Modules and Tensor Products

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Abstract

Some notes on modules and tensor products of modules.

Modules

The Basics

Definition (Modules over a ring). Let R be a ring. A left R-module M is an abelian group (M, +) with a map $R \times M \to M$ (also known as an action of R over M) such that for all $r, s \in R$, $m, n \in M$ we have:

- 1. (r+s)m = rm + sm
- $2. \ r(m+n) = rm + rn$
- 3. (rs)m = r(sm)
- 4. 1m = m (If R contains 1).

We can define right R-modules analogously. Note that when R is a field, then a module over a field is precisely the same thing as a vector space over that field.

Definition (Submodules). Let M be a R-module. A submodule of M is a subgroup N of M that is closed under the ring action, that is, $rn \in N$ for all $r \in R$, $n \in N$ (in the case of left R-modules).

One important example of a module are the \mathbb{Z} -modules:

Example 1 (\mathbb{Z} -Modules). Consider the ring \mathbb{Z} and any abelian group A. Then we can make A into a \mathbb{Z} -module by defining the action $\mathbb{Z} \times A \to A$ by

$$na = \begin{cases} a+a+\cdots+a & \text{(n times)} & n>0\\ 0 & n=0\\ -a-a-\cdots-a & \text{(n times)} & n<0 \end{cases}$$

So any abelian group A is a \mathbb{Z} -module. Conversely, it turns out that every \mathbb{Z} -module is an abelian group.

Now we define the notion of module homomorphisms.

Definition (Module Homomorphisms). Let M and N be R-modules. A function $\varphi: M \to N$ is a module homomorphism if for all $r \in R$ and $x, y \in M$ we have

1.
$$\varphi(x+y) = \varphi(x) + \varphi(y)$$

2.
$$\varphi(rx) = r\varphi(x)$$

Now our goal is to arrive at a definition of tensor products of modules, which will involve free Z-modules, so let's first go over the definition of a free module and an important universal property of free modules.

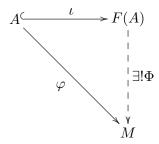
Freely Generated Modules

Definition (Free Modules). An R-module F is free on a subset $A \subseteq F$ if for every nonzero $x \in F$, there are unique nonzero elements $r_1, r_2, \ldots, r_n \in R$ and unique $a_1, a_2, \ldots, a_n \in A$ such that $x = r_1 a_1 + r_2 a_2 + \cdots + r_n a_n$ for some positive integer n. We call A a basis for F and that A is the set of free generators of F.

Notice that when R is a field, then A is the set of basis vectors (will also need linear independence) for the vector space F over the field R.

We now talk about an important universal property of free modules, which is a precursor to the defining universal property of tensor products.

Proposition 1 (Universal Property of Free Modules). For any set A there is a free R-module F(A) on A such that if M is any R-module and $\varphi: A \to M$ is any map of sets, then we have the following commutative diagram:



where Φ is an R-module homomorphism.

Proof. By convention, if $A = \emptyset$ we define $F(A) = \{0\}$. In that case, φ is the unique map of sets $\emptyset \to M$, F(A) is also the empty set and ι is the identity map, which means $\Phi = \varphi$. Otherwise, if A is nonempty, then let F(A) be the collection of all set functions $f: A \to R$ such that f(a) = 0 for all but finitely many $a \in A$. We can make F(A) into an R-module by pointwise addition of functions and pointwise multiplication of ring elements times a function, so we have for all $f, g \in F(A)$ and $r \in R$:

$$(f+g)(a) = f(a) + g(a)$$
$$(rf)(a) = r(f(a))$$

for all $x \in A$.

Let's just check to make sure this indeed gives us an R-module. Let $r, s \in R, f, g \in F(A)$. For each $a \in A$:

- 1. (r+s)f gives (r+s)(f(a)) which equals r(f(a))+s(f(a)) (Since f(a) is an element in R) which finally gives rf+sf. So (r+s)f=rf+sf.
- 2. r(f+g) gives r(f(a)+g(a)) and since f(a),g(a) are elements in R, this gives r(f(a))+r(g(a))=rf+rg.
- 3. (rs)f = (rs)(f(a)) = r(s(f(a))) = r(sf).

