Todo list

Robust Weighting and Matching Techniques for Causal Inference in Observational Studies with Continuous Treatment

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Contents

1 Balancing Weights

1.1 Dual Formulation

Theorem. *The dual of Problem* **??** *is the unconstrained optimization problem*

$$\underset{\lambda \in \mathbb{R}^n}{\text{minimize}} \qquad \frac{1}{n} \sum_{i=1}^{n} T_i \cdot f^*(m_n(\lambda | X_i)) - m_n(\lambda | X_i) \; + \; \langle \delta, |\lambda| \rangle \, ,$$

where

$$f^* : \mathbb{R} \; \to \; \mathbb{R} \, , \qquad x^* \; \mapsto \; x^* \cdot (f')^{-1}(x^*) \; - \; f\left((f')^{-1}(x^*) \right)$$

is the Legendre transformation of f *, the vector* $B(X_i) = [B_1(X_i), \ldots, B_n(X_i)]^\top$ *denotes the* n *basis functions of the covariates of unit* $i \in \{1, \ldots, n\}$ *and* $|\lambda| = [|\lambda_1|, \ldots, |\lambda_n|]^\top$ *, where* $|\cdot|$ *is the absolute value of a real-valued scalar. Moreover, if* $\lambda^\dagger$ *is an optimal solution of the above problem then for all* i *with* $T_i = 1$

$$w_i^\dagger \; = \; (f')^{-1} \left(m_n(\lambda^\dagger | X_i) \right)$$

is part of the optimal solution to Problem **??** *.*

Plan of proof

We bring Problem **??** in the form of ts. Then we apply the results of the ts chapter. We wait to conclude on the weights. Then we eliminate the non-negativity constraints on the dual variables leveraging convexity and optimality.

***Proof*.** We consider the vector of basis functions of the covariates of unit $i \in \{1, \ldots, n\}$, that is,

$$B(X_i) \; := \; [B_1(X_i), \ldots, B_n(X_i)]^\top \; ,$$

the constraints vector

$$d := \begin{bmatrix} 0_n \\ \pm n \\ -n \cdot \delta \pm \sum_{i=1}^n B_k(X_i) \end{bmatrix},$$

the matrix of the basis functions with indicators of treatment

$$T\mathbf{B}(\mathbf{X}) := \Big[T_1 B(X_1), \ldots, T_n B(X_n)\Big]$$

and the constraint matrix with indicator of treatment

$$T\mathbf{A} := \begin{bmatrix} \operatorname{diag}[T_1, \ldots, T_n] \\ \pm[T_1, \ldots, T_n] \\ \pm T\mathbf{B}(\mathbf{X}) \end{bmatrix}.$$

Next we write Problem **??** in the form of ts.

$$\underset{w_1,\ldots,w_n \in \mathbb{R}}{\text{minimize}} \qquad \sum_{i=1}^n T_i \cdot f(w_i)$$
$$\mathbf{A}w \ \geq \ d\,,$$

The convex conjugate of the objective function of Problem **??** is

$$[x_1^*, \ldots, x_n^*]^\top \ \mapsto \sum_{T_i=1} f^*(x_i^*) + \sum_{T_i=0} \chi_{\{0\}}(x_i^*)\,,$$

where we define the characteristic function χ by

$$\chi_{\{0\}}(t) \ = \ \infty \cdot (1 - \mathbf{1}_{\{0\}}(t)) \ = \ \begin{cases} 0, & \text{if } t = 0, \\ \infty, & \text{else}\,. \end{cases}$$

Note that the i-th column of $T\mathbf{A}$, which we denote as $T\mathbf{A}_i$, vanishes if $T_i = 0$. In the subsecquent analysis this prevents the characteristic function from blowing up. We consider the form of the dual in ts to get

$$\sum_{T_i=1} f^*(T\mathbf{A}_i^\top \lambda) + \sum_{T_i=0} \chi_{\{0\}}(T\mathbf{A}_i^\top \lambda) = \sum_{i=1}^n T_i f^*(\mathbf{A}_i^\top \lambda)\,.$$

where $\mathbf{A}$ is as $T\mathbf{A}$ without the indicators of treatment. It is important, that in the final form the indicators of treatment are outside the argument of f^* and we have no singularity for $T_i = 0$. The corresponding dual problem in [TB91] is then

$$\underset{\lambda_1,\ldots,\lambda_K \geq 0}{\text{maximize}} \quad -\sum_{i=1}^n T_i \cdot f^*(\mathbf{A}_i^\top \lambda) \ + \ \langle \lambda, d \rangle\,.$$

Next we want to remove the non-negativity constraints on λ. To this end we write

$$\lambda := \Big[\rho_1, \dots, \rho_n,\ \lambda_0^+, \lambda_0^-,\ \lambda_1^+, \dots, \lambda_n^+,\ \lambda_1^-, \dots, \lambda_n^-\Big]^\top . \tag{1.1}$$

We expand the objective function G of the dual problem.

$$\begin{aligned} G(\rho, \lambda_0^\pm, \lambda^\pm) = &-\sum_{i=1}^n T_i \cdot f^*\left(\rho_i \ + \ \lambda_0^+ - \lambda_0^- \ + \ \langle B(X_i),\, \lambda^+ - \lambda^-\rangle\right) \\ &+ \ n \cdot (\lambda_0^+ - \lambda_0^-) \ + \ \sum_{i=1}^n \langle B(X_i),\, \lambda^+ - \lambda^-\rangle \ - \ n \cdot \langle \delta,\, \lambda^+ + \lambda^-\rangle \end{aligned}$$

To illustrate the procedure, we show for all $i \in \{1, \dots, n\}$

$$\begin{aligned} \text{either} \qquad & \lambda_i^+ > 0 \\ \text{or} \qquad & \lambda_i^- > 0\,. \end{aligned}$$

