Adjoints

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Abstract

We should think about adjunctions as an interesting comparison of two categories that is somewhat more general and of a different nature than an equivalence of categories. Following [Lei14], we'll be looking at three different ways of understanding adjoint functors and showing that they are equivalent.

Hom-Set Definition

Definition (Adjoint Functors)**.** Given a pair of functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$, we say F is left adjoint to G, and G right adjoint to F, written $F \dashv G$ if there is a natural isomorphism $t_{A,B} : \mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$ for each A in $\mathscr{A}$ and B in $\mathscr{B}$. An adjunction between F and G is a choice of natural isomorphism $t_{A,B}$.

So this means for each $g : F(A) \to B$, we have a map $t_{A,B}(g) : A \to G(B)$. We shall call this isomorphism the transpose of g (Leinster denotes this $\overline{g}$) and this process "transposing" g. Similarly, for each $f : A \to G(B)$, we have a map $t^{-1}_{A,B}(f) : F(A) \to B$.

Naturality

Let's take a closer look at what naturality means. In words it would mean that the transpose of a composition of two maps is equal to the composition of the transpose of the two maps. We have four options here:

1. naturality of t with respect to A
2. naturality of t^{-1} with respect to $F(A)$
3. naturality of t^{-1} with respect to B

4. and finally naturality of t with respect to $G(B)$.

Let's first take a look at *naturality of t with respect to A*:

We have the following data (left), and applying $t_{A',B}$ on $g \circ Ff$ and on them separately we get the commutative triangle on the right:

$$
\begin{tikzcd}
A' \arrow[r, mapsto] \arrow[d, "f"'] \arrow[dr, phantom, "\longmapsto"] & F(A') \arrow[d, "Ff"] \\
A \arrow[r, mapsto] & F(A) \arrow[d, "g"] \\
 & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "f"] \arrow[dr, "t_{A',B}(g \circ Ff)"'] & A \arrow[d, "t_{A,B}(g)"] \\
 & G(B)
\end{tikzcd}
$$

So $t_{A',B}(g \circ F(f)) = t_{A,B}(g) \circ f$ (here $t_{A',F(A)}(F(f)) = f$).

Similarly for 2, 3, and 4, we have the following data yielding the following commutative triangles:

naturality of t^{-1} with respect to $F(A)$:

We begin with the map $Ff : F(A') \to F(A)$, and taking the preimage, we get the following data and corresponding commutative triangle:

$$
\begin{tikzcd}
A' \arrow[r, mapsto] \arrow[d, "f"'] & F(A') \arrow[d, "Ff"] \\
{} \arrow[r, phantom, "\longmapsto"] & {} \\
A \arrow[r, mapsto] \arrow[d, "g"] & F(A) \\
G(B) &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A') \arrow[r, "Ff"] \arrow[dr, "t^{-1}_{A',B}(g \circ f)"'] & F(A) \arrow[d, "t^{-1}_{A,B}(g)"] \\
& B
\end{tikzcd}
$$

So $t^{-1}_{A',B}(g \circ f) = t^{-1}_{A,B}(g) \circ Ff$.
naturality of t^{-1} *with respect to* B:

$$
\begin{tikzcd}
& A \arrow[d, "f"] \\
B \arrow[r, mapsto] \arrow[d, "g"'] & G(B) \arrow[d, "Gg"] \\
{} \arrow[r, phantom, "\longmapsto"] & {} \\
B' \arrow[r, mapsto] & G(B')
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \arrow[r, "t^{-1}_{A,B}(f)"] \arrow[dr, "t^{-1}_{A,B'}(Gg \circ f)"'] & B \arrow[d, "g"] \\
& B'
\end{tikzcd}
$$

So $t^{-1}_{A,B'}(Gg \circ f) = g \circ t^{-1}_{A,B}(f)$.
Finally, *naturality of* t *with respect to* $G(B)$:

$$
\begin{tikzcd}
F(A) \arrow[d, "f"'] & \\
B \arrow[r, mapsto] \arrow[d, "g"'] \arrow[dr, phantom, "\longmapsto"] & G(B) \arrow[d, "Gg"] \\
B' \arrow[r, mapsto] & G(B')
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "t_{A,B}(f)"] \arrow[dr, "t_{A,B'}(g \circ f)"'] & G(B) \arrow[d, "Gg"] \\
& G(B')
\end{tikzcd}
$$

So $t_{A,B'}(g \circ f) = Gg \circ t_{A,B}(f)$.

