the map $b':1\to B'$ that picks out g(b(1))=b'. Finally, $t_{B'}(b')$ is precisely this element b' of B'. So in fact the naturality square commutes.

Hence $1_{\mathbf{Set}}$ is represented by H^1 .

Example 3 $(H^1: \mathbf{Cat} \to \mathbf{Set})$. Similarly to the example above, ob: $\mathbf{Cat} \to \mathbf{Set}$ which sends a small category to its underlying set of objects is represented by H^1 where $\mathbf{1}$ is the one-object category. This is because when \mathscr{B} is a category, $H^1(\mathscr{B}) = \mathbf{Cat}(\mathbf{1}, \mathscr{B})$ which picks out objects of \mathscr{B} and is isomorphic to ob \mathscr{B} . The proof that these functors are naturally isomorphic is essentially the same as the one for $1_{\mathbf{Set}}$ and H^1 in the example above.

Adjoints and Representables

Now we establish the claim that any set valued functor with a left adjoint is representable. We first prove the following lemma.

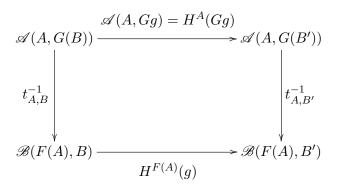
Lemma 1. Let \mathscr{A}, \mathscr{B} be locally small categories with functors $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A}$ such that $F \dashv G$. Fix $A \in \mathscr{A}$, then the functor

$$\mathscr{A}(A,G(-)):\mathscr{B}\to \mathbf{Set}$$

that is, the composition $\mathscr{B} \stackrel{G}{\to} \mathscr{A} \stackrel{H^A}{\to} \mathbf{Set}$ is representable.

Proof. Because $F \dashv G$, we have the isomorphism $t_{A,B}^{-1}: \mathscr{A}(A,G(B)) \to \mathscr{B}(F(A),B)$ for each $B \in \mathscr{B}$. Now $H^{F(A)}$ maps B to $\mathscr{B}(F(A),B)$, and it maps a morphism $g: B \to B'$ to $\mathscr{B}(F(A),g): \mathscr{B}(F(A),B) \to \mathscr{B}(F(A),B')$. So we suspect that the isomorphisms $t_{A,B}^{-1}$ gives rise to the natural isomorphism $t^{-1}: \mathscr{A}(A,G(-)) \Rightarrow H^{F(A)}$. We show that this is in fact the case.

We need to check naturality. Let $g: B \to B'$. We need to check that the following naturality square commutes:



It suffices to consider a map $f:A\to G(B)$ from $\mathscr{A}(A,G(B))$ as it travels around the diagram. Going down from $\mathscr{A}(A,G(B))$, $t_{A,B}^{-1}(f:A\to G(B))$ is a map $t_{A,B}^{-1}(f):F(A)\to B$, and then $H^{F(A)}(g)(t_{A,B}^{-1})=g\circ t_{A,B}^{-1}$ which is a map from F(A) to B'. On the other hand, going right from $\mathscr{A}(A,G(B))$, $H^A(Gg)(f:A\to G(B))=Gg\circ f$ which is a map from A to G(B') and then $t_{A,B'}^{-1}(Gg\circ f)$ is a map from F(A) to B'. So the question is whether $t_{A,B'}^{-1}(Gg\circ f)=g\circ t_{A,B}^{-1}(f)$, but that is precisely what it means for t^{-1} to be natural with respect to B (See [Liu18]), so in fact $\mathscr{A}(A,G(-))\cong H^{F(A)}$ and is therefore representable.

Now we are ready to prove the main claim of this subsection:

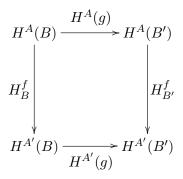
Theorem 1. Any set-valued functor that has a left adjoint is representable.

