Robust Weighting and Matching Techniques for Causal Inference in Observational Studies with Continuous Treatment

Universität Stuttgart

Ioan Scheffel

December 21, 2022

Contents

1 Introduction

2 Balancing Weights

2.1 Introduction

2.2 Estimating the Population Mean of Potential Outcomes

2.3 Application of Convex Optimization

Assumption 2.1. *Assume that the map $f : \mathbb{R} \to \overline{\mathbb{R}}$ has the following properties.*

(i) *f is strictly convex.*

(ii) *f is lower-semicontinuous and continuously differentiable on* $\operatorname{int}(\operatorname{dom}(f))$.

(iii) *The derivative of f on* $\operatorname{int}(\operatorname{dom}(f))$ *is a diffeomorphism.*

(iv) *The Legendre transformation f^* of f is finite.*

(v) *The function $x \mapsto xt - f(x)$ takes its supremum on* $\operatorname{int}(\operatorname{dom}(f))$ *for all $t \in \mathbb{R}$.*

We consider the following optimization problem.

Problem 2.1.

$$\underset{w_1,\ldots,w_n \in \mathbb{R}}{\text{minimize}} \qquad \sum_{i=1}^{n} T_i f(w_i)$$

subject to the constraints

$$w_i T_i \geq 0, \qquad i = 1, \ldots, n,$$
$$\sum_{i=1}^{n} w_i T_i = 1$$
$$\left| \sum_{i=1}^{n} w_i T_i B_k(X_i) - \frac{1}{n} \sum_{i=1}^{n} B_k(X_i) \right| \leq \delta_k, \qquad k = 1, \ldots, K$$

Theorem 2.1. *Under Assumption, the dual of the above Problem is the unconstrained optimization problem*

$$\underset{\lambda\in\mathbb{R}^K}{\text{minimize}} \qquad \frac{1}{n}\sum_{i=1}^{n} nT_i f^*(\langle B(X_i),\lambda\rangle) - \langle B(X_i),\lambda\rangle + \langle \delta, |\lambda|\rangle,$$

where $t \mapsto f^*(t) = t\,(f')^{-1}(t) - f\left((f')^{-1}(t)\right)$ *is the Legendre transformation of* f, $B(X_i) = [B_1(X_i),\ldots,B_K(X_i)]^\top$ *denotes the* K *basis functions of the covariates of unit* $i \in \{1,\ldots,n\}$ *and* $|\lambda| = [|\lambda_1|,\ldots,|\lambda_K|]^\top$, *where* $|\cdot|$ *is the absolute value of a real-valued scalar. Moreover, if* $\lambda^\dagger$ *is an optimal solution then*

$$w_i^* = (f')^{-1}(\langle B(X_i),\lambda^\dagger\rangle), \quad i \in \{1,\ldots,n\} \tag{2.1}$$

are the unique optimal solutions to (P) .

Proof. We prove the following Lemma at the end of the section.

Lemma 2.1. *The dual of the optimization problem is*

$$\underset{\lambda\in\mathbb{R}^{2K}}{\text{minimize}} \qquad \frac{1}{n}\sum_{i=1}^{n} nT_i f^*(\langle Q_{\bullet i},\lambda\rangle) - \langle Q_{\bullet i},\lambda\rangle + \langle d, \lambda\rangle$$

subject to

$$\lambda_k \geq 0 \quad \textit{for all } k \in \{1,\ldots,K\}\,, \tag{2.2}$$

where

$$\mathbf{Q} := \begin{bmatrix} \mathbf{I}_n \\ \mathbf{B}(\mathbf{X}) \\ -\mathbf{B}(\mathbf{X}) \end{bmatrix}, \qquad \mathbf{B}(\mathbf{X}) := \left[B(X_1),\ldots,B(X_n)\right], \qquad \textit{and} \qquad d := \begin{bmatrix} 0_n \\ \delta \\ \delta \end{bmatrix}. \tag{2.3}$$

□

2.4 Application of Matrix Concentration Inequalities

3 Convex Analysis

In our application we want to analyze a convex optimization problem by its dual problem. In particular we want to obtain primal optimal solutions from dual solutions. To accomplish the task we need technical tools from convex analysis, mainly conjugate calculus and some KKT related results.

Our starting point is the support function intersection rule, which we will prove in full detail employing a theorem on convex separation is finite dimensions. To this end, we will have a closer look in relative interiors and support functions. As an application we may prove the conjugate chain and sum rule, which are vital to application of duality. As a simple corollary we will obtain the classical Fenche-Rockafellar Duality theorem which gives general conditions for dual und primal optimal values to be equal.

The material we present is very well known, so we claim no originality. We orient our exposition closly by [Roc70, MMN22] .

We finish the chapter with an exposition of [TB91], where for strictly convex functions we get a dual relationship in terms of the optimal solutions.

3.1 A Convex Analysis Primer

Excursively, we present some well known definitions and facts from convex analysis. For details, see, e.g., [MMN22].

A subset $C \subseteq \mathbb{R}^n$ is called **convex set**, if for all $x, y \in C$ and all $\lambda \in [0, 1]$, we have $\lambda x + (1-\lambda)y \in C$. The Cartesian product of convex sets is convex. The intersection of a collection of convex sets is also convex. Given (not necessary convex) sets $\Omega, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, define the **set addition** and **multiplication** by a real scalar as $\Omega_1 + \Omega_2 := \{x_1 + x_2 \colon x_1 \in \Omega_1, x_2 \in \Omega_2\}$ and $\lambda\Omega := \{\lambda x \colon x \in \Omega\}$. For convex sets the addition and multiplication by a real scalar are convex.

A mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is called **affine mapping** if there exist a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $b \in \mathbb{R}^m$ such that $A(x) = L(x) + b$ for all $x \in \mathbb{R}^n$. The image and inverse image/preimage of convex sets under affine mappings are also convex.

Definition 3.1. (Affine set and hull) *A set $A \subseteq \mathbb{R}^n$ is called **affine**, if*

$$\alpha x + (1-\alpha)y \in A \quad \textit{for all } x, y \in A \textit{ and } \alpha \in \mathbb{R}. \tag{3.1}$$

*The **affine hull** $\operatorname{aff}(\Omega)$ of a set $\Omega \subseteq \mathbb{R}^n$ is the smallest affine set that includes Ω.*

Because the notion of interior is not precise enough for our purposes we define the relative interior which is the interior relative to the affine hull.

Definition 3.2. *Let $\Omega \subseteq \mathbb{R}^n$. We define the **relative interior** of Ω by*

$$\operatorname{ri}(\Omega) := \{x \in \Omega \colon \textit{there exists } \gamma > 0 \textit{ such that } \mathrm{B}_\gamma(x) \cap \operatorname{aff}(\Omega) \in \Omega\}. \tag{3.2}$$

Next we collect some useful properties of relative interiors.

Theorem 3.1. *Let C be a non-empty convex set in $\mathbb{R}^n$. Then we get the representation*

(i) $\mathrm{ri}(C) = \{z \in C\colon$ *for all* $x \in C$ *there exists* $t > 0$ *such that* $z + t(z - x) \in C\}$.

(ii) $\mathrm{cl}(C)$ *and* $\mathrm{ri}(C)$ *are convex sets.*

(iii) $\mathrm{cl}(\mathrm{ri}(C)) = \mathrm{cl}(C)$ *and* $\mathrm{ri}(\mathrm{cl}(C)) = \mathrm{ri}(C)$.

(iv) *Suppose* $\bigcap_{i\in I} C_i \neq \emptyset$ *for a finite index set* I. *Then* $\mathrm{ri}\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} \mathrm{ri}(C_i)$.

(v) *Let* $L : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear mapping. Then* $\mathrm{ri}(L(C)) = L(\mathrm{ri}(C))$. *If additionaly it holds* $L^{-1}(\mathrm{ri}(C)) \neq \emptyset$ *we have* $\mathrm{ri}(L^{-1}(C)) = L^{-1}(\mathrm{ri}(C))$.

(vi) $\mathrm{ri}(C_1 \times C_2) = \mathrm{ri}(C_1) \times \mathrm{ri}(C_2)$.

(vii) $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$ *if and only if* $0 \notin \mathrm{ri}(C_1 - C_2)$.