Assume towards a contradiction that there exists $i \in \{1, \dots, n\}$ such that $\lambda_0^+ > 0$ and $\lambda_0^- > 0$ and that λ is optimal. Consider

$$\tilde\lambda \ := \ \Big[\rho,\ \lambda_0^\pm,\ \lambda_1^+, \dots,\ \lambda_i^+ - (\lambda_i^+ \wedge \lambda_i^-), \dots \lambda_n^+,\ \lambda_1^-, \dots, \lambda_i^- - (\lambda_i^+ \wedge \lambda_i^-), \dots, \lambda_n^-\Big]^\top . \tag{1.2}$$

Since $\lambda_i^\pm - (\lambda_i^+ \wedge \lambda_i^-) \geq 0$, the perturbed vector $\tilde\lambda$ is in the domain of the optimization problem. But

$$G(\tilde\lambda) - G(\lambda) \ = \ 2n \cdot \delta_i \cdot (\lambda_i^+ \wedge \lambda_i^-) \ > \ 0\,, \tag{1.3}$$

which contradicts the optimality of λ. Likewise we can show

$$\begin{aligned} \text{either} \qquad & \lambda_i^+ > 0 \\ \text{or} \qquad & \lambda_i^- > 0\,. \end{aligned}$$

But then $\lambda_i^\pm \geq 0$ collapses to $\lambda_i \in \mathbb{R}$ for $i \in \{0, \dots, n\}$, that is, $\lambda_i = \lambda_i^+ - \lambda_i^-$. Note that $|\lambda_i| = \lambda_i^+ + \lambda_i^-$. Likewise we can see, that $\lambda_0 = \lambda_0^+ - \lambda_0^- \in \mathbb{R}$ removes the constraint on $\lambda_0^\pm$. Let us take this into account for G. We get

$$\begin{aligned} G(\rho, \lambda_0, \lambda) = &-\sum_{i=1}^n T_i \cdot f^*\left(\rho_i \ + \ \lambda_0 \ + \ \langle B(X_i), \lambda\rangle\right) \\ &+ \ n \cdot \lambda_0 \ + \ \sum_{i=1}^n \langle B(X_i), \lambda\rangle \ - \ n \cdot \langle \delta, |\lambda|\rangle\,. \end{aligned}$$

Next we show, that $\rho = 0$. Suppose there exists $i \in \{1, \ldots, n\}$ such that $\rho_i > 0$ and $T_i \cdot (f^{'})^{-1}(\rho_i \ + \ \lambda_0 \ + \ \langle B(X_i), \lambda \rangle) < 0$. It follows

$$G(0, \lambda_0, \lambda) - G(\rho_i, \lambda_0, \lambda) \geq T_i \cdot (f^{'})^{-1}(\rho_i \ + \ \lambda_0 \ + \ \langle B(X_i), \lambda \rangle)(-\rho_i) > 0, \tag{1.4}$$

which contradicts the optimality of λ. Suppose $T_i \cdot (f^{'})^{-1}(\rho_i \ + \ \lambda_0 \ + \ \langle B(X_i), \lambda \rangle) > 0$. Then the claim yields to a perturbation argument as in ts. Thus To eliminate the constraints for ρ we use a similar argument as in the complementary slackness section of the ts chapter. Thus we have complementary slackness of ρ_i and $T_i \cdot (f^{'})^{-1}(\rho_i \ + \ \lambda_0 \ + \ \langle B(X_i), \lambda \rangle)$. But then every optimal solution λ remains optimal by taking $\rho = 0$.

Dividing the optimization problem by n and reversing it, we get

$$\underset{\lambda_0, \ldots, \lambda_n \in \mathbb{R}}{\text{minimize}} \quad \frac{1}{n} \sum_{i=1}^{n} \left[T_i \cdot f^{*}(\lambda_0 + \langle B(X_i), \lambda \rangle) \ - \ (\lambda_0 + \langle B(X_i), \lambda \rangle) \right] \ + \ \langle \delta, |\lambda| \rangle \, .$$

$\square$

2 Convex Analysis

In our application we want to analyse a convex optimization problem by its dual problem. In particular we want to obtain primal optimal solutions from dual solutions. To accomplish the task we need technical tools from convex analysis, mainly conjugate calculus and some KKT related results.

Our starting point is the support function intersection rule [MMN22, Theorem 4.23]. We give the details in the case of finite dimensions and refer for the rest of the proof to the book. The conjugate sum rule is applied to give first conjugate sum and then chain rule, which are vital to calculating convex conjugates. The proofs are omited, since the book is thorough enough. The well known Fenche-Rockafellar Duality theorem is a corollary of conjugate sum and chain rule. It gives general conditions under which dual and primal values coincide. The material we present is very well known, so we claim no originality. We paraphrase the approach of [MMN22] to Duality. As an introduction, we recommend this recently published book together with the classical reference [Roc70].

We finish the chapter with ideas from [TB91]. They provide the high-level ideas to obtain for strictly convex functions a dual relationship between optimal solutions. We will deliver the details that are omited in the paper.

2.1 A Convex Analysis Primer

Excursively, we present some well known definitions and facts from convex analysis. For details, see, e.g., [MMN22].

A subset $C \subseteq \mathbb{R}^n$ is called **convex set**, if for all $x, y \in C$ and all $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in C$. The Cartesian product of convex sets is convex. The intersection of a collection of convex sets is also convex.

Given (not necessary convex) sets $\Omega, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, define the **set addition** and **multiplication** by a real scalar as $\Omega_1 + \Omega_2 := \{x_1 + x_2 \colon x_1 \in \Omega_1, x_2 \in \Omega_2\}$ and $\lambda\Omega := \{\lambda x \colon x \in \Omega\}$. For convex sets the addition and multiplication by a real scalar are convex.

