Robust Weighting and Matching Techniques for Causal Inference in Observational Studies with Continuous Treatment

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2.1 Introduction

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Assumption 2.1. *Assume that the map $f : \mathbb{R} \to \overline{\mathbb{R}}$ has the following properties.*

(i) f *is strictly convex.*

(ii) f *is lower-semicontinuous and continuously differentiable on* $\mathrm{int}(\mathrm{dom}(f))$.

(iii) The derivative of f *on* $\mathrm{int}(\mathrm{dom}(f))$ *is a diffeomorphism.*

(iv) The Legendre transformation f^* *of* f *is finite.*

(v) The function $x \mapsto xt - f(x)$ *takes its supremum on* $\mathrm{int}(\mathrm{dom}(f))$ *for all* $t \in \mathbb{R}$.

We consider the following optimization problem.

Problem 2.1.

$$\underset{w_1,\ldots,w_n \in \mathbb{R}}{\text{minimize}} \qquad \sum_{i=1}^{n} T_i f(w_i)$$

subject to the constraints

$$w_i T_i \geq 0, \qquad i = 1, \ldots, n,$$
$$\sum_{i=1}^{n} w_i T_i = 1$$
$$\left| \sum_{i=1}^{n} w_i T_i B_k(X_i) - \frac{1}{n} \sum_{i=1}^{n} B_k(X_i) \right| \leq \delta_k, \qquad k = 1, \ldots, K$$

Theorem 2.1. *Under Assumption, the dual of the above Problem is the unconstrained optimization problem*

$$\underset{\lambda \in \mathbb{R}^K}{\text{minimize}} \qquad \frac{1}{n}\sum_{i=1}^{n} nT_i f^*(\langle B(X_i), \lambda\rangle) - \langle B(X_i), \lambda\rangle + \langle \delta, |\lambda|\rangle,$$

where $t \mapsto f^*(t) = t\,(f^{'})^{-1}(t) - f\left((f^{'})^{-1}(t)\right)$ *is the Legendre transformation of* f, $B(X_i) = [B_1(X_i), \ldots, B_K(X_i)]^\top$ *denotes the* K *basis functions of the covariates of unit* $i \in \{1, \ldots, n\}$ *and* $|\lambda| = [|\lambda_1|, \ldots, |\lambda_K|]^\top$, *where* $|\cdot|$ *is the absolute value of a real-valued scalar. Moreover, if* $\lambda^\dagger$ *is an optimal solution then*

$$w_i^* = (f^{'})^{-1}(\langle B(X_i), \lambda^\dagger\rangle), \quad i \in \{1, \ldots, n\} \tag{2.1}$$

are the unique optimal solutions to (P) .

Proof. We prove the following Lemma at the end of the section.

Lemma 2.1. *The dual of the optimization problem is*

$$\underset{\lambda \in \mathbb{R}^{2K}}{\text{minimize}} \qquad \frac{1}{n}\sum_{i=1}^{n} nT_i f^*(\langle Q_{\bullet i}, \lambda\rangle) - \langle Q_{\bullet i}, \lambda\rangle + \langle d, \lambda\rangle$$

subject to

$$\lambda_k \geq 0 \quad \textit{for all } k \in \{1, \ldots, K\}, \tag{2.2}$$

where

$$\mathbf{Q} := \begin{bmatrix} \mathbf{I}_n \\ \mathbf{B}(\mathbf{X}) \\ -\mathbf{B}(\mathbf{X}) \end{bmatrix}, \qquad \mathbf{B}(\mathbf{X}) := \left[B(X_1), \ldots, B(X_n)\right], \qquad \textit{and} \qquad d := \begin{bmatrix} 0_n \\ \delta \\ \delta \end{bmatrix}. \tag{2.3}$$

□

2.4 Application of Matrix Concentration Inequalities

3 Convex Analysis

3.1 Basic Notions

3.2 Relative Interior

3.3 Conjugate Calculus

3.4 Tseng Bertsekas

We present the relevant parts of the paper [BT03].

