

## Todo list



# **Solving missing survival times with entropy balancing weights**

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# 1 Introduction

We consider a study population in which we want to test the effect of a treatment. We introduce the **indicator of treatment**  $T \in \{0, 1\}$ . For each treatment level there exist the **marginal potential outcomes**  $(Y(0), Y(1))$ . We would like to estimate  $\mathbf{E}[Y(1)]$ . If we succeed the same technique shall yield an estimate of  $\mathbf{E}[Y(0)]$ . We shall compare  $\mathbf{E}[Y(1)]$  and  $\mathbf{E}[Y(0)]$  and find out something about the effect of the treatment in the population.

The data we acquire is independent and identically distributed. But usually

$$Y(1)|T = 1 \approx Y(1), \quad (1.1)$$

that is,  $T = 1$  carries more information than observing the outcome under treatment. We say that  $Y(1)|T = 1$  is **confounded**. To extract that plus of information from  $T = 1$  and put it where it belongs by collecting more data. We gather it in  $X \in \mathbb{R}^d$  and assume

$$(Y(0), Y(1)) \perp T \mid X, \quad (1.2)$$

that is, **conditional unconfoundedness**. Thus, we end up collecting  $N \in \mathbb{N}$  independent and identically distributed copies of  $(T, X, Y(T))$ . For convenience, we assume that the first  $n \in \mathbb{N}$  copies have  $T = 1$ .

A natural estimator for  $\mathbf{E}[Y(1)]$  is the weighted mean

$$\frac{1}{n} \sum_{i=1}^n w_i Y_i. \quad (1.3)$$

The weights should satisfy (in a broader sense)

$$w_i \cdot Y_i \rightarrow Y(1) \quad \text{for } N \rightarrow \infty. \quad (1.4)$$

One class of such weights has been recently analyzed in [WZ19]. We take ideas and extend.

## The algorithm

**Problem 1.1.**

$$\begin{aligned}
& \underset{w_1, \dots, w_n \in \mathbb{R}}{\text{minimize}} && \sum_{i=1}^n f(w_i) \\
& \text{subject to} && w_i \geq 0 && \text{for all } i \in \{1, \dots, n\}, \\
& && \frac{1}{N} \sum_{i=1}^n w_i = 1 \\
& && \left| \frac{1}{N} \left( \sum_{i=1}^n w_i B_k(X_i) - \sum_{i=1}^N B_k(X_i) \right) \right| \leq \delta_k && \text{for all } k \in \{1, \dots, N\}.
\end{aligned}$$

This is a (convex) optimization problem. We will talk about the **objective function**  $f$  and the **equality** and **inequality constraints**, especially about the **regression basis**  $B$ .

## Objective Function

Strictly speaking, we consider the sum

$$[w_1, \dots, w_n]^\top \mapsto \sum_{i=1}^n f(w_i) \quad (1.5)$$

as the objective function. It is natural to consider the dual formulation of the optimization problem. This involves the **convex conjugate** (cf. Definition ?) of the original objective function. We show in Example that for the sum this is

$$[\lambda_1, \dots, \lambda_n]^\top \mapsto \sum_{i=1}^n f^*(\lambda_i) \quad (1.6)$$

where  $f^*$  is the Legendre transformation of  $f$ .

In the sequel we need  $f$  to be strictly convex and its convex conjugate (or Legendre transformation) to be continuously differentiable and strictly non-decreasing. Two popular choices of  $f$  are the **negative entropy** and the **sample variance**.

## Negative Entropy

We define the negative entropy to be

$$f: [0, \infty) \rightarrow \mathbb{R}, \quad w \mapsto \begin{cases} 0 & \text{if } w = 0, \\ w \log w & \text{else.} \end{cases} \quad (1.7)$$



It is strictly convex. To compute its Legendre transformation we note, that

$$(f')^{-1} = \lambda \mapsto e^{\lambda-1} \quad (1.8)$$

Thus

$$\begin{aligned} f^*(\lambda) &= \lambda \cdot (f')^{-1}(\lambda) - f\left((f')^{-1}(\lambda)\right) \\ &= \lambda \cdot e^{\lambda-1} - e^{\lambda-1} \log\left(e^{\lambda-1}\right) \\ &= e^{\lambda-1}. \end{aligned}$$

Thus  $f^*$  is smooth and strictly non-decreasing.

## Sample Variance

We define the sample variance to be

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad w \mapsto (w - 1/n)^2 \quad (1.9)$$

It is strictly convex. To compute its Legendre transformation we note, that

$$(f')^{-1} = \lambda \mapsto \frac{\lambda}{2} + \frac{1}{n} \quad (1.10)$$

Thus

$$\begin{aligned} f^*(\lambda) &= \lambda \cdot \left(\frac{\lambda}{2} + \frac{1}{n}\right) - \left(\left(\frac{\lambda}{2} + \frac{1}{n}\right) - \frac{1}{n}\right)^2 \\ &= \frac{\lambda^2}{4} + \frac{\lambda}{n}. \end{aligned}$$

Thus  $f^*$  is smooth. To eliminate some variables in the optimization problem, we need  $f^*$  also to be strictly non-decreasing. But the sample variance violates this assumption.

## Constraints

Let's turn our attention to the constraints. The first constraint makes sure we do not extrapolate from the population. The second constraint norms the weights. The third constraint controls the bias of the resulting estimator.

## Regression Basis

We adopt ideas from [GKKW02]. Another angle would be sieve estimates [New97] where the number of basis functions can grow slower than  $N$ . Their notion of (weak)

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consistency [GKKW02, Definition 1.1] for noiseless estimands is

$$\mathbf{E} \left[ \int_{\mathcal{X}} \left| \sum_{k=1}^N B_k(x) \cdot m(X_k) - m(x) \right|^2 \mathbf{P}_X(dx) \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.11)$$

Universal consistency in this sense holds, if this is true for all distributions with  $\mathbf{E}[m(X)^2] < \infty$  (cf. [GKKW02, Definition 1.3]).

We adopt a slightly different notion of consistency. The next theorem dose the translation work.

**Theorem 1.1.** *Assume  $\mathbf{E}[m(X)^2] < \infty$  and the basis function are (weak) universal consistency in the sense of [GKKW02, Definition 1.3]. Then it holds for all  $\varepsilon > 0$*

$$\mathbf{P} \left[ \left| \sum_{k=1}^N B_k(X) \cdot m(X_k) - m(X) \right| \geq \varepsilon \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.12)$$

**Proof.** By Markov's inequality it holds

$$\begin{aligned} & \mathbf{P} \left[ \left| \sum_{k=1}^N B_k(X) \cdot m(X_k) - m(X) \right| \geq \varepsilon \right] \\ & \leq \frac{\mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X) \cdot m(X_k) - m(X) \right|^2 \right]}{\varepsilon^2} \\ & = \frac{\mathbf{E} \left[ \mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X) \cdot m(X_k) - m(X) \right|^2 \middle| X_1, \dots, X_N \right] \right]}{\varepsilon^2} \\ & = \frac{\mathbf{E} \left[ \int_{\mathcal{X}} \left| \sum_{k=1}^N B_k(x) \cdot m(X_k) - m(x) \right|^2 \mathbf{P}_X(dx) \right]}{\varepsilon^2}. \end{aligned}$$

The last equality is due to [GKKW02, (1.2)]. By the weak universal consistency of  $B$  the last expression goes to 0 as  $N \rightarrow \infty$ .  $\square$

Classical choices of the basis functions are **partitioning estimates** and **kernel estimates**(cf. [GKKW02, §4,§5]).

### Partitioning Estimates

We consider a partition  $\mathcal{P}_N = \{A_{N,1}, A_{N,2}, \dots\}$  of  $\mathbb{R}^d$  and define  $A_N(x)$  to be the cell of  $\mathcal{P}_N$  containing  $x$ . We define  $N$  basis functions  $B_k$  of the covariates by

$$B_k(x) := \frac{\mathbf{1}_{X_k \in A_N(x)}}{\sum_{j=1}^N \mathbf{1}_{X_j \in A_N(x)}}, \quad k = 1, \dots, N.$$

The euclidian norm of the basis functions is bounded above by 1 .

$$\|B(x)\|^2 = \sum_{k=1}^n \left( \frac{\mathbf{1}_{X_k \in A_n(x)}}{\sum_{j=1}^n \mathbf{1}_{X_j \in A_n(x)}} \right)^2 \leq \sum_{k=1}^n \frac{\mathbf{1}_{X_k \in A_n(x)}}{\sum_{j=1}^n \mathbf{1}_{X_j \in A_n(x)}} = 1 .$$

Under mild conditions, the basis functions are universally consistent.

**Theorem 1.2.** *If for each sphere  $S$  centered at the origin*

$$\max_{j: A_{N,j} \cap S \neq \emptyset} \text{diam } A_{N,j} \rightarrow 0 \quad \text{for } N \rightarrow \infty \quad (1.13)$$

and

$$\frac{\#\{j: A_{N,j} \cap S \neq \emptyset\}}{N} \rightarrow 0 \quad \text{for } N \rightarrow \infty \quad (1.14)$$

then the partitioning regression function estimate (definition) is universally consistent (definition).

**Proof.** [GKKW02, Theorem 4.2.] □

**Corollary 1.2.1.** *Assume  $\mathbf{E}[m(X)^2] < \infty$  and the basis functions  $B$  belong to a partitioning estimate. Furthermore assume that the conditions of Theorem 1.2 are met. Then it holds for all  $\varepsilon > 0$*

$$\mathbf{P} \left[ \left| \sum_{k=1}^N B_k(X) \cdot m(X_k) - m(X) \right| \geq \varepsilon \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty . \quad (1.15)$$

## Kernel Estimates

Let  $K: \mathbb{R}^d \rightarrow [0, 1]$  (bounded kernel) and  $h_n > 0$  (bandwidth). For examples see [GKKW02, §5.1.]. We define

$$B_k(x) := \frac{K\left(\frac{x-X_k}{h_n}\right)}{\sum_{i=1}^N K\left(\frac{x-X_i}{h_n}\right)} . \quad (1.16)$$

By the boundedness of the kernel it follows  $\|B(x)\| \leq 1$  .