Throughout this section, we shall denote by $B := \{x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n \colon (\sum_{i=1}^n x_i^2)^{1/2} \leq 1\}$

Solve editorial issue with ball.

the **Euclidian unit ball** in $\mathbb{R}^n$. This is a closed convex set. For any $a \in \mathbb{R}^n$, the **ball with radius** $\varepsilon > 0$ **and center** a is given by $\left\{a + x \in \mathbb{R}^n \colon (\sum_{i=1}^n x_i^2)^{1/2} \leq \varepsilon\right\} = a{+}\varepsilon B$. For any set Ω in $\mathbb{R}^n$, the set of points x whose distance from Ω does not exceed ε is $\Omega{+}\varepsilon B$. The **closure** $\mathrm{cl}(\Omega)$ and **interior** $\mathrm{int}(\Omega)$ of Ω can therefore be expressed by $\mathrm{cl}(\Omega) = \bigcap_{\varepsilon>0} \Omega{+}\varepsilon B$ and $\mathrm{int}(\Omega) = \{x \in \Omega \colon \text{there exists } \varepsilon > 0 \text{ such that } x + \varepsilon B \subseteq \Omega\}$.

A set $A \subseteq \mathbb{R}^n$ is called **affine set**, if $\alpha x + (1-\alpha)y \in A$ for all $x, y \in A$ and $\alpha \in \mathbb{R}$. The **affine hull** $\mathrm{aff}(\Omega)$ of a set $\Omega \subseteq \mathbb{R}^n$ is the smallest affine set that includes Ω. A mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is called **affine mapping** if there exist a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $b \in \mathbb{R}^m$ such that $A(x) = L(x) + b$ for all $x \in \mathbb{R}^n$. The image and inverse image/preimage of convex sets under affine mappings are also convex.

Because the notion of interior is not precise enough for our purposes we define the relative interior which is the interior relative to the affine hull. This concept is motivated by the fact that a line segment embedded in $\mathbb{R}^2$ does have a natural interior in $\mathbb{R}$ which is not a true interior in $\mathbb{R}^2$. The relative interior of C is defined as the interior which results when C is regarded as a subset of its affine hull.

Definition. *Let $\Omega \subseteq \mathbb{R}^n$. We define the **relative interior** of Ω by*

$$\mathrm{ri}(\Omega) := \{x \in \Omega \colon \textit{there exists } \varepsilon > 0 \textit{ such that } (x + \varepsilon B) \cap \mathrm{aff}(\Omega) \subset \Omega\}. \tag{2.1}$$

Next we collect some useful properties of relative interiors.

Proposition 2.1. *Let C be a non-empty convex set in $\mathbb{R}^n$. The following holds:*

(i) $\mathrm{ri}(C) \neq \emptyset$ *if and only if* $C \neq \emptyset$

(ii) The sets $\mathrm{cl}\, C$ *and* $\mathrm{ri}\, C$ *are convex*

(iii) $\mathrm{cl}(\mathrm{ri}\, C) = \mathrm{cl}\, C$ *and* $\mathrm{ri}(\mathrm{cl}\, C) = \mathrm{ri}(C)$

(iv) $\mathrm{ri}(C) = \{z \in C \colon \textit{for all } x \in C \textit{ there exists } t > 0 \textit{ such that } z + t(z - x) \in C\}$

(v) Suppose $\bigcap_{i \in I} C_i \neq \emptyset$ *for a finite index set* I*. Then* $\mathrm{ri}\left(\bigcap_{i \in I} C_i\right) = \bigcap_{i \in I} \mathrm{ri}(C_i)$.

(vi) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear function. Then* $\mathrm{ri}\, L(C) = L(\mathrm{ri}\, C)$*. If it also holds* $L^{-1}(\mathrm{ri}\, C) \neq \emptyset$*, we have* $\mathrm{ri}\, L^{-1}(C) = L^{-1}(\mathrm{ri}\, C)$.

(vii) $\mathrm{ri}(C_1 \times C_2) = \mathrm{ri}\, C_1 \times \mathrm{ri}\, C_2$

(viii) $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$ *if and only if* $0 \notin \mathrm{ri}(C_1 - C_2)$.

Proof. For a proof of (i)-(vi) we refer to [Roc70, Theorem 6.2 - 6.7].

To prove (vii) we use (iv). Let $(z_1, z_2) \in \mathrm{ri}(C_1 \times C_2)$. Then for all $(x_1, x_2) \in C_1 \times C_2$ there exists $t > 0$ such that

$$z_i + t(z_i - x_i) \in C_i \qquad \text{for all } i \in \{1, 2\}\,. \tag{2.2}$$

Using (iv) again, we get $\mathrm{ri}(C_1 \times C_2) \subseteq \mathrm{ri}\, C_1 \times \mathrm{ri}\, C_2$. Suppose $(z_1, z_2) \in \mathrm{ri}\, C_1 \times \mathrm{ri}\, C_2$. By (iv), for all $(x_1, x_2) \in C_1 \times C_2$ there exist $(t_1, t_2) > 0$ such that

$$z_i + t_i(z_i - x_i) \in C_i \qquad \text{for all } i \in \{1, 2\}\,. \tag{2.3}$$

If $t_1 = t_2$ we recover (2.2) from (2.3). By (iv) it holds $(z_1, z_2) \in \mathrm{ri}(C_1 \times C_2)$. If $t_1 < t_2$ we define $\theta := \frac{t_1}{t_2} \in (0, 1)$. Consider (2.3) with $i = 2$, together with $z_2 \in C_2$ and the convexity of C_2. It follows

$$z_2 + t_1(z_2 - x_2) \;=\; \theta \cdot (z_2 + t_2(z_2 - x_2)) \;+\; (1 - \theta) \cdot z_2 \in C_2\,. \tag{2.4}$$

Now we consider (2.4) and (2.3) with $i = 1$. This gives (2.2) with $t = t_1$. As before, it follows $(z_1, z_2) \in \mathrm{ri}(C_1 \times C_2)$. If $t_1 > t_2$ similar arguments lead to the same result. We have proven $\mathrm{ri}(C_1 \times C_2) \supseteq \mathrm{ri}\, C_1 \times \mathrm{ri}\, C_2$ and equality. [MMN22, Theorem 2.92] $\square$

We procede with convex separation results which are vital to the subsequent developments.

