# The Yoneda Lemma

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June 18, 2018

#### Abstract

Some notes on the Yoneda Lemma, starting with the notion of representable functors.

#### Representable Functors

We first define a *prototype* of a representable functor out of a locally small category  $\mathscr{A}$ .

**Definition.** Let  $\mathscr{A}$  be a locally small category, and fix  $A \in \mathscr{A}$ . Define the functor  $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$  by the following mapping on

- 1. **objects**: For  $B \in \mathcal{A}$ , define  $H^A(B) = \mathcal{A}(A, B)$ , the hom-set of arrows in  $\mathcal{A}$  from A to B.
- 2. **morphisms**: For  $g: B \to B'$ , define  $H^A(g) = \mathscr{A}(A,g): \mathscr{A}(A,B) \to \mathscr{A}(A,B')$  by  $p \mapsto g \circ p$  for all  $p: A \to B$ , sending morphisms from A to B to morphisms from A to B' by post-composition with g.

This is the covariant version of H, there is a contravariant version, denoted  $H_A$  which does pre-composition instead.

From this prototype of a representable functor, we can now define representable functors:

**Definition** (Representable Functor). Let  $\mathscr{A}$  be a locally small category. A functor  $X : \mathscr{A} \to \mathbf{Set}$  is representable if X is naturally isomorphic to  $H^A$  for some  $A \in \mathscr{A}$ . A representation of X is a choice of an object A along with a natural isomorphism from  $H^A$  to X.

Some examples of representable functors:

**Example 1** (group G regarded as a one object category). Regarding a group G as a one object category  $\mathscr{G}$ , with object  $*, H^* = *(*, -) : \mathscr{G} \to \mathbf{Set}$  is a functor which maps

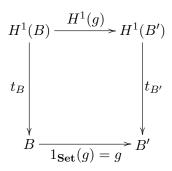
- 1. **objects**: There is only one object in  $\mathscr{G}$ , and  $H^*(*)$  is the set of mappings from \* to \*, otherwise known as the elements of the group G. So  $H^*(*) \cong U(G)$  where  $U : \mathbf{Grp} \to \mathbf{Set}$  is the forgetful functor that returns the underlying set of a group.
- 2. **morphisms**: Let  $g \in G$ , then  $H^*(g)$  maps any element h of G to  $g \circ h$ , which, interpreted in the context of a group is just group mulitiplication on the left, i.e. gh.

Since  $\mathscr{G}$  has only one object, there is only one representable functor on it (up to isomorphism), and the representable is the underlying set G acted on by left multiplication.

**Example 2**  $(H^1: \mathbf{Set} \to \mathbf{Set})$ . Let 1 denote the set with just one element. The functor  $1_{\mathbf{Set}}$  is represented by the functor  $H^1: \mathbf{Set} \to \mathbf{Set}$ . To see this, let us first see what  $H^1$  does and then show that it is naturally isomorphic to  $1_{\mathbf{Set}}$ .

Let  $B \in \mathbf{Set}$ , then  $H^1(B) = \mathbf{Set}(1, B) = \{b : 1 \to B\}$  the set of functions from the singleton set to B that pick out an element of B. In particular,  $H^1(B) \cong B$ . Let  $p : 1 \to B$  and  $g : B \to B'$ , then  $H^1(g) = \mathbf{Set}(1, g) : \mathbf{Set}(1, B) \to \mathbf{Set}(1, B')$  sends functions p that pick out an element of B to functions  $g \circ p$  that pick out an element of B'.

So we have the parallel functors  $H^1, 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$  and we know what they do. Now we construct a natural isomorphism  $t : H^1 \Rightarrow 1_{\mathbf{Set}}$ . We already have that  $H^1(B) \cong B$ , so the components of t are the isomorphisms  $t_B : H^1(B) \to B$  (and of course  $t_B^{-1} : B \to H^1(B)$ ). Now we need to check that the naturality square below commutes:



where  $g: B \to B'$ .

It is enough to consider an element  $b: 1 \to B$  from  $H^1(B)$ . b picks out an element of B, which we will also identify by b(1) = b. Going down from  $H^1(B)$ ,  $t_B(b: 1 \to B)$  gives us this element b, which is then sent by g to some element of B', say b'. On the other hand, going right from  $H^1(B)$ ,  $H^1(g)(b: 1 \to B) = g \circ b$  which gives us the map  $b': 1 \to B'$  that picks out g(b(1)) = b'. Finally,  $t_{B'}(b')$  is precisely this element b' of B'. So in fact the naturality square commutes.