Consider the following optimization problem

$$\underset{x\in\mathbb{R}^m}{\text{minimize}} \qquad f(x)$$

subject to the constraints

$$\mathbf{A}x \geq b, \tag{3.1}$$

Where $f : \mathbb{R}^m \to \overline{\mathbb{R}}$, $\mathbf{A}$ is a given $n \times m$ matrix, and b is a vector in $\mathbb{R}^n$.

Assumption 3.1. *Assume that the map $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ has the following properties.*

(i) f is strictly convex.

(ii) f is lower-semicontinuous and continuous $\mathrm{dom}(f)$.

(iii) The convex conjugate f^* of f is finite.

The dual optimization problem associated with (P) is

$$\underset{p\in\mathbb{R}^n}{\text{maximize}} \qquad q(p)$$

subject to the constraints

$$p \geq 0, \tag{3.2}$$

where $q : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the concave function given by

$$q(p) := \min_{x\in\mathbb{R}^m} f(x) + \langle p, b - \mathbf{A}x\rangle = \langle p, b\rangle - f^*(\mathbf{A}^\top p). \tag{3.3}$$

The dual problem (D) is a concave program with simple nonnegativity constraints. Furthermore, strong duality holds for (P) and (D) , i.e., the optimal value of (P) equals the optimal value of (D).

Since f^* is real-valued and f is strictly convex, f^* and q are continuously differentiable.

Theorem 3.1. [Roc70, Theorem 26.3] *A closed proper convex function is (essentially) strictly convex if and only if its conjugate is essentially smooth.*

We will denote the gradient of q at p by $d(p)$ and its ith coordinate by $d_i(p)$. Since q is continuously differentiable, $d_i(p)$ is continuous, and since q is concave, $d_i(p)$ as nonincreasing in p_i.

By differentiating and by using the chain rule, we obtain the dual cost gradient

$$d(p) = b - \mathbf{A}x, \quad \text{where} \quad x := \nabla f^*(\mathbf{A}^\top p) = \operatorname{argsup}_{\xi \in \mathbb{R}^m} \langle p, \mathbf{A}\xi \rangle - f(\xi). \tag{3.4}$$

The last equality follows from Danskin's Theorem and [Roc70, Theorem 23.5]

Proposition 3.1. (Danskin's Theorem [BT03, page 649]) *Let $Z \subseteq \mathbb{R}^m$ be a non-empty set, and let $\phi : \mathbb{R}^n \times Z \to \mathbb{R}$ be a continuous function such that $\phi(\cdot, z) : \mathbb{R}^n \to \mathbb{R}$, viewed as a function of its first argument, is convex for each $z \in Z$. Then the function*

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto \sup_{z \in Z} \phi(x, z) \tag{3.5}$$

is convex and has directional derivative given by

$$f^{'}(x; y) = \sup_{z \in Z(x)} \phi^{'}(x, z; y), \tag{3.6}$$

where $\phi^{'}(x, z; y)$ is the directional derivative of the function $\phi(\cdot, z)$ at x in the direction y, and

$$Z(x) := \left\{ \overline{z} \in \mathbb{R}^m \colon \phi(x, \overline{z}) = \sup_{z \in Z} \phi(x, z) \right\}. \tag{3.7}$$

In particular, if $Z(x)$ consists of a unique point $\overline{z}$ and $\phi(\cdot, \overline{z})$ is differentiable at x, and $\nabla f(x) = \nabla_x \phi(x, \overline{z})$, where $\nabla_x \phi(x, \overline{z})$ is the vector with coordinates $(\partial\phi/\partial x_i)(x, \overline{z})$

Note that x is the unique vector satisfying

$$\mathbf{A}p \in \partial f(x). \tag{3.8}$$

From the optimality conditions for (D) it follows that a dual vector is an optimal solution of (D) if and only if

$$p = [p + d(p)]^+, \tag{3.9}$$

where $[\ \cdot\ ]^+$ is the projection onto the positive orthant, i.e., $[y]^+ = [0 \vee y_1, \dots 0 \vee y_n,]^\top$.

Given an optimal dual solution p, we may obtain an optimal primal solution from the equation $x = \nabla f^*(\mathbf{A}^\top p)$. To see this, note that

$$\mathbf{A}x \geq b \quad \text{and} \quad p_i = 0 \quad \text{for all } i \text{ such that} \quad \sum_{j=1}^m a_{ij} x_j > b_i. \tag{3.10}$$

We can show that p and x satisfy the KKT conditions and thus x is an optimal solution to (P).

