Lesson 4: Bipartite graphs, Hamiltonian Cycles and Planar Graphs

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4 Planar Graphs

4.1 Bipartite Graphs

4.2 Hamiltonian Cycles

4.3 Planar Graphs

Euler's Relationship for any connected planar graph

$$n - m + r = 2$$

(Th10.2, p190)

If G is planar then order, i.e. number of vertices, $n \geq 3$, size m, then $m \leq 3n - 6$

The boundary of every region must have at least 3 edges.

This is obvious for $n = 3$, so $n \geq 4$ and $m \geq 3$. Suppose G is planar and has r regions.

$$n - m + r = 2$$

Let's sum up all the arcs touching all the regions, N

$$3r \leq N$$

Each arc can only touch two regions

$$N \leq 2r \tag{1}$$
$$2 = n - m + r \tag{2}$$
$$6 = 3n - 3m + 3r \tag{3}$$

but

$$3r \leq 2m$$

So

$$
\begin{align}
6 &\leq 3n - 3m + 2m \tag{4}\\
6 &<= 3n - m \tag{5}\\
m &<= 3n - 6 \tag{6}\\
& \tag{7}
\end{align}
$$

We can go on to say if G is a graph of order at least 3 and size m, $m > 3n - 6$, G is nonplanar. For K_5 that works.

$K_{3,3}$ it doesn't.

Proof that $K_{3,3}$ is not planar:

Using Euler $A = 9$, $N = 6$, so R must be 5.

Pick a region, its boundary must be a cycle and (it is bipartite) it must be an even cycle, at least 4, perhaps 6.

5 regions, $5 * 4$ arcs, each arc separates 2 regions so $(5 * 4/2) = 10$ arcs required ... but we only have 9.

Contradiction

4.4 Ore's Theorem

Let G be a (finite and simple) graph with $n \geq 3$ vertices. We denote by deg_v the degree of a vertex v in G, i.e. the number of incident edges in G to v. Then, Ore's theorem states that if

$deg_v + deg_w \geq n$ for every pair of distinct non-adjacent vertices v and w of G

then G is Hamiltonian.

4.4.1 Proof

It is equivalent to show that every non-Hamiltonian graph G does not obey the condition.

Accordingly, let G be a graph on $n \geq 3$ vertices that is not Hamiltonian, and let H be formed from G by adding edges one at a time that do not create a Hamiltonian cycle, until no more edges can be added.

Let x and y be any two non-adjacent vertices in H. Then adding edge xy to H would create at least one new Hamiltonian cycle, and the edges other than xy in such a cycle must form a Hamiltonian path $v_1 v_2 ... v_n$ in H with $x = v_1$ and $y = v_n$.

For each index i in the range $2 \leq i \leq n$, consider the two possible edges in H from v_1 to v_i and from v_{i-1} to v_n. At most one of these two edges can be present in H, for otherwise the cycle $v_1 v_2 ... v_{i-1} v_n v_{n-1} ... v_i$ would be a Hamiltonian cycle.

Thus, the total number of edges incident to either v_1 or v_n is at most equal to the number of choices of i, which is $n - 1$. Therefore, H does not obey the property, which requires that this total number of edges ($deg_{v_1} + deg_{v_n}$) be greater than or equal to n.

Since the vertex degrees in G are at most equal to the degrees in H, it follows that G also does not obey the property.

[1]

[1]Based on the proof outlined in `https://en.wikipedia.org/wiki/Ore%27s_theorem`.