Definition. *Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^n$. A hyperplane H is said to* ***separate*** *C_1 and C_2 if C_1 is contained in one of the closed half-spaces associated with H and C_2 lies in the opposite closed half-space. It is said to separate C_1 and C_2* ***properly*** *if C_1 and C_2 are not both actually contained in H itselef.*

Theorem 2.1. (Convex separation in finite dimension) *Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^n$. Then C_1 and C_2 can be properly separated if and only if $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$.*

Proof. [Roc70, Theorem 11.3] $\square$

Definition. *Given a nonempty subset $\Omega \subseteq \mathbb{R}^n$, we define the* ***support function*** *of Ω to be*

$$\sigma_\Omega \,:\, \mathbb{R}^n \to \overline{\mathbb{R}}\,, \qquad x^* \;\mapsto\; \sup_{x \in \Omega} \langle x^*, x \rangle\,.$$

Definition 2.1. *Given functions* $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ *for* $i = 1, \ldots, m$, *we define the* ***infimal convolution*** *of these functions to be*

$$f_1 \Box \cdots \Box f_m : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad x \mapsto \inf \left\{ \sum_{i=1}^{m} f_i(x_i) : x_i \in \mathbb{R}^n \text{ and } \sum_{i=1}^{m} x_i = x \right\}.$$

The next result establishes a connection between the support function of the intersection of two convex sets and the infimal convolution of the support functions of the sets taken by themselfes. The proof translates the geometric concept of convex separation to the world of convex functions.

Lemma 2.1. *Let* C_1 *and* C_2 *be two non-empty convex sets in* $\mathbb{R}^n$. *For any* $x^* \in \operatorname{dom} \sigma_{C_1 \cap C_2}$ *the sets*

$$\begin{aligned} \Theta_1 &:= C_1 \times [0, \infty), \\ \Theta_2(x^*) &:= \{(x, \lambda) \in \mathbb{R}^n : x \in C_2 \textit{ and } \lambda \leq \langle x^*, x \rangle - \sigma_{C_1 \cap C_2}(x^*)\} \end{aligned}$$

can by properly separated.

Proof. We fix $x^* \in \operatorname{dom} \sigma_{C_1 \cap C_2}$ and write $\alpha := \sigma_{C_1 \cap C_2}(x^*)$. In order to apply convex separation in finite dimension (Theorem 2.1) to the sets Θ_1 and $\Theta_2(x^*)$, it suffics to show their convexity and $\operatorname{ri} \Theta_1 \cap \operatorname{ri} \Theta_2(x^*) = \emptyset$.

Convexity of Θ_1 and $\Theta_2(x^*)$

Clearly, Θ_1 is convex by the convexity of C_1 and $[0, \infty)$. To see that $\Theta_2(x^*)$ is convex consider the linear function

$$L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \qquad (x, \lambda) \mapsto \langle x^*, x \rangle - \lambda.$$

From the definitions of L and $\Theta_2(x^*)$ we get

$$\Theta_2(x^*) = (C_2 \times \mathbb{R}) \cap L^{-1}[\alpha, \infty).$$

Thus, by Proposition 2.1 (v) and the convexity of C_2 we get the convexity of $L^{-1}[\alpha, \infty)$ and with it that of $\Theta_2(x^*)$.

Relative interiors of Θ_1 and $\Theta_2(x^*)$ are disjoint

We start by calculating the relative interiors. It holds

$$\begin{aligned} \operatorname{ri} \Theta_1 &= \operatorname{ri}(C_1 \times [0, \infty)) = \operatorname{ri} C_1 \times \operatorname{ri} [0, \infty) = \operatorname{ri} C_1 \times (0, \infty), \\ \operatorname{ri} \Theta_2(x^*) &= \operatorname{ri}(L^{-1}[\alpha, \infty)) = L^{-1}(\operatorname{ri} [\alpha, \infty)) = L^{-1}(\alpha, \infty). \end{aligned}$$

Suppose there exists $(\lambda, x) \in \operatorname{ri}\Theta_1 \cap \operatorname{ri}\Theta_2(x^*)$. Then it holds $x \in C_1 \times C_2$ and $\lambda > 0$. We also note, that

$$\alpha \;=\; \sigma_{C_1 \cap C_2}(x^*) \;=\; \sup_{z \in C_1 \cap C_2} \langle x^*, z\rangle \;\geq\; \langle x^*, x\rangle \,.$$

Then it follows

$$\alpha \;<\; \langle x^*, x\rangle - \lambda \;\leq\; \alpha \,,$$

a contradiction. Thus, the relative interiors of Θ_1 and $\Theta_2(x^*)$ are disjoint.

Applying Theorem 2.1 finishes the proof. □

Theorem. *Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^n$ with $\operatorname{ri} C_1 \cap \operatorname{ri} C_2 \neq \emptyset$. Then the support function of the intersection $C_1 \cap C_2$ is represented as*

$$(\sigma_{C_1 \cap C_2})(x^*) = (\sigma_{C_1} \square\, \sigma_{C_2})(x^*) \qquad \textit{for all } x^* \in \mathbb{R}^n. \tag{2.5}$$

Furthermore, for any $x^* \in \operatorname{dom}(\sigma_{C_1 \cap C_2})$ there exist dual elements $x_1^*, x_2^* \in \mathbb{R}^n$ such that $x^* = x_1^* + x_2^*$. and

$$(\sigma_{C_1 \cap C_2})(x^*) = \sigma_{C_1}(x_1^*) + \sigma_{C_2}(x_2^*). \tag{2.6}$$

Proof. Using Lemma 2.1 the rest of the proof is as that of *[MMN22, Theorem 4.23(b)]*. □

Takeaways The support function intersection rule connects the geometric property of convex separation to an identity of support functions This result is central to the analysis of convex conjugates.