Definition 3.1. [Roc70, §28] *By an* ***ordinary convex program*** (P) *we mean an optimization problem of the following form*

$$\underset{x \in C}{\text{minimize}} \qquad f_0(x)$$

subject to the constraints

$$f_1(x) \leq 0, \dots, f_r(x) \leq 0, \qquad f_{r+1}(x) = 0, \dots, f_m(x) = 0, \tag{3.11}$$

where $C \subseteq \mathbb{R}^n$ is a non-empty convex set, f_i is a finite convex function on C for $i \in \{1, \dots, r\}$ and f_i is an affine function on C for $i \in \{r+1, \dots, m\}$.

Definition 3.2. *We define* $[\lambda_1, \ldots, \lambda_m] \in \mathbb{R}^m$ *to be a* ***Karush-Kuhn-Tucker (KKT) vector*** *for* (P)*, if*

(i) $\lambda_i \geq 0$ *for all* $i \in \{1, \ldots, r\}$.

(ii) The infimum of the proper convex function $f_0 + \sum_{i=1}^{m} \lambda_1 f_i$ *is finite and equal to the optimal value in* (P).

Theorem 3.2. (Karush-Kuhn-Tucker conditions) *Let* (P) *be an ordinary convex program,* $\overline{\alpha} \in \mathbb{R}^m$*, and* $\overline{z} \in \mathbb{R}^n$*. Then* $\overline{\alpha}$ *is a KKT vector for* (P) *and* $\overline{z}$ *is an optimal solution to* (P) *if and only if* $\overline{z}$ *and the components* α_i *of* $\overline{\alpha}$ *satisfy the following conditions.*

(i) $\alpha_i \geq 0,\ f_i(\overline{z}) \leq 0,$ *and* $\alpha_i f_i(\overline{z}) = 0$ *for all* $i \in \{1, \ldots, r\}$.

(ii) $f_i(\overline{z}) = 0$ *for* $i \in \{r+1, \ldots, m\}$.

(iii) $0_n \in [\partial f_0(\overline{z}) + \sum_{\alpha_i \neq 0} \alpha_i \partial f_i(\overline{z})]$.

Proof. [Roc70, Theorem 28.3] □

Takeaways For strictly convex functions we can derive duality in terms of the optimal solutions.

4 Random Matrix Inequalities

4.1 Matrix Analysis

The **trace** of a square matrix, denoted by tr, is the sum of its diagonal entries, i.e. $\operatorname{tr}(\mathbf{B}) = \sum_{j=1}^{d} b_{jj}$ for $\mathbf{B} \in \mathbb{M}_d$. The trace is unitarily invariant, i.e. $\operatorname{tr}(\mathbf{B}) = \operatorname{tr}(\mathbf{QBQ}^*)$ for all $\mathbf{B} \in \mathbb{M}_d$ for all unitary $\mathbf{Q} \in \mathbb{M}_d$. In particular, the existence of an eigenvalue value decomposition shows that the trace of a Hermitian matrix equals the sum of its eigenvalues. Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Consider a matrix $\mathbf{A} \in \mathbb{H}_d$ whose eigenvalues are contained in I. We define the matrix $f(\mathbf{A}) \in \mathbb{H}_d$ using an eigenvalue decomposition of $\mathbf{A}$:

$$f(\mathbf{A}) = \mathbf{Q} \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_d) \end{bmatrix} \mathbf{Q}^* \qquad \text{where} \qquad \mathbf{A} = \mathbf{Q} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} \mathbf{Q}^* = \sum_{i=1}^{d} \lambda_i \mathbf{Q}_{\bullet i} \mathbf{Q}_{\bullet i}^*. \tag{4.1}$$

The definition of $f(\mathbf{A})$ does not depend on which eigenvalue decomposition we choose. Any matrix function that arises in this fashion is called a **standard matrix function**.

