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# **Solving missing survival times with entropy balancing weights**

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# 1 Balancing Weights

## 1.1 Introduction

Let  $T_i$  be the survival time and  $C_i$  the censoring time. indepen.  $X_i$  baseline covariates.  $O_i$  indicator if  $(Z_i, \Delta_i)$  are observed.  $Z_i = T_i \wedge C_i$

$$\Delta_i = \mathbf{1}_{T_i \leq C_i}$$

In our sample the first  $n$  units have s. The remaining  $N - n$  do not have survival times. We calculate weights for the observed to estimate the distribution of the survival times. We assume missing at random.

## 1.2 Dual Formulation

**Theorem.** *The dual of Problem ?? is the unconstrained optimization problem*

$$\underset{\lambda \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{n} \sum_{i=1}^n T_i \cdot f^*(m_n(\lambda|X_i)) - m_n(\lambda|X_i) + \langle \delta, |\lambda| \rangle,$$

where

$$f^* : \mathbb{R} \rightarrow \mathbb{R}, \quad x^* \mapsto x^* \cdot (f')^{-1}(x^*) - f\left((f')^{-1}(x^*)\right)$$

is the Legendre transformation of  $f$ , the vector  $B(X_i) = [B_1(X_i), \dots, B_n(X_i)]^\top$  denotes the  $n$  basis functions of the covariates of unit  $i \in \{1, \dots, n\}$  and  $|\lambda| = [|\lambda_1|, \dots, |\lambda_n|]^\top$ , where  $|\cdot|$  is the absolute value of a real-valued scalar. Moreover, if  $\lambda^\dagger$  is an optimal solution of the above problem then for all  $i$  with  $T_i = 1$

$$w_i^\dagger = (f')^{-1}\left(m_n(\lambda^\dagger|X_i)\right)$$

is part of the optimal solution to Problem ?? .

### Plan of proof

We bring Problem ?? in the form of ts. Then we apply the results of the ts chapter. We wait to conclude on the weights. Then we eliminate the non-negativity constraints

on the dual variables leveraging convexity and optimality.

**Proof. Form**

We consider the vector of basis functions of the covariates of unit  $i \in \{1, \dots, n\}$ , that is,

$$B(X_i) := [B_1(X_i), \dots, B_n(X_i)]^\top,$$

the constraints vector

$$d := \begin{bmatrix} 0_n \\ \pm n \\ -n \cdot \delta \pm \sum_{i=1}^n B_k(X_i) \end{bmatrix},$$

the matrix of the basis functions with indicators of treatment

$$T\mathbf{B}(\mathbf{X}) := [T_1 B(X_1), \dots, T_n B(X_n)]$$

and the constraint matrix with indicator of treatment

$$T\mathbf{A} := \begin{bmatrix} \text{diag}[T_1, \dots, T_n] \\ \pm[T_1, \dots, T_n] \\ \pm T\mathbf{B}(\mathbf{X}) \end{bmatrix}.$$

Next we write Problem ?? in the form of ts.

$$\begin{aligned} & \underset{w_1, \dots, w_n \in \mathbb{R}}{\text{minimize}} && \sum_{i=1}^n T_i \cdot f(w_i) \\ & \mathbf{A}w &\geq d, \end{aligned}$$

The convex conjugate of the objective function of Problem ?? is

$$[x_1^*, \dots, x_n^*]^\top \mapsto \sum_{T_i=1} f^*(x_i^*) + \sum_{T_i=0} \chi_{\{0\}}(x_i^*),$$

where we define the characteristic function  $\chi$  by

$$\chi_{\{0\}}(t) = \infty \cdot (1 - \mathbf{1}_{\{0\}}(t)) = \begin{cases} 0, & \text{if } t = 0, \\ \infty, & \text{else.} \end{cases}$$

Note that the  $i$ -th column of  $T\mathbf{A}$ , which we denote as  $T\mathbf{A}_i$ , vanishes if  $T_i = 0$ . In the subsequent analysis this prevents the characteristic function from blowing up. We consider the form of the dual in ts to get

$$\sum_{T_i=1} f^*(T\mathbf{A}_i^\top \lambda) + \sum_{T_i=0} \chi_{\{0\}}(T\mathbf{A}_i^\top \lambda) = \sum_{i=1}^n T_i f^*(\mathbf{A}_i^\top \lambda).$$



where  $\mathbf{A}$  is as  $T\mathbf{A}$  with the indicators of treatment replaced by 1. It is important, that in the final form the indicators of treatment are outside the argument of  $f^*$  and we have no singularity for  $T_i = 0$ . The corresponding dual problem in [TB91] is then

$$\underset{\lambda_1, \dots, \lambda_K \geq 0}{\text{maximize}} \quad - \sum_{i=1}^n T_i \cdot f^*(\mathbf{A}_i^\top \lambda) + \langle \lambda, d \rangle.$$

### Non-negativity constraints

Next we want to remove the non-negativity constraints on  $\lambda$ . To this end we write

$$\lambda := \left[ \rho_1, \dots, \rho_n, \lambda_0^+, \lambda_0^-, \lambda_1^+, \dots, \lambda_n^+, \lambda_1^-, \dots, \lambda_n^- \right]^\top. \quad (1.1)$$

We expand the objective function  $G$  of the dual problem.

$$\begin{aligned} G(\rho, \lambda_0^\pm, \lambda^\pm) &= - \sum_{i=1}^n T_i \cdot f^*(\rho_i + \lambda_0^+ - \lambda_0^- + \langle B(X_i), \lambda^+ - \lambda^- \rangle) \\ &\quad + n \cdot (\lambda_0^+ - \lambda_0^-) + \sum_{i=1}^n \langle B(X_i), \lambda^+ - \lambda^- \rangle - n \cdot \langle \delta, \lambda^+ + \lambda^- \rangle \end{aligned}$$