2.2 Conjugate Calculus

The goal of this section is to establish the tools to calculate convex conjugates. We cite the conjugate sum and chain rule without proof. After some examples, we cite the Fenchel-Rockafellar Theorem.

Definition 2.2. (Convex conjugate) *Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the* ***convex conjugate*** *$f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ of f is defined as*

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} (x^*)^T x - f(x) \tag{2.7}$$

Add comment on nomenclature. What is Legendre transformation in this context?

Note that f in Definition 2.2 does not have to be convex. On the other hand, the convex conjugate is always convex:

Proposition 2.2. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a proper function. Then its convex conjugate $f^* : \mathbb{R}^n \to (-\infty, \infty]$ is convex.*

Proof. [MMN22, Proposition 4.2] $\square$

Theorem 2.2. (Conjugate Chain Rule) *Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map (matrix) and $g : \mathbb{R}^n \to (-\infty, \infty]$ a proper convex function. If $Im(A) \cap ri(dom(g)) \neq \emptyset$ it follows*

$$(g \circ A)^*(x^*) = \inf_{y^* \in (A^*)^{-1}(x^*)} g^*(y^*). \tag{2.8}$$

Furthermore, for any $x^* \in dom(g \circ A)^*$ there exists $y^* \in (A^*)^{-1}(x^*)$ such that $(g \circ A)^*(x^*) = g^*(y^*)$.

Proof. [MMN22, Proposition 4.28] $\square$

Theorem 2.3. *Let $f, g : \mathbb{R}^n \to (-\infty, \infty]$ be proper convex functions and $ri\,(dom(f)) \cap ri\,(dom(g)) \neq \emptyset$. Then we have the conjugate sum rule*

$$(f + g)^*(x^*) = (f^* \Box g^*)(x^*) \tag{2.9}$$

for all $x^* \in \mathbb{R}^n$. Moreover, the infimum in $(f^* \Box g^*)(x^*)$ is attained, i.e., for any $x^* \in dom(f + g)^*$ there exists vectors x_1^*, x_2^* for which

$$(f + g)^*(x^*) = f^*(x_1^*) + g^*(x_2^*), \quad x^* = x_1^* + x_2^*. \tag{2.10}$$

Include lemma on convex conjugates of indicator functions. This should be straightforward.

Streamline example. Provide explanaition in the end. Confer [Roc70, bottom p.337]

Example. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a proper convex function, that is, $\operatorname{dom} f \neq \emptyset$ and f is convex. In steps we apply the conjugate chain and sum rule, together with mathematical induction, to prove the conjugate relationship

$$S_{f,n} : \mathbb{R}^n \to \overline{\mathbb{R}}, \qquad (x_1, \dots, x_n) \mapsto \sum_{i=1}^n f(x_i),$$
$$S_{f,n}^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \qquad (x_1^*, \dots, x_n^*) \mapsto \sum_{i=1}^n f^*(x_i^*)\,.$$

This relationship is very natural and the ensuing calculations serve to confirm our intuition.

First, we work in the projections on the coordinates. For the i-th coordinate, where $i = 1, \ldots, n$, this is

$$p_i : \mathbb{R}^n \to \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto x_i \,. \tag{2.11}$$

All projections p_i are linear function with matrix representation $e_i^\top$, where e_i is i-the coordinate vector. The adjoint of p_i is therefore

$$p_i^* : \mathbb{R} \to \mathbb{R}^n, \quad x \mapsto e_i \cdot x \,. \tag{2.12}$$

For the inverse image of the adjoint of p_i it holds

$$(p_i^*)^{-1} \{(x_1^*, \ldots, x_n^*)\} \;=\; \begin{cases} \{x_i^*\}\,, & \text{if } x_j^* = 0 \text{ for all } j \neq i\,, \\ \emptyset & \text{else.} \end{cases} \tag{2.13}$$

Throughout this example we use the asterisk character * somewhat inconsistently. Note that f^* is the convex conjugate of the function f and p_i^* is the adjoint linear function of the projection on the i-th coordinate. Likewise, we denote dual variables, that is, the arguments of convex conjugates, as x^*.

Next, we employ the conjugate chain rule to establish the conjugate relationship

$$\begin{aligned} f_i &: \mathbb{R}^n \to \overline{\mathbb{R}}, \quad (x_1, \ldots, x_n) \mapsto x_i \mapsto f(x_i)\,, \\ f_i^* &: \mathbb{R}^n \to \overline{\mathbb{R}}, \quad (x_1^*, \ldots, x_n^*) \mapsto \begin{cases} f^*(x_i^*), & \text{if } x_j^* = 0 \text{ for all } j \neq i\,, \\ \infty & \text{else.} \end{cases} \end{aligned}$$

Note, that $f_i = (f \circ p_i)$ and $f_i^* = (f \circ p_i)^*$. Since $\operatorname{Im} p_i = \mathbb{R}$ and $\operatorname{dom} f \neq \emptyset$, it holds $\operatorname{Im} p_i \cap \operatorname{ri}(\operatorname{dom} f) \neq \emptyset$. Then f and p_i conform with the demands of the conjugate chain rule. It follows

$$\begin{aligned} f_i^*(x_1^*, \ldots, x_n^*) \;&=\; (f \circ p_i)^*(x_1^*, \ldots, x_n^*) \;=\; \inf\left\{ f^*(y) \mid y \in (p_i^*)^{-1} \{(x_1^*, \ldots, x_n^*)\}\right\} \\ &= \begin{cases} f^*(x_i^*), & \text{if } x_j^* = 0 \text{ for all } j \neq i\,, \\ \infty & \text{else,} \end{cases} \end{aligned}$$

where we keep to the convention $\inf \emptyset = \infty$.

