Graph Skills

February

Note, this is graph only material - not networks, so this includes Eulerian, Hamiltonian and planar issues, but not Dijkstra, TSP.

1. How many arcs in:
   (a) K_6
   (b) C_7
   (c) $K_{3,4}$
   (d) K_n
   (e) C_n
   (f) $K_{m,n}$
   (g) a MST on a graph with n vertices.
2. Draw example graphs that are:
   (a) simple and connected
   (b) simple but not connected
   (c) connected but not simple
   (d) neither simple nor connected
3. True or False: A minimal spanning tree is a trail.

   Solution: true: Vertices may repeat, arcs cannot.

4. State Euler's formula in terms of a planar graph.
5. Prove this theorem[1]: The sum of all the degrees of the faces/regions of a connected planar graph is equal to twice the number of arcs ($\Sigma degree(f) = 2e$).

   Solution: Case one, an arc is on a region that is a cycle: in which case it has an internal and an external face, inside and outside the face. Case two, it is not on a cycle, in which case it has two exposures to the infinite region.

   Since each arc borders two different faces or will border an infinite face twice, it must contribute 2 to the sum of the number of degree of the faces.

6. If G is a planar simple graph, prove that $e \leq 3v - 6$

   Solution: By Euler's formula, $v - e + f = 2$, and so $f = e - v + 2$.

   By the Handshaking Theorem, $\Sigma degree(f) = 2e$, and so $2e = \Sigma degree(f) \geq 3f$.

   Substituting for f, $2e \geq 3(e - v + 2)$, and so $2e \geq 3e - 3v + 6$, and so $e \leq 3v - 6$.

[1] known as The Handshaking Theorem for Planar Graphs

7. Show that $K_{2,n}$ is planar, for any value of n.

Solution: $K_{2,n}$ is planar, as it can be drawn with the two vertices of degree 2 on the outside, and the n vertices of degree 1 on the inside.

8. State Kuratowksi's Theorem.

Solution: A graph is planar if and only if (iff) it does not contain K_5 or $K_{3,3}$ as a subgraph.

9. Use Kuratowksi's theorem to prove that all complete graphs for K_n where $n \geq 5$ are non-planar.

Solution: A complete graph K_n contains K_5 as a subgraph, and so is non-planar.

10. Prove that a graph is Eulerian if and only if it is connected and every vertex has even degree.

Solution: If a graph is Eulerian, then it has an Eulerian circuit, and so every vertex has even degree.

If a graph is connected and every vertex has even degree, then it has an Eulerian circuit.

11. State Ore's Theorem.

Solution: If G is a simple graph with $n \geq 3$ vertices and for every pair of non-adjacent vertices, the sum of their degrees is greater than or equal to n, then G is Hamiltonian.

12. Prove Ore's Theorem.