Hence  $1_{\mathbf{Set}}$  is represented by  $H^1$ .

**Example 3**  $(H^1: \mathbf{Cat} \to \mathbf{Set})$ . Similarly to the example above, ob:  $\mathbf{Cat} \to \mathbf{Set}$  which sends a small category to its underlying set of objects is represented by  $H^1$  where  $\mathbf{1}$  is the one-object category. This is because when  $\mathscr{B}$  is a category,  $H^1(\mathscr{B}) = \mathbf{Cat}(\mathbf{1}, \mathscr{B})$  which picks out objects of  $\mathscr{B}$  and is isomorphic to ob $\mathscr{B}$ . The proof that these functors are naturally isomorphic is essentially the same as the one for  $1_{\mathbf{Set}}$  and  $H^1$  in the example above.

### Adjoints and Representables

Now we establish the claim that any set valued functor with a left adjoint is representable. We first prove the following lemma.

**Lemma 1.** Let  $\mathscr{A}, \mathscr{B}$  be locally small categories with functors  $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A}$  such that  $F \dashv G$ . Fix  $A \in \mathscr{A}$ , then the functor

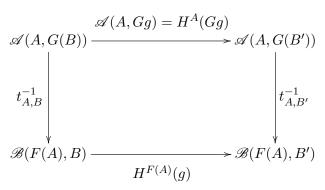
$$\mathscr{A}(A,G(-)):\mathscr{B}\to \mathbf{Set}$$

that is, the composition  $\mathscr{B} \stackrel{G}{\to} \mathscr{A} \stackrel{H^A}{\to} \mathbf{Set}$  is representable.

Proof. Because  $F \dashv G$ , we have the isomorphism  $t_{A,B}^{-1}: \mathscr{A}(A,G(B)) \to \mathscr{B}(F(A),B)$  for each  $B \in \mathscr{B}$ . Now  $H^{F(A)}$  maps B to  $\mathscr{B}(F(A),B)$ , and it maps a morphism  $g: B \to B'$  to  $\mathscr{B}(F(A),g): \mathscr{B}(F(A),B) \to \mathscr{B}(F(A),B')$ . So we suspect that the isomorphisms  $t_{A,B}^{-1}$  gives rise to the natural isomorphism  $t^{-1}: \mathscr{A}(A,G(-)) \Rightarrow H^{F(A)}$ . We show that this is in fact the case.

We need to check naturality. Let  $g: B \to B'$ . We need to check that the following naturality square commutes:

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It suffices to consider a map  $f:A\to G(B)$  from  $\mathscr{A}(A,G(B))$  as it travels around the diagram. Going down from  $\mathscr{A}(A,G(B)),\ t_{A,B}^{-1}(f:A\to G(B))$  is a map  $t_{A,B}^{-1}(f):F(A)\to B$ , and then  $H^{F(A)}(g)(t_{A,B}^{-1})=g\circ t_{A,B}^{-1}$  which is a map from F(A) to B'. On the other hand, going right from  $\mathscr{A}(A,G(B)),\ H^A(Gg)(f:A\to G(B))=Gg\circ f$  which is a map from A to G(B') and then  $t_{A,B'}^{-1}(Gg\circ f)$  is a map from F(A) to B'. So the question is whether  $t_{A,B'}^{-1}(Gg\circ f)=g\circ t_{A,B}^{-1}(f),$  but that is precisely what it means for  $t^{-1}$  to be natural with respect to B (See [Liu18]), so in fact  $\mathscr{A}(A,G(-))\cong H^{F(A)}$  and is therefore representable.

Now we are ready to prove the main claim of this subsection:

**Theorem 1.** Any set-valued functor that has a left adjoint is representable.

*Proof.* Let  $G: \mathscr{A} \to \mathbf{Set}$  be a functor with left adjoint F. Let 1 denote the set with a single element. For all  $A \in \mathscr{A}$ , G(A) is a set and by example 2 above,  $H^1(G(A)) = \mathbf{Set}(1, G(A)) \cong G(A)$  naturally in A where  $H^1$  is our functor from  $\mathbf{Set}$  to  $\mathbf{Set}$  in example 2 above. So  $G \cong \mathbf{Set}(1, G(-))$  and by the lemma above, this means G is representable.

### References

- [Lei14] Tom Leinster. Basic Category Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.
- [Liu18] Stephen Liu. Adjoints, 2018. https://ssyl55.github.io/files/adjoints.pdf.