Proof. [Roc70, Theorem 6.4] th (i) □

Proposition 3.1. *Let $C_1 \subseteq \mathbb{R}^{n_1}$ and $C_2 \subseteq \mathbb{R}^{n_2}$ be two non-empty convex sets. Then it holds*

$$\mathrm{ri}(C_1 \times C_2) = \mathrm{ri}(C_1) \times \mathrm{ri}(C_2). \tag{3.3}$$

Proof. Let $(z_1, z_2) \in \mathrm{ri}(C_1 \times C_2)$. Then for all $(x_1, x_2) \in C_1 \times C_2$ there exists $t > 0$ such that

$$z_i + t(z_i - x_i) \in C_i \qquad \text{for } i \in \{1, 2\}. \tag{3.4}$$

This proves $\subseteq$. Suppose $z_1 \in \mathrm{ri}(C_1)$ and $z_2 \in \mathrm{ri}(C_2)$. Let $(x_1, x_2) \in C_1 \times C_2$ with corresponding $t_1, t_2 > 0$. If $t_1 = t_2$ everything is clear. W.l.o.g. assume $t_1 < t_2$. Define $\theta := \frac{t_1}{t_2} \in (0, 1)$. By the convexity of C_2 it follows

$$z_2 + t_1(z_2 - x_2) = \theta(z_2 + t_2(z_2 - x_2)) + (1 - \theta)z_2 \in C_2. \tag{3.5}$$

Thus $(z_1, z_2) \in \mathrm{ri}(C_1 \times C_2)$. This proves $\supseteq$ and equality. □

We procede with convex separation results which are vital to the subsequent developments.

Definition 3.3. *Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^n$. A hyperplane H is said to* ***separate*** *C_1 and C_2 if C_1 is contained in one of the closed half-spaces associated with H and C_2 lies in the opposite closed half-space. It is said to separate C_1 and C_2* ***properly*** *if C_1 and C_2 are not both actually contained in H itselef.*

Theorem 3.2. *Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^n$. There exists a hyperplane separating C_1 and C_2 properly if and only if there exists a vector $b \in \mathbb{R}^n$ such that*

$$\sup_{x\in C_2} \langle x, b\rangle \leq \inf_{x\in C_1} \langle x, b\rangle \quad \textit{and} \quad \inf_{x\in C_2} \langle x, b\rangle < \sup_{x\in C_1} \langle x, b\rangle. \tag{3.6}$$

Proof. [Roc70, Theorem 11.1] □

Theorem 3.3. (Convex separation in finite dimension) *Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^n$. Then C_1 and C_2 can be properly separated if and only if* $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$.

Proof. [Roc70, Theorem 11.3] $\square$

Definition 3.4. *Given a nonempty subset $\Omega \subseteq \mathbb{R}^n$ the **support function** $\sigma_\Omega : \mathbb{R}^n \to \overline{\mathbb{R}}$ of Ω is defined by*

$$\sigma_\Omega(x^*) := \sup_{x \in \Omega} \langle x^*, x \rangle \qquad \text{for } x^* \in \mathbb{R}^n. \tag{3.7}$$

Definition 3.5. *Given functions $f_i : \mathbb{R}^n \to (-\infty, \infty]$ for $i = 1, \dots, n$ the **infimal convolution** of these functions is defined as*

$$(f_1 \Box \dots \Box f_m)(x) := \inf_{\substack{x_i \in \mathbb{R}^n \\ \sum_{i=1}^m x_i = x}} \sum_{i=1}^m f_i(x_i) \tag{3.8}$$

The next result establishes a connection between the support function of the intersection of two convex sets and the infimal convolution of the support functions of the sets taken by themselfes. The proof translates the geometric concept of convex separation to the world of convex functions.

Theorem 3.4. *Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^n$ with* $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$. *Then the support function of the intersection $C_1 \cap C_2$ is represented as*

$$(\sigma_{C_1 \cap C_2})(x^*) = (\sigma_{C_1} \Box \sigma_{C_2})(x^*) \qquad \text{for all } x^* \in \mathbb{R}^n. \tag{3.9}$$

Furthermore, for any $x^* \in \mathrm{dom}(\sigma_{C_1 \cap C_2})$ there exist dual elements $x_1^*, x_2^* \in \mathbb{R}^n$ such that $x^* = x_1^* + x_2^*$. and

$$(\sigma_{C_1 \cap C_2})(x^*) = \sigma_{C_1}(x_1^*) + \sigma_{C_2}(x_2^*). \tag{3.10}$$

Proof. *[MMN22, Theorem 4.23]* We define

$$\Theta_1 := C_1 \times [0, \infty) \quad \text{and} \quad \Theta_2 := \{(x, \lambda) \in \mathbb{R}^n : x \in C_2 \text{ and } \lambda \leq \langle x^*, x \rangle - \alpha\}. \tag{3.11}$$

Clearly Θ_1 is convex by the convexity of C_1. Consider the affine function

$$\varphi : \mathbb{R}^{n+1} \to \mathbb{R}, \quad (x, \lambda) \mapsto \alpha - \langle x^*, x \rangle - \lambda. \tag{3.12}$$

It holds $\Theta_2 = \varphi^{-1}((-\infty, 0]) \cap (C_2 \times \mathbb{R})$. Thus, by the convexity of the sets $\varphi^{-1}((-\infty, 0])$ and C_2 it follows that Θ_2 is convex. We want to apply convex separation to Θ_1 and Θ_2. To this end we show $\mathrm{ri}(\Theta_1) \cap \mathrm{ri}(\Theta_2) = \emptyset$. First note that

$$\mathrm{ri}(\Theta_1) = \mathrm{ri}(C_1) \times \mathrm{ri}([0, \infty)) \subseteq \mathrm{ri}(C_1) \times (0, \infty). \tag{3.13}$$

Indeed, if $0 \in \mathrm{ri}([0, \infty))$ then there exists $t > 0$ such that $-tx \geq 0$ for some $x > 0$. A contradition. Furthermore

$$\mathrm{ri}(\Theta_2) \subseteq \{(x, \lambda) \in \mathbb{R}^n : x \in \mathrm{ri}(C_2) \text{ and } \lambda < \langle x^*, x \rangle - \alpha\}. \tag{3.14}$$

To see this, assume there is $(x, \lambda) \in \mathrm{ri}(\Theta_2)$ with $\lambda = \langle x^*, x\rangle - \alpha$. Then for some $(y, \mu) \in \Theta_2$ with $\mu < \langle x^*, y\rangle - \alpha$ there exists $t > 0$ such that $(x, \lambda) + t((x, \lambda) - (y, \mu)) \in \Theta_2$. It follows

$$0 \le (1 + t)(\langle x^*, x\rangle - \alpha - \lambda) + t(\mu - \langle x^*, y\rangle + \alpha) < 0, \tag{3.15}$$

a contradiction. The first inequality is due to $(x, \lambda) + t((x, \lambda) - (y, \mu)) \in \Theta_2$ and the second inequality due to $\mu < \langle x^*, y\rangle - \alpha$ and $\lambda = \langle x^*, x\rangle - \alpha$. But then $\mathrm{ri}(\Theta_1) \cap \mathrm{ri}(\Theta_2) = \emptyset$. Indeed, suppose that there exists $(x, \lambda) \in \mathrm{ri}(\Theta_1) \cap \mathrm{ri}(\Theta_2)$. Then it holds $\langle x^*, x\rangle - \alpha \le 0$ and $\lambda > 0$ since $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \subseteq C_1 \cap C_2$. On the other hand

$$0 < \lambda < \langle x^*, x\rangle - \alpha \le 0, \tag{3.16}$$

a contradiction. □

Takeaways This primer is somewhat confusing. Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

3.2 Conjugate Calculus and Fenchel-Rockafellar Theorem

The goal of this section is to establish the tools to calculate convex conjugates. We prove the conjugate sum and chain rule. After some examples we will derive the Fenchel-Rockafellar Theorem.