Next, note that for $n = 1$ we arrive at the result. Thus, for some $n \in \mathbb{N}$ it holds $(S_{f,n})^* = S_{f,n}^*$. In order to apply the conjugate sum rule to $S_{f,n}$ and f_{n+1} we note that

$$\begin{aligned} \operatorname{dom} f_i \;&=\; \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \in \operatorname{dom} f\right\} \;\neq\; \emptyset \qquad \text{for all } i = 1, \ldots, n+1\,, \\ \bigcap_{i=1}^{n+1} \operatorname{dom} f_i \;&=\; \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \in \operatorname{dom} f \text{ for all } i = 1, \ldots, n+1\right\} \;\neq\; \emptyset\,, \end{aligned}$$

and

$$\begin{aligned}\operatorname{ri}(\operatorname{dom} S_{f,n}) \;\cap\; \operatorname{ri}(\operatorname{dom} f_{n+1})\\ = \operatorname{ri}(\operatorname{dom} S_{f,n} \;\cap\; \operatorname{dom} f_{n+1}) \;=\; \operatorname{ri}\left(\bigcap_{i=1}^{n+1} \operatorname{dom} f_i\right) \;\neq\; \emptyset\,.\end{aligned}$$

By the conjugate sum rule it follows

$$\begin{aligned}(S_{f,n+1})^*(x_1^*,\ldots,x_{n+1}^*) = (S_{f,n}+f_{n+1})^*(x_1^*,\ldots,x_{n+1}^*) = ((S_{f,n})^*\square f_{n+1}^*)(x_1^*,\ldots,x_{n+1}^*)\\ (f_1^*\square\cdots\square f_{n+1}^*)(x_1^*,\ldots,x_{n+1}^*) = \sum_{i=1}^{n+1} f_i^*(x_i^*) = S_{f,n+1}^*(x_1^*,\ldots,x_{n+1}^*)\,.\end{aligned}$$

$\diamondsuit$

Find right moment to introduce nomenclature for optimization problem. See also end of Tseng Bertsekas chapter.

Given proper convex functions $f,g:\mathbb{R}^n\to\overline{\mathbb{R}}$ and a matrix $A\in\mathbb{R}^{n\times n}$, we define the primal minimization problem as follows:

Problem 2.1. *(Primal)* Given proper convex functions $f:\mathbb{R}^n\to\overline{\mathbb{R}}$, $g:\mathbb{R}^m\to\overline{\mathbb{R}}$ and a matrix $A\in\mathbb{R}^{m\times n}$ we define the **primal optimization problem** to be

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \qquad f(x)+g(Ax)$$

Remark. Problem 2.1 appears in the unconstrained form. We can impose constraints by controling for the domains of f and g. To incorporate linear constraints $Ax\le 0$ or more general constraints $x\in\Omega$, where Ω is a convex set, we can choose

$$g(x)=\delta_\Omega(x):= \tag{2.14}$$

where $x\notin\Omega$ leads to $f(x)+g(x)=\infty$ and the optimization problem (if feasible) will exclude x from the solutions. $\diamondsuit$

Problem 2.2. *(Dual)* Consider the same setting as in Problem 2.1. Using the convex conjugates of f,g and the transpose of A we define the **dual problem** of Problem 2.1 to be

$$\underset{y^*\in\mathbb{R}^m}{\text{maximize}} \qquad -f^*(A^\top y^*)-g^*(y^*).$$

Theorem 2.4. *Let f and $g:\mathbb{R}^n\to\overline{\mathbb{R}}$ be proper convex functions and*

$$\operatorname{ri}(\operatorname{dom} g)\cap\operatorname{ri}(A\operatorname{dom} f)\;\neq\;\emptyset\,.$$

Then the optimal values of (2.1) *and* (2.2) *are equal, that is,*

$$\inf_{x\in\mathbb{R}^n}\{f(x)\;+\;g(Ax)\} \;=\; \sup_{y\in\mathbb{R}^n}\left\{-f^*\left(A^Ty\right)\;-\;g^*(-y)\right\}\,.$$

Proof. [MMN22, Theorem 4.63] □

Insert lemma in chapter 1.

Lemma 2.2. *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be convex. Then for all* $y \in \mathbb{R}^n$ *and* $\varepsilon > 0$

$$\inf_{\|\Delta\|=\varepsilon} f(y+\Delta) - f(y) \geq 0 \tag{2.15}$$

implies the existence of a global minimum $y^* \in \mathbb{R}^n$ *of* f *satisfying* $\|y^* - y\| \leq \varepsilon$ *.*

Proof. Since $y + \varepsilon B$ is convex, it contains a local minimum of f . Suppose towards a contradiction that $y^* \in y + \varepsilon B$ is a local minimum, but not a global one, and (2.15) is true. Then it holds

$$f(x) < f(y^*) \quad \text{for some } x \in \mathbb{R}^n \setminus (y + \varepsilon B) \,. \tag{2.16}$$

Furthermore, since $y + \varepsilon B$ is compact and contains y^* , the line segment connecting y^* and x intersects the boundary of $y + \mathcal{C}$, that is, there exist $\theta \in (0,1)$ and Δ_x with $\|\Delta_x\| = \varepsilon$ such that

$$\theta x + (1-\theta) y^* = y + \Delta_x \,. \tag{2.17}$$

It follows

$$\begin{aligned} f(y^*) \leq f(y) \leq f(y+\Delta_x) &= f(\theta x + (1-\theta)y^*) \\ &\leq \theta f(x) + (1-\theta) f(y^*) < f(y^*), \end{aligned} \tag{2.18}$$

which is a contradiction. The first inequality is due to y^* being a local minimum of f in $y + \varepsilon B$, the second inequality is due to (2.15) being true, the equality is due to (2.17), the third inequality is due to the convexity of f and the strict inequality is due to (2.16). Thus every local minimum of f in $y + \varepsilon B$ is also a global minimum. □

Takeaways Conjugate sum and chain rule are direct consequences of the support function intersection rule. They are powerful tools, that allow us to compute convex conjugates of difficult expressions as well as proving the Fenchel-Rockafellar Duality theorem.

