Adjoints

Stephen Liu

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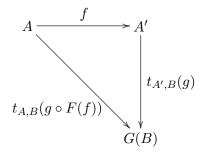
Abstract

We should think about adjunctions as an interesting comparison of two categories that is somewhat more general and of a different nature than an equivalence of categories. Following [Lei14], we'll be looking at three different ways of understanding adjoint functors and showing that they are equivalent.

Hom-Set Definition

Definition (Adjoint Functors). Given a pair of functors $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{A}$, we say F is left adjoint to G, and G right adjoint to F, written $F \dashv G$ if there is a natural isomorphism $t_{A,B}: \mathscr{B}(F(A),B) \to \mathscr{A}(A,G(B))$ for each A in \mathscr{A} and B in \mathscr{B} . An adjunction between F and G is a choice of natural isomorphism $t_{A,B}$.

So this means for each $g: F(A) \to B$, we have a map $t_{A,B}(g): A \to G(B)$. We shall call this isomorphism the transpose of g (Leinster denotes this \overline{g}) and this process "transposing" g. Naturality here means that the transpose of a composition of two maps is equal to the composition of the transpose of the two maps. In symbols, we have the following composition of maps $F(A) \stackrel{F}{\to} F(A') \stackrel{g}{\to} B$ and applying t we get the following commutative triangle:



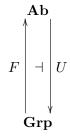
REFERENCES 2

So $t_{A,B}(g \circ F(f)) = t_{A',B}(g) \circ f$ (here $t_{A,A'}(F(f)) = f$). Also note this diagram is naturality in A, we have a similar triangle for naturality in B and probably involving t^{-1} .

We call this understanding of adjoint functors the Hom-Set Definition because the important bit here is this isomorphism between the Hom-Sets of \mathscr{A} and \mathscr{B} .

There are a whole class of examples of adjoint functors that are the forgetful and free functors between algebraic theories. We'll be looking at one of these:

Example (Abelianization of Groups). There is an adjunction



where U is the forgetful inclusion functor from the category of abelian groups to the category of groups, and F is the free functor from the category of groups to the category of abelian groups. For a group G in \mathbf{Grp} , F(G) is the abelianization of the group G, or G/G' where G' is the commutator subgroup of G (see my writeup at [Liu18] for details). This abelianization gives rise to the universal property that for any group homomorphism ϕ out of G to an abelian group A, there is a unique $\overline{\phi}: G/G' \to A$ such that $\phi = \overline{\phi} \circ \pi$ where π is the canonical quotient map from G to G/G'. This universal property is what allows us to specify what $t_{G,A}: \mathbf{Ab}(F(G), A) \to \mathbf{Grp}(G, U(A))$ should do: $t_{G,A}(\overline{\phi}) = \overline{\phi} \circ \pi = \phi$, and $t_{G,A}^{-1}(\phi) = \overline{\phi}$.

Units and Counits Definition

References

- [Lei14] Tom Leinster. Basic Category Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.
- [Liu18] Stephen Liu. Abelianization of groups, 2018. ssyl55.github.io/files/abelianization.pdf.