

Robust Weighting and Matching Techniques for Causal Inference in Observational Studies with Continuous Treatment

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1 Introduction

2 Balancing Weights

2.1 Introduction

2.2 Estimating the Population Mean of Potential Outcomes

2.3 Application of Convex Optimization

Assumption 2.1. Assume that the map $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ has the following properties.

- (i) f is strictly convex.
- (ii) f is lower-semicontinuous and continuously differentiable on $\text{int}(\text{dom}(f))$.
- (iii) The derivative of f on $\text{int}(\text{dom}(f))$ is a diffeomorphism.
- (iv) The Legendre transformation f^* of f is finite.
- (v) The function $x \mapsto xt - f(x)$ takes its supremum on $\text{int}(\text{dom}(f))$ for all $t \in \mathbb{R}$.

We consider the following optimization problem.

Problem 2.1.

$$\underset{w_1, \dots, w_n \in \mathbb{R}}{\text{minimize}} \quad \sum_{i=1}^n T_i f(w_i)$$

subject to the constraints

$$\begin{aligned} w_i T_i &\geq 0, & i &= 1, \dots, n, \\ \sum_{i=1}^n w_i T_i &= 1 \\ \left| \sum_{i=1}^n w_i T_i B_k(X_i) - \frac{1}{n} \sum_{i=1}^n B_k(X_i) \right| &\leq \delta_k, & k &= 1, \dots, K \end{aligned}$$

Theorem 2.1. *Under Assumption, the dual of the above Problem is the unconstrained optimization problem*

$$\underset{\lambda \in \mathbb{R}^K}{\text{minimize}} \quad \frac{1}{n} \sum_{i=1}^n nT_i f^*(\langle B(X_i), \lambda \rangle) - \langle B(X_i), \lambda \rangle + \langle \delta, |\lambda| \rangle,$$

where $t \mapsto f^*(t) = t(f')^{-1}(t) - f((f')^{-1}(t))$ is the Legendre transformation of f , $B(X_i) = [B_1(X_i), \dots, B_K(X_i)]^\top$ denotes the K basis functions of the covariates of unit $i \in \{1, \dots, n\}$ and $|\lambda| = [|\lambda_1|, \dots, |\lambda_K|]^\top$, where $|\cdot|$ is the absolute value of a real-valued scalar. Moreover, if λ^\dagger is an optimal solution then

$$w_i^* = (f')^{-1}(\langle B(X_i), \lambda^\dagger \rangle), \quad i \in \{1, \dots, n\} \quad (2.1)$$

are the unique optimal solutions to (P) .

Proof. We prove the following Lemma at the end of the section.

Lemma 2.1. *The dual of the optimization problem is*

$$\underset{\lambda \in \mathbb{R}^{2K}}{\text{minimize}} \quad \frac{1}{n} \sum_{i=1}^n nT_i f^*(\langle Q_{\bullet i}, \lambda \rangle) - \langle Q_{\bullet i}, \lambda \rangle + \langle d, \lambda \rangle$$

subject to

$$\lambda_k \geq 0 \quad \text{for all } k \in \{1, \dots, K\}, \quad (2.2)$$

where

$$\mathbf{Q} := \begin{bmatrix} \mathbf{I}_n \\ \mathbf{B}(\mathbf{X}) \\ -\mathbf{B}(\mathbf{X}) \end{bmatrix}, \quad \mathbf{B}(\mathbf{X}) := [B(X_1), \dots, B(X_n)], \quad \text{and} \quad d := \begin{bmatrix} 0_n \\ \delta \\ \delta \end{bmatrix}. \quad (2.3)$$

□

2.4 Application of Matrix Concentration Inequalities

3 Convex Analysis

3.1 Basic Notions

3.2 Relative Interior

3.3 Conjugate Calculus

3.4 Tseng Bertsekas

We present the relevant parts of the paper [BT03].

Consider the following optimization problem

$$\underset{x \in \mathbb{R}^m}{\text{minimize}} \quad f(x)$$

subject to the constraints

$$\mathbf{A}x \geq b, \tag{3.1}$$

Where $f : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$, \mathbf{A} is a given $n \times m$ matrix, and b is a vector in \mathbb{R}^n .

Assumption 3.1. Assume that the map $f : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ has the following properties.

- (i) f is strictly convex.
- (ii) f is lower-semicontinuous and continuous on $\text{dom}(f)$.
- (iii) The convex conjugate f^* of f is finite.

The dual optimization problem associated with (P) is

$$\underset{p \in \mathbb{R}^n}{\text{maximize}} \quad q(p)$$

subject to the constraints

$$p \geq 0, \tag{3.2}$$

where $q : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is the concave function given by

$$q(p) := \min_{x \in \mathbb{R}^m} f(x) + \langle p, b - \mathbf{A}x \rangle = \langle p, b \rangle - f^*(\mathbf{A}^\top p). \tag{3.3}$$

The dual problem (D) is a concave program with simple nonnegativity constraints. Furthermore, strong duality holds for (P) and (D), i.e., the optimal value of (P) equals the optimal value of (D).