2.3 Tseng Bertsekas

We present the relevant parts of the paper [BT03].

Problem 2.3.

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & \mathbf{A}x \;\geq\; b\,. \end{array}$$

Assumptions. *Assume that the map* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *has the following properties.*

(i) f *is strictly convex.*

(ii) f *is lower-semicontinuous and continuous on* $\operatorname{dom} f$.

(iii) The convex conjugate f^* *of* f *is finite.*

The dual optimization problem of Problem 2.3 is

Problem 2.4.

$$\begin{array}{ll} \underset{y \in \mathbb{R}^m}{\text{maximize}} & \langle y, b\rangle \;-\; f^*(A^\top y) \\ \text{subject to} & y \;\geq\; 0\,. \end{array}$$

Let q denote the objective function of Problem 2.4, that is,

$$q \,:\, \mathbb{R}^m \to\; \mathbb{R}\,, \qquad y \;\mapsto\; \langle y, b\rangle \;-\; f^*(A^\top y)\,. \tag{2.19}$$

q is concave. The dual problem (D) is a concave program with nonnegativity constraints on the dual variable y_a of the inequality constraints in (P). Furthermore, strong duality holds for (P) and (D), that is, they have the same optimal value.

Since f^* is real-valued and f is strictly convex, f^* and q are continuously differentiable ().

Theorem. *A closed proper convex function is strictly convex if and only if its conjugate is continuously differentiable.*

Proof. cf. [Roc70, Theorem 26.3] semicontinuous and cofinite is closed [Ber09, Proposition 1.1.3] □

We will denote the gradient of q at p by $d(p)$ and its i th coordinate by $d_i(p)$. Since q is continuously differentiable, $d_i(p)$ is continuous.

By differentiating and using the chain rule, we obtain the dual cost gradient

$$d(p) = b - \mathbf{A}x, \quad \text{where} \quad x := \nabla f^*(\mathbf{A}^\top p)\,. \tag{2.20}$$

The last equality follows from Danskin's Theorem and [Roc70, Theorem 23.5]

Proposition. (Danskin's Theorem [BT03, page 649]) *Let $Z \subseteq \mathbb{R}^m$ be a non-empty set, and let $\phi : \mathbb{R}^n \times Z \to \mathbb{R}$ be a continuous function such that $\phi(\cdot, z) : \mathbb{R}^n \to \mathbb{R}$, viewed as a function of its first argument, is convex for each $z \in Z$. Consider the function*

$$f^* : \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto \sup_{z \in Z} \phi(x, z) \tag{2.21}$$

and the set.

$$Z(x) := \{\overline{z} \in \mathbb{R}^m \colon \phi(x, \overline{z}) = f^*(x)\} . \tag{2.22}$$

If $Z(x)$ consists of a unique point $\overline{z}$ and $\phi(\cdot, \overline{z})$ is differentiable at x, then

$$\nabla f^*(x) = \nabla_x \phi(x, \overline{z}) \tag{2.23}$$

where $\nabla_x \phi(x, \overline{z})$ is the vector with coordinates $(\partial\phi/\partial x_i)(x, \overline{z})$.

Theorem. *For any proper convex function f and any vector x , it holds $t \in \partial f(x)$ if and only if $\langle \cdot, t\rangle - f(\cdot)$ achieves its supremum at x .*

Example. We choose f^* to be the convex conjugate of f , that is,

$$f^*(t) = \sup_{x \in \mathbb{R}^n} \langle x, t\rangle - f(x) , \tag{2.24}$$

and accordingly

$$\phi(t, x) = \langle x, t\rangle - f(x) . \tag{2.25}$$

By the strict convexity of f there exists a unique global maximum $\overline{z}$ of $\phi(t, \cdot)$. Since f^* is differentiable by Theorem it follows from the Proposition

$$\nabla f^*(\mathbf{A}^\top p) = \nabla_{\mathbf{A}^\top p} \Big(\langle \mathbf{A}^\top p, \overline{z}\rangle - f(\overline{z}) \Big) = \overline{z} . \tag{2.26}$$

Then by Theorem it follows

$$\mathbf{A}^\top p \in \partial f(\overline{z}) . \tag{2.27}$$

$\diamondsuit$

In the following let p be an optimal solution to Problem 2.4.

Complementary Slackness

We show that for all $i \in \{1, \dots, n\}$ it holds

$$\begin{aligned} \text{either} \qquad p_i = 0 \quad \text{and} \quad d_i(p) \leq 0 \\ \text{or} \qquad p_i > 0 \quad \text{and} \quad d_i(p) = 0\,. \end{aligned}$$

Assume towards a contradiction that $d_i(p) > 0$ for some $i \in \{1, \dots, n\}$. Consider the μ perturbation of p in the i-th coordinate, that is,

$$\tilde{p}(\mu) := p + e_i \cdot \mu\,. \tag{2.28}$$

By the continuity of d_i there exists $\mu > 0$ such that $d_i(\tilde{p}(\mu)) > 0$. By the concavity of q it follows

$$q(\tilde{p}(\mu)) - q(p) \geq d_i(\tilde{p}(\mu)) \cdot \mu > 0\,. \tag{2.29}$$

Since $\tilde{p}(\mu)) \geq 0$ this is a contradiction to the optimality of p in Problem 2.4. Thus $d_i(p) \leq 0$ for all $i \in \{1, \dots, n\}$. Now assume that $p_i > 0$ and $d_i(p) < 0$ for some $i \in \{1, \dots, n\}$. Again, by the continuity of d_i there exists $\mu < 0$ such that $d_i(\tilde{p}(\mu)) < 0$ and $p_i + \mu > 0$. But then it follows (2.29). We have shown the complementary slackness of p and $d(p)$.