Proposition 4.1. *Let $f, g : I \to \mathbb{R}$ be real-valued functions on an interval $I \subseteq \mathbb{R}$, and let $\mathbf{A} \in \mathbb{H}_d$ be a Hermitian matrix whose eigenvalues are contained in I.*

(i) If λ is an eigenvalue of of $\mathbf{A}$, then $f(\lambda)$ is an eigenvalue of $f(\mathbf{A})$.

(ii) $f(a) \leq g(a)$ for all $a \in I$ implies $f(\mathbf{A}) \preccurlyeq g(\mathbf{A})$.

Lemma 4.1. (Mean value trace inequality) *Let I be an interval of the real line. Suppose that $g : I \to \mathbb{R}$ is a weakly increasing function and that $h : I \to \mathbb{R}$ is a function whose derivative h' is convex. Then for all matrices $\mathbf{A}, \mathbf{B} \in \mathbb{H}_d(I)$ it holds*

$$\overline{\operatorname{tr}}[(g(\mathbf{A}) - g(\mathbf{B})) \cdot (h(\mathbf{A}) - h(\mathbf{B}))] \leq \frac{1}{2} \overline{\operatorname{tr}}[(g(\mathbf{A}) - g(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \cdot (h'(\mathbf{A}) + h'(\mathbf{B}))]. \tag{4.2}$$

When h' is concave, the inequality is reversed. The same result holds for the standard trace.

Proof. *[MJC+14, Lemma 3.4]* Fix $a, b \in I$. Since g is weakly increasing, $(g(a) - g(b)) \cdot (a - b) \geq 0$. The fundamental theorem of calculus and the convexity of h' yield the estimate

$$\begin{aligned} (g(a) - g(b)) \cdot (h(a) - h(b)) &= (g(a) - g(b)) \cdot (a - b) \int_0^1 h'(\tau a + (1 - \tau) b) \mathrm{d}\tau && (4.3) \\ &\leq (g(a) - g(b)) \cdot (a - b) \int_0^1 [\tau h'(a) + (1 - \tau) h'(b)] \mathrm{d}\tau && (4.4) \\ &= \frac{1}{2} [(g(a) - g(b)) \cdot (a - b) \cdot (h'(a) + h'(b))]. && (4.5) \end{aligned}$$

The inequality is reversed, if $h^{'}$ is concave. To apply the Kleins inequality we expand the terms. The RHS is

$$\begin{aligned}
&(g(a)-g(b))\cdot(a-b)\cdot(h^{'}(a)+h^{'}(b)) \\
&\quad=[g(a)\cdot a\cdot h^{'}(a)]+[g(a)\cdot a]\cdot h^{'}(b)-b\cdot[h^{'}(a)\cdot g(a)]-[b\cdot h^{'}(b)]\cdot g(a) \\
&\qquad+[\text{ the same as above with } a \text{ and } b \text{ reversed }](a\rightleftarrows b)
\end{aligned}\tag{4.6}$$

Taking the trace yields

$$\begin{aligned}
&\operatorname{tr}[g(\mathbf{A})\cdot\mathbf{A}\cdot(h^{'}(\mathbf{A})+h^{'}(\mathbf{B}))]-\operatorname{tr}[\mathbf{B}\cdot(h^{'}(\mathbf{A})+h^{'}(\mathbf{B}))\cdot g(\mathbf{A})]+(\mathbf{A}\rightleftarrows\mathbf{B}) \\
&\quad=\operatorname{tr}[g(\mathbf{A})\cdot\mathbf{A}\cdot(h^{'}(\mathbf{A})+h^{'}(\mathbf{B}))]-\operatorname{tr}[g(\mathbf{A})\cdot\mathbf{B}\cdot(h^{'}(\mathbf{A})+h^{'}(\mathbf{B}))]+(\mathbf{A}\rightleftarrows\mathbf{B}) \\
&\quad=\operatorname{tr}[g(\mathbf{A})\cdot(\mathbf{A}-\mathbf{B})\cdot(h^{'}(\mathbf{A})+h^{'}(\mathbf{B}))]+(\mathbf{A}\rightleftarrows\mathbf{B}) \\
&\quad=\operatorname{tr}[(g(\mathbf{A})-g(\mathbf{B}))\cdot(\mathbf{A}-\mathbf{B})\cdot(h^{'}(\mathbf{A})+h^{'}(\mathbf{B}))].
\end{aligned}\tag{4.7}$$