**Theorem 1.3.** *Assume that there are balls  $S_{0,r}$  of radius  $r$  and balls  $S_{0,R}$  of radius  $R$  centered at the origin with  $0 < r \leq R$ , and a constant  $b > 0$  such that*

$$\mathbf{1}_{\{x \in S_{0,R}\}} \geq K(x) \geq b \cdot \mathbf{1}_{\{x \in S_{0,r}\}} \quad (1.17)$$

(boxed kernel). Then for bandwidths with  $h_n \rightarrow 0$  and  $n \cdot h_n^d \rightarrow \infty$  as  $n \rightarrow \infty$  the kernel estimate is weakly universally consistent.

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**Corollary 1.3.1.** *Assume  $\mathbf{E}[m(X)^2] < \infty$  and the basis functions  $B$  belong to a kernel estimate. Furthermore assume that the conditions of Theorem 1.3 are met. Then it holds for all  $\varepsilon > 0$*

$$\mathbf{P} \left[ \left| \sum_{k=1}^N B_k(X) \cdot m(X_k) - m(X) \right| \geq \varepsilon \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.18)$$

In the sequel we mainly work with the dual problem.

## Dual Problem

**Theorem.** *The dual of Problem 1.1 is the unconstrained optimization problem*

$$\underset{\lambda_0, \dots, \lambda_N \in \mathbb{R}}{\text{minimize}} \quad \frac{1}{N} \sum_{i=1}^N [T_i \cdot f^*(\lambda_0 + \langle B(X_i), \lambda \rangle) - (\lambda_0 + \langle B(X_i), \lambda \rangle)] + \langle \delta, |\lambda| \rangle.$$

where

$$f^* : \mathbb{R} \rightarrow \mathbb{R}, \quad x^* \mapsto x^* \cdot (f')^{-1}(x^*) - f((f')^{-1}(x^*))$$

is the Legendre transformation of  $f$ , the vector  $B(X_i) = [B_1(X_i), \dots, B_n(X_i)]^\top$  denotes the  $N$  basis functions of the covariates of unit  $i \in \{1, \dots, N\}$  and  $|\lambda| = [|\lambda_1|, \dots, |\lambda_N|]^\top$ , where  $|\cdot|$  is the absolute value of a real-valued scalar. Moreover, if  $\lambda^\dagger$  is an optimal solution of the above problem then the optimal solution to problem Problem 1.1 is given by

$$w_i^\dagger = (f')^{-1}(\langle B(X_i), \lambda^\dagger \rangle + \lambda_0^\dagger) \quad \text{for } i \in \{1 \dots, n\}.$$

## Plan of proof

We want to apply Theorem 4.4. To this end, we find the suitable **matrix notation**. ([WZ19, p.20-22] fail to do so. The problem is, that they divide by 0 in the second display on p.21). Theorem 4.4 covers only parts of the constraints, so we apply the argument in [WZ19, p.19-20] to eliminate the remaining **non-negativity constraints**.

### **Proof. Matrix notation**

We consider the vector of basis functions of the covariates of unit  $i \in \{1, \dots, n\}$ , that is,

$$B(X_i) := [B_1(X_i), \dots, B_N(X_i)]^\top,$$

the constraints vectors

$$\begin{aligned} d &:= \begin{bmatrix} 0_n \\ -N \cdot \delta \pm \sum_{i=1}^N B_k(X_i) \end{bmatrix}, \\ a &:= N \end{aligned}$$

the matrix of the basis functions of the treated

$$\mathbf{B}(\mathbf{X}) := \begin{bmatrix} B(X_1), \dots, B(X_n) \end{bmatrix}$$

and the constraint matrices

$$\begin{aligned} \mathbf{U} &:= \begin{bmatrix} \mathbf{I}_n \\ \pm \mathbf{B}(\mathbf{X}) \end{bmatrix}. \\ \mathbf{A} &:= \mathbf{1}_n \end{aligned}$$

By Example 4.1 the convex conjugate of the objective function of Problem 1.1 is

$$[x_1^*, \dots, x_n^*]^\top \mapsto \sum_{i=1}^n f^*(x_i^*),$$

Before we apply Theorem 4.4 we eliminate the non-negativity constraints. To this end, we consider the objective function  $G$  of the dual problem and update it until we reach its final form. We write

$$\lambda_d =: \begin{bmatrix} \rho \\ \lambda^+ \\ \lambda^- \end{bmatrix} \tag{1.19}$$

$$\begin{aligned} G(\lambda_d, \lambda_0) &= G(\rho, \lambda^+, \lambda^-, \lambda_0) \\ &:= \sum_{i=1}^N -f^*(\rho_i + \lambda_0 + \langle B(X_i), \lambda^+ - \lambda^- \rangle) + (\lambda_0 + \langle B(X_i), \lambda^+ - \lambda^- \rangle) \\ &\quad - N \cdot \langle \delta, \lambda^+ + \lambda^- \rangle \end{aligned}$$

Since we maximize  $G$  and  $f^*$  is strictly non-decreasing,  $\rho = 0$  is optimal. We update  $G$ .

$$\begin{aligned} G(\lambda^+, \lambda^-, \lambda_0) &= \sum_{i=1}^N -f^*(\lambda_0 + \langle B(X_i), \lambda^+ - \lambda^- \rangle) + (\lambda_0 + \langle B(X_i), \lambda^+ - \lambda^- \rangle) \\ &\quad - N \cdot \langle \delta, \lambda^+ + \lambda^- \rangle \end{aligned}$$

### Non-negativity constraints

Next we want to remove the non-negativity constraints on  $\lambda^\pm$ . We show for all  $i \in \{1, \dots, N\}$

$$\begin{aligned} \text{either} \quad & \lambda_i^+ > 0 \\ \text{or} \quad & \lambda_i^- > 0. \end{aligned}$$

Assume towards a contradiction that there exists  $i \in \{1, \dots, N\}$  such that  $\lambda_i^+ > 0$  and  $\lambda_i^- > 0$  and that  $\lambda^\pm$  is optimal. Consider

$$\tilde{\lambda} := \left[ \lambda_1^+, \dots, \lambda_i^+ - (\lambda_i^+ \wedge \lambda_i^-), \dots, \lambda_N^+, \lambda_1^-, \dots, \lambda_i^- - (\lambda_i^+ \wedge \lambda_i^-), \dots, \lambda_N^-, \lambda_0 \right]^\top. \quad (1.20)$$

Since  $\lambda_i^\pm - (\lambda_i^+ \wedge \lambda_i^-) \geq 0$ , the perturbed vector  $\tilde{\lambda}$  is in the domain of the optimization problem. But

$$G(\tilde{\lambda}) - G(\lambda) = 2N \cdot \delta_i \cdot (\lambda_i^+ \wedge \lambda_i^-) > 0, \quad (1.21)$$

which contradicts the optimality of  $\lambda$ . But then  $\lambda_i^\pm \geq 0$  collapses to  $\lambda_i \in \mathbb{R}$  for all  $i \in \{0, \dots, N\}$ , that is,  $\lambda_i = \lambda_i^+ - \lambda_i^-$ . Note that  $|\lambda_i| = \lambda_i^+ + \lambda_i^-$ .

We update the objective function one more time. Multiplying with  $-1/N$  and introducing  $T$  we get

$$\underset{\lambda_0, \dots, \lambda_N \in \mathbb{R}}{\text{minimize}} \quad \frac{1}{N} \sum_{i=1}^N [T_i \cdot f^*(\lambda_0 + \langle B(X_i), \lambda \rangle) - (\lambda_0 + \langle B(X_i), \lambda \rangle)] + \langle \delta, |\lambda| \rangle.$$

We apply Theorem 4.4 to finish the proof. □

We have gathered all the tools to tackle consistency of the weighted mean.

## 2 Consistency

Throughout this section assume the existence of an optimal solution  $(\lambda^\dagger, \lambda_0^\dagger)$ . We use a hint from the last display of [WZ19, p.22]. The high-level idea is, to connect the optimality of a dual solution to being in the neighborhood of an oracle parameter by looking at the objective function of the dual. We deliver the omitted technical details.

### Neighbourhood of Oracle Parameter

Let  $\lambda^*$  denote the vector with coordinates

$$\lambda_i^* := f'(1/\pi_i) - \lambda_0^\dagger, \quad (2.1)$$

where  $\pi_i = \mathbf{P}[T_i = 1|X_i]$  is the **propensity score** of the  $i$ -th unit.

**Theorem 2.1.** *For all  $\varepsilon > 0$  it holds*

$$\mathbf{P} \left[ \left\| \lambda^\dagger - \lambda^* \right\| \geq \varepsilon \right] \rightarrow 0 \quad \text{for } N \rightarrow \infty. \quad (2.2)$$

We want to leverage the convexity of the objective function of the dual to get

$$\mathbf{P} \left[ \left\| \lambda^\dagger - \lambda^* \right\| \leq \varepsilon \right] = \mathbf{P} \left[ \inf_{\|(\Delta, \Delta_0)\|=\varepsilon} G(\lambda^* + \Delta, \lambda_0^\dagger + \Delta_0) - G(\lambda^*, \lambda_0^\dagger) \geq 0 \right].$$

We learned about a similar idea from [WZ19, p.22]. The next Lemma makes this rigorous.

**Lemma 2.1.** *Let  $m \in \mathbb{N}$  and  $g : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$  be convex. Then for all  $y \in \mathbb{R}^m$  and  $\varepsilon > 0$*

$$\inf_{\|\Delta\|=\varepsilon} g(y + \Delta) - g(y) \geq 0 \quad (2.3)$$

*implies the existence of a global minimum  $y^* \in \mathbb{R}^m$  of  $g$  satisfying  $\|y^* - y\| \leq \varepsilon$ .*

**Proof.** Since  $y + \varepsilon B$  is convex, it contains a local minimum of  $g$ . Suppose towards a contradiction that  $y^* \in y + \varepsilon B$  is a local minimum, but not a global one, and (2.3) is true. Then it holds

$$g(x) < g(y^*) \quad \text{for some } x \in \mathbb{R}^m \setminus (y + \varepsilon B). \quad (2.4)$$

## 2 Consistency

Furthermore, since  $y + \varepsilon B$  is compact and contains  $y^*$ , the line segment connecting  $y^*$  and  $x$  intersects the boundary of  $y + \mathcal{C}$ , that is, there exist  $\theta \in (0, 1)$  and  $\Delta_x$  with  $\|\Delta_x\| = \varepsilon$  such that

$$\theta x + (1 - \theta)y^* = y + \Delta_x. \quad (2.5)$$

It follows

$$\begin{aligned} g(y^*) &\leq g(y) \leq g(y + \Delta_x) = g(\theta x + (1 - \theta)y^*) \\ &\leq \theta g(x) + (1 - \theta)g(y^*) < g(y^*), \end{aligned} \quad (2.6)$$

which is a contradiction. The first inequality is due to  $y^*$  being a local minimum of  $g$  in  $y + \varepsilon B$ , the second inequality is due to (2.3) being true, the equality is due to (2.5), the third inequality is due to the convexity of  $g$  and the strict inequality is due to (2.4). Thus every local minimum of  $g$  in  $y + \varepsilon B$  is also a global minimum.  $\square$

**Proof.** The objective function  $G$  of the dual satisfies

$$G(\lambda, \lambda_0) := \frac{1}{N} \sum_{i=1}^N [T_i \cdot f^*(\lambda_0 + \langle B(X_i), \lambda \rangle) - (\lambda_0 + \langle B(X_i), \lambda \rangle)] + \langle \delta, |\lambda| \rangle.$$

Without the last term, this is a differentiable convex function.