Proof. Let $G: \mathscr{A} \to \mathbf{Set}$ be a functor with left adjoint F. Let 1 denote the set with a single element. For all $A \in \mathscr{A}$, G(A) is a set and by example 2 above, $H^1(G(A)) = \mathbf{Set}(1, G(A)) \cong G(A)$ naturally in A where H^1 is our functor from \mathbf{Set} to \mathbf{Set} in example 2 above. So $G \cong \mathbf{Set}(1, G(-))$ and by the lemma above, this means G is representable.

Covariant Embedding

Proposition 1. Let $f: A' \to A$, then f induces a natural transformation $H^f: H^A \Rightarrow H^{A'}$ with components $H^f_B: H^A(B) \to H^{A'}(B)$ so that a map $p: A \to B$ in $H^A(B)$ gets mapped via precomposition with f to $p \circ f: A' \to B$.

Proof. We need to show that this map H^f is indeed a natural transformation. Let $g: B \to B'$. This means checking that the naturality square:



commutes.

It is sufficient to consider a map $p:A\to B$ in $H^A(B)$ as it travels around the square. Going down, we have first $H_B^f(p)=p\circ f$, and then to the right we have $H^{A'}(g)(p\circ f)=g\circ (p\circ f)$. Going to the right we have $H^A(g)(p)=g\circ p$, and then going down we have $H_{B'}^f(g\circ p)=(g\circ p)\circ f$. However, since morphism composition is associative, these two things are equal and so the square indeed commutes.

With this natural transformation between covariant representables, we have the following definition:

Definition. The *covariant embedding* is the functor $H^{\bullet}: \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$ defined by the following mapping on:

- 1. **objects:** For object A in \mathscr{A} , $H^{\bullet}(A) = H^{A}$.
- 2. morphisms: For morphism $f: A' \to A, H^{\bullet}(f) = H^f$.

We can do many of the same things with the dual case.

Contravariant Representable Functors

Again, we start with a definition of the *prototypical* contravariant representable functor:

Definition. Let \mathscr{A} be a locally small category with object A. Define the functor $H_A : \mathscr{A}(-, A) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ by the following mapping on

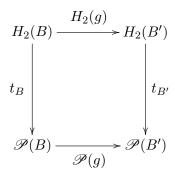
- 1. **objects:** For object B in \mathscr{A} , define $H_A(B) = \mathscr{A}(B,A)$.
- 2. **morphisms:** For morphism $g: B' \to B$, define $H_A(g) = \mathscr{A}(g, A): \mathscr{A}(B, A) \to \mathscr{A}(B', A)$ by $p: B \to A \mapsto p \circ g: B' \to B \to A$.

Similar to the covariant case, any functor $F: \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ that is naturally isomorphic to a *prototypical* contravariant representable functor is called contravariantly representable.

Example 4. The functor $\mathscr{P}: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ which maps sets to their powersets and maps functions $g: B' \to B$ to $\mathscr{P}(g) = g^{-1}U$ for all $U \in \mathscr{P}(B)$ where $g^{-1}U = \{x' \in B' | g(x') \in U\}$ is naturally isomorphic to $H_2: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ where 2 denotes the set with two elements.

To show this, we first need to realize that a subset of a set B is just a boolean mapping from the set to 2 (f(b) = 1 means b is in that subset). So $H_2(B) = \mathbf{Set}(B,2) \cong \mathscr{P}(B)$. Define $t_B : H_2(B) \to \mathscr{P}(B)$ with this isomorphism. We show that the t_B 's are natural in B and therefore define

the components of a natural transformation $t: H_2 \Rightarrow \mathscr{P}$. Let B, B' be sets, and let g be a map from B' to B. To do this, we show that the following naturality square commutes:



Consider a map $u: B \to 2$ that picks out an associated subset $U \subseteq B$. $t_B(u)$ gives precisely this subset U, and $\mathscr{P}(g)(U)$ gives the subset U' of B' that maps along g into U. On the other hand, $H_2(g)(u)$ precomposes g with u to give the mapping $u \circ g: B' \to B \to 2$ which picks out the subset U' of B' that, when mapped along g gives back U. And $t'_B(u \circ g)$ gives this U'. So the diagram commutes.