On the LHS we have only products of two factors which commute under the trace operation. Thus we may use the same expression as in the scalar case without further calculations. The result follows immediately from the Klein inequality. □

Proposition 4.2. (Generalized Klein inequality) *Let $u_1,\dots,u_n$ and $v_1,\dots,v_n$ be real-valued functions on an interval I of the real line. Suppose*

$$\sum_{k=1}^{n} u_k(a)v_k(b)\geq 0 \qquad \textit{for all } a,b\in I. \tag{4.8}$$

Then

$$\overline{\operatorname{tr}}\left(\sum_{k=1}^{n} u_k(\mathbf{A})v_k(\mathbf{B})\right)\geq 0 \qquad \textit{for all } \mathbf{A},\mathbf{B}\in\mathbb{H}_d(I). \tag{4.9}$$

Proof. *[Pet94, Proposition 3]* Let $\mathbf{A}=\sum_{i=1}^{d}\lambda_i\mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^{*}$ and $\mathbf{B}=\sum_{j=1}^{d}\mu_j\mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^{*}$ be the orthonormal decompositions of $\mathbf{A}$ and $\mathbf{B}$. Then

$$\overline{\operatorname{tr}}\left(\sum_{k=1}^{n} u_k(\mathbf{A})v_k(\mathbf{B})\right)=\sum_{k=1}^{n}\sum_{i,j=1}^{d}\overline{\operatorname{tr}}\left(u_k(\lambda_i)\mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^{*}v_k(\mu_j)\mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^{*}\right) \tag{4.10}$$

$$=\sum_{i,j=1}^{d}\overline{\operatorname{tr}}\left(\mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^{*}\mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^{*}\right)\sum_{k=1}^{n}u_k(\lambda_i)v_k(\mu_j)\geq 0 \tag{4.11}$$

by the hypothesis. To see that $\operatorname{tr}\left(\mathbf{P}_{\bullet i}\mathbf{P}_{\bullet i}^{*}\mathbf{Q}_{\bullet j}\mathbf{Q}_{\bullet j}^{*}\right)$ is non-negative for all $i,j\in\{1,\dots,d\}$, we apply a well known extension of von Neumann's trace inequality [Ruh70, Lemma 1], namely

$$\operatorname{tr}(\mathbf{PQ})\geq\sum_{i=1}^{d}p_i q_{d-i+1}\geq 0 \qquad \text{for all } \mathbf{P},\mathbf{Q}\in\mathbb{H}_d([0,\infty)), \tag{4.12}$$

where the eigenvalues $p_1\geq\ldots\geq p_d$ and $q_1\geq\ldots\geq q_d$ are sorted decreasingly. □

Proposition 4.3. (Hölder inequality for trace) *Let p and q be Hölder conjugate indices. Then*

$$\operatorname{tr}(\mathbf{BC})\leq\|\mathbf{B}\|_p\|\mathbf{C}\|_q \qquad \textit{for all } \mathbf{B},\mathbf{C}\in\mathbb{M}_d. \tag{4.13}$$

Proof. [Bha97, Corollary IV.2.6] □

4.2 Matrix Concentration Inequalities via the Method of Exchangeable Pairs

Definition 4.1. *Let Z and $Z^{'}$ random variables taking values in a Polish space $\mathcal{Z}$. We say that $(Z, Z^{'})$ is an **exchangable pair** if it has the same distribution as $(Z^{'}, Z)$. In particular, Z and $Z^{'}$ must share the same distribution.*

Definition 4.2. *Let $(Z, Z^{'})$ be an exchangable pair of random variables taking values in a Polish space $\mathcal{Z}$, and let $\mathbf{\Psi} : \mathcal{Z} \to \mathbb{H}_d$ be a measurable function. Define the random Hermitian matrices*

$$\mathbf{X} := \mathbf{\Psi}(Z) \quad \text{and} \quad \mathbf{X}^{'} := \mathbf{\Psi}(Z^{'}). \tag{4.14}$$

