Robust Weighting and Matching Techniques for Causal Inference in Observational Studies with Continuous Treatment

Universität Stuttgart

Ioan Scheffel

October 28, 2022

Contents

Chapter 1

Introduction

Chapter 2

Chapter One Title

Assumption 1. *Assume, the following conditions hold:*
(i) The minimizer $\lambda_0 = \arg\min_{\lambda \in \Theta} \mathbb{E}\left[-Tn\rho\left(B(X)^T\lambda\right) + B(X)^T\lambda\right]$ *is unique, where* $\Theta \subseteq \mathbb{R}^n$ *is the parameter space for* λ.
(ii) The parameter space $\Theta \subseteq \mathbb{R}^n$ *is compact compact with diameter* $diam(\Theta) < \infty$.
(iii) $\lambda_0 \in int(\Theta)$, *where* $int(\cdot)$ *stands for the interior of a set.*
(iv) There exists $\lambda_1^* \in \Theta$ *such that* $\left\|m^*(\cdot) - B(\cdot)^T\lambda_1^*\right\|_\infty \leq \varphi_{m^*}$, *where* $m^*(\cdot) := \left(\rho'\right)^{-1}\left(\frac{1}{n\pi(\cdot)}\right)$.
(v) There exists a constant $\varphi_{\rho' \vee \pi} \in \left(0, \frac{1}{2}\right)$ *such that* $n\rho(v) \in (\varphi_{\rho' \vee \pi}, 1 - \varphi_{\rho' \vee \pi})$ *for* $v = B(x)^T\lambda$ *with* $\lambda \in int(\Theta)$ ***or*** $\pi(x) \in (\varphi_{\rho' \vee \pi}, 1 - \varphi_{\rho' \vee \pi})$.
(vi) There exists $\varphi_{\rho''} > 0$ *such that* $-\rho'' \geq \varphi_{\rho''} > 0$
(vii) There exists $\varphi_{B(x)B(x)^T} > 0$ *such that* $B(x)B(x)^T \succcurlyeq \varphi_{B(x)B(x)^T} I$
(viii) There exists $\varphi_{\|B\|} > 0$ *such that* $\sup_{x \in \mathcal{X}} \|B(x)\|_2 \leq \varphi_{\|B\|}$.

We study the following problem:

$$\begin{aligned}
&\underset{w \in \mathbb{R}^n}{\text{minimize}} \quad && \sum_{i=1}^n T_i f(w_i) \\
&\text{subject to} && \left|\sum_{i=1}^n w_i T_i B_k(X_i) - \frac{1}{n}\sum_{i=1}^n B_k(X_i)\right| \leq \delta_k,\ k = 1, \ldots, K
\end{aligned} \tag{2.1}$$

We aim to prove that the solution to Problem (2.1) is asymptotical consistent with the propensity score, i.e.

Theorem 2.1. *Under some (non-optimal) Assumptions, there exist constants $c_1, c_2 > 0$ and decreasing sequences $(\varepsilon_n^1), (\varepsilon_n^2) \subset (0,1]$ that converge to 0 such that for all $\tau \in (0,1]$ there exists a constant $c_\tau \in [0,\infty)$ only depending on τ such that for all $n \geq 1$ and $\tau \in (0,1]$ it holds*

$$\mathbb{P}\left(\left\|w_i^* - \frac{1}{n\pi(X_i)}\right\|_\infty \leq c_1 c_\tau \varepsilon_n^1\right) \geq 1-\tau,$$
$$\left\|w_i^* - \frac{1}{n\pi(X_i)}\right\|_{\mathbb{P},2} \leq c_2 \varepsilon_n^2,$$

where w^* is the solution to Problem (2.1).

2.1 Plan of proof

It is easier to study the dual of Problem (2.1). Thus we employ results from convex analysis to establish

Proposition 2.1. *The dual of Problem* (2.1) *is equivalent to the unconstrained optimization problem*

$$\underset{\lambda\in\mathbb{R}^K}{\text{minimize}} \quad \frac{1}{n}\sum_{j=1}^{n}\left[-T_j n\rho\left(B(X_j)^T\lambda\right) + B(X_j)^T\lambda\right] + |\lambda|^T\delta, \tag{2.2}$$

where $B(X_j) = (B_k(X_j))_{1\leq k\leq K}$ denotes the K basis functions of the covariates, $\rho(t) := \frac{t}{n} - t(h')^{-1}(t) + h((h')^{-1}(t))$ with $h(x) := f\left(\frac{1}{n} - x\right)$ and $|\lambda| := (|\lambda_k|)_{1\leq k\leq K}$. Moreover, the primal solution w_j^* satisfies

$$w_j^* = \rho'\left(B(X_j)^T\lambda^\dagger\right) \tag{2.3}$$

for $j = 1,\ldots,n$, where $\lambda^\dagger$ is the solution to the dual optimization problem.