To illustrate the procedure, we show for all  $i \in \{1, \dots, n\}$

$$\begin{aligned} \text{either} \quad & \lambda_i^+ > 0 \\ \text{or} \quad & \lambda_i^- > 0. \end{aligned}$$

Assume towards a contradiction that there exists  $i \in \{1, \dots, n\}$  such that  $\lambda_i^+ > 0$  and  $\lambda_i^- > 0$  and that  $\lambda$  is optimal. Consider

$$\tilde{\lambda} := \left[ \rho, \lambda_0^\pm, \lambda_1^+, \dots, \lambda_i^+ - (\lambda_i^+ \wedge \lambda_i^-), \dots, \lambda_n^+, \lambda_1^-, \dots, \lambda_i^- - (\lambda_i^+ \wedge \lambda_i^-), \dots, \lambda_n^- \right]^\top. \quad (1.2)$$

Since  $\lambda_i^\pm - (\lambda_i^+ \wedge \lambda_i^-) \geq 0$ , the perturbed vector  $\tilde{\lambda}$  is in the domain of the optimization problem. But

$$G(\tilde{\lambda}) - G(\lambda) = 2n \cdot \delta_i \cdot (\lambda_i^+ \wedge \lambda_i^-) > 0, \quad (1.3)$$

which contradicts the optimality of  $\lambda$ . Likewise we can show

$$\begin{aligned} \text{either} \quad & \lambda_i^+ > 0 \\ \text{or} \quad & \lambda_i^- > 0. \end{aligned}$$

## 1 Balancing Weights

Let now

$$\tilde{\lambda} := \left[ \rho, \lambda_0^\pm - (\lambda_0^+ \wedge \lambda_0^-), \lambda_1^+, \dots, \lambda_n^+, \lambda_1^-, \dots, \lambda_n^- \right]^\top. \quad (1.4)$$

We get

$$G(\tilde{\lambda}) - G(\lambda) = 2n \cdot \delta_i \cdot (\lambda_i^+ \wedge \lambda_i^-) > 0, \quad (1.5)$$

But then  $\lambda_i^\pm \geq 0$  collapses to  $\lambda_i \in \mathbb{R}$  for  $i \in \{0, \dots, n\}$ , that is,  $\lambda_i = \lambda_i^+ - \lambda_i^-$ . Note that  $|\lambda_i| = \lambda_i^+ + \lambda_i^-$ . Likewise we can see, that  $\lambda_0 = \lambda_0^+ - \lambda_0^- \in \mathbb{R}$  removes the constraint on  $\lambda_0^\pm$ . Let us take this into account for  $G$ . We get

$$\begin{aligned} G(\rho, \lambda_0, \lambda) &= - \sum_{i=1}^n T_i \cdot f^*(\rho_i + \lambda_0 + \langle B(X_i), \lambda \rangle) \\ &\quad + n \cdot \lambda_0 + \sum_{i=1}^n \langle B(X_i), \lambda \rangle - n \cdot \langle \delta, |\lambda| \rangle. \end{aligned}$$

Next we show, that  $\rho = 0$ . Suppose there exists  $i \in \{1, \dots, n\}$  such that  $\rho_i > 0$  and  $T_i \cdot (f')^{-1}(\rho_i + \lambda_0 + \langle B(X_i), \lambda \rangle) < 0$ . It follows

$$G(0, \lambda_0, \lambda) - G(\rho_i, \lambda_0, \lambda) \geq T_i \cdot (f')^{-1}(\rho_i + \lambda_0 + \langle B(X_i), \lambda \rangle) (-\rho_i) > 0, \quad (1.6)$$

which contradicts the optimality of  $\lambda$ . Suppose  $T_i \cdot (f')^{-1}(\rho_i + \lambda_0 + \langle B(X_i), \lambda \rangle) > 0$ . Then the claim yields to a perturbation argument as in ts. Thus To eliminate the constraints for  $\rho$  we use a similar argument as in the complementary slackness section of the ts chapter. Thus we have complementary slackness of  $\rho_i$  and  $T_i \cdot (f')^{-1}(\rho_i + \lambda_0 + \langle B(X_i), \lambda \rangle)$ . But then every optimal solution  $\lambda$  remains optimal by taking  $\rho = 0$ .

Dividing the optimization problem by  $n$  and reversing it, we get

$$\underset{\lambda_0, \dots, \lambda_n \in \mathbb{R}}{\text{minimize}} \quad \frac{1}{n} \sum_{i=1}^n [T_i \cdot f^*(\lambda_0 + \langle B(X_i), \lambda \rangle) - (\lambda_0 + \langle B(X_i), \lambda \rangle)] + \langle \delta, |\lambda| \rangle.$$

□

## 2 Convex Analysis

In our application we want to analyse a convex optimization problem by its dual problem. In particular we want to obtain primal optimal solutions from dual solutions. To accomplish the task we need technical tools from convex analysis, mainly conjugate calculus and some KKT related results.

Our starting point is the support function intersection rule [MMN22, Theorem 4.23]. We give the details in the case of finite dimensions and refer for the rest of the proof to the book. The conjugate sum rule is applied to give first conjugate sum and then chain rule, which are vital to calculating convex conjugates. The proofs are omitted, since the book is thorough enough. The well known Fenchel-Rockafellar Duality theorem is a corollary of conjugate sum and chain rule. It gives general conditions under which dual and primal values coincide. The material we present is very well known, so we claim no originality. We paraphrase the approach of [MMN22] to Duality. As an introduction, we recommend this recently published book together with the classical reference [Roc70].