*We say that $(\mathbf{X}, \mathbf{X}^{'})$ is a **matrix Stein pair** if there is a constant $\alpha \in (0, 1]$ for which*

$$\mathbf{E}[\mathbf{X} - \mathbf{X}^{'} | Z] = \alpha \mathbf{X} \qquad \text{almost surely.} \tag{4.15}$$

*The constant α is called the **scale factor** of the pair. We always assume $\mathbf{E}\left[\|\mathbf{X}\|^2\right] < \infty$.*

Lemma 4.2. *Suppose that $(\mathbf{X}, \mathbf{X}^{'})$ is a matrix Stein pair with scale factor α. Let $\mathbf{F} : \mathbb{H}_d \to \mathbb{H}_d$ be a measurable function that satisfies the regularity condition $\mathbf{E}\left[\left\|(\mathbf{X} - \mathbf{X}^{'})\mathbf{F}(\mathbf{X})\right\|\right] < \infty$. Then*

$$\mathbf{E}[\mathbf{X} \cdot \mathbf{F}(\mathbf{X})] = \frac{1}{2\alpha} \mathbf{E}[(\mathbf{X} - \mathbf{X}^{'})(\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}^{'}))]. \tag{4.16}$$

Proof. *[MJC+14, Lemma 2.4]* Suppose that $(\mathbf{X}, \mathbf{X}^{'})$ constructed from an auxiliary exchangable pair $(Z, Z^{'})$. The defining property implies

$$\alpha \cdot \mathbf{E}[\mathbf{X} \cdot \mathbf{F}(\mathbf{X})] = \mathbf{E}[\mathbf{E}[\mathbf{X} - \mathbf{X}^{'} | Z] \cdot \mathbf{F}(\mathbf{X})] = \mathbf{E}[(\mathbf{X} - \mathbf{X}^{'})\mathbf{F}(\mathbf{X})] \tag{4.17}$$

□

4.3 Matrix Khintchin Inequality

Theorem 4.1. (Matrix BDG inequality) *Let $p = 1$ or $p \geq 3/2$. Suppose that $(\mathbf{X}, \mathbf{X}^{'})$ is a matrix Stein pair where $\mathbf{E}[\|\mathbf{X}\|_{2p}^{2p}] < \infty$. Then*

$$\mathbf{E}[\|\mathbf{X}\|_{2p}^{2p}]^{1/(2p)} \leq \sqrt{2p-1}\ \mathbf{E}[\|\mathbf{\Delta}_{\mathbf{X}}\|_{p}^{p}]^{1/(2p)}, \tag{4.18}$$

where $\mathbf{\Delta}_{\mathbf{X}}$ is the conditional variance .

Proof. *[MJC+14, §7.3]* Apply method of exchangeable pairs, generalized Klein inequality, trace Hölder □

Theorem 4.2. [MJC+14, Corollary 7.3] *Suppose that $p = 1$ or $p \geq 3/2$. Consider a finite sequence $(\mathbf{Y}_k)_{k \geq 1}$ of independent, random, Hermitian matrices and a deterministic sequence $(\mathbf{A}_k)_{k \geq 1}$ for which*

$$\mathbf{E}[\mathbf{Y}_k] = 0 \quad \text{and} \quad \mathbf{Y}_k^2 \preccurlyeq \mathbf{A}_k^2 \qquad \text{almost surely for all } k \geq 1. \tag{4.19}$$

Then

$$\mathbf{E}\left[\left\|\sum_{k \geq 1} \mathbf{Y}_k\right\|_{2p}^{2p}\right]^{1/(2p)} \leq \sqrt{p - \frac{1}{2}} \left\|\left(\sum_{k \geq 1}(\mathbf{A}_k^2 + \mathbf{E}[\mathbf{Y}_k^2])\right)^{1/2}\right\|_{2p}. \tag{4.20}$$

In particular, when $(\xi_k)_{k\geq 1}$ is an independent sequence of Rademacher random variables,

$$\mathbf{E}\left[\left\|\sum_{k\geq 1}\xi_k\mathbf{A}_k\right\|_{2p}^{2p}\right]^{1/(2p)} \leq \sqrt{2p-1}\left\|\left(\sum_{k\geq 1}\mathbf{A}_k^2\right)^{1/2}\right\|_{2p}. \tag{4.21}$$