Hence, \mathscr{P} is represented by H_2 .

Contravariant Embedding

Similar to the covariant case, the map $f: A \to A'$ induces a natural transformation $H_f: H_A \Rightarrow H_{A'}$ with components $H_f^B: H_A(B) \to H_{A'}(B)$ such that a map $p: B \to A \mapsto f \circ p: B \to A \to A'$.

This gives rise to the following contravariant embedding (which we will call the Yoneda Embedding)

Definition (Yoneda Embedding). Let \mathscr{A} be a locally small category. The Yoneda Embedding is the functor $H_{\bullet}: \mathscr{A} \to [\mathscr{A}^{op}, \mathbf{Set}]$ defined by the following mapping on:

- 1. **objects:** For object A in \mathscr{A} , $H_{\bullet}(A) = H_A$.
- 2. morphisms: For morphism $f: A \to A'$, $H_{\bullet}(f) = H_f$.

Proposition 2. H_{\bullet} is injective on isomorphism classes of objects.

Proof. Let A, A' be objects in \mathscr{A} such that $H_A \cong H_{A'}$. We need to show $A \cong A'$, that is, find maps $f: A \to A'$, $g: A' \to A$ that are mutually inverse. Since $H_A \cong H_{A'}$ that means there are maps (natural transformations) α :

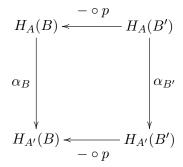
 $H_A \Rightarrow H_{A'}$ and $\beta: H_{A'} \Rightarrow H_A$ such that $\alpha \circ \beta = 1_{H_{A'}}$ and $\beta \circ \alpha = 1_{H_A}$. We need to use α and β to construct the f and g we need.

First let's consider the components of α . Let B be an object in \mathscr{A} . Then α_B is a map from $H_A(B) \to H_{A'}(B)$. Letting B = A, we have $\alpha_A : H_A(A) \to H_{A'}(A)$. Now, we don't know what morphisms there are in $H_A(A)$, but we certainly know that 1_A exists. And applying α_A to 1_A , we get a map $\alpha_A(1_A) : A \to A'$.

Similarly, considering $\beta_{A'}$ applied to $(1_{A'})$, we get a map $\beta_{A'}(1_{A'}): A' \to A$.

So now we have our maps between A and A'. We need to show that they are mutually inverse.

Naturality of α in B means that for every $p: B \to B'$ the following naturality square holds (note the contravariance):



In particular, if $x: B' \to A$ is a map $H_A(B')$, then $\alpha_B(x \circ p) = \alpha_{B'}(x) \circ p$. Again, what we need is a specialization of this general phenomenon. Letting $x = 1_A$, which means making B = A' and B' = A, our p becomes a map from $A' \to A$ and by naturality above we have $\alpha_{A'}(1_A \circ p) = \alpha_A(1_A) \circ p$. Now p is a map from $A' \to A$, so letting $p = \beta_{A'}(1_{A'})$ we have that $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = \alpha_{A'}(1_A \circ \beta_{A'}(1_{A'})) = \alpha_{A'}(\beta_{A'}(1_{A'}))$. But since $\alpha_{A'}$ and $\beta_{A'}$ are mutually inverse, this just means $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = 1_{A'}$, which is precisely what we wanted.

Similarly, we can show using the naturality of β and specializing to $1_{A'}$ that $\beta_{A'}(1_{A'}) \circ \alpha_A(1_A) = 1_A$.

Thus
$$A \cong A'$$
.

The Yoneda Lemma

Theorem 2 (The Yoneda Lemma). Let \mathscr{A} be a locally small category. Then $[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \cong X(A)$ naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, \mathbf{Set}]$.

REFERENCES 8

References

[Lei14] Tom Leinster. Basic Category Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.

[Liu18] Stephen Liu. Adjoints, 2018. https://ssyl55.github.io/files/adjoints.pdf.