Definition 3.6. (Convex conjugate) *Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the **convex conjugate** $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ of f is defined as*

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} (x^*)^T x - f(x) \tag{3.17}$$

Note that f in Definition 3.6 does not have to be convex. On the other hand, the convex conjugate is always convex:

Proposition 3.2. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a proper function. Then its convex conjugate $f^* : \mathbb{R}^n \to (-\infty, \infty]$ is convex.*

Lemma 3.1. *For any proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ we have*

$$f^*(x^*) = \sigma_{\mathrm{epi}(f)}(x^*, -1) \qquad \text{for } x^* \in \mathbb{R}^n. \tag{3.18}$$

Proof. Let $x^* \in \mathbb{R}^n$ and $(x, \lambda) \in \mathrm{epi}(f)$. Then $x \in \mathrm{dom}(f)$ and $f(x) \le \lambda$. Thus

$$\langle x^*, x\rangle - f(x) \ge \langle x^*, x\rangle - \lambda \qquad \text{for all } (x, \lambda) \in \mathrm{epi}(f). \tag{3.19}$$

On the other hand $(x, f(x)) \in \operatorname{epi}(f)$ for all $x \in \operatorname{dom}(f)$. It follows

$$\langle x^*, x\rangle - f(x) \leq \sup_{(x,\lambda)\in\operatorname{epi}(f)} \langle x^*, x\rangle - \lambda \qquad \text{for all } x \in \operatorname{dom}(f). \tag{3.20}$$

Taking the supremum in the last two displays yields

$$\begin{aligned} f^*(x^*) = \sup_{x\in\operatorname{dom}(f)} \langle x^*, x\rangle - f(x) &= \sup_{(x,\lambda)\in\operatorname{epi}(f)} \langle x^*, x\rangle - \lambda & (3.21)\\ &= \sup_{(x,\lambda)\in\operatorname{epi}(f)} \langle (x^*, -1), (x, \lambda)\rangle = \sigma_{\operatorname{epi}(f)}(x^*, -1). & (3.22)\end{aligned}$$

□

Theorem 3.5. (Conjugate Chain Rule) *Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map (matrix) and $g : \mathbb{R}^n \to (-\infty, \infty]$ a proper convex function. If Im$(A) \cap$ ri(dom$(g)) \neq \emptyset$ it follows*

$$(g \circ A)^*(x^*) = \inf_{y^*\in(A^*)^{-1}(x^*)} g^*(y^*). \tag{3.23}$$

Furthermore, for any $x^* \in$ dom$(g\circ A)^*$ there exists $y^* \in (A^*)^{-1}(x^*)$ such that $(g\circ A)^*(x^*) = g^*(y^*)$.

Theorem 3.6. *Let $f, g : \mathbb{R}^n \to (-\infty, \infty]$ be proper convex functions and ri(dom$(f)) \cap$ ri(dom$(g)) \neq \emptyset$. Then we have the conjugate sum rule*

$$(f + g)^*(x^*) = (f^*\Box g^*)(x^*) \tag{3.24}$$

for all $x^* \in \mathbb{R}^n$. Moreover, the infimum in $(f^*\Box g^*)(x^*)$ is attained, i.e., for any $x^* \in$ dom$(f + g)^*$ there exists vectors x_1^*, x_2^* for which

$$(f + g)^*(x^*) = f^*(x_1^*) + g^*(x_2^*), \quad x^* = x_1^* + x_2^*. \tag{3.25}$$

Proof. Let $x^* \in \mathbb{R}^n$ and fix $x_1^*, x_2^* \in \mathbb{R}^n$ such that $x^* = x_1^* + x_2^*$. We get

$$\begin{aligned} f^*(x_1^*) + g^*(x_2^*) &= \sup_{x\in\mathbb{R}^n} \langle x_1^*, x\rangle - f(x) + \sup_{x\in\mathbb{R}^n} \langle x_2^*, x\rangle - g(x)\\ &\geq \sup_{x\in\mathbb{R}^n} \langle x_1^*, x\rangle - f(x) + \langle x_2^*, x\rangle - g(x) = \sup_{x\in\mathbb{R}^n} \langle x_1^* + x_2^*, x\rangle - (f(x) + g(x))\\ &= \sup_{x\in\mathbb{R}^n} \langle x^*, x\rangle - (f + g)(x) = (f + g)^*(x^*)\end{aligned}$$

Taking the infimum over $x_1^*, x_2^* \in \mathbb{R}^n$ in the above display gives $(f^*\Box g^*)(x^*) \geq (f+g)^*(x^*)$. Let us prove now $\leq$ under the condition $\operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g)) \neq \emptyset$. The only case we need to consider is $(f + g)^*(x^*) < \infty$. Define two convex sets by

$$\begin{aligned} \Omega_1 &:= \left\{(x, \alpha, \beta) \in \mathbb{R}^{n+2} \colon \alpha \geq f(x)\right\} = \operatorname{epi}(f) \times \mathbb{R}, & (3.26)\\ \Omega_2 &:= \left\{(x, \alpha, \beta) \in \mathbb{R}^{n+2} \colon \beta \geq g(x)\right\}. & (3.27)\end{aligned}$$

Similar to Lemma we get the representation

$$(f + g)^*(x^*) = \sigma_{\Omega_1\cap\Omega_2}(x^*, -1, -1). \tag{3.28}$$

Indeed, the only thing we need to verify is $\operatorname{dom}(f) \cap \operatorname{dom}(g) = \operatorname{dom}(f + g)$. The inclusion $\subseteq$ is clear. Assume towards a contradiction that $(f + g)(x) < \infty$ and $f(x) = \infty$. Since $g(x) > -\infty$ it holds

$$\infty = \infty + g(x) = f(x) + g(x) = (f + g)(x) < \infty. \tag{3.29}$$

This is a contradiction. The same holds for f and g reversed. It follows the inclusion $\supseteq$ and equality. By the support function intersection rule there exist triples

$$(x_1^*, -\alpha_1, -\beta_1), (x_2^*, -\alpha_2, -\beta_2) \in \mathbb{R}^{n+2} \quad \text{such that} \quad (x^*, -1, -1) = (x_1^* + x_2^*, -(\alpha_1 + \alpha_2), -(\beta_1 + \beta_2)) \tag{3.30}$$

and

$$(f+g)^*(x^*) = \sigma_{\Omega_1 \cap \Omega_2}(x^*, -1, -1) = \sigma_{\Omega_1}(x_1^*, -\alpha_1, -\beta_1) + \sigma_{\Omega_2}(x_2^*, -\alpha_2, -\beta_2). \tag{3.31}$$

Next we show $\beta_1 = \alpha_2 = 0$. Suppose towards a contradiction that $\beta_1 \neq 0$. We fix $(\overline{x}, \overline{\alpha}) \in \operatorname{epi}(f)$. Then

$$\sigma_{\Omega_1}(x_1^*, -\alpha_1, -\beta_1) = \sup_{(x,\alpha,\beta) \in \operatorname{epi}(f) \times \mathbb{R}} \langle x^*, x \rangle - \alpha\alpha_1 - \beta\beta_1 \geq \sup_{\beta \in \mathbb{R}} \langle x^*, \overline{x} \rangle - \overline{\alpha}\alpha_1 - \beta\beta_1 = \infty. \tag{3.32}$$

This contradicts $(f+g)^*(x^*) < \infty$. In a similar fashion we can derive a contradiction for $\alpha_2 \neq 0$. Employing Lemma and taking into account the structures of the sets Ω_1 and Ω_2 this implies

$$\begin{aligned} (f+g)^*(x^*) &= \sigma_{\Omega_1 \cap \Omega_2}(x^*, -1, -1) = \sigma_{\Omega_1}(x_1^*, -1, 0) + \sigma_{\Omega_2}(x_2^*, 0, -1) & (3.33) \\ &= \sigma_{\operatorname{epi}(f)}(x_1^*, -1) + \sigma_{\operatorname{epi}(g)}(x_2^*, -1) = f^*(x_1^*) + g^*(x_2^*) \geq (f^* \Box g^*)(x^*). & (3.34) \end{aligned}$$

This finishes the proof. $\square$

Given proper convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a matrix $A \in \mathbb{R}^{n \times n}$, we define the primal minimization problem as follows:

Problem 3.1. (Primal) *Given proper convex functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ and a matrix $A \in \mathbb{R}^{m \times n}$ we define the* ***primal optimization problem*** *to be*

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \qquad f(x) + g(Ax)$$

Remark 3.1. *Problem 3.1 appears in the unconstrained form. We can impose constraints by controling for the domains of f and g. To incorporate linear constraints $Ax \leq 0$ or more general constraints $x \in \Omega$, where Ω is a convex set, we can choose*

$$g(x) = \delta_\Omega(x) := \tag{3.35}$$

where $x \notin \Omega$ leads to $f(x) + g(x) = \infty$ and the optimization problem (if feasible) will exclude x from the solutions. $\diamondsuit$

Problem 3.2. (Dual) *Consider the same setting as in Problem 3.1. Using the convex conjugates of f, g and the transpose of A we define the* ***dual problem*** *of Problem 3.1 to be*

$$\underset{y^* \in \mathbb{R}^m}{\text{maximize}} \qquad -f^*(A^\top y^*) - g^*(y^*).$$

Proposition 3.3. *Consider the optimization problem 3.1 and its dual 3.2, where the functions f and g are not assumed to be convex. Define the* ***optimal values*** *of these problems by*

$$\widehat{p} := \inf_{x \in \mathbb{R}^n} f(x) + g(Ax) \quad \text{and} \quad \widehat{d} := \sup_{y \in \mathbb{R}^m} -f^*(A^\top y) - g^*(y).$$

Then we have the relationship $\widehat{d} \leq \widehat{p}$.