So F(A) is indeed an R-module. Now we need to show F(A) is freely generated by A. We define the map $\iota: A \to F(A)$ by $a \mapsto f_a$ where

$$f_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases}$$

Since ι is injective (Let $a, b \in A$ such that $f_a = f_b$, then f_a and f_b both take the value 1 at the same point x which is both equal to a and equal to a, so a = b, we see that a can be seen as an embedding of a in a

allows us to view F(A) as all finite R-linear combinations of elements of A in the following way:

Let $f: A \to R$ be a nonzero element of F(A). Then by definition of F(A), f takes on a nonzero value (in R) for finitely many points in A, say a_1, a_2, \ldots, a_n . So at each a_i , f takes on a nonzero value, say r_i . That means we can uniquely write f as the R-linear combination $r_1f_{a_i}+r_2f_{a_2}+\cdots+r_nf_{a_n}$. Hence, F(A) is indeed freely generated by A.

Now given the map on sets $\varphi: A \to M$, we define $\Phi: F(A) \to M$ by $\sum_{i=1}^n r_i f_{a_i} \mapsto \sum_{i=1}^n r_i \varphi(a_i)$. Let's verify that Φ is indeed a well-defined R-module homomorphism. Let $r \in R$ and $f, g \in F(A)$. We have just established that f can be written uniquely as $\sum_{i=1}^n r_i f_{a_i}$ and likewise g can be written uniquely as $\sum_{k=1}^m s_k g_{b_k}$. Then:

- 1. well-defined: Since elements $f \in F(A)$ are written uniquely as formal R-linear sums of f_a 's and φ is well-defined, Φ is well-defined.
- 2. $\Phi(rf) = \Phi(r \sum_{i=1}^{n} r_i f_{a_i}) = r \sum_{i=1}^{n} r_i \varphi(a_i) = r \Phi(f)$
- 3. $\Phi(f+g) = \Phi(\sum_{i=1}^{n} r_i f_{a_i} + \sum_{k=1}^{m} s_k g_{b_k}) = \Phi(r_1 f_{a_1} + r_2 f_{a_2} + \dots + r_n f_{a_n} + s_1 g_{b_1} + s_2 g_{b_2} + \dots + s_m g_{b_m}) = r_1 \varphi(a_1) + r_2 \varphi(a_2) + \dots + r_n \varphi(a_n) + s_1 \varphi(b_1) + s_2 \varphi(b_2) + \dots + s_m \varphi(b_m) = \sum_{i=1}^{n} r_i \varphi(a_i) + \sum_{k=1}^{m} s_k \varphi(b_k) = \Phi(f) + \Phi(g).$

Hence, Φ is a well-defined R-module homomorphism and by definition, Φ restricted to $A \subseteq F(A)$ is φ . Finally, since F(A) is generated by A, which means the elements of F(A) are uniquely written as formal R-linear sums of elements of A, once we know the values of φ on A, φ 's values on elements of F(A) are uniquely determined. So Φ is the unique extension of φ to all of F(A).

Tensor Products of Modules

Basic Definition

We now have the algebraic framework we need to define the tensor product of modules:

Definition (Tensor Product of Modules). Let R be a ring with right R-module M and left R-module N. Then the free \mathbb{Z} -module on the set $M \times N$, which we will write $\mathbb{Z}(M \times N)$ is the set of formal \mathbb{Z} -linear sums of elements $(m,n) \in M \times N$. Since this is a free \mathbb{Z} -module, it is an abelian group. Quotienting out the subgroup H generated by elements of the form

$$(m, (n_1 + n_2)) - (m, n_1) - (m, n_2)$$

 $((m_1 + m_2), n) - (m_1, n) - (m_2, n)$
 $(mr, n) - (m, rn)$

produces the abelian quotient group $\mathbb{Z}(M\times N)/H$ which we call the tensor product of M and N over R, written $M\bigotimes_R N$. We write cosets (m,n) in this abelian group as $m\otimes n$ and call them simple tensors in the tensor product. Elements of $M\bigotimes_R N$ are formal \mathbb{Z} -linear sums of simple tensors.

Note that quotienting out by that particular subgroup basically enforces the following relations (which we write with tensor notation now):

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$mr \otimes n = m \otimes rn$$

Using these relations we can look at the following examples:

Example 2. For any tensor product $M \bigotimes_R N$ we have $m \otimes 0 = m \otimes (0+0) = m \otimes 0 + m \otimes 0$, so $m \otimes 0 = 0$. Similarly, $0 \otimes n = (0+0) \otimes n = 0 \otimes n + 0 \otimes n$, so $0 \otimes n = 0$.

Example 3. $\mathbb{Z}/n \bigotimes_R \mathbb{Z}/m = 0$ whenever n, m are relatively prime. This is because since n, m are relatively prime, for any $a \in \mathbb{Z}/n$, ma = a, so for any $a \in \mathbb{Z}/n$, $b \in \mathbb{Z}/m$, $a \otimes b = ma \otimes b = a \otimes mb = a \otimes 0 = 0$.