It is well known that a differentiable convex function  $g$  satisfies

$$g(x) - g(y) \geq \nabla g(y)^\top (x - y) \quad \text{for all } x, y. \quad (2.7)$$

The gradient of

$$g := (\lambda, \lambda_0) \mapsto \frac{1}{N} \sum_{i=1}^N [T_i \cdot f^*(\lambda_0 + \langle B(X_i), \lambda \rangle) - (\lambda_0 + \langle B(X_i), \lambda \rangle)] \quad (2.8)$$

is

$$\nabla g = (\lambda, \lambda_0) \mapsto \frac{1}{N} \sum_{i=1}^N \left[ T_i \cdot (f')^{-1}(\lambda_0 + \langle B(X_i), \lambda \rangle) - 1 \right] [B(X_i)^\top, 1]^\top \quad (2.9)$$

Thus

$$\begin{aligned} &G(\lambda^* + \Delta, \lambda_0^\dagger + \Delta_0) - G(\lambda^*, \lambda_0^\dagger) \\ &\geq -\frac{1}{N} \sum_{i=1}^N \left[ B(X_i)^\top, 1 \right] \cdot \begin{bmatrix} \Delta \\ \Delta_0 \end{bmatrix} \left( 1 - T_i \cdot (f')^{-1} \left( \langle B(X_i), \lambda^* \rangle + \lambda_0^\dagger \right) \right) \\ &\quad + \langle \delta, |\lambda^* + \Delta| - |\lambda^*| \rangle. \end{aligned} \quad (2.10)$$

We fix  $\tilde{\varepsilon} > 0$  and establish the lower bound  $-\tilde{\varepsilon}$  with probability going to 1 as  $N \rightarrow \infty$ .

We control the **first term** by (what?) and the **second term** by  $\|\delta\|$ .



## First Term

We note, that by  $\|B(x)\| \leq 1$  and the Cauchy-Schwarz inequality it holds

$$\left[ B(X_i)^\top, 1 \right] \cdot \begin{bmatrix} \Delta \\ \Delta_0 \end{bmatrix} \lesssim \|(\Delta, \Delta_0)\| = \varepsilon. \quad (2.11)$$

Next, we see that

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N \left( 1 - T_i \cdot (f')^{-1} \left( \langle B(X_i), \lambda^* \rangle + \lambda_0^\dagger \right) \right) \\ & \lesssim \frac{1}{N} \sum_{i=1}^N \left| 1 - \frac{T_i}{\pi_i} \right| + \frac{1}{N} \sum_{i=1}^N \left| \langle B(X_i), \lambda^* \rangle + \lambda_0^\dagger - f' \left( \frac{1}{\pi_i} \right) \right| \\ & =: S_N + M_N. \end{aligned} \quad (2.12)$$

With  $\tilde{\varepsilon} > 0$  fixed previously, we want to establish the upper bound  $\tilde{\varepsilon}/(2\varepsilon)$  with probability going to 1 as  $N \rightarrow \infty$ .

First, we bound  $S_N$ . By the properties of conditional expectation it holds

$$\mathbf{E} \left[ \frac{T}{\pi(X)} \right] = \mathbf{E} \left[ \frac{\mathbf{E}[T|X]}{\pi(X)} \right] = 1.$$

By the weak law of large numbers (L1 version ? some assumption on  $1/\pi$ ?)

$$\mathbf{P}[S_N \geq \tilde{\varepsilon}/(4\varepsilon)] \rightarrow 0 \quad \text{for } N \rightarrow \infty. \quad (2.13)$$

Next, we bound  $M_N$ . Recall that  $\sum_{k=1}^N B_k(x) = 1$ . Thus

$$\langle B(X), \lambda^* \rangle + \lambda_0^\dagger = \sum_{k=1}^N B_k(X) \left( f' \left( \frac{1}{\pi_k} \right) - \lambda_0^\dagger \right) + \lambda_0^\dagger = \sum_{k=1}^N B_k(X) \cdot f' \left( \frac{1}{\pi_k} \right).$$

By Markov's inequality it holds

$$\begin{aligned} & \mathbf{P}[M_N \geq \tilde{\varepsilon}/(4\varepsilon)] \\ & \leq \frac{4\varepsilon}{\tilde{\varepsilon}} \frac{1}{N} \sum_{i=1}^N \mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X_i) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) \right| \right] \\ & \leq \frac{4\varepsilon}{\tilde{\varepsilon}} \mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi(X)} \right) \right| \right] \\ & \leq \frac{4\varepsilon}{\tilde{\varepsilon}} \mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi(X)} \right) \right|^2 \right]^{1/2} \rightarrow 0 \quad \text{for } N \rightarrow \infty. \end{aligned}$$

The convergence is due to the universal consistency of  $B$ . This establishes the desired bound of  $\tilde{\varepsilon}/(2\varepsilon)$  in (2.12). Together with (2.11) we conclude that the **first term** in (2.10) is bounded below by  $-\tilde{\varepsilon}/2$  with probability going to 1 as  $N \rightarrow \infty$ .

## Second Term

It holds

$$|x + y| - |x| \geq -|y| \quad \text{for all } x, y.$$

Since  $\delta \geq 0$  we get

$$\begin{aligned} & \langle \delta, |\lambda^* + \Delta| - |\lambda^*| \rangle \\ & \geq -\langle \delta, |\Delta| \rangle \geq -\|\delta\| \|\Delta\| \geq -\|\delta\| \|(\Delta, \Delta_0)\| \geq -\|\delta\| \varepsilon \geq -\tilde{\varepsilon}/2, \end{aligned}$$

with probability going to 1 as  $N \rightarrow \infty$ . The convergence is due to  $\|\delta\|$  converging to 0 in probability.

## Conclusion

With the analysis of the **first** and **second term** in (2.10) we conclude

$$G(\lambda^* + \Delta, \lambda_0^\dagger + \Delta_0) - G(\lambda^*, \lambda_0^\dagger) \geq -\tilde{\varepsilon} \quad (2.14)$$

with probability going to 1 as  $N \rightarrow \infty$ . Since this holds true for all  $\tilde{\varepsilon} > 0$  we get

$$G(\lambda^* + \Delta, \lambda_0^\dagger + \Delta_0) - G(\lambda^*, \lambda_0^\dagger) \geq 0 \quad (2.15)$$

with probability going to 1 as  $N \rightarrow \infty$ . But this holds for all  $(\Delta, \Delta_0)$  with  $\|(\Delta, \Delta_0)\| = \varepsilon$ . Thus

$$\inf_{\|(\Delta, \Delta_0)\|=\varepsilon} G(\lambda^* + \Delta, \lambda_0^\dagger + \Delta_0) - G(\lambda^*, \lambda_0^\dagger) \geq 0 \quad (2.16)$$

with probability going to 1 as  $N \rightarrow \infty$ . Thus by Lemma 2.1

$$\mathbf{P} \left[ \left\| \lambda^\dagger - \lambda^* \right\| \geq \varepsilon \right] \rightarrow 0 \quad \text{for } N \rightarrow \infty. \quad (2.17)$$

Finally, note that this holds for all  $\varepsilon > 0$ . This finishes the proof.  $\square$

## Consistency for Inverse Propensity Score

**Theorem 2.2.** *For all  $\varepsilon > 0$  it holds*

$$\mathbf{P} [|w(X) - 1/\pi(X)| \geq \varepsilon] \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

*Furthermore, it holds*

$$\mathbf{E} \left[ |w(X) - 1/\pi(X)|^2 \right]^{1/2} \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

**Proof.** We employ the consistency of the dual variable, the universal consistency and boundedness of the regression basis and the constraint on the arithmetic mean of the weights.

$$\begin{aligned}
\left| w(X) - \frac{1}{\pi(X)} \right| &= \left| (f')^{-1} \left( \langle B(X), \lambda^\dagger \rangle + \lambda_0^\dagger \right) - \frac{1}{\pi(X)} \right| \\
&\lesssim \left| \langle B(X), \lambda^\dagger - \lambda^* \rangle \right| + \left| \langle B(X), \lambda^\dagger \rangle + \lambda_0^\dagger - f' \left( \frac{1}{\pi(X)} \right) \right| \\
&\lesssim \left\| \lambda^\dagger - \lambda^* \right\| + \left| \sum_{k=1}^N B_k(X) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi(X)} \right) \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,
\end{aligned} \tag{2.18}$$

with probability going to 1 as  $N \rightarrow \infty$ .