Proof. It holds

$$\begin{aligned}
-f^*(A^\top y^*) - g^*(y^*) &= -\sup_{x\in\mathbb{R}^n} \langle A^\top y^*, x\rangle - f(x) - \sup_{y\in\mathbb{R}^m} \langle -y^*, y\rangle - g(y) \\
&= \inf_{x\in\mathbb{R}^n} f(x) - \langle y^*, Ax\rangle + \inf_{y\in\mathbb{R}^m} g(y) + \langle y^*, y\rangle \\
&\leq \inf_{x\in\mathbb{R}^n} f(x) - \langle y^*, Ax\rangle + \inf_{x\in\mathbb{R}^n} g(Ax) + \langle y^*, Ax\rangle \\
&\leq \inf_{x\in\mathbb{R}^n} f(x) - \langle y^*, Ax\rangle + g(Ax) + \langle y^*, Ax\rangle \\
&= \inf_{x\in\mathbb{R}^n} f(x) + g(Ax) = \widehat{p}
\end{aligned}$$

The first equality is due to the definition of convex conjugates, the second equality due to $\langle A^\top y, x\rangle = \langle y, Ax\rangle$ and $\inf\{-B\} = -\sup\{B\}$ for all $B \subseteq \overline{\mathbb{R}}$ and the first inequality due to $\mathrm{Im}(A) \subseteq \mathbb{R}^m$. Taking the supremum with respect to all $y^* \in \mathbb{R}^m$ yields the result. □

Theorem 3.7. *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions and $0 \in ri(dom(g) - A(dom(f)))$. Then the optimal values of (3.1) and (3.2) are equal, i.e.*

$$\inf_{x\in\mathbb{R}^n} \{f(x) + g(Ax)\} = \sup_{y\in\mathbb{R}^n} \left\{-f^*\left(A^T y\right) - g^*(-y)\right\}. \tag{3.36}$$

Lemma 3.2. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be convex. Then for all $y \in \mathbb{R}^n$ and $C > 0$*

$$\inf_{\|\Delta\|=C} f(y+\Delta) - f(y) \geq 0 \quad \Longrightarrow \quad \exists y^* \in \mathbb{R}^n \colon y^* \textit{ is global minimum of } f \textit{ and } \|y^* - y\| \leq C. \tag{3.37}$$

Proof. Since $\mathcal{C} := \{\|\Delta\| \leq C\}$ is convex f has a local minimum in $y+\mathcal{C} := \{y + \Delta \mid \|\Delta\| \leq C\}$. Suppose towards a contradiction that $y^* \in y + \mathcal{C}$ is a local minimum, but not a global minimum and the left-hand side of (3.37) is true. Then it holds

$$f(x) < f(y^*) \quad \text{for some } x \in \mathbb{R}^n \setminus y + \mathcal{C}. \tag{3.38}$$

Furthermore since $y + \mathcal{C}$ is compact and contains y^*, the line segment $\mathcal{L}[y^*, x]$ contains a point on the boundary of $y + \mathcal{C}$, i.e.

$$\theta x + (1-\theta)y^* = y + \Delta_x \quad \text{for some } \theta \in (0,1) \text{ and } \Delta_x \text{ with } \|\Delta_x\| = C. \tag{3.39}$$

It follows

$$\begin{aligned}
f(y^*) \leq f(y) \leq f(y + \Delta_x) &= f(\theta x + (1-\theta)y^*) \\
&\leq \theta f(x) + (1-\theta) f(y^*) < f(y^*),
\end{aligned} \tag{3.40}$$

which is a contradiction. Thus every local minimum of f in $y+\mathcal{C}$ is also a global minimum. The first inequality is due to y^* being a local minimum of f in $y+\mathcal{C}$, the second inequality is due to the left-hand side of (3.37) being true, the equality is due to (3.39), the third inequality is due to the convexity of f and the strict inequality is due to (3.38). □

Takeaways Almost there Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

3.3 Tseng Bertsekas

We present the relevant parts of the paper [BT03].

Consider the following optimization problem

$$\underset{x\in\mathbb{R}^m}{\text{minimize}} \qquad f(x)$$

subject to the constraints

$$\mathbf{A}x \geq b, \tag{3.41}$$

Where $f : \mathbb{R}^m \to \overline{\mathbb{R}}$, $\mathbf{A}$ is a given $n \times m$ matrix, and b is a vector in $\mathbb{R}^n$.

Assumption 3.1. *Assume that the map $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ has the following properties.*

(i) f is strictly convex.

(ii) f is lower-semicontinuous and continuous dom(f).

(iii) The convex conjugate f^* of f is finite.

The dual optimization problem associated with (P) is

$$\underset{p\in\mathbb{R}^n}{\text{maximize}} \qquad q(p)$$

subject to the constraints

$$p \geq 0, \tag{3.42}$$

where $q : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the concave function given by

$$q(p) := \min_{x\in\mathbb{R}^m} f(x) + \langle p, b - \mathbf{A}x\rangle = \langle p, b\rangle - f^*(\mathbf{A}^\top p). \tag{3.43}$$

The dual problem (D) is a concave program with simple nonnegativity constraints. Furthermore, strong duality holds for (P) and (D) , i.e., the optimal value of (P) equals the optimal value of (D).

Since f^* is real-valued and f is strictly convex, f^* and q are continuously differentiable.

Theorem 3.8. [Roc70, Theorem 26.3] *A closed proper convex function is (essentially) strictly convex if and only if its conjugate is essentially smooth.*

We will denote the gradient of q at p by $d(p)$ and its ith coordinate by $d_i(p)$. Since q is continuously differentiable, $d_i(p)$ is continuous, and since q is concave, $d_i(p)$ as nonincreasing in p_i.

By differentiating and by using the chain rule, we obtain the dual cost gradient

$$d(p) = b - \mathbf{A}x, \quad \text{where} \quad x := \nabla f^*(\mathbf{A}^\top p) = \operatorname{argsup}_{\xi \in \mathbb{R}^m} \langle p, \mathbf{A}\xi \rangle - f(\xi). \tag{3.44}$$

The last equality follows from Danskin's Theorem and [Roc70, Theorem 23.5]

Proposition 3.4. (Danskin's Theorem [BT03, page 649]) *Let $Z \subseteq \mathbb{R}^m$ be a non-empty set, and let $\phi : \mathbb{R}^n \times Z \to \mathbb{R}$ be a continuous function such that $\phi(\cdot, z) : \mathbb{R}^n \to \mathbb{R}$, viewed as a function of its first argument, is convex for each $z \in Z$. Then the function*

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto \sup_{z \in Z} \phi(x, z) \tag{3.45}$$

is convex and has directional derivative given by

$$f^{'}(x; y) = \sup_{z \in Z(x)} \phi^{'}(x, z; y), \tag{3.46}$$

where $\phi^{'}(x, z; y)$ is the directional derivative of the function $\phi(\cdot, z)$ at x in the direction y, and

$$Z(x) := \left\{ \overline{z} \in \mathbb{R}^m \colon \phi(x, \overline{z}) = \sup_{z \in Z} \phi(x, z) \right\}. \tag{3.47}$$

In particular, if $Z(x)$ consists of a unique point $\overline{z}$ and $\phi(\cdot, \overline{z})$ is differentiable at x, and $\nabla f(x) = \nabla_x \phi(x, \overline{z})$, where $\nabla_x \phi(x, \overline{z})$ is the vector with coordinates $(\partial \phi / \partial x_i)(x, \overline{z})$

Note that x is the unique vector satisfying

$$\mathbf{A}p \in \partial f(x). \tag{3.48}$$

From the optimality conditions for (D) it follows that a dual vector is an optimal solution of (D) if and only if

$$p = [p + d(p)]^+, \tag{3.49}$$

where $[\ \cdot\ ]^+$ is the projection onto the positive orthant, i.e., $[y]^+ = [0 \vee y_1, \dots 0 \vee y_n,]^\top$.

Given an optimal dual solution p, we may obtain an optimal primal solution from the equation $x = \nabla f^*(\mathbf{A}^\top p)$. To see this, note that

$$\mathbf{A}x \geq b \quad \text{and} \quad p_i = 0 \quad \text{for all } i \text{ such that} \quad \sum_{j=1}^m a_{ij} x_j > b_i. \tag{3.50}$$

We can show that p and x satisfy the KKT conditions and thus x is an optimal solution to (P).

Definition 3.7. [Roc70, §28] *By an* ***ordinary convex program*** (P) *we mean an optimization problem of the following form*

$$\underset{x \in C}{\text{minimize}} \qquad f_0(x)$$

subject to the constraints

$$f_1(x) \le 0, \dots, f_r(x) \le 0, \qquad f_{r+1}(x) = 0, \dots, f_m(x) = 0, \tag{3.51}$$

where $C \subseteq \mathbb{R}^n$ *is a non-empty convex set,* f_i *is a finite convex function on* C *for* $i \in \{1, \dots, r\}$ *and* f_i *is an affine function on* C *for* $i \in \{r+1, \dots, m\}$.

