# The Yoneda Lemma

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#### Abstract

Some notes on the Yoneda Lemma, starting with the notion of representable functors.

#### Covariant Representable Functors

We first define a *prototype* of a covariant representable functor out of a locally small category  $\mathscr{A}$ .

**Definition.** Let  $\mathscr{A}$  be a locally small category, and fix  $A \in \mathscr{A}$ . Define the functor  $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$  by the following mapping on

- 1. **objects**: For  $B \in \mathcal{A}$ , define  $H^A(B) = \mathcal{A}(A, B)$ , the hom-set of arrows in  $\mathcal{A}$  from A to B.
- 2. **morphisms**: For  $g: B \to B'$ , define  $H^A(g) = \mathscr{A}(A,g): \mathscr{A}(A,B) \to \mathscr{A}(A,B')$  by  $p \mapsto g \circ p$  for all  $p: A \to B$ , sending morphisms from A to B to morphisms from A to B' by post-composition with g.

From this prototype of a representable functor, we can now define covariant representable functors:

**Definition** (Covariant Representable Functor). Let  $\mathscr{A}$  be a locally small category. A functor  $X : \mathscr{A} \to \mathbf{Set}$  is representable if X is naturally isomorphic to  $H^A$  for some  $A \in \mathscr{A}$ . A representation of X is a choice of an object A along with a natural isomorphism from  $H^A$  to X.

Some examples of representable functors:

**Example 1** (group G regarded as a one object category). Regarding a group G as a one object category  $\mathscr{G}$ , with object  $*, H^* = *(*, -) : \mathscr{G} \to \mathbf{Set}$  is a functor which maps

- 1. **objects**: There is only one object in  $\mathscr{G}$ , and  $H^*(*)$  is the set of mappings from \* to \*, otherwise known as the elements of the group G. So  $H^*(*) \cong U(G)$  where  $U : \mathbf{Grp} \to \mathbf{Set}$  is the forgetful functor that returns the underlying set of a group.
- 2. **morphisms**: Let  $g \in G$ , then  $H^*(g)$  maps any element h of G to  $g \circ h$ , which, interpreted in the context of a group is just group mulitiplication on the left, i.e. gh.

Since  $\mathscr{G}$  has only one object, there is only one representable functor on it (up to isomorphism), and the representable is the underlying set G acted on by left multiplication.

**Example 2** ( $H^1: \mathbf{Set} \to \mathbf{Set}$ ). Let 1 denote the set with just one element. The functor  $1_{\mathbf{Set}}$  is represented by the functor  $H^1: \mathbf{Set} \to \mathbf{Set}$ . To see this, let us first see what  $H^1$  does and then show that it is naturally isomorphic to  $1_{\mathbf{Set}}$ .

Let  $B \in \mathbf{Set}$ , then  $H^1(B) = \mathbf{Set}(1,B) = \{b: 1 \to B\}$  the set of functions from the singleton set to B that pick out an element of B. In particular,  $H^1(B) \cong B$ . Let  $p: 1 \to B$  and  $g: B \to B'$ , then  $H^1(g) = \mathbf{Set}(1,g)$ :  $\mathbf{Set}(1,B) \to \mathbf{Set}(1,B')$  sends functions p that pick out an element of B to functions  $g \circ p$  that pick out an element of B'.

So we have the parallel functors  $H^1, 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$  and we know what they do. Now we construct a natural isomorphism  $t : H^1 \Rightarrow 1_{\mathbf{Set}}$ . We already have that  $H^1(B) \cong B$ , so the components of t are the isomorphisms  $t_B : H^1(B) \to B$  (and of course  $t_B^{-1} : B \to H^1(B)$ ). Now we need to check that the naturality square below commutes:

$$H^{1}(B) \xrightarrow{H^{1}(g)} H^{1}(B')$$

$$t_{B} \downarrow \qquad \qquad \downarrow t_{B'}$$

$$B \xrightarrow{1_{\mathbf{Set}}(g) = g} B'$$

where  $g: B \to B'$ .