We finish the chapter with ideas from [TB91]. They provide the high-level ideas to obtain for strictly convex functions a dual relationship between optimal solutions. We will deliver the details that are omitted in the paper.

### 2.1 A Convex Analysis Primer

Excursively, we present some well known definitions and facts from convex analysis. For details, see, e.g., [MMN22].

A subset  $C \subseteq \mathbb{R}^n$  is called **convex set**, if for all  $x, y \in C$  and all  $\lambda \in [0, 1]$ , we have  $\lambda x + (1 - \lambda)y \in C$ . The Cartesian product of convex sets is convex. The intersection of a collection of convex sets is also convex.

Given (not necessary convex) sets  $\Omega, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ , define the **set addition** and **multiplication** by a real scalar as  $\Omega_1 + \Omega_2 := \{x_1 + x_2 : x_1 \in \Omega_1, x_2 \in \Omega_2\}$  and  $\lambda\Omega := \{\lambda x : x \in \Omega\}$ . For convex sets the addition and multiplication by a real scalar are convex.

Throughout this section, we shall denote by  $B := \{x = [x_1, \dots, x_n]^\top \in \mathbb{R}^n : (\sum_{i=1}^n x_i^2)^{1/2} \leq 1\}$

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the **Euclidian unit ball** in  $\mathbb{R}^n$ . This is a closed convex set. For any  $a \in \mathbb{R}^n$ , the **ball with radius  $\varepsilon > 0$  and center  $a$**  is given by  $\{a + x \in \mathbb{R}^n : (\sum_{i=1}^n x_i^2)^{1/2} \leq \varepsilon\} = a + \varepsilon B$ . For any set  $\Omega$  in  $\mathbb{R}^n$ , the set of points  $x$  whose distance from  $\Omega$  does not exceed  $\varepsilon$  is  $\Omega + \varepsilon B$ . The **closure**  $\text{cl}(\Omega)$  and **interior**  $\text{int}(\Omega)$  of  $\Omega$  can therefore be expressed by  $\text{cl}(\Omega) = \bigcap_{\varepsilon > 0} \Omega + \varepsilon B$  and  $\text{int}(\Omega) = \{x \in \Omega : \text{there exists } \varepsilon > 0 \text{ such that } x + \varepsilon B \subseteq \Omega\}$ .

A set  $A \subseteq \mathbb{R}^n$  is called **affine set**, if  $\alpha x + (1 - \alpha)y \in A$  for all  $x, y \in A$  and  $\alpha \in \mathbb{R}$ . The **affine hull**  $\text{aff}(\Omega)$  of a set  $\Omega \subseteq \mathbb{R}^n$  is the smallest affine set that includes  $\Omega$ . A mapping  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called **affine mapping** if there exist a linear mapping  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and a vector  $b \in \mathbb{R}^m$  such that  $A(x) = L(x) + b$  for all  $x \in \mathbb{R}^n$ . The image and inverse image/preimage of convex sets under affine mappings are also convex.

Because the notion of interior is not precise enough for our purposes we define the relative interior which is the interior relative to the affine hull. This concept is motivated by the fact that a line segment embedded in  $\mathbb{R}^2$  does have a natural interior in  $\mathbb{R}$  which is not a true interior in  $\mathbb{R}^2$ . The relative interior of  $C$  is defined as the interior which results when  $C$  is regarded as a subset of its affine hull.

**Definition.** Let  $\Omega \subseteq \mathbb{R}^n$ . We define the **relative interior** of  $\Omega$  by

$$\text{ri}(\Omega) := \{x \in \Omega : \text{there exists } \varepsilon > 0 \text{ such that } (x + \varepsilon B) \cap \text{aff}(\Omega) \subseteq \Omega\}. \quad (2.1)$$

Next we collect some useful properties of relative interiors.

**Proposition 2.1.** Let  $C$  be a non-empty convex set in  $\mathbb{R}^n$ . The following holds:

- (i)  $\text{ri}(C) \neq \emptyset$  if and only if  $C \neq \emptyset$
- (ii) The sets  $\text{cl } C$  and  $\text{ri } C$  are convex
- (iii)  $\text{cl}(\text{ri } C) = \text{cl } C$  and  $\text{ri}(\text{cl } C) = \text{ri}(C)$
- (iv)  $\text{ri}(C) = \{z \in C : \text{for all } x \in C \text{ there exists } t > 0 \text{ such that } z + t(z - x) \in C\}$
- (v) Suppose  $\bigcap_{i \in I} C_i \neq \emptyset$  for a finite index set  $I$ . Then  $\text{ri}(\bigcap_{i \in I} C_i) = \bigcap_{i \in I} \text{ri}(C_i)$ .
- (vi) Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear function. Then  $\text{ri } L(C) = L(\text{ri } C)$ . If it also holds  $L^{-1}(\text{ri } C) \neq \emptyset$ , we have  $\text{ri } L^{-1}(C) = L^{-1}(\text{ri } C)$ .
- (vii)  $\text{ri}(C_1 \times C_2) = \text{ri } C_1 \times \text{ri } C_2$
- (viii)  $\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset$  if and only if  $0 \notin \text{ri}(C_1 - C_2)$ .

