The Yoneda Lemma

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Abstract

Some notes on the Yoneda Lemma, starting with the notion of representable functors.

Representable Functors

We first define a *prototype* of a representable functor out of a locally small category $\mathscr{A}$.

Definition. Let $\mathscr{A}$ be a locally small category, and fix $A \in \mathscr{A}$. Define the functor $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ by the following mapping on

1. **objects**: For $B \in \mathscr{A}$, define $H^A(B) = \mathscr{A}(A, B)$, the hom-set of arrows in $\mathscr{A}$ from A to B.

2. **morphisms**: For $g : B \to B'$, define $H^A(g) = \mathscr{A}(A, g) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$ by $p \mapsto g \circ p$ for all $p : A \to B$, sending morphisms from A to B to morphisms from A to B' by post-composition with g.

This is the covariant version of H, there is a contravariant version, denoted H_A which does pre-composition instead.

From this prototype of a representable functor, we can now define representable functors:

Definition (Representable Functor)**.** Let $\mathscr{A}$ be a locally small category. A functor $X : \mathscr{A} \to \mathbf{Set}$ is representable if X is naturally isomorphic to H^A for some $A \in \mathscr{A}$. A representation of X is a choice of an object A along with a natural isomorphism from H^A to X.

Some examples of representable functors:

Example 1 (group G regarded as a one object category)**.** Regarding a group G as a one object category $\mathscr{G}$, with object $*$, $H^* = *(*, -) : \mathscr{G} \to \mathbf{Set}$ is a functor which maps

1. **objects**: There is only one object in $\mathscr{G}$, and $H^*(*)$ is the set of mappings from $*$ to $*$, otherwise known as the elements of the group G. So $H^*(*) \cong U(G)$ where $U : \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor that returns the underlying set of a group.

2. **morphisms**: Let $g \in G$, then $H^*(g)$ maps any element h of G to $g \circ h$, which, interpreted in the context of a group is just group mulitiplication on the left, i.e. gh.

Since $\mathscr{G}$ has only one object, there is only one representable functor on it (up to isomorphism), and the representable is the underlying set G acted on by left multiplication.

Example 2 ($H^1 : \mathbf{Set} \to \mathbf{Set}$)**.** Let 1 denote the set with just one element. The functor $1_{\mathbf{Set}}$ is represented by the functor $H^1 : \mathbf{Set} \to \mathbf{Set}$. To see this, let us first see what H^1 does and then show that it is naturally isomorphic to $1_{\mathbf{Set}}$.

Let $B \in \mathbf{Set}$, then $H^1(B) = \mathbf{Set}(1, B) = \{b : 1 \to B\}$ the set of functions from the singleton set to B that pick out an element of B. In particular, $H^1(B) \cong B$. Let $p : 1 \to B$ and $g : B \to B'$, then $H^1(g) = \mathbf{Set}(1, g) : \mathbf{Set}(1, B) \to \mathbf{Set}(1, B')$ sends functions p that pick out an element of B to functions $g \circ p$ that pick out an element of B'.

So we have the parallel functors $H^1, 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ and we know what they do. Now we construct a natural isomorphism $t : H^1 \Rightarrow 1_{\mathbf{Set}}$. We already have that $H^1(B) \cong B$, so the components of t are the isomorphisms $t_B : H^1(B) \to B$ (and of course $t_B^{-1} : B \to H^1(B)$). Now we need to check that the naturality square below commutes:

$$\begin{tikzcd} H^1(B) \arrow[r, "H^1(g)"] \arrow[d, "t_B"'] & H^1(B') \arrow[d, "t_{B'}"] \\ B \arrow[r, "1_{\mathbf{Set}}(g) = g"'] & B' \end{tikzcd}$$

where $g : B \to B'$.

It is enough to consider an element $b : 1 \to B$ from $H^1(B)$. b picks out an element of B, which we will also identify by $b(1) = b$. Going down from $H^1(B)$, $t_B(b : 1 \to B)$ gives us this element b, which is then sent by g to some element of B', say b'. On the other hand, going right from $H^1(B)$, $H^1(g)(b : 1 \to B) = g \circ b$ which gives us the map $b' : 1 \to B'$ that picks out $g(b(1)) = b'$. Finally, $t_{B'}(b')$ is precisely this element b' of B'. So in fact the naturality square commutes.

Hence $1_{\mathbf{Set}}$ is represented by H^1.