If we prove boundedness,  $L_2$ -convergence follows readily.

$$\begin{aligned}
\left| w(X) - \frac{1}{\pi(X)} \right|^2 &\leq \left| w(X) - \frac{1}{C_\pi} \right|^2 \\
&\lesssim \left| \langle B(X), \lambda^\dagger \rangle + \lambda_0^\dagger - f' \left( \frac{1}{C_\pi} \right) \right|^2 \\
&\lesssim \left( \text{diam } \Theta + f' \left( \frac{1}{C_\pi} \right) \right)^2
\end{aligned}$$

□

## Consistency of the Weighted Mean

**Theorem 2.3.** For all  $\varepsilon > 0$  it holds

$$\mathbf{P} \left[ \left| \frac{1}{N} \sum_{i=1}^n w_i Y_i - \mathbf{E}[Y(1)] \right| \geq \varepsilon \right] \rightarrow 0 \quad \text{for } N \rightarrow \infty,$$

that is, the weighted mean is a consistent estimator. Furthermore, it holds for all  $p \in [1, \infty)$

$$\mathbf{E} \left| \frac{1}{N} \sum_{i=1}^n w_i Y_i - \mathbf{E}[Y(1)] \right|^p \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

**Proof.** Let  $\mathbf{Y}(1)$  be the vector with  $i$ -th coordinate  $Y_i(1)$ , that is, the vector of marginal potential outcomes under treatment. Note, that  $\mathbf{Y}(1)$  is usually unknown. Nevertheless, we can leverage its existence in the following error decomposition. Also

## 2 Consistency

note, that for  $i > n$  we may choose  $w_i = 1/\pi_i$ .

$$\begin{aligned}
\left| \frac{1}{N} \sum_{i=1}^n w_i Y_i - \mathbf{E}[Y(1)] \right| &\leq \left| \frac{1}{N} \left( \sum_{i=1}^n w_i B(X_i) - \sum_{i=1}^N B(X_i) \right)^\top \mathbf{Y}(1) \right| \\
&\quad + \left| \frac{1}{N} \sum_{i=1}^N (T_i \cdot w_i - 1) (\mathbf{E}[Y(1)|X_i] - \langle B(X_i), \mathbf{Y}(1) \rangle) \right| \\
&\quad + \left| \frac{1}{N} \sum_{i=1}^N T_i (w_i - 1/\pi_i) (Y_i - \mathbf{E}[Y(1)|X_i]) \right| \\
&\quad + \left| \frac{1}{N} \sum_{i=1}^N T_i / \pi_i (Y_i - \mathbf{E}[Y(1)|X_i]) + (\mathbf{E}[Y(1)|X_i] - \mathbf{E}[Y(1)]) \right| \\
&=: R_1 + R_2 + R_3 + R_4
\end{aligned}$$

### Analysis of $R_1$

By the Cauchy-Schwarz inequality it holds

$$\left| \frac{1}{N} \left( \sum_{i=1}^n w_i B(X_i) - \sum_{i=1}^N B(X_i) \right)^\top \mathbf{Y}(1) \right| \leq \|\delta\| \|\mathbf{Y}(1)\| \lesssim \|\delta\| N \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

### Analysis of $R_2$

This calculation will be central to the asymptotic normality.

$$\begin{aligned}
&\mathbf{P}[R_2 \geq \varepsilon] \\
&\leq \varepsilon^{-1} \frac{1}{N} \sum_{i=1}^N \mathbf{E}[|(T_i \cdot w_i - 1) (\mathbf{E}[Y(1)|X_i] - \langle B(X_i), \mathbf{Y}(1) \rangle)|] \\
&\leq \varepsilon^{-1} \mathbf{E}[|w(X) - 1/\pi(X)|^2]^{1/2} \mathbf{E}[|\mathbf{E}[Y(1)|X_i] - \langle B(X_i), \mathbf{Y}(1) \rangle|^2]^{1/2} \rightarrow 0
\end{aligned}$$

for  $N \rightarrow \infty$ . Notice that the rates multiply. This is important for later.

### Analysis of $R_3$

By Theorem? it holds

$$\mathbf{P}[R_3 \geq \varepsilon] \leq \frac{\mathbf{E}[|w(X) - 1/\pi(X)|^2]^{1/2}}{\varepsilon} \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

## Analysis of $R_4$

$$\begin{aligned}
\mathbf{E}[Y(T) \cdot (T / \pi(X))] &= \mathbf{E}[Y(1) / \pi(X) | T = 1] \cdot \mathbf{P}[T = 1] \\
&= \int_{\mathcal{X}} \mathbf{E}[Y(1) | X = x, T = 1] \cdot (\mathbf{P}[T = 1] / \pi(x)) \mathbf{P}_{X|T}(dx | 1) \\
&= \int_{\mathcal{X}} [Y(1) | X = x] \mathbf{P}_X(dx) \\
&= \mathbf{E}[Y(1)].
\end{aligned} \tag{2.19}$$

The first equality holds because of the definition of the weights. The second, third and last equality stem from  $T \in \{0, 1\}$ , and the law of total expectation, applied with  $T$  and  $X$ . The fourth equality is justified by the assumption of conditional unconfoundedness. The density transformation is due to Bayes's Theorem. Thus the expectation of the summands in  $R_4$  is 0. Convergence follows from the WLLN.

## Conclusion

We conclude that

$$\left| \frac{1}{N} \sum_{i=1}^n w_i Y_i - \mathbf{E}[Y(1)] \right| \leq \varepsilon \tag{2.20}$$

with probability going to 1 as  $N \rightarrow \infty$ . To show the second statement we calculate

$$\left| \frac{1}{N} \sum_{i=1}^n w_i Y_i - \mathbf{E}[Y(1)] \right|^p = \left| \frac{1}{N} \sum_{i=1}^n w_i (Y_i - \mathbf{E}[Y(1)]) \right|^p \tag{2.21}$$

$$\leq \frac{1}{N} \sum_{i=1}^n w_i |Y_i - \mathbf{E}[Y(1)]|^p \leq (2M)^p \tag{2.22}$$

□



### 3 Asymptotic Normality and Convergence to Gaussian Bridge

**Theorem 3.1.** *Under conditions the partition estimate has*

$$\mathbf{E} \|m_N - m\|^2 \leq C_{\mathbf{P}} N^{-\frac{2}{d+2}} \quad (3.1)$$

**Theorem 3.2.** *Under conditions the kernel estimate has*

$$\mathbf{E} \|m_N - m\|^2 \leq C_{\mathbf{P}} N^{-\frac{2}{d+2}} \quad (3.2)$$

#### Learning Rates for the Dual

**Theorem 3.3.** *Under conditions*

$$\mathbf{P} \left[ \left\| \lambda^\dagger - \lambda^* \right\| \leq C_{\mathbf{P}} C_\tau \varepsilon_n \right] \geq 1 - \tau, \quad (3.3)$$

where  $\varepsilon_n$  is the square root of the basis function Learning rate and  $C_\tau$  depends on the Concentration Inequality. We need bernstein confidence  $\sqrt{\log(1/\tau)}$  to preserve minimal Learning rate for  $d = 1$ .

#### Plan of Proof

The path is similar to that of the previous chapter. We will use strong convexity to obtain a quadratic term. This gives us the flexibility to obtain learning rates.

**Theorem 3.4.** (Bernstein's inequality) *Let  $(\Omega, \mathcal{A}, \mathbf{P})$  be a probability space,  $B > 0$  and  $\sigma > 0$  be real numbers, and  $n \geq 1$  be an integer. Furthermore, let  $X_1, \dots, X_n : \Omega \rightarrow \mathbb{R}$  be independent random variables satisfying  $\mathbf{E}[X_i] = 0$ ,  $\|X_i\|_\infty \leq B$  and  $\mathbf{E}[X_i^2] \leq \sigma^2$  for all  $i = 1, \dots, n$ . Then we have*

$$\mathbf{P} \left[ \left| \frac{1}{n} \sum_{i=1}^n X_i \right| \leq \sqrt{\frac{2\sigma^2 \log(e/\tau)}{n}} + \frac{2B \log(e/\tau)}{3n} \right] \geq 1 - \tau \quad \text{for all } \tau > 0.$$

**Proof.** See [SC08, Theorem 6.12] for the one-sided version. The two-sided version, as stated in the above theorem, is an easy consequence. We omit the details.  $\square$

### Strong Convexity

For simpler notation we define  $B_0(x) := 1$ . The Hessian matrix of  $g$  as in ?? is

$$\nabla^2 g(\lambda_0^\dagger, \lambda^\dagger) = \left[ \frac{1}{N} \sum_{i=1}^n \left( (f')^{-1} \right)' \left( \lambda_0^\dagger + \langle B(X_i), \lambda^\dagger \rangle \right) B_k(X_i) \cdot B_l(X_i) \right]_{0 \leq k, l \leq N}. \quad (3.4)$$

Since  $\left( (f')^{-1} \right)' > 0$  by the strict convexity of  $f$ ,  $(\lambda_0^\dagger, \lambda^\dagger) \in \Theta$ , where  $\Theta$  is compact parameter space, and the support of  $X$  is compact it holds

$$\left( (f')^{-1} \right)' \left( \lambda_0^\dagger + \langle B(X_i), \lambda^\dagger \rangle \right) > \frac{1}{C}. \quad (3.5)$$

Consider

$$\mathbf{B}(X) := \begin{bmatrix} 1 & B_1(X_1) & \cdots & B_N(X_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & B_1(X_n) & \cdots & B_N(X_n) \end{bmatrix} \in \mathbb{R}^{n \times (N+1)} \quad (3.6)$$

Since  $B_k(x) \geq 0$  for all  $k$  it holds

$$\nabla^2 g(\lambda_0^\dagger, \lambda^\dagger) \succcurlyeq \frac{1}{C} N^{-1} \mathbf{B}(X)^\top \mathbf{B}(X) \succcurlyeq \frac{\lambda_{\min}(N^{-1} \mathbf{B}(X)^\top \mathbf{B}(X))}{C} \cdot \mathbf{I} \succcurlyeq \frac{1}{C \sqrt{\log(1/\tau)}} \cdot \mathbf{I} \quad (3.7)$$

with probability greater than  $1 - \tau$ . Thus  $g$  is strongly convex with parameter  $\left( C \sqrt{\log(1/\tau)} \right)^{-1}$  with probability greater than  $1 - \tau$ . We analyze  $1 - T_i/\pi_i$ . It holds  $\mathbf{E}[1 - T_i/\pi_i] = 0$ . Furthermore,

$$\left| 1 - \frac{T_i}{\pi_i} \right| \leq 1 \wedge \frac{1 - \pi_i}{\pi_i} \leq 1 + \frac{1 - \pi_i}{\pi_i} = \frac{1}{\pi_i} \leq \frac{1}{C_\pi} \quad \text{almost surely.} \quad (3.8)$$

Also

$$\mathbf{E} \left[ \left| 1 - \frac{T_i}{\pi_i} \right|^2 \right] = 1 - 2\mathbf{E} \left[ \frac{T_i}{\pi_i} \right] + \mathbf{E} \left[ \frac{T_i^2}{\pi_i^2} \right] = \mathbf{E} \left[ \frac{1}{\pi_i} \right] - 1 \leq \frac{1}{C_\pi}. \quad (3.9)$$