More examples to come...

Universal Property of Tensor Products

There is a canonical map $\iota: M \times N \to M \bigotimes_R N$ defined by $(m, n) \mapsto m \otimes n$.

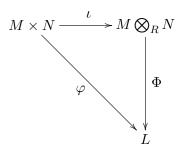
Definition (R-balanced map). Let M be a right R-module, N be a left R-module and L be an abelian group. Then a map $\varphi: M \times N \to L$ is R-balanced if it is linear in each variable and additionally, $\varphi(mr, n) = \varphi(m, rn)$ for all $m \in M, n \in N, r \in R$.

So the canonical map ι is R-balanced.

We now have the analogous universal property of the tensor product:

Proposition 2 (Universal Property of the Tensor Product). Let M be a right R-module, N be a left R-module and L be any abelian group. Then

there is a bijection between R-balanced maps $\varphi: M \times N \to L$ and group homomorphisms $\Phi: M \bigotimes_R N \to L$ that satisfies the commutative triangle:



Proof. In the first direction, let Φ be a group homomorphism from $M \bigotimes_R N \to L$. Then defining $\varphi = \Phi \circ \iota$, we have a map from $M \times N \to L$. We need to check that φ is in fact R-balanced. Let $m_1, m_2 \in M$ and $n \in N$. Then $\varphi(m_1+m_2,n) = \Phi(\iota(m_1+m_2,n)) = \Phi((m_1+m_2)\otimes n) = \Phi(m_1\otimes n+m_2\otimes n)$. Since Φ is a group homomorphism, we have $\Phi(m_1\otimes n+m_2\otimes n) = \Phi(m_1\otimes n) + \Phi(m_2\otimes n) = \Phi(\iota(m_1,n)) + \Phi(\iota(m_2,n)) = \varphi(m_1,n) + \varphi(m_2,n)$. So φ is linear in M. Similarly we can show that φ is linear in N. Let $r \in R$, then $\varphi(mr,n) = \Phi(\iota(mr,n)) = \Phi(mr\otimes n) = \Phi(\iota(m,rn)) = \varphi(m,rn)$. So φ is R-balanced in $M \times N$.

In the other direction, using Proposition 1 the R-balanced map φ defines a \mathbb{Z} -module homomorphism φ' from the free \mathbb{Z} module $\mathbb{Z}(M\times N)$ to L such that $\varphi'(m,n)=\varphi(m,n)$. Since φ is R-balanced, φ' maps elements of the form that generated the subgroup H in our definition of the tensor product to 0, so the kernel of φ' contains H. Hence, φ' induces a group homomorphism Φ on the quotient group $M\bigotimes_R N$ to L by $\Phi(m\otimes n)=\varphi'(m,n)=\varphi(m,n)$. Since the elements $m\otimes n$ generate $M\bigotimes_R N$, Φ is uniquely determined by this equation. \square

We now have to consider the module structure of the tensor product. Notice that if R is a commutative ring, then since rm = mr, $M \bigotimes_R N$ is a left R-module given by:

$$r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn)$$

In this case, Proposition 2 gives a bijection between R-bilinear maps $M \times N \to L$ (no longer needed to be R-balanced because of this commutativity relation above) and R-module homomorphisms $M \bigotimes_R N \to L$.

We have the following fact about the tensor product:

Proposition 3 (Tensor Product is Associative). $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$.

We actually have two proofs of this fact. The first uses only the universal property of tensor products, while the second uses an application of the Yoneda Lemma (see [Liu18]).

Proof using Universal Property. For each $x \in X$, we define a map

$$\phi: Y \times Z \to (X \bigotimes Y) \bigotimes Z$$
$$(y, z) \mapsto (x \otimes y) \otimes z$$

This map is bilinear because $\phi(y_1 + y_2, z) = (x \otimes (y_1 + y_2)) \otimes z = ((x \otimes y_1) + (x \otimes y_2)) \otimes z = (x \otimes y_1) \otimes z + (x \otimes y_2) \otimes z$ and similarly for the z-coordinate.