**Proof.** For a proof of (i)-(vi) we refer to [Roc70, Theorem 6.2 - 6.7].

To prove (vii) we use (iv). Let  $(z_1, z_2) \in \text{ri}(C_1 \times C_2)$ . Then for all  $(x_1, x_2) \in C_1 \times C_2$  there exists  $t > 0$  such that

$$z_i + t(z_i - x_i) \in C_i \quad \text{for all } i \in \{1, 2\}. \quad (2.2)$$

Using (iv) again, we get  $\text{ri}(C_1 \times C_2) \subseteq \text{ri } C_1 \times \text{ri } C_2$ . Suppose  $(z_1, z_2) \in \text{ri } C_1 \times \text{ri } C_2$ . By (iv), for all  $(x_1, x_2) \in C_1 \times C_2$  there exist  $(t_1, t_2) > 0$  such that

$$z_i + t_i(z_i - x_i) \in C_i \quad \text{for all } i \in \{1, 2\}. \quad (2.3)$$

If  $t_1 = t_2$  we recover (2.2) from (2.3). By (iv) it holds  $(z_1, z_2) \in \text{ri}(C_1 \times C_2)$ . If  $t_1 < t_2$  we define  $\theta := \frac{t_1}{t_2} \in (0, 1)$ . Consider (2.3) with  $i = 2$ , together with  $z_2 \in C_2$  and the convexity of  $C_2$ . It follows

$$z_2 + t_1(z_2 - x_2) = \theta \cdot (z_2 + t_2(z_2 - x_2)) + (1 - \theta) \cdot z_2 \in C_2. \quad (2.4)$$

Now we consider (2.4) and (2.3) with  $i = 1$ . This gives (2.2) with  $t = t_1$ . As before, it follows  $(z_1, z_2) \in \text{ri}(C_1 \times C_2)$ . If  $t_1 > t_2$  similar arguments lead to the same result. We have proven  $\text{ri}(C_1 \times C_2) \supseteq \text{ri } C_1 \times \text{ri } C_2$  and equality. [MMN22, Theorem 2.92]  $\square$

We proceed with convex separation results which are vital to the subsequent developments.

**Definition.** Let  $C_1$  and  $C_2$  be two non-empty convex sets in  $\mathbb{R}^n$ . A hyperplane  $H$  is said to **separate**  $C_1$  and  $C_2$  if  $C_1$  is contained in one of the closed half-spaces associated with  $H$  and  $C_2$  lies in the opposite closed half-space. It is said to **separate**  $C_1$  and  $C_2$  **properly** if  $C_1$  and  $C_2$  are not both actually contained in  $H$  itself.

**Theorem 2.1.** (Convex separation in finite dimension) Let  $C_1$  and  $C_2$  be two non-empty convex sets in  $\mathbb{R}^n$ . Then  $C_1$  and  $C_2$  can be properly separated if and only if  $\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset$ .

**Proof.** [Roc70, Theorem 11.3]  $\square$

**Definition.** Given a nonempty subset  $\Omega \subseteq \mathbb{R}^n$ , we define the **support function** of  $\Omega$  to be

$$\sigma_\Omega : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, \quad x^* \mapsto \sup_{x \in \Omega} \langle x^*, x \rangle.$$

**Definition 2.1.** Given functions  $f_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  for  $i = 1, \dots, m$ , we define the *infimal convolution* of these functions to be

$$f_1 \square \dots \square f_m : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, \quad x \mapsto \inf \left\{ \sum_{i=1}^m f_i(x_i) : x_i \in \mathbb{R}^n \text{ and } \sum_{i=1}^m x_i = x \right\}.$$

The next result establishes a connection between the support function of the intersection of two convex sets and the infimal convolution of the support functions of the sets taken by themselves. The proof translates the geometric concept of convex separation to the world of convex functions.

**Lemma 2.1.** Let  $C_1$  and  $C_2$  be two non-empty convex sets in  $\mathbb{R}^n$ . For any  $x^* \in \text{dom } \sigma_{C_1 \cap C_2}$  the sets

$$\begin{aligned} \Theta_1 &:= C_1 \times [0, \infty), \\ \Theta_2(x^*) &:= \{(x, \lambda) \in \mathbb{R}^n : x \in C_2 \text{ and } \lambda \leq \langle x^*, x \rangle - \sigma_{C_1 \cap C_2}(x^*)\} \end{aligned}$$

can be properly separated.

**Proof.** We fix  $x^* \in \text{dom } \sigma_{C_1 \cap C_2}$  and write  $\alpha := \sigma_{C_1 \cap C_2}(x^*)$ . In order to apply convex separation in finite dimension (Theorem 2.1) to the sets  $\Theta_1$  and  $\Theta_2(x^*)$ , it suffices to show their convexity and  $\text{ri } \Theta_1 \cap \text{ri } \Theta_2(x^*) = \emptyset$ .

### Convexity of $\Theta_1$ and $\Theta_2(x^*)$

Clearly,  $\Theta_1$  is convex by the convexity of  $C_1$  and  $[0, \infty)$ . To see that  $\Theta_2(x^*)$  is convex consider the linear function

$$L : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, \lambda) \mapsto \langle x^*, x \rangle - \lambda.$$

From the definitions of  $L$  and  $\Theta_2(x^*)$  we get

$$\Theta_2(x^*) = (C_2 \times \mathbb{R}) \cap L^{-1}[\alpha, \infty).$$

Thus, by Proposition 2.1 (v) and the convexity of  $C_2$  we get the convexity of  $L^{-1}[\alpha, \infty)$  and with it that of  $\Theta_2(x^*)$ .