Thus, by Bernstein's inequality, it holds

$$\mathbf{P} \left[ \left| \frac{1}{n} \sum_{i=1}^n 1 - \frac{T_i}{\pi_i} \right| \lesssim \sqrt{\frac{\log(e/\tau)}{N}} \right] \geq 1 - \tau \quad \text{for all } \tau > 0. \quad (3.10)$$

Next, we analyze

$$\sum_{k=1}^N B_k(X_i) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) - \mathbf{E}[\cdots]. \quad (3.11)$$



To this end,

$$\begin{aligned}
& \left| \sum_{k=1}^N B_k(X_i) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) \right| \\
& \leq \left| \sum_{k=1}^N B_k(X_i) \cdot \left( f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) \right) \right| \leq \sum_{k=1}^N B_k(X_i) \cdot \left| f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) \right| \\
& \leq 2f' \left( \frac{1}{C_\pi} \right).
\end{aligned}$$

Also

$$\begin{aligned}
& \mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X_i) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) - \mathbf{E}[\dots] \right|^2 \right]^{1/2} \\
& \leq \mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X_i) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) \right|^2 \right]^{1/2} + \left| \mathbf{E} \left[ \sum_{k=1}^N B_k(X_i) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) \right] \right| \\
& \lesssim N^{-1/(d+2)} + 2f' \left( \frac{1}{C_\pi} \right).
\end{aligned}$$

Thus, by Bernstein's inequality, it holds

$$\mathbf{P} \left[ \left| \frac{1}{N} \sum_{i,k=1}^N B_k(X_i) \cdot f' \left( \frac{1}{\pi_k} \right) - f' \left( \frac{1}{\pi_i} \right) - \mathbf{E}[\dots] \right| \lesssim \sqrt{\frac{\log(e/\tau)}{N}} \right] \geq 1 - \tau \quad \text{for all } \tau > 0. \quad (3.12)$$

Similar to the analysis of the first and second term we get

$$G(\lambda^* + \Delta, \lambda_0^\dagger + \Delta_0) - G(\lambda^*, \lambda_0^\dagger) \geq \|\Delta\| \left( \frac{\|\Delta\|}{C\sqrt{\log(1/\tau)}} - \sqrt{\frac{\log(1/\tau)}{N}} - N^{-1/(d+2)} \right) \quad (3.13)$$

with probability greater than  $1 - \tau$ . Thus for

$$\|\Delta\| = C \log(1/\tau) N^{-1/(d+2)} \quad (3.14)$$

it holds

$$G(\lambda^* + \Delta, \lambda_0^\dagger + \Delta_0) - G(\lambda^*, \lambda_0^\dagger) \geq 0 \quad (3.15)$$

with probability greater than  $1 - \tau$ . We conclude

$$\mathbf{P} \left[ \left\| \lambda^\dagger - \lambda^* \right\| \lesssim \log(1/\tau) N^{-1/(d+2)} \right] \geq 1 - \tau. \quad (3.16)$$

### Learning Rates for the Primal

**Theorem 3.5.** *Under conditions the weights satisfy*

$$\mathbf{E}[|w^\dagger(X) - 1/\pi(X)|^2]^{1/2} \leq C_{\mathbf{P}} \sqrt{\log(n)} n^{-1/(2+d)} \quad (3.17)$$

where  $\text{varepsilon}_n$  depends on the Learning rate of the basis functions and the confidence of the dual.  $C_{\mathbf{P}}$  depends on the size of the parameter space.

**Proof.**

$$\mathbf{E}[|w^\dagger(X) - 1/\pi(X)|^2]^{1/2} = \mathbf{E} \left[ \left| (f')^{-1} \left( \langle B(X), \lambda^\dagger \rangle + \lambda_0^\dagger \right) - 1/\pi(X) \right|^2 \right]^{1/2} \quad (3.18)$$

$$\leq \left| (f')^{-1} \right|_L (I_1 + I_2) \quad (3.19)$$

where

$$I_1 := \left( \mathbf{E} \left\| \lambda^\dagger - \lambda^* \right\|^2 \right)^{1/2} \quad (3.20)$$

$$I_2 := \mathbf{E} \left[ \left| \sum_{k=1}^N B_k(X) \cdot f'(X_k) - f'(X) \right|^2 \right]^{1/2} \quad (3.21)$$

It holds  $I_2 \leq n^{-1/(d+2)}$  by the lr of the basis. To analyse  $I_1$  we use the lr of the dual.

$$I_1 \leq C_\tau n^{-1/(d+2)} + \sqrt{\tau} \cdot \text{diam } \Theta \quad (3.22)$$

Note that the Markov confidence  $1/\sqrt{\tau}$  is insufficient. With Bernstein confidence, bounded diameter and  $\tau = n^{-2/(d+2)}$  we get

$$I_1 \lesssim \log(N) \cdot N^{-1/(2+d)} \quad (3.23)$$

Thus

$$\mathbf{E}[|w^\dagger(X) - 1/\pi(X)|^2]^{1/2} \lesssim \log(N) \cdot N^{-1/(2+d)} \quad (3.24)$$

□

### Asymptotic Normality of the Weighted Mean

**Theorem 3.6.** *The estimate*

$$\sqrt{N} \left( \frac{1}{N} \sum_{i=1}^n w_i \mathbf{1}_{\{Y_i \leq t\}} - \mathbf{P}[Y(1) \leq t] \right) \quad (3.25)$$

is Asymptotically normal. Furthermore, the stochastic process

$$\sqrt{N} \left( \frac{1}{N} \sum_{i=1}^n w_i \mathbf{1}_{\{Y_i \leq z\}} - \mathbf{P}[Y(1) \leq z] \right)_{z \in \mathbb{R}} \quad (3.26)$$

converges to a Gaussian process with mean 0 and covariance ??.

**Proof.**

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^n w_i \mathbf{1}_{\{Y_i \leq t\}} - \mathbf{P}[Y(1) \leq t] \\ &= \frac{1}{N} \left( \sum_{i=1}^n w_i B(X_i) - \sum_{i=1}^N B(X_i) \right)^\top \mathbf{Y}(1) \\ &+ \frac{1}{N} \sum_{i=1}^N (T_i \cdot w_i - 1) (\mathbf{P}[Y(1) \leq t | X_i] - \langle B(X_i), \mathbf{Y}(1) \rangle) \\ &+ \frac{1}{N} \sum_{i=1}^N T_i (w_i - 1/\pi_i) (\mathbf{1}_{\{Y_i \leq t\}} - \mathbf{P}[Y(1) \leq t | X_i]) \\ &+ \frac{1}{N} \sum_{i=1}^N T_i / \pi_i (\mathbf{1}_{\{Y_i \leq t\}} - \mathbf{E}[Y(1) \leq t | X_i]) + (\mathbf{P}[Y(1) \leq t | X_i] - \mathbf{P}[Y(1) \leq t]) \\ &=: R_1 + R_2 + R_3 + R_4 \end{aligned}$$

The term

$$\left| \frac{1}{N} \sum_{i=1}^N (T_i w_i - 1) (\mathbf{P}[Y(1) \leq t | X_i] - \langle B(X_i), \lambda^* \rangle) \right| \quad (3.27)$$

gives back the expectation. We control its rate with the factor of basis rates and primal weights. Choose  $d$  to be faster than  $\sqrt{n}$ . The rest follows with standard empirical theory. Expectations are 0.  $\|\delta\|$  has to converge fast enough.

$$\begin{aligned} & \sqrt{N} \cdot \mathbf{E}[(T \cdot w(X) - 1) (\mathbf{P}[Y(1) \leq t | X] - \langle B(X), \lambda^* \rangle)] \\ &= \sqrt{N} \cdot \mathbf{E}[\pi(X) \cdot (w(X) - 1/\pi(X)) (\mathbf{P}[Y(1) \leq t | X] - \langle B(X), \lambda^* \rangle)] \\ &\leq \sqrt{N} \cdot \mathbf{E} \left[ |w(X) - 1/\pi(X)|^2 \right]^{1/2} \mathbf{E} \left[ |\mathbf{P}[Y(1) \leq t | X] - \langle B(X), \lambda^* \rangle|^2 \right]^{1/2} \\ &\lesssim \sqrt{N} \log(N) N^{-1/(d+2)} \cdot N^{-1/(d+2)} \\ &\lesssim \log(N) N^{-1/6} \rightarrow 0 \quad \text{for } N \rightarrow \infty \end{aligned}$$

Note, that we get no convergence for  $d > 1$ . Also note that

$$\mathbf{E}[T \cdot w(X) - 1 | X, X_1, \dots, X_N] = \mathbf{E}[T | X] w(X) - 1 = \pi(X) (w(X) - 1/\pi(X))$$

□

### 3 Asymptotic Normality and Convergence to Gaussian Bridge

Let

$$B_{\mathcal{F}} := \{f \in \mathcal{F} : \|f\|_{\infty} \leq 1\} \quad (3.28)$$

be the intersection of the unit ball with  $\mathcal{F}$ .

**Lemma 3.1.** *Let  $(\varepsilon_N)$  be a sequence converging to 0 and let  $(\mathcal{F}_N) := (\varepsilon_N \cdot B_{\mathcal{F}})$  denote the sequence of scaled unit balls in  $\mathcal{F}$ . Assume that there exists  $k < 2$  such that the covering number of the unit ball in  $\mathcal{F}$  satisfies*

$$\log N_{[]}(\varepsilon, B_{\mathcal{F}}, L_2(\mathbf{P})) \leq \left(\frac{1}{\varepsilon}\right)^k \quad \text{for all } \varepsilon > 0. \quad (3.29)$$

Then it holds  $\|G_N\|_{\mathcal{F}_N}^* \xrightarrow{\mathbf{P}} 0$  for  $N \rightarrow \infty$ .