Since ϕ is bilinear, by the universal property it induces a linear map

$$\Phi_x: Y \bigotimes Z \to (X \bigotimes Y) \bigotimes Z$$
$$y \otimes z \mapsto (x \otimes y) \otimes z$$

Now we define a map

$$\delta: X \times (Y \otimes Z) \to (X \bigotimes Y) \bigotimes Z$$
$$(x, \sum_{i=1}^{n} y_i \otimes z_i) \mapsto \Phi_x(\sum_{i=1}^{n} y_i \otimes z_i) = \sum_{i=1}^{n} (x \otimes y_i) \otimes z_i$$

Again, this map is bilinear (because of linearity of Φ_x and properties of the tensor product), so by the universal property it induces a linear map

$$\Delta: X \bigotimes (Y \otimes Z) \to (X \bigotimes Y) \bigotimes Z$$
$$x \otimes (\sum_{i=1}^{n} y_i \otimes z_i) = \sum_{i=1}^{n} x \otimes (y_i \otimes z_i) \mapsto \sum_{i=1}^{n} (x \otimes y_i) \otimes z_i$$

We construct the inverse by fixing $z \in \mathbb{Z}$ and proceeding in a similar way to get

$$\Gamma: (X \bigotimes Y) \otimes Z \to X \bigotimes (Y \bigotimes Z)$$
$$\sum_{i=1}^{n} (x_i \otimes y_i) \otimes z \mapsto \sum_{i=1}^{n} x_i \otimes (y_i \otimes z)$$

Notice that in Δ there is only one x and many z and in Γ there is only one z and many x. One might think that this is a problem but we check that Δ, Γ are indeed mutually inverse:

$$\Gamma(\Delta(\sum_{i=1}^{n} x \otimes (y_i \otimes z_i))) = \Gamma(\sum_{i=1}^{n} (x \otimes y_i) \otimes z_i)$$

$$= \sum_{i=1}^{n} \Gamma((x \otimes y_i) \otimes z_i) \quad \text{By linearity of } \Gamma$$

$$= \sum_{i=1}^{n} x \otimes (y_i \otimes z_i)$$

and

$$\Delta(\Gamma(\sum_{i=1}^{n} (x_i \otimes y_i) \otimes z)) = \Delta(\sum_{i=1}^{n} x_i \otimes (y_i \otimes z))$$

$$= \sum_{i=1}^{n} \Delta(x_i \otimes (y_i \otimes z)) \quad \text{By linearity of } \Delta$$

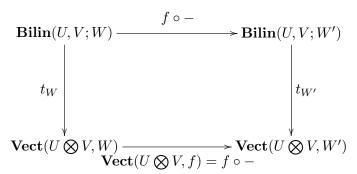
$$= \sum_{i=1}^{n} (x_i \otimes y_i) \otimes z$$

Proof using Yoneda Lemma. By the Yoneda Lemma, if two representable functors $H^A, H^{A'}$ are isomorphic, then that means $A \cong A'$. We first show that the functor

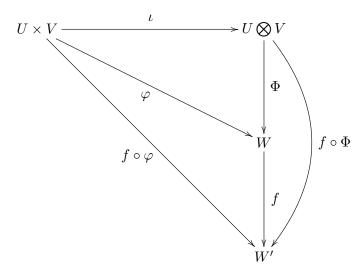
$$\mathbf{Bilin}(U,V;-): \mathbf{Vect}_k \to \mathbf{Set}$$
 Vector space $W \mapsto \mathbf{set}$ of bilinear maps $U \times V \to W$

is representable by showing that it is naturally isomorphic to the functor $H^{U \bigotimes V} = \mathbf{Vect}(U \bigotimes V, -)$

We define the mapping $t_W: \mathbf{Bilin}(U,V;W) \to \mathbf{Vect}(U \bigotimes V,W)$ by $t_W(\varphi) = \Phi$ where Φ is the linear map out of the tensor product uniquely determined by φ given by the universal property of tensor products. We define $t_W^{-1}: \mathbf{Vect}(U \bigotimes V, W) \to \mathbf{Bilin}(U,V;W)$ by $t_W^{-1}(\Phi) = \varphi$. Clearly t_W, t_W^{-1} are inverse to each other. Let $f: W \to W'$. We need to show that the following naturality square holds:



This clearly holds because of the following universal property:



So we have the following chain of isomorphisms:

$$\begin{split} \mathbf{Vect}(X \bigotimes (Y \bigotimes Z), -) &\cong \mathbf{Bilin}(X, Y \bigotimes Z; -) \\ &\cong \mathbf{3\text{-}lin}(X, Y, Z; -) \\ &\cong \mathbf{Bilin}(X \bigotimes Y, Z; -) \\ &\cong \mathbf{Vect}((X \bigotimes Y) \bigotimes Z, -) \end{split}$$

REFERENCES 10

References

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