To obtain the primal solution by a dual solution we use results of Karush-Kuhn-Tucker.

Definition. [Roc70, §28] *By an* ***ordinary convex program*** (P) *we mean an optimization problem of the following form*

$$\underset{x \in C}{\text{minimize}} \qquad f_0(x)$$

subject to the constraints

$$f_1(x) \leq 0, \dots, f_r(x) \leq 0, \tag{2.30}$$

where $C \subseteq \mathbb{R}^n$ *is a non-empty convex set and* f_i *is a finite convex function on* C *for* $i \in \{1, \dots, r\}$.

Theorem 2.5. (Karush-Kuhn-Tucker conditions) *Let* (P) *be an ordinary convex program,* $\overline{\alpha} \in \mathbb{R}^m$, *and* $\overline{z} \in \mathbb{R}^n$. *Then* $\overline{z}$ *is an optimal solution to* (P) *if* $\overline{z}$ *and the components* α_i *of* $\overline{\alpha}$ *satisfy the following conditions.*

(i) $\alpha_i \geq 0$, $f_i(\overline{z}) \leq 0$, *and* $\alpha_i f_i(\overline{z}) = 0$ *for all* $i \in \{1, \dots, r\}$.

(ii) $0_n \in [\partial f_0(\overline{z}) + \sum_{\alpha_i \neq 0} \alpha_i \partial f_i(\overline{z})]$.

Proof. [Roc70, Theorem 28.3] □

Karush-Kuhn-Tucker conditions

Problem 2.3 is an ordinary convex program with $f_0 = f$ and $f_i(x) = b_i - \sum_{j=1}^{n} a_{ij} x_j$. From (2.20) it follows $f_i(x) = d_i(p) \leq 0$. From the complementary slackness it follows $p_i \cdot f_i(x) = p_i \cdot d_i(p) = 0$. Let $x^* \in \mathbb{R}^n$. It holds

$$\langle -\mathbf{A}^\top e_i, x^* - x \rangle = \langle e_i, \mathbf{A}x \rangle - \langle e_i, \mathbf{A}x^* \rangle = f_i(x^*) - f_i(x) . \tag{2.31}$$

Thus $-\mathbf{A}^\top e_i$ is a subgradient of f_i at x, that is, $-\mathbf{A}^\top e_i \in \partial f_i(x)$. By $\mathbf{A}^\top p \in \partial f(x)$ it follows

$$0 = \mathbf{A}^\top p - \sum_{p_i \neq 0} p_i \mathbf{A}^\top e_i \in [\partial f_0(\overline{x}) + \sum_{p_i \neq 0} p_i \partial f_i(\overline{x})] . \tag{2.32}$$

Thus, by Theorem 2.5,

$$x = \nabla f^*(\mathbf{A}^\top p) \tag{2.33}$$

is an optimal solution to Problem 2.3. Since f is strictly convex, this solution is the only one.

Takeaways Employing the Karush-Kuhn-Tucker conditions, we derive a dual relationship between optimal solutions for strictly convex functions.

3 Simple yet useful Calculations

Theorem 3.1. (Multivariate Taylor Theorem) *Let* $f \in C^2(\mathbb{R}^n, \mathbb{R})$ *. Then for all* $x, \Delta \in \mathbb{R}^n$ *there exists* $\xi \in [0,1]$ *such that it holds*

$$\begin{aligned} f(x+\Delta) = f(x) &+ \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i}\Delta_i + \sum_{\substack{i,j=1 \\ i\neq j}} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i \partial x_j}\Delta_i\Delta_j \\ &+ \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i^2}\Delta_i^2 \end{aligned} \tag{3.1}$$

Corollary 3.1.1. *Let* $f \in C^2(\mathbb{R})$ *. Then for all* $a, x, \Delta \in \mathbb{R}^n$ *there exist* $\xi \in [0,1]$ *such that it holds*

$$f(a^T(x+\Delta)) - f(a^T x) = f^{'}(a^T x)\, \Delta^T a + \frac{1}{2} f^{''}(a^T(x+\xi\Delta))\, \Delta^T A\ \Delta, \tag{3.2}$$

where $A := aa^T \in \mathbb{R}^{n\times n}$ *.*

Proof. By the chain rule we have for all $a, x, \Delta \in \mathbb{R}^n$ and $\xi \in [0,1]$

$$\frac{\partial^2 f(a^T(x+\xi\Delta))}{\partial x_i \partial x_j} = f^{''}(a^T(x+\xi\Delta))\, a_i a_j. \tag{3.3}$$

Since $A := aa^T$ is symmetric we have

$$\Delta^T A\ \Delta = 2\sum_{\substack{i,j=1 \\ i\neq j}}^{n} a_i a_j \Delta_i \Delta_j + \sum_{i=1}^{n} a_i^2 \Delta_i^2. \tag{3.4}$$

Plugging (3.3) and (3.4) into (3.1) yields (3.2). □

Proposition 3.1. *For all* $x, y \in \mathbb{R}$ *it holds*

$$|x+y| - |x| \geq -|y| \tag{3.5}$$

Proof. Checking all 6 combinations of $x+y, x, y$ being nonnegative or negative yields the result. □

Notation Index

$\#A$ cardinality of the set A

$\mathbf{E}[X|Y]$ conditional expectation of the random variable X with respect to $\sigma(Y)$

$\mathbf{E}[X]$ expectation of the random variable X

$\mathbf{Var}[X]$ variance of the random variable X

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ extension of the real numbers

$\xrightarrow{\mathcal{D}}$ convergence of distributions

$\mathbf{P}$ generic probability measure

$\mathbf{P}_X = \mathbf{P} \circ X^{-1}$ distribution of the random variable X

$\mathbb{R}$ set of real numbers

$x \vee y, x \wedge y, x^+, x^-$ maximum, minimum, positive part, negative part of real numbers

$X \sim \mu$ the random variable has distribution μ

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