Proposition 2.2. *There exists a solution $\lambda^\dagger$ to* (2.2) *such that*

$$\mathbb{P}\left(\left\|\lambda^\dagger - \lambda_1^*\right\|_2 \leq C_{\mathbb{P}} C_\tau \varepsilon_n\right) \geq 1-\tau. \tag{2.4}$$

We employ Theorem 3.2 together with the box constraints in Problem (2.1) to obtain Proposition 2.1.

To prove Proposition 2.2 we employ Proposition 5.1 and Corollary 5.1.1 to get

$$\begin{aligned}
&G(\lambda_1^* + \Delta) - G(\lambda_1^*) \\
&\geq \frac{1}{n}\sum_{j=1}^{n}\left[-T_j n \rho^{'}\left(B(X_j)^T\lambda_1^*\right) + 1\right]\Delta^T B(X_j) \\
&+ \frac{1}{2}\sum_{j=1}^{n} -T_j \rho^{''}\left(B(X_j)^T(\lambda_1^* + \xi\Delta)\right)\Delta^T\left(B(X_j)B(X_j)^T\right)\Delta \\
&- |\Delta|^T\delta \\
&\geq -\left\|\Delta\right\|_2\left(\left\|\frac{1}{n}\sum_{j=1}^{n}\left[-T_j n \rho^{'}\left(B(X_j)^T\lambda_1^*\right) + 1\right]B(X_j)\right\|_2 + \left\|\delta\right\|_2\right) \\
&+ n\left\|\Delta\right\|_2^2 \varphi_{\rho''}\underline{\varphi_{aa^T}}
\end{aligned} \tag{2.5}$$

Next we employ Bernstein inequality 4.1 to bound

$$\left\|\frac{1}{n}\sum_{j=1}^{n}\left[-T_j n \rho^{'}\left(B(X_j)^T\lambda_1^*\right) + 1\right]B(X_j)\right\|_2 \leq C_{\mathbb{P}} C_\tau \varepsilon_n \tag{2.6}$$

with probability $1 - \tau$. Then for $\left\|\Delta\right\|_2$ large enough it holds

$$G(\lambda_1^* + \Delta) - G(\lambda_1^*) > 0 \tag{2.7}$$

with probability $1 - \tau$. Thus by Proposition 5.1

$$\mathbb{P}\left(\left\|\lambda^\dagger - \lambda_1^*\right\|_2 \leq \left\|\Delta\right\|_2\right) \geq 1 - \tau. \tag{2.8}$$

It is then straightforward to prove

Theorem 2.2. *Let $\lambda^\dagger$ be the solution to Problem 2.2 and $w^*(x) = \rho^{'}\left(B(x)^T\lambda^\dagger\right)$. Then under the conditions in Assumption 1 it holds*

$$\left\|w^*(\cdot) - \frac{1}{n\pi(\cdot)}\right\|_{\mathbb{P},2} \leq \textit{stuff} \tag{2.9}$$

and

$$\mathbb{P}\left(\left\|w^*(\cdot) - \frac{1}{n\pi(\cdot)}\right\|_\infty \leq \textit{stuff}\right) \geq 1 - \tau. \tag{2.10}$$

Proof. Motivated by Proposition 5.1 we set $\|\Delta\|_2 = C$ and consider

$$G(\lambda) := \frac{1}{n}\sum_{j=1}^{n}\left[-T_j n\rho\left(B(X_j)^T\lambda\right) + B(X_j)^T\lambda\right] + |\lambda|^T\delta. \tag{2.11}$$