We call this understanding of adjoint functors the Hom-Set Definition because the important bit here is this isomorphism between the Hom-Sets of $\mathscr{A}$ and $\mathscr{B}$.

There are a whole class of examples of adjoint functors that are the forgetful and free functors between algebraic theories. We'll be looking at one of these:

Example (Abelianization of Groups)**.** There is an adjunction

$$
\begin{tikzcd}
\mathbf{Ab} \arrow[d, shift left=2, "U"] \\
\mathbf{Grp} \arrow[u, shift left=2, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

where U is the forgetful inclusion functor from the category of abelian groups to the category of groups, and F is the free functor from the category of groups to the category of abelian groups. For a group G in **Grp**, $F(G)$ is the abelianization of the group G, or G/G' where G' is the commutator subgroup of G (see my writeup at [Liu18] for details). This abelianization gives rise to the universal property that for any group homomorphism ϕ out of G to an abelian group A, there is a unique $\overline{\phi} : G/G' \to A$ such that

$\phi = \overline{\phi} \circ \pi$ where π is the canonical quotient map from G to G/G'. This universal property is what allows us to specify what $t_{G,A} : \mathbf{Ab}(F(G), A) \to \mathbf{Grp}(G, U(A))$ should do: $t_{G,A}(\overline{\phi}) = \overline{\phi} \circ \pi = \phi$, and $t^{-1}_{G,A}(\phi) = \overline{\phi}$.

Units and Counits Definition

Definition (Unit and Counit of an Adjunction)**.** Given $A \in \mathscr{A}$ and the identity map $1_{F(A)}$, $t_{A,F(A)}(1_{F(A)})$ defines the isomorphism $\eta_A : A \to GF(A)$. Similarly, given $B \in \mathscr{B}$ and the identity map $1_{G(B)}$, $t^{-1}_{G(B),B}(1_{G(B)})$ defines the isomorphism $\varepsilon_B : FG(B) \to B$. Together, η_A and ε_B define the natural transformations

$$\eta : 1_{\mathscr{A}} \to G \circ F, \qquad \varepsilon : F \circ G \to 1_{\mathscr{B}}$$

called the unit and counit of the adjunction, respectively.

We have important triangle identities associated with the unit and counit.

Proposition 1 (Triangle Identities)**.** *Given an adjunction $F \dashv G$ with unit η and counit ε, the triangles*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

commute.

Proof. We prove the equivalent statement that the triangles

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "1_{F(A)}"'] & FGF(A) \arrow[d, "\varepsilon_{F(A)}"] \\ & F(A) \end{tikzcd} \qquad \begin{tikzcd} G(B) \arrow[r, "\eta_{G(B)}"] \arrow[dr, "1_{G(B)}"'] & GFG(B) \arrow[d, "G(\varepsilon_B)"] \\ & G(B) \end{tikzcd}$$

commute for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

For the triangle on the left, we use *naturality of* t^{-1} *with respect to* $F(A)$ that we explained above where we replace f with η_A and g with $1_{GF(A)}$. So we have the following data giving rise to the commutative triangle on the right:

$$
\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\longmapsto"] & F(A) \arrow[d, "F(\eta_A)"] \\
GF(A) \arrow[r, mapsto] \arrow[d, "1_{GF(A)}"'] & FGF(A) \\
GF(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "t^{-1}_{A,F(A)}(1_{GF(A)} \circ \eta_A)"'] & FGF(A) \arrow[d, "t^{-1}_{GF(A),F(A)}(1_{GF(A)})"] \\
& F(A)
\end{tikzcd}
$$