**Proof.** By maximal inequalities it holds

$$\begin{aligned} \mathbf{E}^* \left[ \|G_N\|_{\mathcal{F}_N} \right] &\lesssim J_{[]}(\varepsilon_N, \mathcal{F}_N, L_2(\mathbf{P})) \\ &= \int_0^{\varepsilon_N} \sqrt{\log N_{[]}(\varepsilon, \mathcal{F}_N, L_2(\mathbf{P}))} d\varepsilon \\ &= \int_0^{\varepsilon_N} \sqrt{\log N_{[]}(\varepsilon/\varepsilon_N, B_{\mathcal{F}}, L_2(\mathbf{P}))} d\varepsilon \\ &\leq \int_0^{\varepsilon_N} \left(\frac{\varepsilon_N}{\varepsilon}\right)^{k/2} d\varepsilon \\ &\lesssim \varepsilon_N^{k/2} \frac{1}{1-k/2} \varepsilon_N^{1-k/2} \\ &\lesssim \varepsilon_N \\ &\rightarrow 0 \quad \text{for } N \rightarrow \infty. \end{aligned}$$

Note, that  $k < 2$ . By the boundedness of  $\mathbf{E}^*$  there is no measurability problem. By Markov's Inequality it holds

$$\mathbf{P} \left[ \|G_N\|_{\mathcal{F}_N}^* \geq \varepsilon \right] \leq \varepsilon^{-1} \mathbf{E}^* \left[ \|G_N\|_{\mathcal{F}_N} \right] \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

□

We define  $f^\dagger$  by

$$f^\dagger(T, X, Y) := T \left( (f')^{-1} \left( \langle B(X), \lambda^\dagger \rangle + \lambda_0^\dagger \right) - \frac{1}{\pi(X)} \right) (Y(T) - \mathbf{E}[Y(1)|X]) \quad (3.30)$$

**Lemma 3.2.** *Assume that  $f^\dagger \in \mathcal{F}$  almost surely for some function class  $\mathcal{F}$  satisfying the requirements of the previous lemma. It then holds  $|G_N f^\dagger| \xrightarrow{\mathbf{P}} 0$  for  $N \rightarrow \infty$ .*

**Proof.** Let  $(\varepsilon_N)$  be the lr of the primal then

$$\mathbf{P} \left[ f^\dagger \in \mathcal{F}_N \right] \rightarrow 1 \quad \text{for } N \rightarrow \infty. \quad (3.31)$$

For all  $\varepsilon > 0$  it holds

$$\begin{aligned} \mathbf{P} \left[ \left| G_N f^\dagger \right| \leq \varepsilon \right] &\geq \mathbf{P} \left[ \left| G_N f^\dagger \right| \leq \|G_N\|_{\mathcal{F}_N}^* \leq \varepsilon \right] \\ &\geq \mathbf{P} \left[ f^\dagger \in \mathcal{F}_N \text{ and } \|G_N\|_{\mathcal{F}_N}^* \leq \varepsilon \right] \\ &\geq \mathbf{P} \left[ f^\dagger \in \mathcal{F}_N \right] - \mathbf{P} \left[ \|G_N\|_{\mathcal{F}_N}^* \geq \varepsilon \right] \\ &\rightarrow 1. \end{aligned}$$

The last step is due to Lemma.

The convergence of the stochastic process: [vdV00, Example 19.6] Slutsky for stochastic process?  $\square$

### Gaussian Bridge

We can even view  $\frac{1}{\sqrt{n}} \sum_{i=1}^n S_i$  as an empirical process  $\mathbb{G}_n f$  indexed over

$$f_\Phi(T, X, Y) = \frac{T}{\pi(X)} (\Phi(Y) - \mathbf{E}[\Phi(Y)|X]) + \mathbf{E}[\Phi(Y)|X]. \quad (3.32)$$

If  $\mathcal{F} = \{f_\Phi : \Phi \in \text{some set}\}$  is  $\mathbf{P}$ -Donsker, the empirical process converges to a tight gaussian process. Then the functional delta Method is applicable.

## 3.1 Application to Plug In Estimators

A plethora of applications of the delta method to estimates of the distribution function are to be found in [vdV00] and [vdvW13]. This includes Quantile estimation [vdV00, §21] [vdvW13, §3.9.21/24], survival analysis via Nelson-Aalen and Kaplan-Meier estimator [vdvW13, §3.9.19/31], Wilcoxon Test [vdvW13, §3.9.4.1], and much more. Maybe Booststrapping from the weighted distribution is also sensible .



## 4 Convex Analysis

In our application we want to analyse a convex optimization problem by its dual problem. In particular we want to obtain primal optimal solutions from dual solutions. To accomplish the task we need technical tools from convex analysis, mainly conjugate calculus and some KKT related results.

Our starting point is the support function intersection rule [MMN22, Theorem 4.23]. We give the details in the case of finite dimensions and refer for the rest of the proof to the book. The support function intersection rule is applied to give first conjugate sum and then chain rule, which are vital to calculating convex conjugates. The proofs are omitted, since the book is thorough enough. The material we present is very well known. As an introduction, we recommend the recent book [MMN22] and classical reference [Roc70]. We finish the chapter with ideas from [TB91]. They provide the high-level ideas to obtain for strictly convex functions a dual relationship between optimal solutions. We will deliver the details that are omitted in the paper.

### 4.1 A Convex Analysis Primer

Throughout this section let  $n \in \mathbb{N}$ .

#### Sets

A subset  $C \subseteq \mathbb{R}^n$  is called **convex set**, if for all  $x, y \in C$  and all  $\theta \in [0, 1]$ , we have  $\theta x + (1 - \theta)y \in C$ . Many set operations preserve convexity. Among them forming the **Cartesian product** of two convex sets, **intersection** of a collection of convex sets and taking the **inverse image under linear functions**.

The classical theory evolves around the question if convex sets can be separated.

**Definition.** Let  $C_1$  and  $C_2$  be two non-empty convex sets in  $\mathbb{R}^n$ . A hyperplane  $H$  is said to **separate**  $C_1$  and  $C_2$  if  $C_1$  is contained in one of the closed half-spaces associated with  $H$  and  $C_2$  lies in the opposite closed half-space. It is said to separate  $C_1$  and  $C_2$  **properly** if  $C_1$  and  $C_2$  are not both contained in  $H$ .

We need a refined concept of interiors, since some convex sets have empty interior. To this end, we call a set  $A \subseteq \mathbb{R}^n$  **affine set**, if  $\alpha x + (1 - \alpha)y \in A$  for all  $x, y \in A$  and  $\alpha \in \mathbb{R}$ . The **affine hull**  $\text{aff}(\Omega)$  of a set  $\Omega \subseteq \mathbb{R}^n$  is the smallest affine set that includes  $\Omega$ . We define the **relative interior**  $\text{ri}\Omega$  of a set  $\Omega \subseteq \mathbb{R}^n$  to be the interior relative to the affine hull, that is,

$$\text{ri}(\Omega) := \{x \in \Omega \mid \exists \varepsilon > 0 : (x + \varepsilon B_{\mathbb{R}^n}) \cap \text{aff}(\Omega) \subset \Omega\}. \quad (4.1)$$

**Theorem 4.1.** (Convex separation in finite dimension) *Let  $C_1$  and  $C_2$  be two non-empty convex sets in  $\mathbb{R}^n$ . Then  $C_1$  and  $C_2$  can be properly separated if and only if  $\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset$ .*

**Proof.** [Roc70, Theorem 11.3] □

We collect some useful properties of relative interiors before we get on to convex functions.

**Proposition 4.1.** *Let  $C$  be a non-empty convex set in  $\mathbb{R}^n$ . The following holds:*

- (i)  $\text{ri}(C) \neq \emptyset$  if and only if  $C \neq \emptyset$
- (ii)  $\text{cl}(\text{ri } C) = \text{cl } C$  and  $\text{ri}(\text{cl } C) = \text{ri}(C)$
- (iii)  $\text{ri}(C) = \{z \in C : \text{for all } x \in C \text{ there exists } t > 0 \text{ such that } z + t(z - x) \in C\}$
- (iv) Suppose  $\bigcap_{i \in I} C_i \neq \emptyset$  for a finite index set  $I$ . Then  $\text{ri}(\bigcap_{i \in I} C_i) = \bigcap_{i \in I} \text{ri}(C_i)$ .
- (v) Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear function. Then  $\text{ri } L(C) = L(\text{ri } C)$ . If it also holds  $L^{-1}(\text{ri } C) \neq \emptyset$ , we have  $\text{ri } L^{-1}(C) = L^{-1}(\text{ri } C)$ .
- (vi)  $\text{ri}(C_1 \times C_2) = \text{ri } C_1 \times \text{ri } C_2$

**Proof.** For a proof of (i)-(v) we refer to [Roc70, Theorem 6.2 - 6.7].

To prove (vi) we use (iii). Let  $(z_1, z_2) \in \text{ri}(C_1 \times C_2)$ . Then for all  $(x_1, x_2) \in C_1 \times C_2$  there exists  $t > 0$  such that

$$z_i + t(z_i - x_i) \in C_i \quad \text{for all } i \in \{1, 2\}. \quad (4.2)$$

Using (iii) again, we get  $\text{ri}(C_1 \times C_2) \subseteq \text{ri } C_1 \times \text{ri } C_2$ . Suppose  $(z_1, z_2) \in \text{ri } C_1 \times \text{ri } C_2$ . By (iii), for all  $(x_1, x_2) \in C_1 \times C_2$  there exist  $(t_1, t_2) > 0$  such that

$$z_i + t_i(z_i - x_i) \in C_i \quad \text{for all } i \in \{1, 2\}. \quad (4.3)$$



If  $t_1 = t_2$  we recover (4.2) from (4.3). By (iii) it holds  $(z_1, z_2) \in \text{ri}(C_1 \times C_2)$ . If  $t_1 < t_2$  we define  $\theta := \frac{t_1}{t_2} \in (0, 1)$ . Consider (4.3) with  $i = 2$ , together with  $z_2 \in C_2$  and the convexity of  $C_2$ . It follows

$$z_2 + t_1(z_2 - x_2) = \theta \cdot (z_2 + t_2(z_2 - x_2)) + (1 - \theta) \cdot z_2 \in C_2. \quad (4.4)$$

Now we consider (4.4) and (4.3) with  $i = 1$ . This gives (4.2) with  $t = t_1$ . As before, it follows  $(z_1, z_2) \in \text{ri}(C_1 \times C_2)$ . If  $t_1 > t_2$  similar arguments lead to the same result. We have proven  $\text{ri}(C_1 \times C_2) \supseteq \text{ri } C_1 \times \text{ri } C_2$  and equality.  $\square$

## Functions

A function  $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is called **convex function**, if the area above its graph, that is, its epigraph(cf. [MMN22, §2.4.1]), is convex. We shall often use an equivalent definition. To this end, a function  $f$  is convex if and only if

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and all } \theta \in [0, 1]. \quad (4.5)$$

This definition extends to convex combinations  $\theta_1, \dots, \theta_m \in [0, 1]$  with  $\sum_{i=1}^m \theta_i = 1$ , that is, a function  $f$  is convex if and only if

$$f\left(\sum_{i=1}^m \theta_i x_i\right) \leq \sum_{i=1}^m \theta_i f(x_i) \quad \text{for all } x_1, \dots, x_m \in \mathbb{R}^n. \quad (4.6)$$

We call a function **strictly convex** if the inequality in (4.5) is strict.