Since $\rho \in C^2(\mathbb{R})$ we can employ Proposition 5.1, Corollary 5.1.1 and Proposition 5.2 to get

$$\begin{aligned}
&G(\lambda_1^* + \Delta) - G(\lambda_1^*)\\
&\geq \frac{1}{n}\sum_{j=1}^{n}\left[-T_j n\rho^{'}\left(B(X_j)^T\lambda_1^*\right) + 1\right]\Delta^T B(X_j)\\
&+ \frac{1}{2}\sum_{j=1}^{n} -T_j\rho^{''}\left(B(X_j)^T(\lambda_1^* + \xi\Delta)\right)\Delta^T\left(B(X_j)B(X_j)^T\right)\Delta\\
&- |\Delta|^T\delta\\
&\geq -\|\Delta\|_2\left(\left\|\frac{1}{n}\sum_{j=1}^{n}\left[-T_j n\rho^{'}\left(B(X_j)^T\lambda_1^*\right) + 1\right]B(X_j)\right\|_2 + \|\delta\|_2\right)\\
&+ n\|\Delta\|_2^2\,\varphi_{\rho''}\underline{\varphi_{aa^T}}\\
&:= -\|\Delta\|_2\left(I_1 + \|\delta\|_2\right) + \|\Delta\|_2^2\, I_2.
\end{aligned} \tag{2.12}$$

The second inequality is due to the Cauchy-Schwarz-Inequality and Assumptions 1(vi) and 1(vii) .

Analysis of I_1

We want to use Assumption 1(iii). Thus we perform the following split:

$$I_1 \leq \left\|\sum_{j=1}^{n} T_j\left[\rho^{'}\left(B(X_j)^T\lambda_1^*\right) - \frac{1}{n\pi(X_j)}\right]B(X_j)\right\|_2 \tag{2.13}$$

$$+ \left\|\frac{1}{n}\sum_{j=1}^{n}\left[\frac{T_j}{\pi(X_j)} - 1\right]B(X_j)\right\|_2 \tag{2.14}$$

$$=: J_1 + J_2 \tag{2.15}$$

Analysis of J_1

By the Lipschitz-continuity of $\rho^{'}$, Assumption 1$(viii)$ and Assumption 1(iv), $T \in \{0,1\}$ and the triangle inequality we have

$$J_1 \leq nL_{\rho'}\varphi_{\|B(x)\|}\varphi_{m^*} \tag{2.16}$$

Analysis of J_2

We employ Bernstein Inequality for matrices To this end we define

$$A_j := \frac{1}{n}\left[\frac{T_j}{\pi(X_j)} - 1\right]B(X_j) \tag{2.17}$$

$\mathbb{E}A_j = 0$

It holds

$$\mathbb{E}\left[\frac{T_j}{\pi(X_j)}B(X_j)\right] = \mathbb{E}\left[\mathbb{E}\left[T_j \mid X_j\right]\frac{1}{\pi(X_j)}B(X_j)\right] = \mathbb{E}[B(X_j)]. \tag{2.18}$$

Thus $\mathbb{E}[A_j] = 0$.

L

Since

$$\left|\frac{T_j}{\pi(X_j)} - 1\right| \leq 1 + \frac{1-\varphi_\pi}{\varphi_\pi} = \frac{1}{\varphi_\pi} \tag{2.19}$$

by Assumption 1(v), we can employ Assumption 1$(viii)$ to get

$$\|A_j\|_2 \leq \frac{\varphi_{\|B\|}}{n\varphi_\pi} =: L. \tag{2.20}$$

v(S)

Since

$$\mathbb{E}\left[A_jA_j^T\right] \leq \left(\frac{1}{n\varphi_\pi}\right)^2 \mathbb{E}\left[B(X)B(X)^T\right] \tag{2.21}$$

and

$$\mathbb{E}\left[A_j^T A_j\right] \leq \left(\frac{\varphi_{\|B\|}}{n\varphi_\pi}\right)^2 \tag{2.22}$$

we have

$$v(S) \leq \frac{|\lambda_{\max}| + \varphi_{\|B\|}^2}{n\varphi_\pi^2}, \tag{2.23}$$

where $\lambda_{\max}$ is the maximal eigenvalue of $\mathbb{E}\left[B(X)B(X)^T\right]$. Then by Bernsteins inequality 4.1 we get

$$\mathbb{E}[J_2] \leq \sqrt{\frac{2\log(K+1)\left(|\lambda_{\max}| + \varphi_{\|B\|}^2\right)}{n\varphi_\pi^2}} + \frac{\log(K+1)\varphi_{\|B\|}}{3n\varphi_\pi} \tag{2.24}$$

and by the Markov-inequality

$$\mathbb{P}\left(J_2 \leq \frac{1}{\tau}\mathbb{E}[J_2]\right) \geq 1-\tau \tag{2.25}$$