Definition 3.8. *We define* $[\lambda_1, \dots, \lambda_m] \in \mathbb{R}^m$ *to be a* ***Karush-Kuhn-Tucker (KKT) vector*** *for* (P), *if*

(i) $\lambda_i \ge 0$ *for all* $i \in \{1, \dots, r\}$.

(ii) The infimum of the proper convex function $f_0 + \sum_{i=1}^m \lambda_1 f_i$ *is finite and equal to the optimal value in* (P).

Theorem 3.9. (Karush-Kuhn-Tucker conditions) *Let* (P) *be an ordinary convex program,* $\overline{\alpha} \in \mathbb{R}^m$, *and* $\overline{z} \in \mathbb{R}^n$. *Then* $\overline{\alpha}$ *is a KKT vector for* (P) *and* $\overline{z}$ *is an optimal solution to* (P) *if and only if* $\overline{z}$ *and the components* α_i *of* $\overline{\alpha}$ *satisfy the following conditions.*

(i) $\alpha_i \ge 0$, $f_i(\overline{z}) \le 0$, *and* $\alpha_i f_i(\overline{z}) = 0$ *for all* $i \in \{1, \dots, r\}$.

(ii) $f_i(\overline{z}) = 0$ *for* $i \in \{r+1, \dots, m\}$.

(iii) $0_n \in [\partial f_0(\overline{z}) + \sum_{\alpha_i \neq 0} \alpha_i \partial f_i(\overline{z})]$.

Proof. [Roc70, Theorem 28.3] □

Takeaways For strictly convex functions we can derive duality in terms of the optimal solutions.

4 Random Matrix Inequalities

In our application we want to bound moments of vector-valued random variables. For this we choose the theory of random matrix inequalities which lately received a lot of attention. In particular an approach via the method of exchangable pairs [MJC+14] has been fruitful in simplifying the proofs of long standing results such as the matrix Khintchin inequality. We base our exposition on [MJC+14]. A lot will be exact copy of this paper, so no originality is claimed. Where it seemed fit, we conducted some calculations in more detail than presented in the paper.

We will first introduce the method of exchangable pairs and derive auxiliary theorems to establish the matrix Khintchin inequality. Then we will derive inequalities for moments of matrices, first for psd matrices and then via the Hermitian dilataition for general rectangular matrices. In a last step we will introduce the notion of intrinsic dimension to improve the bounds.

4.1 A Matrix Analysis Primer

The **trace** of a square matrix, denoted by tr, is the sum of its diagonal entries, i.e. $\operatorname{tr}(\mathbf{B}) = \sum_{j=1}^{d} b_{jj} \quad \text{for } \mathbf{B} \in \mathbb{M}_d$. The trace is unitarily invariant, i.e. $\operatorname{tr}(\mathbf{B}) = \operatorname{tr}(\mathbf{QBQ}^*) \quad \text{for all } \mathbf{B} \in \mathbb{M}_d$ for all unitary $\mathbf{Q} \in \mathbb{M}_d$. In particular, the existence of an eigenvalue value decomposition shows that the trace of a Hermitian matrix equals the sum of its eigenvalues. Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Consider a matrix $\mathbf{A} \in \mathbb{H}_d$ whose eigenvalues are contained in I. We define the matrix $f(\mathbf{A}) \in \mathbb{H}_d$ using an eigenvalue decomposition of $\mathbf{A}$:

$$f(\mathbf{A}) = \mathbf{Q} \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_d) \end{bmatrix} \mathbf{Q}^* \qquad \text{where} \qquad \mathbf{A} = \mathbf{Q} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} \mathbf{Q}^* = \sum_{i=1}^{d} \lambda_i \mathbf{Q}_{\bullet i} \mathbf{Q}^*_{\bullet i}. \tag{4.1}$$

The definition of $f(\mathbf{A})$ does not depend on which eigenvalue decomposition we choose. Any matrix function that arises in this fashion is called a **standard matrix function**.

For each $p \geq 1$ the **Schatten p-norm** is defined as $\|\mathbf{B}\|_p := (\operatorname{tr}(|\mathbf{B}|^p))^{1/p}$ for $\mathbf{B} \in \mathbb{M}_d$. In this setting, $|\mathbf{B}| := (\mathbf{B}^*\mathbf{B})^{1/2}$. The **spectral norm** of an Hermitian matrix $\mathbf{A}$ is defined by the relation $\|\mathbf{A}\| := \lambda_{\max}(\mathbf{A}) \vee (-\lambda_{\min}(\mathbf{B}))$. For a general matrix $\mathbf{B}$, the spectral norm is defined to be the largest singular value: $\|\mathbf{B}\| := \sigma_1(\mathbf{B})$. The Schatten p-norm dominates the spectral norm for all $p \geq 1$.

Proposition 4.1. *Let $f, g : I \to \mathbb{R}$ be real-valued functions on an interval $I \subseteq \mathbb{R}$, and let $\mathbf{A} \in \mathbb{H}_d$ be a Hermitian matrix whose eigenvalues are contained in I.*

(i) If λ is an eigenvalue of of $\mathbf{A}$, then $f(\lambda)$ is an eigenvalue of $f(\mathbf{A})$.

(ii) $f(a) \leq g(a) \quad$ for all $a \in I \quad$ implies $\quad f(\mathbf{A}) \preccurlyeq g(\mathbf{A})$.

Takeaways This Primer is not a prim number. Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

4.2 The Method of Exchangeable Pairs

We first define an exchangable pair.

Definition 4.1. *Let Z and $Z^{'}$ random variables taking values in a Polish space $\mathcal{Z}$. We say that $(Z, Z^{'})$ is an **exchangable pair** if it has the same distribution as $(Z^{'}, Z)$. In particular, Z and $Z^{'}$ must share the same distribution.*

The following approach originates in the work of Charles Stein [Ste72] on normal approximation for a sum of dependent random variable. We will explain how some central ideas of this theory extends to matrices.

We can obtain a lot of information about the fluctuation of a random matrix $\mathbf{X}$ if we can construct a good exchangable pair $(\mathbf{X}, \mathbf{X}^{'})$. With this motivation in mind, let us introduce a special class of exchangable pairs.

Definition 4.2. *Let $(Z, Z^{'})$ be an exchangable pair of random variables taking values in a Polish space $\mathcal{Z}$, and let $\mathbf{\Psi} : \mathcal{Z} \to \mathbb{H}_d$ be a measurable function. Define the random Hermitian matrices*

$$\mathbf{X} := \mathbf{\Psi}(Z) \quad \textit{and} \quad \mathbf{X}^{'} := \mathbf{\Psi}(Z^{'}). \tag{4.2}$$

*We say that $(\mathbf{X}, \mathbf{X}^{'})$ is a **matrix Stein pair** if there is a constant $\alpha \in (0, 1]$ for which*

$$\mathbf{E}[\mathbf{X} - \mathbf{X}^{'} | Z] = \alpha \mathbf{X} \qquad \textit{almost surely.} \tag{4.3}$$

*The constant α is called the **scale factor** of the pair. We always assume $\mathbf{E}\left[\|\mathbf{X}\|^2\right] < \infty$.*

A matrix Stein pair $(\mathbf{X}, \mathbf{X}^{'})$ has several useful propreties. First, $(\mathbf{X}, \mathbf{X}^{'})$ always forms an exchangable pair. Second, it must be the case that $\mathbf{E}[\mathbf{X}] = \mathbf{0}$. Indeed,

$$\mathbf{E}[\mathbf{X}] = \frac{1}{\alpha}\mathbf{E}[\mathbf{E}[\mathbf{X} - \mathbf{X}^{'} | Z]] = \frac{1}{\alpha}\mathbf{E}[\mathbf{X} - \mathbf{X}^{'} |] = \mathbf{0}.$$

A well-chosen matrix Stein pair $(\mathbf{X}, \mathbf{X}^{'})$ provides a surprisingly powerful tool for studying the random matrix $\mathbf{X}$. The technique depends on a fundamental technical lemma.

Lemma 4.1. *Suppose that $(\mathbf{X}, \mathbf{X}^{'})$ is a matrix Stein pair with scale factor α. Let $\mathbf{F} : \mathbb{H}_d \to \mathbb{H}_d$ be a measurable function that satisfies the regularity condition $\mathbf{E}\left[\left\|(\mathbf{X} - \mathbf{X}^{'})\mathbf{F}(\mathbf{X})\right\|\right] < \infty$. Then*

$$\mathbf{E}[\mathbf{X} \cdot \mathbf{F}(\mathbf{X})] = \frac{1}{2\alpha}\mathbf{E}[(\mathbf{X} - \mathbf{X}^{'})(\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}^{'}))]. \tag{4.4}$$

In short, the randomness in the Stein pair furnishes an alternative expression for the expected product of $\mathbf{X}$ and a function $\mathbf{F}$. It allows us to estimate the expectation using the smoothness properties of the function $\mathbf{F}$ and the discrepancy between $\mathbf{X}$ and $\mathbf{X}'$.

