# CS-E4895 Gaussian Processes
# Lecture 0: Example

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# Agenda for today

1. Motivation for Gaussian processes
2. Course content, format, and evaluation
3. Warm up for Gaussian processes: Review of the multivariate Gaussian distribution
4. First assignment

# The multivariate Gaussian distribution

- **Definition** A random vector $\boldsymbol{x} = [x_1, x_2, \cdots, x_D]^T$ is said to have the multivariate Gaussian distribution if all linear combinations of $\boldsymbol{x}$ are Gaussian distributed:

$$y = \boldsymbol{a^T x} = a_1 x_1 + a_2 x_2 + \cdots + a_D x_D \sim \mathcal{N}(m, v) \tag{1}$$

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- The multivariate Gaussian density for a variable $\boldsymbol{x} \in \mathbb{R}^D$:

$$(|,) = (2\pi)^{-\frac{D}{2}} ||^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(-)^{T-1}(-)\right] \quad \in \mathbb{R}_{\geq 0} \tag{2}$$

$$\log(|,) = -\frac{D}{2}\log 2\pi - \frac{1}{2}\log|| - \frac{1}{2}(-)^{T-1}(-) \quad \in \mathbb{R} \tag{3}$$

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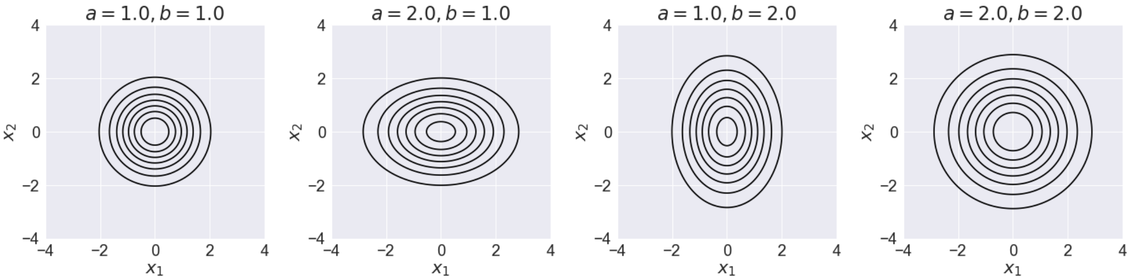
- $(\boldsymbol{\Sigma})_{ij}$ is the covariance between the $i$'th and $j$'th elements $x_i$ and $x_j$ of $\boldsymbol{x}$

# Interpretation of the covariance matrix - 2D examples

The diagonal of the covariance controls the scaling/marginal variances

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \tag{4}$$



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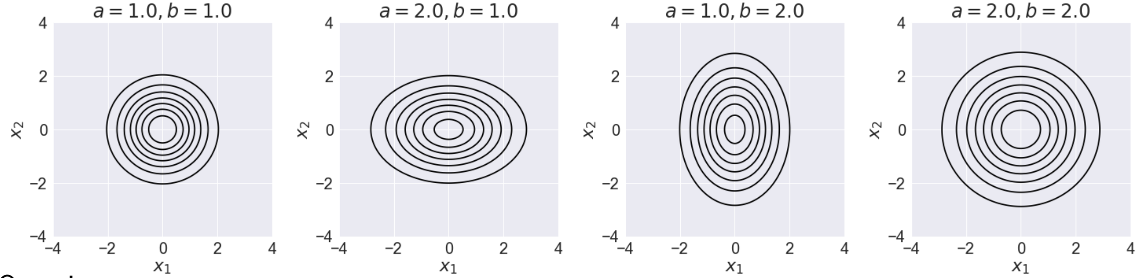


Questions:

1. If $\boldsymbol{\Sigma}$ is diagonal, then $x_1$ and $x_2$ are uncorrelated? True or false?
2. If $\boldsymbol{\Sigma}$ is diagonal, then $x_1$ and $x_2$ are independent? True or false?
3. What is the volume (integral) of density?
4. Which of the four densities has the highest peak and why?

# More slides..

# The end of todays lecture

- Next thursday 14th, 10pm
  - We will introduce Gaussian processes more formally
  - Read Chapter 1 & 2 of the Gaussian process book `gaussianprocess.org/gpml`

- Time to work: first assignment
  - Released today, deadline jan 20th, 12:00 (midday)
  - Reviews the basics of Bayesian inference and Gaussians
  - **Must be** handed in through MyCourses
  - Q&A sessions on 20th and 22th (grants extra point for being present!)