It is enough to consider an element  $b: 1 \to B$  from  $H^1(B)$ . b picks out an element of B, which we will also identify by b(1) = b. Going down from  $H^1(B)$ ,  $t_B(b: 1 \to B)$  gives us this element b, which is then sent by g to

some element of B', say b'. On the other hand, going right from  $H^1(B)$ ,  $H^1(g)(b:1\to B)=g\circ b$  which gives us the map  $b':1\to B'$  that picks out g(b(1))=b'. Finally,  $t_{B'}(b')$  is precisely this element b' of B'. So in fact the naturality square commutes.

Hence  $1_{\mathbf{Set}}$  is represented by  $H^1$ .

**Example 3**  $(H^1: \mathbf{Cat} \to \mathbf{Set})$ . Similarly to the example above, ob:  $\mathbf{Cat} \to \mathbf{Set}$  which sends a small category to its underlying set of objects is represented by  $H^1$  where  $\mathbf{1}$  is the one-object category. This is because when  $\mathscr{B}$  is a category,  $H^1(\mathscr{B}) = \mathbf{Cat}(\mathbf{1}, \mathscr{B})$  which picks out objects of  $\mathscr{B}$  and is isomorphic to ob $\mathscr{B}$ . The proof that these functors are naturally isomorphic is essentially the same as the one for  $1_{\mathbf{Set}}$  and  $H^1$  in the example above.

#### Adjoints and Representables

Now we establish the claim that any set valued functor with a left adjoint is representable. We first prove the following lemma.

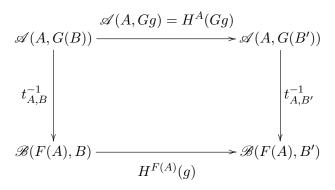
**Lemma 1.** Let  $\mathscr{A}, \mathscr{B}$  be locally small categories with functors  $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A}$  such that  $F \dashv G$ . Fix  $A \in \mathscr{A}$ , then the functor

$$\mathscr{A}(A,G(-)):\mathscr{B}\to \mathbf{Set}$$

that is, the composition  $\mathscr{B} \stackrel{G}{\to} \mathscr{A} \stackrel{H^A}{\to} \mathbf{Set}$  is representable.

Proof. Because  $F \dashv G$ , we have the isomorphism  $t_{A,B}^{-1}: \mathscr{A}(A,G(B)) \to \mathscr{B}(F(A),B)$  for each  $B \in \mathscr{B}$ . Now  $H^{F(A)}$  maps B to  $\mathscr{B}(F(A),B)$ , and it maps a morphism  $g: B \to B'$  to  $\mathscr{B}(F(A),g): \mathscr{B}(F(A),B) \to \mathscr{B}(F(A),B')$ . So we suspect that the isomorphisms  $t_{A,B}^{-1}$  gives rise to the natural isomorphism  $t^{-1}: \mathscr{A}(A,G(-)) \Rightarrow H^{F(A)}$ . We show that this is in fact the case.

We need to check naturality. Let  $g: B \to B'$ . We need to check that the following naturality square commutes:



It suffices to consider a map  $f:A\to G(B)$  from  $\mathscr{A}(A,G(B))$  as it travels around the diagram. Going down from  $\mathscr{A}(A,G(B))$ ,  $t_{A,B}^{-1}(f:A\to G(B))$  is a map  $t_{A,B}^{-1}(f):F(A)\to B$ , and then  $H^{F(A)}(g)(t_{A,B}^{-1})=g\circ t_{A,B}^{-1}$  which is a map from F(A) to B'. On the other hand, going right from  $\mathscr{A}(A,G(B))$ ,  $H^A(Gg)(f:A\to G(B))=Gg\circ f$  which is a map from A to G(B') and then  $t_{A,B'}^{-1}(Gg\circ f)$  is a map from F(A) to B'. So the question is whether  $t_{A,B'}^{-1}(Gg\circ f)=g\circ t_{A,B}^{-1}(f)$ , but that is precisely what it means for  $t^{-1}$  to be natural with respect to B (See [Liu18]), so in fact  $\mathscr{A}(A,G(-))\cong H^{F(A)}$  and is therefore representable.