Finish

If we choose for $\gamma > 0$

$$\begin{aligned} \|\Delta\|_2 &= \frac{\frac{1}{\tau}\mathbb{E}[J_2] + nL_{\rho'}\varphi_{\|B(x)\|}\varphi_{m^*} + \|\delta\|_2}{\varphi_{\rho''}\underline{\varphi_{BB^T}}}(1+\gamma) && (2.26)\\ &=: C && (2.27) \end{aligned}$$

we have

$$\begin{aligned} \mathbb{P}\left(\left\|\lambda^\dagger - \lambda_1^*\right\|_2 \leq C\right) &= \mathbb{P}\left(\inf_{\|\Delta\|_2 = C} G(\lambda_1^* + \Delta) - G(\lambda_1^*) > 0\right) && (2.28)\\ &\geq 1-\tau && (2.29) \end{aligned}$$

Finish 2

$$\begin{aligned} \left\|w^*(X) - \frac{1}{n\pi(X)}\right\|_{\mathbb{P},2} &\leq L_{\rho'}\left[\left\|B(X)^T\left(\lambda^\dagger - \lambda_1^*\right)\right\|_{\mathbb{P},2}\right. && (2.30)\\ &\quad \left. + \left\|m^*(X) - B(X)^T\lambda_1^*\right\|_{\mathbb{P},2}\right] && (2.31)\\ &\leq L_{\rho'}\left(\varphi_{\|B\|}\sqrt{C^2(1-\tau) + \mathrm{diam}(\Theta)^2\tau} + \varphi_{m^*}\right) && (2.32) \end{aligned}$$

$$\left\| w^*(\cdot) - \frac{1}{n\pi(\cdot)} \right\|_\infty \leq L_{\rho'} \left[\left\| B(\cdot)^T \left(\lambda^\dagger - \lambda_1^* \right) \right\|_\infty \right. \tag{2.33}$$

$$\left. + \left\| m^*(\cdot) - B(\cdot)^T \lambda_1^* \right\|_\infty \right] \tag{2.34}$$

$$\leq L_{\rho'} \left(\varphi_{\|B\|} C + \varphi_{m^*} \right) \tag{2.35}$$

with probabitity greater than $1 - \tau$. $\square$

Chapter 3

Convex Analysis

We begin by defining convex sets

Definition 3.1. *A subset* $\Omega \subseteq \mathbb{R}^n$ *is called CONVEX if we have* $\lambda x + (1 - \lambda)y \in \Omega$ *for all* $x, y \in \Omega$ *and* $\lambda \in (0, 1)$.

Clearly, the line segment $[a, b] := \{\lambda a + (1 - \lambda)b \mid \lambda \in [0, 1]\}$ is contained in Ω for all $a, b \in \Omega$ if and only if Ω is a convex set.

Next we define convex functions.

The concept of convex functions is closely related to convex sets.

The line segment between two points on the graph of a convex function lies on or above and does not intersect the graph.

In other words: The area above the graph of a convex function f is a convex set, i.e. the *epigraph* $\operatorname{epi}(f) := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \alpha\}$ is a convex set in $\mathbb{R}^{n+1}$.

Often an equivalent characterisation of convex functions is more useful.

Theorem 3.1. *The convexity of a function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *on* $\mathbb{R}^n$ *is equivalent to the following statement:*

For all $x, y \in \mathbb{R}^n$ *and* $\lambda \in (0, 1)$ *we have*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{3.1}$$

Definition 3.2. proper convex function

Definition 3.3. convex conjugate

Given proper convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a matrix $A \in \mathbb{R}^{n \times n}$, we define the primal minimization problem as follows:

$$\text{minimize} \quad f(x) + g(Ax) \quad \text{subject to} \quad x \in \mathbb{R}^n. \tag{3.2}$$

The Fenchel dual problem is then

$$\text{maximize} \quad -f^*\left(A^T y\right) - g^*(-y) \quad \text{subject to} \quad y \in \mathbb{R}^n. \tag{3.3}$$

Theorem 3.2. *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions and $0 \in ri(dom(g) - A(dom(f)))$. Then the optimal values of* (3.2) *and* (3.3) *are equal, i.e.*

$$\inf_{x \in \mathbb{R}^n} \{f(x) + g(Ax)\} = \sup_{y \in \mathbb{R}^n} \left\{-f^*\left(A^T y\right) - g^*(-y)\right\}. \tag{3.4}$$