### Relative interiors of $\Theta_1$ and $\Theta_2(x^*)$ are disjoint

We start by calculating the relative interiors. It holds

$$\begin{aligned} \text{ri } \Theta_1 &= \text{ri}(C_1 \times [0, \infty)) = \text{ri } C_1 \times \text{ri } [0, \infty) = \text{ri } C_1 \times (0, \infty), \\ \text{ri } \Theta_2(x^*) &= \text{ri}(L^{-1}[\alpha, \infty)) = L^{-1}(\text{ri } [\alpha, \infty)) = L^{-1}(\alpha, \infty). \end{aligned}$$

Suppose there exists  $(\lambda, x) \in \text{ri } \Theta_1 \cap \text{ri } \Theta_2(x^*)$ . Then it holds  $x \in C_1 \times C_2$  and  $\lambda > 0$ . We also note, that

$$\alpha = \sigma_{C_1 \cap C_2}(x^*) = \sup_{z \in C_1 \cap C_2} \langle x^*, z \rangle \geq \langle x^*, x \rangle.$$

Then it follows

$$\alpha < \langle x^*, x \rangle - \lambda \leq \alpha,$$

a contradiction. Thus, the relative interiors of  $\Theta_1$  and  $\Theta_2(x^*)$  are disjoint.

Applying Theorem 2.1 finishes the proof.  $\square$

**Theorem.** Let  $C_1$  and  $C_2$  be two non-empty convex sets in  $\mathbb{R}^n$  with  $\text{ri } C_1 \cap \text{ri } C_2 \neq \emptyset$ . Then the support function of the intersection  $C_1 \cap C_2$  is represented as

$$(\sigma_{C_1 \cap C_2})(x^*) = (\sigma_{C_1} \square \sigma_{C_2})(x^*) \quad \text{for all } x^* \in \mathbb{R}^n. \quad (2.5)$$

Furthermore, for any  $x^* \in \text{dom}(\sigma_{C_1 \cap C_2})$  there exist dual elements  $x_1^*, x_2^* \in \mathbb{R}^n$  such that  $x^* = x_1^* + x_2^*$ . and

$$(\sigma_{C_1 \cap C_2})(x^*) = \sigma_{C_1}(x_1^*) + \sigma_{C_2}(x_2^*). \quad (2.6)$$

**Proof.** Using Lemma 2.1 the rest of the proof is as that of [MMN22, Theorem 4.23(b)].  $\square$

**Takeaways** The support function intersection rule connects the geometric property of convex separation to an identity of support functions. This result is central to the analysis of convex conjugates.

## 2.2 Conjugate Calculus

The goal of this section is to establish the tools to calculate convex conjugates. We cite the conjugate sum and chain rule without proof. After some examples, we cite the Fenchel-Rockafellar Theorem.

**Definition 2.2.** (Convex conjugate) Given a function  $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ , the **convex conjugate**  $f^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  of  $f$  is defined as

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} (x^*)^T x - f(x) \quad (2.7)$$

Note that  $f$  in Definition 2.2 does not have to be convex. On the other hand, the convex conjugate is always convex:

**Proposition 2.2.** *Let  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be a proper function. Then its convex conjugate  $f^* : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is convex.*

**Proof.** [MMN22, Proposition 4.2] □

**Theorem 2.2.** *Let  $f, g : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be proper convex functions and  $\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g)) \neq \emptyset$ . Then we have the **conjugate sum rule***

$$(f + g)^*(x^*) = (f^* \square g^*)(x^*) \quad (2.8)$$

for all  $x^* \in \mathbb{R}^n$ . Moreover, the infimum in  $(f^* \square g^*)(x^*)$  is attained, i.e., for any  $x^* \in \text{dom}(f + g)^*$  there exists vectors  $x_1^*, x_2^*$  for which

$$(f + g)^*(x^*) = f^*(x_1^*) + g^*(x_2^*), \quad x^* = x_1^* + x_2^*. \quad (2.9)$$

**Proof.** [MMN22, Theorem 4.27(c)] □

**Theorem 2.3.** *Let  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a linear map (matrix) and  $g : \mathbb{R}^n \rightarrow (-\infty, \infty]$  a proper convex function. If  $\text{Im}(A) \cap \text{ri}(\text{dom}(g)) \neq \emptyset$  it follows the **conjugate chain rule***

$$(g \circ A)^*(x^*) = \inf_{y^* \in (A^*)^{-1}(x^*)} g^*(y^*). \quad (2.10)$$

Furthermore, for any  $x^* \in \text{dom}(g \circ A)^*$  there exists  $y^* \in (A^*)^{-1}(x^*)$  such that  $(g \circ A)^*(x^*) = g^*(y^*)$ .

**Proof.** [MMN22, Theorem 4.28(c)] □

**Example.** Let  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be a proper convex function, that is,  $\text{dom } f \neq \emptyset$  and  $f$  is convex. In steps we apply the conjugate chain and sum rule, together with mathematical induction, to prove the conjugate relationship

$$\begin{aligned} S_{f,n} : \mathbb{R}^n &\rightarrow \overline{\mathbb{R}}, & (x_1, \dots, x_n) &\mapsto \sum_{i=1}^n f(x_i), \\ S_{f,n}^* : \mathbb{R}^n &\rightarrow \overline{\mathbb{R}}, & (x_1^*, \dots, x_n^*) &\mapsto \sum_{i=1}^n f^*(x_i^*). \end{aligned}$$

This relationship is very natural and the ensuing calculations serve to confirm our intuition.

First, we work in the projections on the coordinates. For the  $i$ -th coordinate, where  $i = 1, \dots, n$ , this is

$$p_i : \mathbb{R}^n \rightarrow \mathbb{R}, \quad (x_1, \dots, x_n) \mapsto x_i. \quad (2.11)$$



All projections  $p_i$  are linear function with matrix representation  $e_i^\top$ , where  $e_i$  is  $i$ -the coordinate vector. The adjoint of  $p_i$  is therefore

$$p_i^* : \mathbb{R} \rightarrow \mathbb{R}^n, \quad x \mapsto e_i \cdot x. \quad (2.12)$$

For the inverse image of the adjoint of  $p_i$  it holds

$$(p_i^*)^{-1} \{(x_1^*, \dots, x_n^*)\} = \begin{cases} \{x_i^*\}, & \text{if } x_j^* = 0 \text{ for all } j \neq i, \\ \emptyset & \text{else.} \end{cases} \quad (2.13)$$

Throughout this example we use the asterisk character  $*$  somewhat inconsistently. Note that  $f^*$  is the convex conjugate of the function  $f$  and  $p_i^*$  is the adjoint linear function of the projection on the  $i$ -th coordinate. Likewise, we denote dual variables, that is, the arguments of convex conjugates, as  $x^*$ .