Now we are ready to prove the main claim of this subsection:

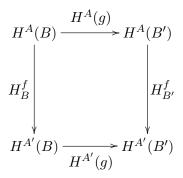
**Theorem 1.** Any set-valued functor that has a left adjoint is representable.

*Proof.* Let  $G: \mathscr{A} \to \mathbf{Set}$  be a functor with left adjoint F. Let 1 denote the set with a single element. For all  $A \in \mathscr{A}$ , G(A) is a set and by example 2 above,  $H^1(G(A)) = \mathbf{Set}(1, G(A)) \cong G(A)$  naturally in A where  $H^1$  is our functor from  $\mathbf{Set}$  to  $\mathbf{Set}$  in example 2 above. So  $G \cong \mathbf{Set}(1, G(-))$  and by the lemma above, this means G is representable.

#### Covariant Embedding

**Proposition 1.** Let  $f: A' \to A$ , then f induces a natural transformation  $H^f: H^A \Rightarrow H^{A'}$  with components  $H^f_B: H^A(B) \to H^{A'}(B)$  so that a map  $p: A \to B$  in  $H^A(B)$  gets mapped via precomposition with f to  $p \circ f: A' \to B$ .

*Proof.* We need to show that this map  $H^f$  is indeed a natural transformation. Let  $g: B \to B'$ . This means checking that the naturality square:



commutes.

It is sufficient to consider a map  $p:A\to B$  in  $H^A(B)$  as it travels around the square. Going down, we have first  $H_B^f(p)=p\circ f$ , and then to the right we have  $H^{A'}(g)(p\circ f)=g\circ (p\circ f)$ . Going to the right we have  $H^A(g)(p)=g\circ p$ , and then going down we have  $H_{B'}^f(g\circ p)=(g\circ p)\circ f$ . However, since morphism composition is associative, these two things are equal and so the square indeed commutes.

With this natural transformation between covariant representables, we have the following definition:

**Definition.** The *covariant embedding* is the functor  $H^{\bullet}: \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$  defined by the following mapping on:

- 1. **objects:** For object A in  $\mathscr{A}$ ,  $H^{\bullet}(A) = H^{A}$ .
- 2. morphisms: For morphism  $f: A' \to A, H^{\bullet}(f) = H^f$ .

We can do many of the same things with the dual case.

#### Contravariant Representable Functors

Again, we start with a definition of the *prototypical* contravariant representable functor:

**Definition.** Let  $\mathscr{A}$  be a locally small category with object A. Define the functor  $H_A : \mathscr{A}(-, A) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$  by the following mapping on

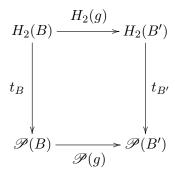
- 1. **objects:** For object B in  $\mathscr{A}$ , define  $H_A(B) = \mathscr{A}(B,A)$ .
- 2. **morphisms:** For morphism  $g: B' \to B$ , define  $H_A(g) = \mathscr{A}(g, A): \mathscr{A}(B, A) \to \mathscr{A}(B', A)$  by  $p: B \to A \mapsto p \circ g: B' \to B \to A$ .

Similar to the covariant case, any functor  $F: \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$  that is naturally isomorphic to a *prototypical* contravariant representable functor is called contravariantly representable.

**Example 4.** The functor  $\mathscr{P}: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$  which maps sets to their powersets and maps functions  $g: B' \to B$  to  $\mathscr{P}(g) = g^{-1}U$  for all  $U \in \mathscr{P}(B)$  where  $g^{-1}U = \{x' \in B' | g(x') \in U\}$  is naturally isomorphic to  $H_2: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$  where 2 denotes the set with two elements.