4.4 Matrix Moment Inequality

Theorem 4.3. *Assume $n \geq 3$*

(i) Suppose that $p \geq 1$, and fix $r \geq p \vee 2\log(n)$. Consider a finite sequence $(\mathbf{S}_k)_{k\geq 1}$ of independent, random, positive-semidefinite matrices with dimension $n \times n$. Then

$$\mathbf{E}\left[\left\|\sum_{k\geq 1}\mathbf{S}_k\right\|^p\right]^{1/p} \leq \left[\left\|\sum_{k\geq 1}\mathbf{E}[\mathbf{S}_k]\right\|^{1/2} + 2\sqrt{er}\mathbf{E}[\max_{k\geq 1}\|\mathbf{S}_k\|^p]^{1/(2p)}\right]^2. \tag{4.22}$$

(ii) Suppose that $p \geq 2$, and fix $r \geq p \vee 2\log(n)$. Consider a finite sequence $(\mathbf{Y}_k)_{k\geq 1}$ of independent, symmetric, random, self-adjoint matrices with dimension $n \times n$. Then

$$\mathbf{E}\left[\left\|\sum_{k\geq 1}\mathbf{Y}_k\right\|^p\right]^{1/p} \leq \sqrt{er}\left\|\left(\sum_{k\geq 1}\mathbf{E}[\mathbf{Y}_k^2]\right)^{1/2}\right\| + 2er\mathbf{E}[\max_{k\geq 1}\|\mathbf{S}_k\|^p]^{1/p}. \tag{4.23}$$

4.5 Intrinsic Dimension

Definition 4.3. *For a positive-semidefinite matrix $\mathbf{S}$, the* ***intrinic dimension*** *is the quantity*

$$\mathrm{intdim}(\mathbf{A}) := \frac{\mathrm{tr}\mathbf{A}}{\|\mathbf{A}\|}.$$

Lemma 4.3. (Intrinsic dimenision) *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a convex function with $\varphi(0) = 0$. For any positive-semidefinite matrix $\mathbf{S}$ it holds that*

$$\mathrm{tr}(\varphi(\mathbf{S})) \leq \mathrm{intdim}(\mathbf{S}) \cdot \varphi(\|\mathbf{S}\|).$$

Proof. *[Tro15, Lemma 7.5.1]* Since φ is convex on any interval $[0, L]$ with $L > 0$ and $\varphi(0) = 0$, it holds

$$\varphi(a) \leq \left(1 - \frac{a}{L}\right)\varphi(0) + \frac{a}{L}\varphi(L) = \frac{a}{L}\varphi(L) \qquad \text{for all } a \in [0, L]. \tag{4.24}$$

Since $\mathbf{S}$ is positive-semidefinite, the eigenvalues of $\mathbf{S}$ fall in the interval $[0, L]$, where $L = \|\mathbf{S}\|$.

$$\mathrm{tr}(\varphi(\mathbf{S})) = \sum_{i=1}^{d}\varphi(\lambda_i) \leq \frac{\sum_{i=1}^{d}\lambda_i}{\|\mathbf{S}\|}\varphi(\|\mathbf{S}\|) = \frac{\mathrm{tr}(\mathbf{S})}{\|\mathbf{S}\|}\varphi(\|\mathbf{S}\|) = \mathrm{intdim}(\mathbf{S}) \cdot \varphi(\|\mathbf{S}\|). \tag{4.25}$$

□

5 Empirical Processes

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $(\mathcal{X}, \Sigma)$ a measurable space. Let $X_j : (\Omega, \mathcal{A}, \mathbf{P}) \to (\mathcal{X}, \Sigma), j = 1, \ldots, n$ be independent and identically-distributed (i.i.d.) random variables with probability distribution $\mathbf{P}_X$ and $\mathcal{F}$ a family of measurable functions $f : (\mathcal{X}, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider the map

$$f \mapsto G_n f := \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbf{P}_X f \right), \tag{5.1}$$

where $\mathbf{P}_X f := \int_{\mathcal{X}} f \mathrm{d}\mathbf{P}_X$. We call $(G_n f)_{f \in \mathcal{F}}$ the empirical process indexed by $\mathcal{F}$. Furthermore

$$\|G_n f\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |G_n f| \, . \tag{5.2}$$