Proof. *[MJC+14, Lemma 2.4]* Suppose that $(\mathbf{X}, \mathbf{X}')$ constructed from an auxiliary exchangable pair (Z, Z'). The defining property implies

$$\alpha \cdot \mathbf{E}[\mathbf{X} \cdot \mathbf{F}(\mathbf{X})] = \mathbf{E}[\mathbf{E}[\mathbf{X} - \mathbf{X}'|Z] \cdot \mathbf{F}(\mathbf{X})] = \mathbf{E}[(\mathbf{X} - \mathbf{X}')\mathbf{F}(\mathbf{X})] \tag{4.5}$$

□

To each matrix Stein pair $(\mathbf{X}, \mathbf{X}')$, we may associate a random matrix called the conditional variance of $\mathbf{X}$. The purpose of this section is to argue that the spectral norm of $\mathbf{X}$ is unlikely to be large, when the conditional variance is small.

Definition 4.3. *Suppose that $(\mathbf{X}, \mathbf{X}')$, is a matrix Stein pair, constructed from an auxiliary exchangeable pair (Z, Z'). The **conditional variance** is the random matrix*

$$\mathbf{\Delta}_{\mathbf{X}} := \mathbf{\Delta}_{\mathbf{X}}(Z) := \frac{1}{2\alpha}\mathbf{E}[(\mathbf{X} - \mathbf{X}')^2|Z], \tag{4.6}$$

where α is the scale factor of the pair. We may take any version of the conditional expectation in this definition.

The conditional variance $\mathbf{\Delta}_{\mathbf{X}}$ can be regarded as a stochastic estimate for the variance of the random matrix $\mathbf{X}$. To see this, assume the second moment of $\mathbf{X}$ exists. Then it follows from Lemma with $\mathbf{F}(\mathbf{X}) = \mathbf{X}$

$$\mathbf{E}[\mathbf{\Delta}_{\mathbf{X}}] = \mathbf{E}[\mathbf{X}^2]. \tag{4.7}$$

To verify the regularity condition, note that

$$\mathbf{E}[\|(\mathbf{X} - \mathbf{X}')\mathbf{X}\|] \leq \mathbf{E}[\|\mathbf{X}\|^2] + \mathbf{E}[\|\mathbf{X}\| \cdot \|\mathbf{X}'\|] \leq 2\mathbf{E}[\|\mathbf{X}\|^2] < \infty. \tag{4.8}$$

Example 4.1. [MJC+14, Example 2.4] ♢

Takeaways The conditional variance is cool. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

4.3 Matrix Khintchin Inequality and Applications

The goal of this section is to derive the matrix Khintchin inequality and show some important applications. For this we need an auxiliary theorem which is an extension of the *Burkholder-Davis-Gundy (BDG) inequality* from classical martingale theory [Bur73]. We prepare for the proof of this theorem by assembling some analytic tools.

Proposition 4.2. (Generalized Klein inequality) *Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be real-valued functions on an interval I of the real line. Suppose*

$$\sum_{k=1}^{n} u_k(a)v_k(b) \geq 0 \qquad \textit{for all } a, b \in I. \tag{4.9}$$

Then

$$\overline{\mathrm{tr}}\left(\sum_{k=1}^{n} u_k(\mathbf{A})v_k(\mathbf{B})\right) \geq 0 \qquad \textit{for all } \mathbf{A}, \mathbf{B} \in \mathbb{H}_d(I). \tag{4.10}$$

Proof. *[Pet94, Proposition 3]* Let $\mathbf{A} = \sum_{i=1}^{d} \lambda_i \mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^*$ and $\mathbf{B} = \sum_{j=1}^{d} \mu_j \mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^*$ be the orthonormal decompositions of $\mathbf{A}$ and $\mathbf{B}$. Then

$$\overline{\mathrm{tr}}\left(\sum_{k=1}^{n} u_k(\mathbf{A})v_k(\mathbf{B})\right) = \sum_{k=1}^{n}\sum_{i,j=1}^{d} \overline{\mathrm{tr}}\left(u_k(\lambda_i)\mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^* v_k(\mu_j)\mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^*\right) \tag{4.11}$$

$$= \sum_{i,j=1}^{d} \overline{\mathrm{tr}}\left(\mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^*\mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^*\right)\sum_{k=1}^{n} u_k(\lambda_i)v_k(\mu_j) \geq 0 \tag{4.12}$$

by the hypothesis. To see that $\mathrm{tr}\left(\mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^*\mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^*\right)$ is non-negative for all $i, j \in \{1, \ldots, d\}$, we apply a well known extension of von Neumann's trace inequality [Ruh70, Lemma 1], namely

$$\mathrm{tr}(\mathbf{PQ}) \geq \sum_{i=1}^{d} p_i q_{d-i+1} \geq 0 \qquad \text{for all } \mathbf{P}, \mathbf{Q} \in \mathbb{H}_d([0, \infty)), \tag{4.13}$$

where the eigenvalues $p_1 \geq \ldots \geq p_d$ and $q_1 \geq \ldots \geq q_d$ are sorted decreasingly. □

Lemma 4.2. (Mean value trace inequality) *Let I be an interval of the real line. Suppose that $g : I \to \mathbb{R}$ is a weakly increasing function and that $h : I \to \mathbb{R}$ is a function whose derivative $h^{'}$ is convex. Then for all matrices $\mathbf{A}, \mathbf{B} \in \mathbb{H}_d(I)$ it holds*

$$\overline{\mathrm{tr}}[(g(\mathbf{A}) - g(\mathbf{B})) \cdot (h(\mathbf{A}) - h(\mathbf{B}))] \leq \frac{1}{2}\,\overline{\mathrm{tr}}[(g(\mathbf{A}) - g(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \cdot (h^{'}(\mathbf{A}) + h^{'}(\mathbf{B}))]. \tag{4.14}$$

When $h^{'}$ is concave, the inequality is reversed. The same result holds for the standard trace.

Proof. *[MJC⁺14, Lemma 3.4]* Fix $a, b \in I$. Since g is weakly increasing, $(g(a) - g(b)) \cdot (a-b) \geq 0$. The fundamental theorem of calculus and the convexity of $h^{'}$ yield the estimate

$$(g(a) - g(b)) \cdot (h(a) - h(b)) = (g(a) - g(b)) \cdot (a - b)\int_0^1 h^{'}(\tau a + (1-\tau)b)\mathrm{d}\tau \tag{4.15}$$

$$\leq (g(a) - g(b)) \cdot (a - b)\int_0^1 [\tau h^{'}(a) + (1-\tau)h^{'}(b)]\mathrm{d}\tau \tag{4.16}$$

$$= \frac{1}{2}\left[(g(a) - g(b)) \cdot (a - b) \cdot (h^{'}(a) + h^{'}(b))\right]. \tag{4.17}$$

The inequality is reversed, if h' is concave. To apply the Kleins inequality we expand the terms. The RHS is

$$\begin{aligned}
&(g(a) - g(b)) \cdot (a - b) \cdot (h'(a) + h'(b)) \\
&\quad = [g(a) \cdot a \cdot h'(a)] + [g(a) \cdot a] \cdot h'(b) - b \cdot [h'(a) \cdot g(a)] - [b \cdot h'(b)] \cdot g(a) \\
&\qquad + [\text{ the same as above with } a \text{ and } b \text{ reversed }](a \rightleftarrows b)
\end{aligned} \tag{4.18}$$

Taking the trace yields

$$\begin{aligned}
&\operatorname{tr}[g(\mathbf{A}) \cdot \mathbf{A} \cdot (h'(\mathbf{A}) + h'(\mathbf{B}))] - \operatorname{tr}[\mathbf{B} \cdot (h'(\mathbf{A}) + h'(\mathbf{B})) \cdot g(\mathbf{A})] + (\mathbf{A} \rightleftarrows \mathbf{B}) \\
&\quad = \operatorname{tr}[g(\mathbf{A}) \cdot \mathbf{A} \cdot (h'(\mathbf{A}) + h'(\mathbf{B}))] - \operatorname{tr}[g(\mathbf{A}) \cdot \mathbf{B} \cdot (h'(\mathbf{A}) + h'(\mathbf{B}))] + (\mathbf{A} \rightleftarrows \mathbf{B}) \\
&\quad = \operatorname{tr}[g(\mathbf{A}) \cdot (\mathbf{A} - \mathbf{B}) \cdot (h'(\mathbf{A}) + h'(\mathbf{B}))] + (\mathbf{A} \rightleftarrows \mathbf{B}) \\
&\quad = \operatorname{tr}[(g(\mathbf{A}) - g(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \cdot (h'(\mathbf{A}) + h'(\mathbf{B}))].
\end{aligned} \tag{4.19}$$

On the LHS we have only products of two factors which commute under the trace operation. Thus we may use the same expression as in the scalar case without further calculations. The result follows immediately from the Klein inequality. □

Proposition 4.3. (Hölder inequality for trace) *Let p and q be Hölder conjugate indices. Then*

$$\operatorname{tr}(\mathbf{BC}) \leq \|\mathbf{B}\|_p \|\mathbf{C}\|_q \qquad \textit{for all } \mathbf{B}, \mathbf{C} \in \mathbb{M}_d. \tag{4.20}$$

Proof. [Bha97, Corollary IV.2.6] □

We are now ready to prove the auxiliary theorem.

