Abelianization of Groups

Stephen Liu

June 11, 2018

Definition (Commutator)**.** Let G be a group and $x, y \in G$. Then the commutator of x and y, is $[x, y] = x^{-1}y^{-1}xy$.

It is easy to see from this definition that $xy = yxx^{-1}y^{-1}xy = yx[x, y]$ and that $xy = yx$ if and only if $[x, y] = 1$.

Definition (Commutator Subgroup)**.** Suppose A, B are nonempty subsets of G, then $[A, B] = \langle [a, b] | a \in A, b \in B \rangle$ is the group generated by commutators of elements from A and from B. In particular, $G' = [G, G] = \langle [x, y] | x, y \in G \rangle$ is called the commutator subgroup of G.

This commutator subgroup has nice properties which allow us to "abelianize" G, that is, get an abelian group from G. Since we saw above that x, y commute if and only if $[x, y] = 1$, this suggests that to get an abelian group from G, we need some way of *setting* $[x, y]$ to 1, which suggests taking the quotient by G'. And indeed we have the following proposition:

Proposition 1. *G/G' is abelian.*

Proof. Let $xG', yG' \in G/G'$, since $xy = yx[x, y]$ and $[x, y] \in G'$, we have $(xG')(yG') = (xyG') = (yx[x, y]G') = (yxG') = (yG')(xG')$. □

So we were able to "abelianize" G. However, G' is also nice in that it has the property that it is the smallest normal subgroup of G such that G/G' is abelian. In other words, G/G' is the largest abelian quotient of G. Precisely, we have the following proposition:

Proposition 2. *$H \trianglelefteq G$ and G/H abelian if and only if $G' \leq H$.*

Proof. First suppose $H \trianglelefteq G$ and G/H is abelian.

So $1H = (xH)^{-1}(yH)^{-1}(xH)(yH) = (x^{-1}y^{-1}xyH) = ([x, y]H)$ which means $[x, y] \in H$ for all $x, y \in G$, so $G' \leq H$.

Now suppose $G' \leq H$, then since G/G' is abelian, every subgroup is normal and so $G/H \trianglelefteq G/G'$ and by the lattice isomorphism theorem that means $H \trianglelefteq G$. Also, by the third isomorphism theorem, we have $G/H \cong (G/G')/(H/G')$ which means G/H is abelian since it is the quotient of an abelian group (G/G'). $\square$

Additionally we have this nice universal property regarding G/G' which is that for any abelian group A and group homomorphism $\phi : G \to A$, $G' \subseteq \text{Ker}\phi$ and the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[dr, "\phi"'] & G/G' \arrow[d, dashed, "\exists! \overline{\phi}"] \\ & A \end{tikzcd}$$

Proof. It is easy enough to show $G' \subseteq \text{Ker}\phi$

since $\phi(x^{-1}y^{-1}xy) = \phi(x^{-1})\phi(y^{-1})\phi(x)\phi(y) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y)$ and since A is abelian, this equals 1_A. We define $\overline{\phi} : G/G' \to A$ by $\overline{\phi}(xG') = \phi(x)$. Now we need to show $\overline{\phi}$ exists and is unique and commutes according to the diagram above. To show that it exists, we show that it is a well defined group homomorphism. Suppose $xG' = yG'$, for $\overline{\phi}$ to be well defined, we need $\overline{\phi}(xG') = \overline{\phi}(yG')$. Since $xG' = yG'$, $xy^{-1} \in G' \subseteq \text{Ker}\phi$, so $\phi(x)\phi(y)^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1}) = 1_A$, so $\phi(x) = \phi(y)$, which by the definition of $\overline{\phi}$, means $\overline{\phi}(xG') = \overline{\phi}(yG')$. $\overline{\phi}$ is indeed a group homomorphism because $\overline{\phi}(xG'yG') = \overline{\phi}(xyG') = \phi(xy) = \phi(x)\phi(y) = \overline{\phi}(xG')\overline{\phi}(yG')$ and $\overline{\phi}$ is unique because it is completely determined and defined by ϕ. Finally, $\phi = \overline{\phi} \circ \eta$ because $\overline{\phi}(\eta(x)) = \overline{\phi}(xG') = \phi(x)$. $\square$

References

[DF04] David Dummit and Richard Foote. *Abstract Algebra*. John Wiley & Sons, Inc., third edition, 2004.