Chapter 4

Random Matrix Inequality

Theorem 4.1. *Let $(A_k)_{1\le k\le n} \subseteq \mathbb{R}^{d_1\times d_2}$ be a finite sequence of independent, random matrices. Assume that*

$$\mathbb{E}(A_k) = 0 \quad \textit{and} \quad \|A_k\| \le L \quad \textit{for each} \quad k \in \{1,\dots,n\}. \tag{4.1}$$

Introduce the random matrix

$$S := \sum_{k=1}^{n} A_k. \tag{4.2}$$

Let $v(S)$ be the matrix variance statistic of the sum:

$$\begin{aligned} v(S) : &= \max\left\{\left\|\mathbb{E}(SS^T)\right\|, \left\|\mathbb{E}(S^TS)\right\|\right\} &(4.3)\\ &= \max\left\{\left\|\sum_{k=1}^{n}\mathbb{E}(A_kA_k^T)\right\|, \left\|\sum_{k=1}^{n}\mathbb{E}(A_k^TA_k)\right\|\right\}. &(4.4)\end{aligned}$$

Then

$$\mathbb{E}\,\|S\| \le \sqrt{2v(S)\log(d_1+d_2)} + \frac{1}{3}L\log(d_1+d_2). \tag{4.5}$$

Furthermore, for all $t \ge 0$,

$$\mathbb{P}(\|S\| \ge t) \ge (d_1+d_2)\exp\left(\frac{-t^2/2}{v(S)+Lt/3}\right). \tag{4.6}$$

Chapter 5

Simple yet useful Calculations

Proposition 5.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous such that a minimum x^* exists and is unique. Then for all $y \in \mathbb{R}^n$ and $C > 0$ it follows*

$$\inf_{\|\Delta\|=C} f(y+\Delta) - f(y) > 0 \qquad \Rightarrow \qquad \|x^* - y\| \leq C. \tag{5.1}$$

Proof. Since $\mathcal{C} := \{\|\Delta\| \leq C\}$ is compact and

$$f(x^*) \leq f(y) < \inf_{\|\Delta\|=C} f(y+\Delta),$$

the continious function $f(y + \cdot\,)$ has a minimum in $\mathrm{int}(\mathcal{C}) := \{\|\Delta\| < C\}$. Since x^* is the unique minimum of f there exists $\Delta^* \in \mathrm{int}(\mathcal{C})$ such that $x^* - y = \Delta^*$. We conclude that $\|x^* - y\| \leq C$. □

Theorem 5.1. (Multivariate Taylor Theorem) *Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$. Then for all $x, \Delta \in \mathbb{R}^n$ there exists $\xi \in [0,1]$ such that it holds*

$$\begin{aligned} f(x+\Delta) = f(x) + \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i}\Delta_i + \sum_{\substack{i,j=1 \\ i\neq j}} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i \partial x_j}\Delta_i\Delta_j \\ + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i^2}\Delta_i^2 \end{aligned} \tag{5.2}$$

Corollary 5.1.1. *Let $f \in C^2(\mathbb{R})$. Then for all $a, x, \Delta \in \mathbb{R}^n$ there exist $\xi \in [0,1]$ such that it holds*

$$f(a^T(x+\Delta)) - f(a^Tx) = f^{'}(a^Tx)\,\Delta^T a + \frac{1}{2} f^{''}(a^T(x+\xi\Delta))\,\Delta^T A\;\Delta, \tag{5.3}$$

where $A := aa^T \in \mathbb{R}^{n\times n}$.

Proof. By the chain rule we have for all $a, x, \Delta \in \mathbb{R}^n$ and $\xi \in [0,1]$

$$\frac{\partial^2 f(a^T(x+\xi\Delta))}{\partial x_i \partial x_j} = f^{''}(a^T(x+\xi\Delta))\, a_i a_j. \tag{5.4}$$

Since $A := aa^T$ is symmetric we have

$$\Delta^T A\ \Delta = 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j \Delta_i \Delta_j + \sum_{i=1}^{n} a_i^2 \Delta_i^2. \tag{5.5}$$

Plugging (5.4) and (5.5) into (5.2) yields (5.3). □

Proposition 5.2. *For all $x, y \in \mathbb{R}$ it holds*

$$|x+y| - |x| \geq -|y| \tag{5.6}$$

Proof. Checking all 6 combinations of $x+y$, x, y being nonnegative or negative yields the result. □

Bibliography