Theorem 4.1. (Matrix BDG inequality) *Let $p = 1$ or $p \geq 3/2$. Suppose that $(\mathbf{X}, \mathbf{X}')$ is a matrix Stein pair where $\mathbf{E}[\|\mathbf{X}\|_{2p}^{2p}] < \infty$. Then*

$$\mathbf{E}[\|\mathbf{X}\|_{2p}^{2p}]^{1/(2p)} \leq \sqrt{2p - 1}\ \mathbf{E}[\|\mathbf{\Delta_X}\|_p^p]^{1/(2p)}, \tag{4.21}$$

where $\mathbf{\Delta_X}$ is the conditional variance .

Proof. *[MJC⁺14, §7.3]* Suppose that $(\mathbf{X}, \mathbf{X}')$ is a matrix Stein pair with scale factor α. First, observe that the result for $p = 1$ already follows from $\mathbf{E}[\mathbf{\Delta_X}] = \mathbf{E}[\mathbf{X}^2]$. Therefore we may assume that $p \geq 3/2$. We introduce the notation for the quantity of interest,

$$E := \mathbf{E}[\|\mathbf{X}\|_{2p}^{2p}] = \mathbf{E}[\operatorname{tr}(|\mathbf{X}|^{2p})]. \tag{4.22}$$

We rewrite the expression for E by peeling off a copy of $|\mathbf{X}|$. This yields

$$E = \mathbf{E}[\operatorname{tr}(|\mathbf{X}| \cdot |\mathbf{X}|^{2p-1})] = \mathbf{E}[\operatorname{tr}(\mathbf{X} \cdot \operatorname{sgn}(\mathbf{X}) \cdot |\mathbf{X}|^{2p-1})]. \tag{4.23}$$

Apply the method of exchangable pairs with $\mathbf{F}(\mathbf{X}) = \operatorname{sgn}(\mathbf{X}) \cdot |\mathbf{X}|^{2p-1}$ to reach

$$E = \frac{1}{2\alpha} \mathbf{E}[\operatorname{tr}((\mathbf{X} - \mathbf{X}') \cdot (\operatorname{sgn}(\mathbf{X}) \cdot |\mathbf{X}|^{2p-1} - \operatorname{sgn}(\mathbf{X}') \cdot |\mathbf{X}'|^{2p-1}))] \tag{4.24}$$

Apply method of exchangeable pairs, generalized Klein inequality, trace Hölder □

Theorem 4.2. [MJC+14, Corollary 7.3] *Suppose that $p = 1$ or $p \geq 3/2$. Consider a finite sequence $(\mathbf{Y}_k)_{k\geq 1}$ of independent, random, Hermitian matrices and a deterministic sequence $(\mathbf{A}_k)_{k\geq 1}$ for which*

$$\mathbf{E}[\mathbf{Y}_k] = 0 \quad \text{and} \quad \mathbf{Y}_k^2 \preccurlyeq \mathbf{A}_k^2 \qquad \text{almost surely for all } k \geq 1. \tag{4.25}$$

Then

$$\mathbf{E}\left[\left\|\sum_{k\geq 1} \mathbf{Y}_k\right\|_{2p}^{2p}\right]^{1/(2p)} \leq \sqrt{p - \frac{1}{2}} \left\| \left(\sum_{k\geq 1} (\mathbf{A}_k^2 + \mathbf{E}[\mathbf{Y}_k^2])\right)^{1/2} \right\|_{2p}. \tag{4.26}$$

In particular, when $(\xi_k)_{k\geq 1}$ is an independent sequence of Rademacher random variables,

$$\mathbf{E}\left[\left\|\sum_{k\geq 1} \xi_k \mathbf{A}_k\right\|_{2p}^{2p}\right]^{1/(2p)} \leq \sqrt{2p-1} \left\| \left(\sum_{k\geq 1} \mathbf{A}_k^2\right)^{1/2} \right\|_{2p}. \tag{4.27}$$

Theorem 4.3. *Assume $n \geq 3$*

(i) Suppose that $p \geq 1$, and fix $r \geq p \vee 2\log(n)$. Consider a finite sequence $(\mathbf{S}_k)_{k\geq 1}$ of independent, random, positive-semidefinite matrices with dimension $n \times n$. Then

$$\mathbf{E}\left[\left\|\sum_{k\geq 1} \mathbf{S}_k\right\|^{p}\right]^{1/p} \leq \left[\left\|\sum_{k\geq 1} \mathbf{E}[\mathbf{S}_k]\right\|^{1/2} + 2\sqrt{er}\,\mathbf{E}[\max_{k\geq 1} \|\mathbf{S}_k\|^p]^{1/(2p)}\right]^2. \tag{4.28}$$

(ii) Suppose that $p \geq 2$, and fix $r \geq p \vee 2\log(n)$. Consider a finite sequence $(\mathbf{Y}_k)_{k\geq 1}$ of independent, symmetric, random, self-adjoint matrices with dimension $n \times n$. Then

$$\mathbf{E}\left[\left\|\sum_{k\geq 1} \mathbf{Y}_k\right\|^{p}\right]^{1/p} \leq \sqrt{er} \left\| \left(\sum_{k\geq 1} \mathbf{E}[\mathbf{Y}_k^2]\right)^{1/2} \right\| + 2er\,\mathbf{E}[\max_{k\geq 1} \|\mathbf{S}_k\|^p]^{1/p}. \tag{4.29}$$

Takeaways This is so amazing Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

4.4 Generalzed Inequalities by Hermitian Dilataition

Definition 4.4. (Hermitian Dilation) *The Hermitian dilation*

$$\mathfrak{H} : \mathbb{C}^{d_1 \times d_2} \to \mathbb{H}_{d_1 \times d_2}$$

is a map from a general matrix to an Hermitian matrix defined by

$$\mathfrak{H}(B) := \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} \tag{4.30}$$

Theorem 4.4. (Matrix Rosenthal-Pinelis) *Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be independent, random matrices with dimension $d_1 \times d_2$. Introduce the random matrix*

$$\mathbf{S} := \sum_{k=1}^{n} \mathbf{A}_k.$$

Let $v(\mathbf{S})$ be the matrix variance statistic of the sum:

$$v(\mathbf{S}) := \left\|\mathbf{E}[\mathbf{S}\mathbf{S}^\top]\right\| \vee \left\|\mathbf{E}[\mathbf{S}^\top\mathbf{S}]\right\| = \left\|\sum_{k=1}^{n} \mathbf{E}[\mathbf{A}_k\mathbf{A}_k^\top]\right\| \vee \left\|\sum_{k=1}^{n} \mathbf{E}[\mathbf{A}_k^T\mathbf{A}_k]\right\|. \tag{4.31}$$

Then

$$\left(\mathbf{E}\left[\|\mathbf{S}\|^2\right]\right)^{\frac{1}{2}} \leq \sqrt{2ev(\mathbf{S})\log(d_1+d_2)} + 4e\left(\mathbf{E}[\max_{k\leq n}\|\mathbf{A}_k\|^2]\right)^{\frac{1}{2}} \log(d_1+d_2). \tag{4.32}$$

Remark 4.1. *Since $\mathbf{E}[\|S\|] \leq \mathbf{E}[\|S\|^2]^{\frac{1}{2}}$ by the Cauchy-Schwarz inequality, Theorem 4.4 also holds with $\mathbf{E}[\|S\|]$ on the left-hand side of* (4.32)*. To obtain a tail bound we can employ the Markov inequality and Theorem 4.4:*

$$\mathbf{P}[\,\|S\| \geq t]$$
$$\leq \frac{\mathbf{E}[\|S\|]}{t} \leq \frac{1}{t}\left(\sqrt{2ev(\mathbf{S})\log(d_1+d_2)} + 4e\left(\mathbf{E}[\max_{k\leq n}\|\mathbf{A}_k\|^2]\right)^{\frac{1}{2}}\log(d_1+d_2)\right) \quad \text{for } t > 0. \tag{4.33}$$