To show this, we first need to realize that a subset of a set B is just a boolean mapping from the set to 2 (f(b) = 1 means b is in that subset). So  $H_2(B) = \mathbf{Set}(B,2) \cong \mathscr{P}(B)$ . Define  $t_B : H_2(B) \to \mathscr{P}(B)$  with this isomorphism. We show that the  $t_B$ 's are natural in B and therefore define

the components of a natural transformation  $t: H_2 \Rightarrow \mathscr{P}$ . Let B, B' be sets, and let g be a map from B' to B. To do this, we show that the following naturality square commutes:



Consider a map  $u: B \to 2$  that picks out an associated subset  $U \subseteq B$ .  $t_B(u)$  gives precisely this subset U, and  $\mathscr{P}(g)(U)$  gives the subset U' of B' that maps along g into U. On the other hand,  $H_2(g)(u)$  precomposes g with u to give the mapping  $u \circ g: B' \to B \to 2$  which picks out the subset U' of B' that, when mapped along g gives back U. And  $t'_B(u \circ g)$  gives this U'. So the diagram commutes.

Hence,  $\mathscr{P}$  is represented by  $H_2$ .

#### Contravariant Embedding

Similar to the covariant case, the map  $f: A \to A'$  induces a natural transformation  $H_f: H_A \Rightarrow H_{A'}$  with components  $H_f^B: H_A(B) \to H_{A'}(B)$  such that a map  $p: B \to A \mapsto f \circ p: B \to A \to A'$ .

This gives rise to the following contravariant embedding (which we will call the Yoneda Embedding)

**Definition** (Yoneda Embedding). Let  $\mathscr{A}$  be a locally small category. The Yoneda Embedding is the functor  $H_{\bullet}: \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$  defined by the following mapping on:

- 1. **objects:** For object A in  $\mathscr{A}$ ,  $H_{\bullet}(A) = H_A$ .
- 2. morphisms: For morphism  $f: A \to A'$ ,  $H_{\bullet}(f) = H_f$ .

**Proposition 2.**  $H_{\bullet}$  is injective on isomorphism classes of objects.

*Proof.* Let A, A' be objects in  $\mathscr{A}$  such that  $H_A \cong H_{A'}$ . We need to show  $A \cong A'$ , that is, find maps  $f: A \to A'$ ,  $g: A' \to A$  that are mutually inverse. Since  $H_A \cong H_{A'}$  that means there are maps (natural transformations)  $\alpha$ :

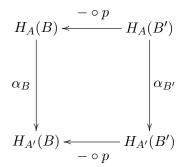
 $H_A \Rightarrow H_{A'}$  and  $\beta: H_{A'} \Rightarrow H_A$  such that  $\alpha \circ \beta = 1_{H_{A'}}$  and  $\beta \circ \alpha = 1_{H_A}$ . We need to use  $\alpha$  and  $\beta$  to construct the f and g we need.

First let's consider the components of  $\alpha$ . Let B be an object in  $\mathscr{A}$ . Then  $\alpha_B$  is a map from  $H_A(B) \to H_{A'}(B)$ . Letting B = A, we have  $\alpha_A : H_A(A) \to H_{A'}(A)$ . Now, we don't know what morphisms there are in  $H_A(A)$ , but we certainly know that  $1_A$  exists. And applying  $\alpha_A$  to  $1_A$ , we get a map  $\alpha_A(1_A) : A \to A'$ .

Similarly, considering  $\beta_{A'}$  applied to  $(1_{A'})$ , we get a map  $\beta_{A'}(1_{A'}): A' \to A$ .

So now we have our maps between A and A'. We need to show that they are mutually inverse.

Naturality of  $\alpha$  in B means that for every  $p: B \to B'$  the following naturality square holds (note the contravariance):



In particular, if  $x: B' \to A$  is a map  $H_A(B')$ , then  $\alpha_B(x \circ p) = \alpha_{B'}(x) \circ p$ . Again, what we need is a specialization of this general phenomenon. Letting  $x = 1_A$ , which means making B = A' and B' = A, our p becomes a map from  $A' \to A$  and by naturality above we have  $\alpha_{A'}(1_A \circ p) = \alpha_A(1_A) \circ p$ . Now p is a map from  $A' \to A$ , so letting  $p = \beta_{A'}(1_{A'})$  we have that  $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = \alpha_{A'}(1_A \circ \beta_{A'}(1_{A'})) = \alpha_{A'}(\beta_{A'}(1_{A'}))$ . But since  $\alpha_{A'}$  and  $\beta_{A'}$  are mutually inverse, this just means  $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = 1_{A'}$ , which is precisely what we wanted.