We define the **domain**  $\text{dom } f$  of a convex function  $f$  to be the set where  $f$  is finite, that is,

$$\text{dom } f := \{x \in \mathbb{R}^n : f(x) < \infty\}. \quad (4.7)$$

The domain of a convex function is convex. We say that  $f$  is a **proper function** if  $\text{dom } f \neq \emptyset$ .

For any  $\bar{x} \in \text{dom } f$  we call  $x^* \in \mathbb{R}^n$  a **subgradient** of  $f$  at  $\bar{x}$  if for all  $x \in \mathbb{R}^n$  it holds

$$\langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}). \quad (4.8)$$

We denote the collection of all subgradients at  $\bar{x}$ , that is, the **subdifferential** of  $f$  at  $\bar{x}$ , as  $\partial f(\bar{x})$ . If  $f$  is differentiable at  $\bar{x}$  it holds  $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$  and thus

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq f(x) - f(\bar{x}). \quad (4.9)$$

We call a differentiable function  $f$  **strongly convex** with parameter  $m > 0$  if for all  $x, y \in \text{dom } f$  it holds

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \frac{m}{2} \|y - x\|^2. \quad (4.10)$$

If  $f$  is twice continuously differentiable, then it is strongly convex with parameter  $m > 0$  if and only if the matrix

$$\nabla^2 f(x) - m \cdot \mathbf{I} \quad \text{is positive semi-definite for all } x \in \text{dom } f, \quad (4.11)$$

where  $\nabla^2 f$  is the Hessian Matrix.

One important application of convex functions is in optimization. There we often analyse a dual problem instead, which relies on the notion of **convex conjugate**  $f^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  of  $f$  defined by

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} \langle x^*, x \rangle - f(x). \quad (4.12)$$

Even for arbitrary functions, the convex conjugate is convex(cf.). Like in differential calculus, there exist sum and chain rule for computing the convex conjugate.

## 4.2 Conjugate Calculus

The goal of this section is to establish the tools to calculate convex conjugates. We cite the conjugate sum and chain rule without proof. After some examples, we cite the Fenchel-Rockafellar Theorem.

**Definition 4.1.** (Convex conjugate) *Given a function  $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ , the **convex conjugate**  $f^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  of  $f$  is defined as*

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} (x^*)^T x - f(x) \quad (4.13)$$

Note that  $f$  in Definition ?? does not have to be convex. On the other hand, the convex conjugate is always convex:

**Proposition 4.2.** *Let  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be a proper function. Then its convex conjugate  $f^* : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is convex.*

**Proof.** [MMN22, Proposition 4.2] □

**Theorem 4.2.** *Let  $f, g : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be proper convex functions and  $\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g)) \neq \emptyset$ . Then we have the **conjugate sum rule***

$$(f + g)^*(x^*) = (f^* \square g^*)(x^*) \quad (4.14)$$

for all  $x^* \in \mathbb{R}^n$ . Moreover, the infimum in  $(f^* \square g^*)(x^*)$  is attained, i.e., for any  $x^* \in \text{dom}(f + g)^*$  there exists vectors  $x_1^*, x_2^*$  for which

$$(f + g)^*(x^*) = f^*(x_1^*) + g^*(x_2^*), \quad x^* = x_1^* + x_2^*. \quad (4.15)$$

**Proof.** [MMN22, Theorem 4.27(c)] □

**Theorem 4.3.** Let  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a linear map (matrix) and  $g : \mathbb{R}^n \rightarrow (-\infty, \infty]$  a proper convex function. If  $\text{Im}(A) \cap \text{ri}(\text{dom}(g)) \neq \emptyset$  it follows the **conjugate chain rule**

$$(g \circ A)^*(x^*) = \inf_{y^* \in (A^*)^{-1}(x^*)} g^*(y^*). \quad (4.16)$$

Furthermore, for any  $x^* \in \text{dom}(g \circ A)^*$  there exists  $y^* \in (A^*)^{-1}(x^*)$  such that  $(g \circ A)^*(x^*) = g^*(y^*)$ .

**Proof.** [MMN22, Theorem 4.28(c)] □

**Example 4.1.** Let  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be a proper convex function, that is,  $\text{dom } f \neq \emptyset$  and  $f$  is convex. In steps we apply the conjugate chain and sum rule, together with mathematical induction, to prove the conjugate relationship

$$\begin{aligned} S_{f,n} : \mathbb{R}^n &\rightarrow \overline{\mathbb{R}}, & (x_1, \dots, x_n) &\mapsto \sum_{i=1}^n f(x_i), \\ S_{f,n}^* : \mathbb{R}^n &\rightarrow \overline{\mathbb{R}}, & (x_1^*, \dots, x_n^*) &\mapsto \sum_{i=1}^n f^*(x_i^*). \end{aligned}$$

This relationship is very natural and the ensuing calculations serve to confirm our intuition.

First, we work in the projections on the coordinates. For the  $i$ -th coordinate, where  $i = 1, \dots, n$ , this is

$$p_i : \mathbb{R}^n \rightarrow \mathbb{R}, \quad (x_1, \dots, x_n) \mapsto x_i. \quad (4.17)$$

All projections  $p_i$  are linear function with matrix representation  $e_i^\top$ , where  $e_i$  is  $i$ -the coordinate vector. The adjoint of  $p_i$  is therefore

$$p_i^* : \mathbb{R} \rightarrow \mathbb{R}^n, \quad x \mapsto e_i \cdot x. \quad (4.18)$$

For the inverse image of the adjoint of  $p_i$  it holds

$$(p_i^*)^{-1} \{(x_1^*, \dots, x_n^*)\} = \begin{cases} \{x_i^*\}, & \text{if } x_j^* = 0 \text{ for all } j \neq i, \\ \emptyset & \text{else.} \end{cases} \quad (4.19)$$

Throughout this example we use the asterisk character  $*$  somewhat inconsistently. Note that  $f^*$  is the convex conjugate of the function  $f$  and  $p_i^*$  is the adjoint linear function of the projection on the  $i$ -th coordinate. Likewise, we denote dual variables, that is, the arguments of convex conjugates, as  $x^*$ .

Next, we employ the conjugate chain rule to establish the conjugate relationship

$$\begin{aligned} f_i : \mathbb{R}^n &\rightarrow \overline{\mathbb{R}}, & (x_1, \dots, x_n) &\mapsto x_i \mapsto f(x_i), \\ f_i^* : \mathbb{R}^n &\rightarrow \overline{\mathbb{R}}, & (x_1^*, \dots, x_n^*) &\mapsto \begin{cases} f^*(x_i^*), & \text{if } x_j^* = 0 \text{ for all } j \neq i, \\ \infty & \text{else.} \end{cases} \end{aligned}$$

Note, that  $f_i = (f \circ p_i)$  and  $f_i^* = (f \circ p_i)^*$ . Since  $\text{Im } p_i = \mathbb{R}$  and  $\text{dom } f \neq \emptyset$ , it holds  $\text{Im } p_i \cap \text{ri}(\text{dom } f) \neq \emptyset$ . Then  $f$  and  $p_i$  conform with the demands of the conjugate chain rule. It follows

$$\begin{aligned} f_i^*(x_1^*, \dots, x_n^*) &= (f \circ p_i)^*(x_1^*, \dots, x_n^*) = \inf \{ f^*(y) \mid y \in (p_i^*)^{-1} \{ (x_1^*, \dots, x_n^*) \} \} \\ &= \begin{cases} f^*(x_i^*), & \text{if } x_j^* = 0 \text{ for all } j \neq i, \\ \infty & \text{else,} \end{cases} \end{aligned}$$

where we keep to the convention  $\inf \emptyset = \infty$ . In the same way it follows

$$(S_{f,n} \circ p_{\{1, \dots, n\}})^*(x_1^*, \dots, x_{n+1}^*) = \begin{cases} S_{f,n}^*(x_1^*, \dots, x_n^*) & \text{if } x_{n+1}^* = 0, \\ \infty & \text{else,} \end{cases} \quad (4.20)$$

Next, note that for  $n = 1$  we arrive at the result. Thus, for some  $n \in \mathbb{N}$  it holds  $(S_{f,n})^* = S_{f,n}^*$ . In order to apply the conjugate sum rule to  $S_{f,n}$  and  $f_{n+1}$  we note that

$$\begin{aligned} \text{dom } f_i &= \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \in \text{dom } f \} \neq \emptyset \quad \text{for all } i = 1, \dots, n+1, \\ \bigcap_{i=1}^{n+1} \text{dom } f_i &= \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \in \text{dom } f \text{ for all } i = 1, \dots, n+1 \} \neq \emptyset, \end{aligned}$$

and

$$\begin{aligned} \text{ri}(\text{dom}(S_{f,n} \circ p_{\{1, \dots, n\}})) &\cap \text{ri}(\text{dom } f_{n+1}) \\ &= \text{ri}(\text{dom}(S_{f,n} \circ p_{\{1, \dots, n\}}) \cap \text{dom } f_{n+1}) = \text{ri}\left(\bigcap_{i=1}^{n+1} \text{dom } f_i\right) \neq \emptyset. \end{aligned}$$

By the conjugate sum rule it follows

$$\begin{aligned} (S_{f,n+1})^* &= (S_{f,n} \circ p_{\{1, \dots, n\}} + f_{n+1})^* = (S_{f,n} \circ p_{\{1, \dots, n\}})^* \square f_{n+1}^* \\ &= S_{f,n}^* \circ p_{\{1, \dots, n\}} + f_{n+1}^* = S_{f,n+1}^*. \end{aligned}$$

◇

**Takeaways** Conjugate sum and chain rule are direct consequences of the support function intersection rule. They are powerful tools, that allow us to compute convex conjugates of difficult expressions as well as proving the Fenchel-Rockafellar Duality theorem.