Next, we employ the conjugate chain rule to establish the conjugate relationship

$$\begin{aligned} f_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, \quad (x_1, \dots, x_n) &\mapsto x_i \mapsto f(x_i), \\ f_i^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, \quad (x_1^*, \dots, x_n^*) &\mapsto \begin{cases} f^*(x_i^*), & \text{if } x_j^* = 0 \text{ for all } j \neq i, \\ \infty & \text{else.} \end{cases} \end{aligned}$$

Note, that  $f_i = (f \circ p_i)$  and  $f_i^* = (f \circ p_i)^*$ . Since  $\text{Im } p_i = \mathbb{R}$  and  $\text{dom } f \neq \emptyset$ , it holds  $\text{Im } p_i \cap \text{ri}(\text{dom } f) \neq \emptyset$ . Then  $f$  and  $p_i$  conform with the demands of the conjugate chain rule. It follows

$$\begin{aligned} f_i^*(x_1^*, \dots, x_n^*) &= (f \circ p_i)^*(x_1^*, \dots, x_n^*) = \inf \{f^*(y) \mid y \in (p_i^*)^{-1} \{(x_1^*, \dots, x_n^*)\}\} \\ &= \begin{cases} f^*(x_i^*), & \text{if } x_j^* = 0 \text{ for all } j \neq i, \\ \infty & \text{else,} \end{cases} \end{aligned}$$

where we keep to the convention  $\inf \emptyset = \infty$ . In the same way it follows

$$(S_{f,n} \circ p_{\{1, \dots, n\}})^*(x_1^*, \dots, x_{n+1}^*) = \begin{cases} S_{f,n}^*(x_1^*, \dots, x_n^*) & \text{if } x_{n+1}^* = 0, \\ \infty & \text{else,} \end{cases} \quad (2.14)$$

Next, note that for  $n = 1$  we arrive at the result. Thus, for some  $n \in \mathbb{N}$  it holds  $(S_{f,n})^* = S_{f,n}^*$ . In order to apply the conjugate sum rule to  $S_{f,n}$  and  $f_{n+1}$  we note that

$$\begin{aligned} \text{dom } f_i &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \in \text{dom } f\} \neq \emptyset \quad \text{for all } i = 1, \dots, n+1, \\ \bigcap_{i=1}^{n+1} \text{dom } f_i &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \in \text{dom } f \text{ for all } i = 1, \dots, n+1\} \neq \emptyset, \end{aligned}$$

and

$$\begin{aligned} \text{ri}(\text{dom}(S_{f,n} \circ p_{\{1,\dots,n\}})) \cap \text{ri}(\text{dom } f_{n+1}) \\ = \text{ri}(\text{dom}(S_{f,n} \circ p_{\{1,\dots,n\}}) \cap \text{dom } f_{n+1}) = \text{ri}\left(\bigcap_{i=1}^{n+1} \text{dom } f_i\right) \neq \emptyset. \end{aligned}$$

By the conjugate sum rule it follows

$$\begin{aligned} (S_{f,n+1})^* &= (S_{f,n} \circ p_{\{1,\dots,n\}} + f_{n+1})^* = (S_{f,n} \circ p_{\{1,\dots,n\}})^* \square f_{n+1}^* \\ &= S_{f,n}^* \circ p_{\{1,\dots,n\}} + f_{n+1}^* = S_{f,n+1}^*. \end{aligned}$$

◇

Insert lemma in chapter 1.

**Lemma 2.2.** *Let  $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  be convex. Then for all  $y \in \mathbb{R}^n$  and  $\varepsilon > 0$*

$$\inf_{\|\Delta\|=\varepsilon} f(y + \Delta) - f(y) \geq 0 \quad (2.15)$$

*implies the existence of a global minimum  $y^* \in \mathbb{R}^n$  of  $f$  satisfying  $\|y^* - y\| \leq \varepsilon$ .*

**Proof.** Since  $y + \varepsilon B$  is convex, it contains a local minimum of  $f$ . Suppose towards a contradiction that  $y^* \in y + \varepsilon B$  is a local minimum, but not a global one, and (2.15) is true. Then it holds

$$f(x) < f(y^*) \quad \text{for some } x \in \mathbb{R}^n \setminus (y + \varepsilon B). \quad (2.16)$$

Furthermore, since  $y + \varepsilon B$  is compact and contains  $y^*$ , the line segment connecting  $y^*$  and  $x$  intersects the boundary of  $y + \mathcal{C}$ , that is, there exist  $\theta \in (0, 1)$  and  $\Delta_x$  with  $\|\Delta_x\| = \varepsilon$  such that

$$\theta x + (1 - \theta)y^* = y + \Delta_x. \quad (2.17)$$

It follows

$$\begin{aligned} f(y^*) &\leq f(y) \leq f(y + \Delta_x) = f(\theta x + (1 - \theta)y^*) \\ &\leq \theta f(x) + (1 - \theta)f(y^*) < f(y^*), \end{aligned} \quad (2.18)$$

which is a contradiction. The first inequality is due to  $y^*$  being a local minimum of  $f$  in  $y + \varepsilon B$ , the second inequality is due to (2.15) being true, the equality is due to (2.17), the third inequality is due to the convexity of  $f$  and the strict inequality is due to (2.16). Thus every local minimum of  $f$  in  $y + \varepsilon B$  is also a global minimum.  $\square$

**Takeaways** Conjugate sum and chain rule are direct consequences of the support function intersection rule. They are powerful tools, that allow us to compute convex conjugates of difficult expressions as well as proving the Fenchel-Rockafellar Duality theorem.