Similarly, we can show using the naturality of  $\beta$  and specializing to  $1_{A'}$  that  $\beta_{A'}(1_{A'}) \circ \alpha_A(1_A) = 1_A$ .

Thus 
$$A \cong A'$$
.

## The Yoneda Lemma

**Theorem 2** (The Yoneda Lemma). Let  $\mathscr{A}$  be a locally small category. Then  $[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \cong X(A)$  naturally in  $A \in \mathscr{A}$  and  $X \in [\mathscr{A}^{op}, \mathbf{Set}]$ .

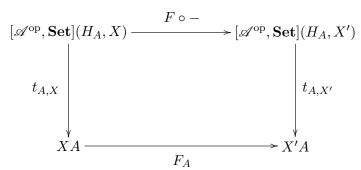
*Proof.* First let's give a quick outline of the proof:

- 1. Produce a mapping  $t_{A,X} : [\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \to XA$ .
- 2. Produce a mapping  $t_{A,X}^{-1}: XA \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$ .
- 3. Show that  $t, t^{-1}$  are mutually inverse.
- 4. Show that t is natural in A.
- 5. Show that t is natural in X.

We're not going to do these steps in order, instead we'll do these steps the way I did it when I came up with the proof, so you'll see my motivation for each successive step.

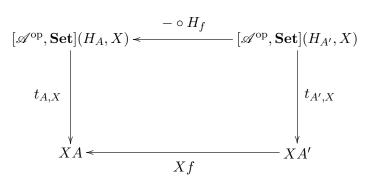
Let's begin with step 1. Let  $\alpha: H_A \Rightarrow X$  be a natural transformation. Looking at  $\alpha$ 's components,  $\alpha_B: H_A(B) \to XB$  sends a map  $g: B \to A$  to the element of XB,  $\alpha_B(g)$ .  $t_{A,X}$  needs to map  $\alpha$  to one specific element of XA. Consider the special case where B = A. Then we have the component  $\alpha_A: H_A(A) \to XA$ . Since the only map we are certain exists in  $H_A(A)$  is the identity map  $1_A$ , this suggests we need to define  $t_{A,X}(\alpha) = \alpha_A(1_A)$ .

To check that this mapping is the correct one, let's first check that  $t_{A,X}$  is natural in X. Let  $F: X \Rightarrow X'$  be a natural transformation.  $t_{A,X}$  natural in X means that the following naturality square holds:



Let  $\alpha: H_A \Rightarrow X$  be a natural transformation. Then going down first and then to the right via  $F_A$  we have  $F_A(t_{A,X}(\alpha)) = F_A(\alpha_A(1_A))$ . On the other hand going right first via  $F \circ -$  and then down we have  $t_{A,X'}(F \circ \alpha) = (F \circ \alpha)_A(1_A)$ . And by compositionality,  $(F \circ \alpha)_A(1_A) = F_A \circ \alpha_A(1_A) = F_A(\alpha_A(1_A))$ . So  $t_{A,X}$  is indeed natural in X.