Example 3 **($H^{\mathbf{1}}$: Cat $\to$ Set).** Similarly to the example above, ob : **Cat** $\to$ **Set** which sends a small category to its underlying set of objects is represented by $H^{\mathbf{1}}$ where $\mathbf{1}$ is the one-object category. This is because when $\mathscr{B}$ is a category, $H^{\mathbf{1}}(\mathscr{B}) = \mathbf{Cat}(\mathbf{1}, \mathscr{B})$ which picks out objects of $\mathscr{B}$ and is isomorphic to ob$\mathscr{B}$. The proof that these functors are naturally isomorphic is essentially the same as the one for $1_{\mathbf{Set}}$ and H^1 in the example above.

Adjoints and Representables

Now we establish the claim that any set valued functor with a left adjoint is representable. We first prove the following lemma.

Lemma 1. *Let $\mathscr{A}, \mathscr{B}$ be locally small categories with functors $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A}$ such that $F \dashv G$. Fix $A \in \mathscr{A}$, then the functor*

$$\mathscr{A}(A, G(-)) : \mathscr{B} \to \boldsymbol{Set}$$

that is, the composition $\mathscr{B} \xrightarrow{G} \mathscr{A} \xrightarrow{H^A} \boldsymbol{Set}$ is representable.

Proof. Because $F \dashv G$, we have the isomorphism $t^{-1}_{A,B} : \mathscr{A}(A, G(B)) \to \mathscr{B}(F(A), B)$ for each $B \in \mathscr{B}$. Now $H^{F(A)}$ maps B to $\mathscr{B}(F(A), B)$, and it maps a morphism $g : B \to B'$ to $\mathscr{B}(F(A), g) : \mathscr{B}(F(A), B) \to \mathscr{B}(F(A), B')$. So we suspect that the isomorphisms $t^{-1}_{A,B}$ gives rise to the natural isomorphism $t^{-1} : \mathscr{A}(A, G(-)) \Rightarrow H^{F(A)}$. We show that this is in fact the case.

We need to check naturality. Let $g : B \to B'$. We need to check that the following naturality square commutes:

$$\begin{tikzcd}
\mathscr{A}(A, G(B)) \arrow[r, "\mathscr{A}(A, Gg) = H^A(Gg)"] \arrow[d, "t^{-1}_{A,B}"'] & \mathscr{A}(A, G(B')) \arrow[d, "t^{-1}_{A,B'}"] \\
\mathscr{B}(F(A), B) \arrow[r, "H^{F(A)}(g)"'] & \mathscr{B}(F(A), B')
\end{tikzcd}$$

It suffices to consider a map $f : A \to G(B)$ from $\mathscr{A}(A, G(B))$ as it travels around the diagram. Going down from $\mathscr{A}(A, G(B))$, $t^{-1}_{A,B}(f : A \to G(B))$ is a map $t^{-1}_{A,B}(f) : F(A) \to B$, and then $H^{F(A)}(g)(t^{-1}_{A,B}) = g \circ t^{-1}_{A,B}$ which is a map from $F(A)$ to B'. On the other hand, going right from $\mathscr{A}(A, G(B))$, $H^A(Gg)(f : A \to G(B)) = Gg \circ f$ which is a map from A to $G(B')$ and then $t^{-1}_{A,B'}(Gg \circ f)$ is a map from $F(A)$ to B'. So the question is whether $t^{-1}_{A,B'}(Gg \circ f) = g \circ t^{-1}_{A,B}(f)$, but that is precisely what it means for t^{-1} to be natural with respect to B (See [Liu18]), so in fact $\mathscr{A}(A, G(-)) \cong H^{F(A)}$ and is therefore representable. □

Now we are ready to prove the main claim of this subsection:

Theorem 1. *Any set-valued functor that has a left adjoint is representable.*

Proof. Let $G : \mathscr{A} \to \mathbf{Set}$ be a functor with left adjoint F. Let 1 denote the set with a single element. For all $A \in \mathscr{A}$, $G(A)$ is a set and by example 2 above, $H^1(G(A)) = \mathbf{Set}(1, G(A)) \cong G(A)$ naturally in A where H^1 is our functor from **Set** to **Set** in example 2 above. So $G \cong \mathbf{Set}(1, G(-))$ and by the lemma above, this means G is representable. □

References

[Lei14] Tom Leinster. *Basic Category Theory.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.

[Liu18] Stephen Liu. Adjoints, 2018. https://ssyl55.github.io/files/adjoints.pdf.