Now by definition $t^{-1}_{GF(A),F(A)}(1_{GF(A)}) = \varepsilon_{F(A)}$, and $t^{-1}_{A,F(A)}(1_{GF(A)} \circ \eta_A) = t^{-1}_{A,F(A)}(\eta_A)$ and by definition, $t_{A,F(A)}(1_{F(A)}) = \eta_A$, so $t^{-1}_{A,F(A)}(\eta_A) = 1_{F(A)}$. So from the triangle we get $1_{F(A)} = \varepsilon_{F(A)} \circ F(\eta_A)$, proving the commutative triangle.

Similarly, for the triangle on the right, we use *naturality of* t *with respect to* $G(B)$ that we explained above where we replace f with $1_{FG(B)}$ and g with ε_B. So we from the resulting commutative triangle we have

$$t_{G(B),B}(\varepsilon_B \circ 1_{FG(B)}) = G(\varepsilon_B) \circ t_{G(B),FG(B)}(1_{FG(B)}).$$

And again, by definition $t_{G(B),FG(B)}(1_{FG(B)}) = \eta_{G(B)}$ and $t_{G(B),B}(\varepsilon_B \circ 1_{FG(B)}) = t_{G(B),B}(\varepsilon_B) = 1_{G(B)}$, so $1_{G(B)} = G\varepsilon_B \circ \eta_{G(B)}$, proving the identity. $\square$

It turns out the unit and counit determine the whole adjunction.

Proposition 2. *Given an adjunction* $t_{A,B} : \mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$ *for any* $g : F(A) \to B$, $t_{A,B}(g) = Gg \circ \eta_A$, *and for any* $f : A \to G(B)$, $t^{-1}_{A,B}(f) = \varepsilon_B \circ Ff$.

Proof. For any $g : F(A) \to B$, by naturality, $t_{A,B}(g) = t_{A,B}(g \circ 1_{F(A)})$ which by naturality of t with respect to $GF(A)$, is equal to $Gg \circ \eta_A$. Similarly, for any $f : A \to G(B)$, $t^{-1}_{A,B}(f) = t^{-1}_{A,B}(1_{G(B)} \circ f)$ which by naturality of t^{-1} with respect to $FG(B)$, is equal to $\varepsilon_B \circ Ff$. □

Using this fact, we can equivalently define adjunctions by specifying pairs of units and counits.

Theorem 1. *Given functors $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$, there is a bijection between adjunctions $F \dashv G$ and pairs of units and counits (η, ε) that satisfy the triangle identities.*

Proof. We have already shown that given an adjunction t, we can define natural transformations η, ε that satisfy the triangle identites. Now, we just need to show that given unit and counit η, ε, we can uniquely define a natural isomorphism $t_{A,B} : \mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$ for all $A \in \mathscr{A}$, for all $B \in \mathscr{B}$.

Given $g : F(A) \to B$, define $t_{A,B}(g) = Gg \circ \eta_A$, and given $f : A \to G(B)$, define $t^{-1}_{A,B}(f) = \varepsilon_B \circ Ff$. We need to show that t and t^{-1} are well-defined, mutually inverse, natural, and that η, ε are in fact their unit and counit.

Well-Defined Let $g : F(A) \to B$, $h : F(A) \to B$ with $g = h$. Since G is well-defined, $t_{A,B}(g) = Gg \circ \eta_A = Gh \circ \eta_A = t_{A,B}(h)$, so t is well-defined. Similarly for t^{-1}.

Isomorphism □

References

[Lei14] Tom Leinster. *Basic Category Theory.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.

[Liu18] Stephen Liu. Abelianization of groups, 2018. ssyl55.github.io/files/abelianization.pdf.