Now let us check that  $t_{A,X}$  is natural in A. We won't succeed yet but this will give us a hint as to how we should define  $t_{A,X}^{-1}$ . Let f be a map from A to A'. We need to check that the following naturality square commutes:



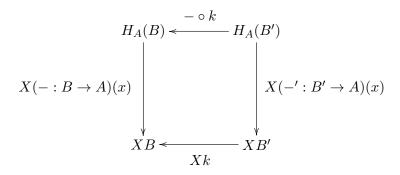
Let  $\alpha': H_{A'} \Rightarrow X$  be a natural transformation. Then going down first and then left via Xf we have  $Xf(t_{A',X}(\alpha')) = Xf(\alpha'_{A'}(1_{A'}))$ . On the other hand, going to the left first via  $-\circ H_f$  and then down, we have  $t_{A,X}(\alpha'\circ H_f) = (\alpha'_A\circ H_f^A)(1_A) = \alpha'_A(f)$  (Because  $H_f^A(1_A) = f\circ 1_A$ ) = f. So we need to have the equality

$$Xf(\alpha'_{A'}(1_{A'})) = \alpha'_{A}(f)$$

Which we can't prove just yet. However, this does give us a hint as to how we should define  $t_{A,X}^{-1}$ . Notice that  $\alpha'_{A'}(1'_A)$  is an element of XA' and that for every  $g: B \to A' \in H_{A'}(B)$ ,  $g \mapsto Xg(\alpha'_{A'}(1'_A))$  defines a mapping from  $H_{A'}(B)$  to XB, which is beginning to look like the components of a natural transformation  $H_{A'} \Rightarrow X$ .

With that let us define our mapping  $t_{A,X}^{-1}: XA \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$ . Let x be an element of XA. We define  $t_{A,X}^{-1}(x) = (X(f:B\to A)(x))_{B\in\mathscr{A}}$ . So for each B, we have a family of morphisms  $H_A(B) \to XB$  given by  $f \mapsto Xf(x)$ . Now we need to check that this family of morphisms does indeed form a natural transformation  $H_A \Rightarrow X$ .

Let  $k: B \to B'$ . We need to check that the following naturality square commutes:

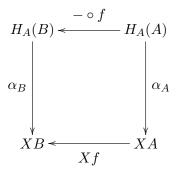


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Let  $p: B' \to A$ . Then going left and down we first have  $p \circ k: B \to A$  which then becomes  $X(p \circ k)(x)$ . On the other hand, going down first and then left we have Xk(Xp(x)). These expressions are equal by the (co)functoriality of X. So  $t_{A,X}^{-1}(x)$  does indeed yield a natural transformation.

Now we need to check that  $t_{A,X}$  and  $t_{A,X}^{-1}$  are in fact mutually inverse. Let  $x \in XA$ . Then  $t_{A,X}(t_{A,X}^{-1}(x)) = t_{A,X}(X(-)(x)) = (X(-)(x))_A(1_A) = X1_A(x) = 1_{XA}(x) = x$ . So  $t_{A,X} \circ t_{A,X}^{-1}$  does indeed equal  $1_{XA}$ . The other direction is a little bit harder:

Let  $\alpha$  be a natural transformation from  $H_A \Rightarrow X$ .  $t_{A,X}^{-1}(t_{A,X}(\alpha)) = t_{A,X}^{-1}(\alpha_A(1_A)) = X(-)(\alpha_A(1_A))$ . Let  $f: B \to A$ . For  $X(-)(\alpha_A(1_A)) = \alpha$ , we need to have  $Xf(\alpha_A(1_A)) = \alpha_B(f)$  for every f. The only tool we have at our disposal is naturality, either naturality of X(-)(x) which we just established, or the naturality of  $\alpha$ . When I first went about trying to prove this I thought we ought to use the naturality of X(-)(x), however it turns out that the correct thing to do is use the naturality of  $\alpha$ . (Thank you [Lei14]!) Let's write out the naturality square for  $\alpha$  with f:



Considering the image of  $1_A$  as it moves around the square, we have  $\alpha_B(1_A \circ f) = \alpha_B(f) = Xf(\alpha_A(1_A))$ , which is exactly what we needed.

So t and  $t^{-1}$  are mutually inverse, and in fact this last step is also what we needed to prove that  $t_{A,X}$  is natural in A, so we are done.

## References

- [Lei14] Tom Leinster. Basic Category Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.
- [Liu18] Stephen Liu. Adjoints, 2018. https://ssyl55.github.io/files/adjoints.pdf.