### 4.3 Duality of Optimal Solutions

We consider a general convex optimization problem with matrix equality and inequality constraints. For this problem there exists a related problem, which we call its dual. With ideas from [TB91] we establish a functional relationship between the optimal solution of the original problem and optimal solutions of the dual. The main assumption is that in the original problem we have a strictly convex objective function with continuously differentiable convex conjugate (cf. Definition 4.1).

**Theorem 4.4.** *Consider the optimization problem*

$$\begin{aligned} & \underset{w \in \mathbb{R}^n}{\text{minimize}} && f(w) \\ & \text{subject to} && \mathbf{U}w \geq d, \\ & && \mathbf{A}w = a, \end{aligned} \tag{4.21}$$

*and its dual problem*

$$\begin{aligned} & \underset{\lambda_d \in \mathbb{R}^r, \lambda_a \in \mathbb{R}^s}{\text{maximize}} && \langle \lambda_d, d \rangle + \langle \lambda_a, a \rangle - f^*(\mathbf{U}^\top \lambda_d + \mathbf{A}^\top \lambda_a) \\ & \text{subject to} && \lambda_d \geq 0. \end{aligned} \tag{4.22}$$

*Let  $(\lambda_d^\dagger, \lambda_a^\dagger)$  be an optimal solution to (4.22). If the objective function  $f$  of (4.21) is strictly convex and its convex conjugate  $f^*$  is continuously differentiable, then the unique optimal solution to (4.21) is given by*

$$w^\dagger = \nabla f^*(\mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger). \tag{4.23}$$

#### Plan of Proof

We show that  $w^\dagger$  and  $(\lambda_d^\dagger, \lambda_a^\dagger)$  meet the Karush-Kuhn-Tucker conditions for 4.21, that is, **complementary slackness**

$$\langle \lambda_d^\dagger, d - \mathbf{U}w^\dagger \rangle = 0, \tag{4.24}$$

#### primal and dual feasibility

$$\mathbf{U}w^\dagger \geq d, \tag{4.25}$$

$$\begin{aligned} \mathbf{A}w^\dagger &= a, \\ \lambda_d^\dagger &\geq 0, \end{aligned} \tag{4.26}$$

and **stationarity**

$$0_n \in [\partial f(w^\dagger) + \partial(w \mapsto d - \mathbf{U}w)(w^\dagger) \cdot \lambda_d^\dagger + \partial(w \mapsto a - \mathbf{A}w)(w^\dagger) \cdot \lambda_a^\dagger]. \quad (4.27)$$

Applying the well know result [Roc70, Theorem 28.3] finishes the proof. Apart from elementary calculations, our main tools are the strict convexity of  $f$ , the smoothness of  $f^*$  and

**Proposition 4.3.** [Roc70, Theorem 23.5(a)-(b)]. *For any proper convex function  $g$  and any vector  $w$ , it holds  $t \in \partial f(w)$  if and only if  $x \mapsto \langle x, t \rangle - f(x)$  achieves its supremum at  $w$ .*

**Proof.** Let  $(\lambda_d^\dagger, \lambda_a^\dagger)$  be an optimal solution to (4.22).

### Complementary Slackness

We fix  $\lambda_a^\dagger$  and work with the objective function  $G$  of the dual problem, that is,

$$G(\lambda_d) := \langle \lambda_d, d \rangle + \langle \lambda_a^\dagger, a \rangle - f^*(\mathbf{U}^\top \lambda_d + \mathbf{A}^\top \lambda_a^\dagger).$$

Since  $f^*$  is continuously differentiable, so is  $G$ . Thus

$$\nabla G(\lambda_d^\dagger) := d - \mathbf{U} \cdot \nabla f^*(\mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger) = d - \mathbf{U} w^\dagger.$$

Let  $\lambda_{d,i}^\dagger$  be the  $i$ -th coordinate of  $\lambda_d^\dagger$  and  $\nabla G_i(\lambda_d^\dagger)$  be the  $i$ -th coordinate of  $\nabla G(\lambda_d^\dagger)$ . To establish (4.24) we will show for all coordinates

$$\begin{aligned} \text{either} \quad & \lambda_{d,i}^\dagger = 0 \quad \text{and} \quad \nabla G_i(\lambda_d^\dagger) \leq 0 \\ \text{or} \quad & \lambda_{d,i}^\dagger > 0 \quad \text{and} \quad \nabla G_i(\lambda_d^\dagger) = 0. \end{aligned}$$

It is well know that a concave functions  $g$  satisfies

$$g(x) - g(y) \geq \nabla g(x)^\top (x - y) \quad \text{for all } x, y. \quad (4.28)$$

But  $G$  is concave by the convexity of  $f^*$  (cf. Proposition 4.2).

First, we show

$$\nabla G_i(\lambda_d^\dagger) \leq 0 \quad \text{for all } i \in \{1, \dots, s\}. \quad (4.29)$$

Assume towards a contradiction that  $\nabla G_i(\lambda_d^\dagger) > 0$  for some  $i \in \{1, \dots, s\}$ . By the continuity of  $\nabla G$  there exists  $\varepsilon > 0$  such that  $\nabla G_i(\lambda_d^\dagger + e_i \cdot \varepsilon) > 0$ . It follows from (4.28)

$$G(\lambda_d^\dagger + e_i \cdot \varepsilon) - G(\lambda_d^\dagger) \geq \nabla G_i(\lambda_d^\dagger + e_i \cdot \varepsilon) \cdot \varepsilon > 0,$$

which contradicts the optimality of  $\lambda_d^\dagger$  for (4.22). It follows (4.29).

Next, we assume that  $\lambda_{d,i}^\dagger > 0$  and  $\nabla G_i(\lambda_d^\dagger) < 0$  for some  $i \in \{1, \dots, s\}$ . Again, by the continuity of  $\nabla G$  there exists  $\varepsilon > 0$  such that  $\nabla G_i(\lambda_d^\dagger - e_i \cdot \varepsilon) < 0$  and  $\varepsilon - \lambda_{d,i}^\dagger < 0$ . Thus

$$G(\lambda_d^\dagger - e_i \cdot \varepsilon) - G(\lambda_d^\dagger) \geq \nabla G_i(\lambda_d^\dagger - e_i \cdot \varepsilon) \cdot (-\varepsilon) > 0,$$

which contradicts the optimality of  $\lambda_d^\dagger$ . It follows (4.24), that is, we proved complementary slackness.

### Primal Feasibility

Since  $f^*$  is continuously differentiable it holds

$$\nabla G(\lambda_d^\dagger) = d - \mathbf{U} \cdot \nabla f^* \left( \mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger \right) = d - \mathbf{U} w^\dagger.$$

Thus, by (4.29),  $w^\dagger$  satisfies the inequality constraints in (4.21). To prove this for the equality constraints, we view  $G$  from a different angel. Let for fixed  $\lambda_d^\dagger$

$$G(\lambda_a) := \langle \lambda_a, a \rangle - \left( f^* \left( \mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a \right) - \langle \lambda_d^\dagger, d \rangle \right) =: \langle \lambda_a, a \rangle - g(\lambda_a).$$

The function  $g$  inherits convexity and differentiability from  $f^*$ . From the optimality of  $\lambda_a^\dagger$  we know that  $G$  takes its maximum there. But then by Proposition 4.3 and the differentiability of  $g$  it holds

$$a \in \partial g(\lambda_a^\dagger) = \left\{ \mathbf{A} \cdot \nabla f^* \left( \mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger \right) \right\} = \left\{ \mathbf{A} w^\dagger \right\}. \quad (4.30)$$

Thus  $a = \mathbf{A} w^\dagger$ . But then  $w^\dagger$  satisfies also the equality constraints. We proved (4.25).

### Stationarity

First we show

$$\mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger \in \partial f(w^\dagger). \quad (4.31)$$

By Proposition 4.3 it suffices to show that

$$w \mapsto \langle w, \mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger \rangle - f(w)$$

achieves its supremum at  $w^\dagger$ . Since  $f$  is strictly convex there exists a unique vector  $x^\dagger$  where the above expression achieves its maximum. Since  $f^*$  is differentiable it holds

$$w^\dagger = \nabla f^* \left( \mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger \right) = \nabla \left( \lambda \mapsto \langle x^\dagger, \lambda \rangle - f(x^\dagger) \right) \left( \mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger \right) = x^\dagger.$$



It follows (4.31). Next we show

$$-\mathbf{U}^\top \in \partial(w \mapsto d - \mathbf{U}w)(w^\dagger) \quad \text{and} \quad -\mathbf{A}^\top \in \partial(w \mapsto d - \mathbf{A}w)(w^\dagger). \quad (4.32)$$

To this end, note that

$$\langle -\mathbf{U}^\top e_i, w - w^\dagger \rangle = (d - \mathbf{U}w)_i - (d - \mathbf{U}w^\dagger)_i \quad \text{for all } i \in \{1, \dots, r\}.$$

Thus  $-\mathbf{U}^\top \in \partial(w \mapsto d - \mathbf{U}w)(w^\dagger)$ . In the same way it follows  $-\mathbf{A}^\top \in \partial(w \mapsto d - \mathbf{A}w)(w^\dagger)$ .

From (4.31) and (4.32) we conclude

$$\begin{aligned} 0_n &= \left( \mathbf{U}^\top \lambda_d^\dagger + \mathbf{A}^\top \lambda_a^\dagger \right) - \mathbf{U}^\top \lambda_d^\dagger - \mathbf{A}^\top \lambda_a^\dagger \\ &\in [\partial f(w^\dagger) + \partial(w \mapsto d - \mathbf{U}w)(w^\dagger) \cdot \lambda_d^\dagger + \partial(w \mapsto a - \mathbf{A}w)(w^\dagger) \cdot \lambda_a^\dagger]. \end{aligned}$$

We have proved (4.27), that is, stationarity.

### Dual Feasibility and Conclusion

Dual feasibility (4.26) follows immediately from the optimality of  $\lambda_d^\dagger$  for (4.22). Thus,  $(\lambda_d^\dagger, \lambda_a^\dagger)$  and  $w^\dagger$  satisfy the Karush-Kuhn-Tucker conditions for (4.21). Applying [Roc70, Theorem 28.3] finishes the proof.  $\square$

**Takeaways** For strictly convexity objective functions with continuously differentiable convex conjugate we get a functional relationship of primal and dual solutions via the Karush-Kuhn-Tucker conditions.



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