## 2.3 Tseng Bertsekas

**Theorem 2.4.** *Consider the optimization problem*

$$\begin{aligned} & \underset{w \in \mathbb{R}^n}{\text{minimize}} && f(w) \\ & \text{subject to} && \mathbf{U}w \geq d \\ & && \mathbf{A}w = a, \end{aligned} \tag{2.19}$$

*and its dual problem*

$$\begin{aligned} & \underset{\lambda_d \in \mathbb{R}^r, \lambda_a \in \mathbb{R}^s}{\text{maximize}} && \langle \lambda_d, d \rangle + \langle \lambda_a, a \rangle - f^*(\mathbf{U}^\top \lambda_d + \mathbf{A}^\top \lambda_a) \\ & \text{subject to} && \lambda_d \geq 0. \end{aligned} \tag{2.20}$$

*Let  $(\lambda_d^*, \lambda_a^*)$  be an optimal solution to (2.20). If the objective function  $f$  of (2.19) is strictly convex and its convex conjugate  $f^*$  is continuously differentiable, then the unique optimal solution to (2.19) is given by*

$$w^* = \nabla f^*(\mathbf{U}^\top \lambda_d^* + \mathbf{A}^\top \lambda_a^*). \tag{2.21}$$

### Plan of Proof

We show that for  $w^*$  and  $(\lambda_d^*, \lambda_a^*)$  the Karush-Kuhn-Tucker conditions are met. They are **complementary slackness**

$$\langle \lambda_d^*, d - \mathbf{U}w^* \rangle = 0, \tag{2.22}$$

**primal and dual feasibility**

$$\mathbf{U}w^* \geq d \tag{2.23}$$

$$\mathbf{A}w^* = a$$

$$\lambda_d^* \geq 0 \tag{2.24}$$

and **stationarity**

$$0 \in [\partial f(w^*) + \partial(w \mapsto d - \mathbf{U}w)(w^*) \cdot \lambda_d^* + \partial(w \mapsto a - \mathbf{A}w)(w^*) \cdot \lambda_a^*]. \tag{2.25}$$

Then by [Roc70, Theorem 28.3] it follows the result. We need strict convexity of  $f$  to obtain stationarity. The other conditions follow from continuous differentiability of  $f^*$ . Also we will use

**Proof.** Let  $(\lambda_d^*, \lambda_a^*)$  be an optimal solution to (2.20).

### Stationarity

First we show

$$\mathbf{U}^\top \lambda_d^* + \mathbf{A}^\top \lambda_a^* \in \partial f(w^*), \quad (2.26)$$

where

$$w^* := \nabla f^* \left( \mathbf{U}^\top \lambda_d^* + \mathbf{A}^\top \lambda_a^* \right).$$

To this end we need

**Proposition 2.3.** [Roc70, Theorem 23.5(a)-(b)]. *For any proper convex function  $f$  and any vector  $x$ , it holds  $t \in \partial f(x)$  if and only if  $\langle \cdot, t \rangle - f(\cdot)$  achieves its supremum at  $x$ .*

By Proposition 2.3 it suffices to show that

$$w \mapsto \langle w, \mathbf{U}^\top \lambda_d^* + \mathbf{A}^\top \lambda_a^* \rangle - f(w) \quad (2.27)$$

achieves its supremum at  $w^*$ . Since  $f$  is strictly convex there exists a unique vector  $x^*$  where the above expression achieves its maximum. Since  $f^*$  is differentiable it holds

$$w^* = \nabla f^* \left( \mathbf{U}^\top \lambda_d^* + \mathbf{A}^\top \lambda_a^* \right) = \nabla (\lambda \mapsto \langle x^*, \lambda \rangle - f(x^*)) \left( \mathbf{U}^\top \lambda_d^* + \mathbf{A}^\top \lambda_a^* \right) = x^*.$$

It follows (2.26) Next we show

$$-\left( \mathbf{U}^\top + \mathbf{A}^\top \right) \in [\partial (w \mapsto d - \mathbf{U}w)(w^*) + \partial (w \mapsto a - \mathbf{A}w)(w^*)]. \quad (2.28)$$

To this end, note that

$$\langle -\mathbf{U}^\top \cdot e_i, w - w^* \rangle = \left( d - \mathbf{U}^\top \cdot w \right)_i - \left( d - \mathbf{U}^\top \cdot w^* \right)_i. \quad (2.29)$$

Thus  $-\mathbf{U}^\top \in \partial (w \mapsto d - \mathbf{U}w)(w^*)$ . In the same way it follows  $-\mathbf{A}^\top \in \partial (w \mapsto a - \mathbf{A}w)(w^*)$ .