Lemma 5.1. (Bernstein Inequality for Empirical Processes) *For any bounded, measurable function f it holds for all $t > 0$*

$$\mathbf{P}\left(|G_n f| > t\right) \leq 2 \exp\left(-\frac{1}{4} \frac{t^2}{\mathbf{P}_X(f^2) + t \, \|f\|_\infty / \sqrt{n}} \right) \tag{5.3}$$

Proof. By the Markov inequality it holds for all $\lambda > 0$

$$\mathbf{P}\left(G_n f > t\right) \leq e^{-\lambda t} \mathbf{E} \exp\left(\lambda G_n f\right) \tag{5.4}$$

$\square$

Lemma 5.2. *For any finite class $\mathcal{F}$ of bounded, measurable, square-integrable functions, with $|\mathcal{F}|$ elements, it holds*

$$\mathbf{E} \|G_n f\|_{\mathcal{F}} \lesssim \max_{f \in \mathcal{F}} \frac{\|f\|_\infty}{\sqrt{n}} \log\left(1 + |\mathcal{F}|\right) + \max_{f \in \mathcal{F}} \|f\|_{\mathbf{P},2} \sqrt{\log\left(1 + |\mathcal{F}|\right)}. \tag{5.5}$$

6 Simple yet useful Calculations

Theorem 6.1. (Multivariate Taylor Theorem) *Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$. Then for all $x, \Delta \in \mathbb{R}^n$ there exists $\xi \in [0,1]$ such that it holds*

$$\begin{aligned} f(x+\Delta) = f(x) &+ \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i}\Delta_i + \sum_{\substack{i,j=1 \\ i\neq j}} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i \partial x_j}\Delta_i\Delta_j \\ &+ \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f(x+\xi\Delta)}{\partial x_i^2}\Delta_i^2 \end{aligned} \tag{6.1}$$

Corollary 6.1.1. *Let $f \in C^2(\mathbb{R})$. Then for all $a, x, \Delta \in \mathbb{R}^n$ there exist $\xi \in [0,1]$ such that it holds*

$$f(a^T(x+\Delta)) - f(a^T x) = f^{'}(a^T x)\,\Delta^T a + \frac{1}{2} f^{''}(a^T(x+\xi\Delta))\,\Delta^T A\ \Delta, \tag{6.2}$$

where $A := aa^T \in \mathbb{R}^{n\times n}$.

Proof. By the chain rule we have for all $a, x, \Delta \in \mathbb{R}^n$ and $\xi \in [0,1]$

$$\frac{\partial^2 f(a^T(x+\xi\Delta))}{\partial x_i \partial x_j} = f^{''}(a^T(x+\xi\Delta))\,a_i a_j. \tag{6.3}$$

Since $A := aa^T$ is symmetric we have

$$\Delta^T A\ \Delta = 2\sum_{\substack{i,j=1 \\ i\neq j}}^{n} a_i a_j \Delta_i \Delta_j + \sum_{i=1}^{n} a_i^2 \Delta_i^2. \tag{6.4}$$

Plugging (6.3) and (6.4) into (6.1) yields (6.2). □

Proposition 6.1. *For all $x, y \in \mathbb{R}$ it holds*

$$|x+y| - |x| \geq -|y| \tag{6.5}$$

Proof. Checking all 6 combinations of $x+y, x, y$ being nonnegative or negative yields the result. □

Notation Index

$\#A$ cardinality of the set A

$\mathbf{E}[X|Y]$ conditional expectation of the random variable X with respect to $\sigma(Y)$

$\mathbf{E}[X]$ expectation of the random variable X

$\mathbf{Var}[X]$ variance of the random variable X

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ extension of the real numbers

$\xrightarrow{\mathcal{D}}$ convergence of distributions

$\mathbf{P}$ generic probability measure

$\mathbf{P}_X = \mathbf{P} \circ X^{-1}$ distribution of the random variable X

$\mathbb{R}$ set of real numbers

$x \vee y, x \wedge y, x^+, x^-$ maximum, minimum, positive part, negative part of real numbers

$X \sim \mu$ the random variable has distribution μ

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