It might be possible to improve the log *term employing an intrinsic dimension argument.* ◊

Takeaways Dilataition is so deep. Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

4.5 Intrinsic Dimension

Definition 4.5. *For a positive-semidefinite matrix* $\mathbf{S}$*, the* ***intrinic dimension*** *is the quantity*

$$\operatorname{intdim}(\mathbf{A}) := \frac{\operatorname{tr}\mathbf{A}}{\|\mathbf{A}\|}.$$

Lemma 4.3. (Intrinsic dimenision) *Let* $\varphi : [0, \infty) \to \mathbb{R}$ *be a convex function with* $\varphi(0) = 0$*. For any positive-semidefinite matrix* $\mathbf{S}$ *it holds that*

$$\operatorname{tr}(\varphi(\mathbf{S})) \leq \operatorname{intdim}(\mathbf{S}) \cdot \varphi(\|\mathbf{S}\|).$$

Proof. *[Tro15, Lemma 7.5.1]* Since φ is convex on any interval $[0, L]$ with $L > 0$ and $\varphi(0) = 0$, it holds

$$\varphi(a) \leq \left(1 - \frac{a}{L}\right)\varphi(0) + \frac{a}{L}\varphi(L) = \frac{a}{L}\varphi(L) \qquad \text{for all } a \in [0, L]. \tag{4.34}$$

Since $\mathbf{S}$ is positive-semidefinite, the eigenvalues of $\mathbf{S}$ fall in the interval $[0, L]$, where $L = \|\mathbf{S}\|$.

$$\operatorname{tr}(\varphi(\mathbf{S})) = \sum_{i=1}^{d} \varphi(\lambda_i) \leq \frac{\sum_{i=1}^{d} \lambda_i}{\|\mathbf{S}\|}\varphi(\|\mathbf{S}\|) = \frac{\operatorname{tr}(\mathbf{S})}{\|\mathbf{S}\|}\varphi(\|\mathbf{S}\|) = \operatorname{intdim}(\mathbf{S}) \cdot \varphi(\|\mathbf{S}\|). \tag{4.35}$$

□

Takeaways Is it intrinsic or extrinsic? Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

5 Empirical Processes

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $(\mathcal{X}, \Sigma)$ a measurable space. Let $X_j : (\Omega, \mathcal{A}, \mathbf{P}) \to (\mathcal{X}, \Sigma), j = 1, \ldots, n$ be independent and identically-distributed (i.i.d.) random variables with probability distribution $\mathbf{P}_X$ and $\mathcal{F}$ a family of measurable functions $f : (\mathcal{X}, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider the map

$$f \mapsto G_n f := \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbf{P}_X f \right), \tag{5.1}$$

where $\mathbf{P}_X f := \int_{\mathcal{X}} f \mathrm{d}\mathbf{P}_X$. We call $(G_n f)_{f \in \mathcal{F}}$ the empirical process indexed by $\mathcal{F}$. Furthermore

$$\|G_n f\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |G_n f| \,. \tag{5.2}$$

Lemma 5.1. (Bernstein Inequality for Empirical Processes) *For any bounded, measurable function f it holds for all $t > 0$*

$$\mathbf{P}\left(|G_n f| > t\right) \leq 2 \exp\left(-\frac{1}{4} \frac{t^2}{\mathbf{P}_X(f^2) + t \|f\|_\infty / \sqrt{n}} \right) \tag{5.3}$$

Proof. By the Markov inequality it holds for all $\lambda > 0$

$$\mathbf{P}\left(G_n f > t\right) \leq e^{-\lambda t} \mathbf{E} \exp\left(\lambda G_n f\right) \tag{5.4}$$

□

Lemma 5.2. *For any finite class $\mathcal{F}$ of bounded, measurable, square-integrable functions, with $|\mathcal{F}|$ elements, it holds*

$$\mathbf{E} \|G_n f\|_{\mathcal{F}} \lesssim \max_{f \in \mathcal{F}} \frac{\|f\|_\infty}{\sqrt{n}} \log\left(1 + |\mathcal{F}|\right) + \max_{f \in \mathcal{F}} \|f\|_{\mathbf{P},2} \sqrt{\log\left(1 + |\mathcal{F}|\right)}. \tag{5.5}$$

6 Simple yet useful Calculations

Theorem 6.1. (Multivariate Taylor Theorem) *Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$. Then for all $x, \Delta \in \mathbb{R}^n$ there exists $\xi \in [0,1]$ such that it holds*

$$\begin{aligned} f(x+\Delta) = f(x) &+ \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i}\Delta_i + \sum_{\substack{i,j=1\\ i\neq j}} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i \partial x_j}\Delta_i\Delta_j \\ &+ \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i^2}\Delta_i^2 \end{aligned} \tag{6.1}$$

Corollary 6.1.1. *Let $f \in C^2(\mathbb{R})$. Then for all $a, x, \Delta \in \mathbb{R}^n$ there exist $\xi \in [0,1]$ such that it holds*

$$f(a^T(x+\Delta)) - f(a^T x) = f^{'}(a^T x)\,\Delta^T a + \frac{1}{2} f^{''}(a^T(x+\xi\Delta))\,\Delta^T A\ \Delta, \tag{6.2}$$

where $A := aa^T \in \mathbb{R}^{n\times n}$.

Proof. By the chain rule we have for all $a, x, \Delta \in \mathbb{R}^n$ and $\xi \in [0,1]$

$$\frac{\partial^2 f(a^T(x+\xi\Delta))}{\partial x_i \partial x_j} = f^{''}(a^T(x+\xi\Delta))\, a_i a_j. \tag{6.3}$$

Since $A := aa^T$ is symmetric we have

$$\Delta^T A\ \Delta = 2\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i a_j \Delta_i \Delta_j + \sum_{i=1}^{n} a_i^2 \Delta_i^2. \tag{6.4}$$

Plugging (6.3) and (6.4) into (6.1) yields (6.2). □

Proposition 6.1. *For all $x, y \in \mathbb{R}$ it holds*

$$|x+y| - |x| \geq -|y| \tag{6.5}$$

Proof. Checking all 6 combinations of $x+y, x, y$ being nonnegative or negative yields the result. □

Notation Index

$\#A$ cardinality of the set A

$\mathbf{E}[X|Y]$ conditional expectation of the random variable X with respect to $\sigma(Y)$

$\mathbf{E}[X]$ expectation of the random variable X

$\mathbf{Var}[X]$ variance of the random variable X

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ extension of the real numbers

$\xrightarrow{\mathcal{D}}$ convergence of distributions

$\mathbf{P}$ generic probability measure

$\mathbf{P}_X = \mathbf{P} \circ X^{-1}$ distribution of the random variable X

$\mathbb{R}$ set of real numbers

$x \vee y, x \wedge y, x^+, x^-$ maximum, minimum, positive part, negative part of real numbers

$X \sim \mu$ the random variable has distribution μ

Bibliography

[Bha97] Rajendra Bhatia. *Matrix Analysis*, volume 169 of *Graduate Texts in Mathematics*. Springer, New York, NY, 1997.

[BT03] Dimitri P. Bertsekas and John N. Tsitsiklis. Parallel and Distributed Computation:Numerical Methods. November 2003.

[Bur73] D. L. Burkholder. Distribution Function Inequalities for Martingales. *The Annals of Probability*, 1(1):19–42, 1973.

[MJC+14] Lester Mackey, Michael I. Jordan, Richard Y. Chen, Brendan Farrell, and Joel A. Tropp. Matrix concentration inequalities via the method of exchangeable pairs. *The Annals of Probability*, 42(3), May 2014.

[MMN22] Boris S. Mordukhovich and Nguyen Mau Nam. *Convex Analysis and Beyond: Volume I: Basic Theory*. Springer Series in Operations Research and Financial Engineering. Springer International Publishing, Cham, 2022.

[Pet94] Dénes Petz. A survey of certain trace inequalities. *Banach Center Publications*, 30(1):287–298, 1994.

[Roc70] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[Ruh70] Axel Ruhe. Perturbation bounds for means of eigenvalues and invariant subspaces. *BIT Numerical Mathematics*, 10(3):343–354, September 1970.

[Ste72] Charles Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory*, 6.2:583–603, January 1972.

[TB91] Paul Tseng and Dimitri P. Bertsekas. Relaxation Methods for Problems with Strictly Convex Costs and Linear Constraints. *Mathematics of Operations Research*, 16(3):462–481, 1991.

[Tro15] Joel A. Tropp. An Introduction to Matrix Concentration Inequalities, January 2015.