We conclude that

$$0 \in [\partial f(w^*) + \partial (w \mapsto d - \mathbf{U}w)(w^*) \cdot \lambda_d^* + \partial (w \mapsto a - \mathbf{A}w)(w^*) \cdot \lambda_a^*]. \quad (2.30)$$

We fix  $\lambda_a^*$  and work with the objective function  $G$  of the dual problem, that is,

$$G(\lambda_d) := \langle \lambda_d, d \rangle + \langle \lambda_a, a \rangle - f^* \left( \mathbf{U}^\top \lambda_d + \mathbf{A}^\top \lambda_a \right). \quad (2.31)$$

Since  $f^*$  is continuously differentiable, so is  $G$ . Since  $f^*$  is convex,  $G$  is concave. For concave differentiable functions it holds

$$G(x) - G(y) \geq \nabla G(x)^\top (x - y). \quad (2.32)$$

## 2 Convex Analysis

Let  $\lambda_{d,i}^*$  be the  $i$ -th coordinate of  $\lambda_d^*$  and for fixed  $\lambda_d^*$  let  $\nabla G_i(\lambda_d^*)$  be the  $i$ -th coordinate of  $\nabla G(\lambda_d^*)$ . We show that for all  $i \in \{1, \dots, s\}$  it holds

$$\begin{aligned} \text{either} \quad & \lambda_{d,i}^* = 0 \quad \text{and} \quad \nabla G_i(\lambda_d^*) \leq 0 \\ \text{or} \quad & \lambda_{d,i}^* > 0 \quad \text{and} \quad \nabla G_i(\lambda_d^*) = 0. \end{aligned}$$

Assume towards a contradiction that  $\nabla G_i(\lambda_d^*) > 0$  for some  $i \in \{1, \dots, s\}$ . By the continuity of  $\nabla G$  there exists  $\varepsilon > 0$  such that  $\nabla G_i(\lambda_d^* + e_i \cdot \varepsilon) > 0$ . Thus

$$G(\lambda_d^* + e_i \cdot \varepsilon) - G(\lambda_d^*) \geq \nabla G_i(\lambda_d^* + e_i \cdot \varepsilon) \cdot \varepsilon > 0, \quad (2.33)$$

which contradicts the optimality of  $\lambda_d^*$ . Now assume that  $\lambda_{d,i}^* > 0$  and  $\nabla G_i(\lambda_d^*) < 0$  for some  $i \in \{1, \dots, s\}$ . Again, by the continuity of  $\nabla G$  there exists  $\varepsilon > 0$  such that  $\nabla G_i(\lambda_d^* - e_i \cdot \varepsilon) < 0$  and  $\varepsilon - \lambda_{d,i}^* < 0$ . Thus

$$G(\lambda_d^* - e_i \cdot \varepsilon) - G(\lambda_d^*) \geq \nabla G_i(\lambda_d^* - e_i \cdot \varepsilon) \cdot (\varepsilon - \lambda_{d,i}^*) > 0, \quad (2.34)$$

which contradicts the optimality of  $\lambda_d^*$ . We have shown the complementary slackness. By the Karush-Kuhn-Tucker conditions we finish the proof.  $\square$

### 3 Simple yet useful Calculations

**Theorem 3.1.** (Multivariate Taylor Theorem) *Let  $f \in C^2(\mathbb{R}^n, \mathbb{R})$ . Then for all  $x, \Delta \in \mathbb{R}^n$  there exists  $\xi \in [0, 1]$  such that it holds*

$$\begin{aligned} f(x + \Delta) = & f(x) + \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} \Delta_i + \sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{\partial^2 f(x + \xi \Delta)}{\partial x_i \partial x_j} \Delta_i \Delta_j \\ & + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 f(x + \xi \Delta)}{\partial x_i^2} \Delta_i^2 \end{aligned} \quad (3.1)$$

**Corollary 3.1.1.** *Let  $f \in C^2(\mathbb{R})$ . Then for all  $a, x, \Delta \in \mathbb{R}^n$  there exist  $\xi \in [0, 1]$  such that it holds*

$$f(a^T(x + \Delta)) - f(a^T x) = f'(a^T x) \Delta^T a + \frac{1}{2} f''(a^T(x + \xi \Delta)) \Delta^T A \Delta, \quad (3.2)$$

where  $A := aa^T \in \mathbb{R}^{n \times n}$ .

**Proof.** By the chain rule we have for all  $a, x, \Delta \in \mathbb{R}^n$  and  $\xi \in [0, 1]$

$$\frac{\partial^2 f(a^T(x + \xi \Delta))}{\partial x_i \partial x_j} = f''(a^T(x + \xi \Delta)) a_i a_j. \quad (3.3)$$

Since  $A := aa^T$  is symmetric we have

$$\Delta^T A \Delta = 2 \sum_{\substack{i,j=1 \\ i \neq j}}^n a_i a_j \Delta_i \Delta_j + \sum_{i=1}^n a_i^2 \Delta_i^2. \quad (3.4)$$

Plugging (3.3) and (3.4) into (3.1) yields (3.2).  $\square$

**Proposition 3.1.** *For all  $x, y \in \mathbb{R}$  it holds*

$$|x + y| - |x| \geq -|y| \quad (3.5)$$

**Proof.** Checking all 6 combinations of  $x + y, x, y$  being nonnegative or negative yields the result.  $\square$





# Notation Index

$\#A$  cardinality of the set  $A$

$\mathbf{E}[X|Y]$  conditional expectation of the random variable  $X$  with respect to  $\sigma(Y)$

$\mathbf{E}[X]$  expectation of the random variable  $X$

$\mathbf{Var}[X]$  variance of the random variable  $X$

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  extension of the real numbers

$\xrightarrow{\mathcal{D}}$  convergence of distributions

$\mathbf{P}$  generic probability measure

$\mathbf{P}_X = \mathbf{P} \circ X^{-1}$  distribution of the random variable  $X$

$\mathbb{R}$  set of real numbers

$x \vee y, x \wedge y, x^+, x^-$  maximum, minimum, positive part, negative part of real numbers

$X \sim \mu$  the random variable has distribution  $\mu$



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