Preliminaries

February

Prelims, inc sets, pigeonhole principle, arrangement, selection, include-exclude

1. Sets:

   (a) Find the least element of $\{n \in \mathbb{N} : n^2 - 3 \geq 2\}$

   (b) Find $n|A \cap B|$ when $A = \{x \in \mathbb{N} : x \leq 20\}$ and $B = \{x \in \mathbb{N} : x \text{ is prime}\}$.

   (c) How many of the first 100 positive integers are divisible by 2, 3, or 5?

   (d) How many subsets does a set with 10 elements have?

   (e) Consider a universal set U containing elements that belong to four subsets: A, B, C, and D, with the number of elements in each subset is as follows:

   - Subset A contains 20 elements.
   - Subset B contains 30 elements.
   - Subset C contains 25 elements.
   - Subset D contains 15 elements.

   Additionally, the intersections of these subsets have the following counts:

   - $A \cap B$ contains 10 elements.
   - $A \cap C$ contains 8 elements.
   - $A \cap D$ contains 5 elements.
   - $B \cap C$ contains 12 elements.
   - $B \cap D$ contains 7 elements.
   - $C \cap D$ contains 6 elements.
   - $A \cap B \cap C$ contains 4 elements.
   - $A \cap B \cap D$ contains 3 elements.
   - $A \cap C \cap D$ contains 2 elements.
   - $B \cap C \cap D$ contains 5 elements.

   (The assumption can be made that the intersection of all four subset is empty.)

   How many elements are there in the union of these four subsets, $A \cup B \cup C \cup D$? [1]

2. Arrangements and selections:

   (a) How many times does the digit 1 appear in the numbers from 1 to 10000? In how many different numbers?

   (b) Some positive integers are chosen, how many must be chosen for at least two of the squares of these integers to have a difference that is a multiple of 10? What about a multiple of 5?

   (c) How many ways are there to arrange the letters of the word 'MISSISSIPPI'?

   (d) You're organising a Secret Santa, for this set (of 7 students) who decide to exchange gifts. However, each person intends to give their gift to someone other than themselves. How many ways are there for the friends to exchange gifts such that no one receives their own gift? And if I was involved, to make 8?

   (e) nge) There are 9 points on a circle and lines connect all pairs of points. At how many places inside the circle do these lines intersect?

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[1] This was ChatGPT, can you tell?

3. Pigeonhole principle:

   (a) Justify or criticise this statement: "The Pigeonhole Principle tells us that if we have $n + 1$ pigeons and n holes, since $n + 1 > n$, each box will have at least one pigeon."

   (b) The Pigeonhole Principle tells us that with n pigeons and k holes each hole can have at most $\lceil n/k \rceil$ pigeons.

   (c) Three people are running for student government. There are 202 people who vote. What is the minimum number of votes needed for someone to win the election?

   (d) Suppose you have 8 points placed inside a square with sides of length 3 units. Prove that there must be at least one pair of points that are less than or equal to $\sqrt{2}$ units apart.

Further reading

https://discrete.openmathbooks.org/dmoi3/sec_intro-sets.html

   But not https://www.sanfoundry.com/discrete-mathematics-questions-answers-pigeonhole-principle/, e.g. q10.