Since f^* is real-valued and f is strictly convex, f^* and q are continuously differentiable.

Theorem 3.1. [Roc70, Theorem 26.3] *A closed proper convex function is (essentially) strictly convex if and only if its conjugate is essentially smooth.*

We will denote the gradient of q at p by $d(p)$ and its i th coordinate by $d_i(p)$. Since q is continuously differentiable, $d_i(p)$ is continuous, and since q is concave, $d_i(p)$ is nonincreasing in p_i .

By differentiating and by using the chain rule, we obtain the dual cost gradient

$$d(p) = b - \mathbf{A}x, \quad \text{where} \quad x := \nabla f^*(\mathbf{A}^\top p) = \operatorname{argsup}_{\xi \in \mathbb{R}^m} \langle p, \mathbf{A}\xi \rangle - f(\xi). \quad (3.4)$$

The last equality follows from Danskin's Theorem and [Roc70, Theorem 23.5]

Proposition 3.1. (Danskin's Theorem [BT03, page 649]) *Let $Z \subseteq \mathbb{R}^m$ be a non-empty set, and let $\phi : \mathbb{R}^n \times Z \rightarrow \mathbb{R}$ be a continuous function such that $\phi(\cdot, z) : \mathbb{R}^n \rightarrow \mathbb{R}$, viewed as a function of its first argument, is convex for each $z \in Z$. Then the function*

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \mapsto \sup_{z \in Z} \phi(x, z) \quad (3.5)$$

is convex and has directional derivative given by

$$f'(x; y) = \sup_{z \in Z(x)} \phi'(x, z; y), \quad (3.6)$$

where $\phi'(x, z; y)$ is the directional derivative of the function $\phi(\cdot, z)$ at x in the direction y , and

$$Z(x) := \left\{ \bar{z} \in \mathbb{R}^m : \phi(x, \bar{z}) = \sup_{z \in Z} \phi(x, z) \right\}. \quad (3.7)$$

In particular, if $Z(x)$ consists of a unique point \bar{z} and $\phi(\cdot, \bar{z})$ is differentiable at x , and $\nabla f(x) = \nabla_x \phi(x, \bar{z})$, where $\nabla_x \phi(x, \bar{z})$ is the vector with coordinates $(\partial \phi / \partial x_i)(x, \bar{z})$

Note that x is the unique vector satisfying

$$\mathbf{A}p \in \partial f(x). \quad (3.8)$$

From the optimality conditions for (D) it follows that a dual vector is an optimal solution of (D) if and only if

$$p = [p + d(p)]^+, \quad (3.9)$$

where $[\cdot]^+$ is the projection onto the positive orthant, i.e., $[y]^+ = [0 \vee y_1, \dots, 0 \vee y_n]^\top$.

Given an optimal dual solution p , we may obtain an optimal primal solution from the equation $x = \nabla f^*(\mathbf{A}^\top p)$. To see this, note that

$$\mathbf{A}x \geq b \quad \text{and} \quad p_i = 0 \quad \text{for all } i \text{ such that} \quad \sum_{j=1}^m a_{ij}x_j > b_i. \quad (3.10)$$

We can show that p and x satisfy the KKT conditions and thus x is an optimal solution to (P).

Definition 3.1. [Roc70, §28] *By an **ordinary convex program** (P) we mean an optimization problem of the following form*

$$\underset{x \in C}{\text{minimize}} \quad f_0(x)$$

subject to the constraints

$$f_1(x) \leq 0, \dots, f_r(x) \leq 0, \quad f_{r+1}(x) = 0, \dots, f_m(x) = 0, \quad (3.11)$$

where $C \subseteq \mathbb{R}^n$ is a non-empty convex set, f_i is a finite convex function on C for $i \in \{1, \dots, r\}$ and f_i is an affine function on C for $i \in \{r+1, \dots, m\}$.

Definition 3.2. We define $[\lambda_1, \dots, \lambda_m] \in \mathbb{R}^m$ to be a **Karush-Kuhn-Tucker (KKT) vector** for (P) , if

- (i) $\lambda_i \geq 0$ for all $i \in \{1, \dots, r\}$.
- (ii) The infimum of the proper convex function $f_0 + \sum_{i=1}^m \lambda_i f_i$ is finite and equal to the optimal value in (P) .

Theorem 3.2. (Karush-Kuhn-Tucker conditions) Let (P) be an ordinary convex program, $\bar{\alpha} \in \mathbb{R}^m$, and $\bar{z} \in \mathbb{R}^n$. Then $\bar{\alpha}$ is a KKT vector for (P) and \bar{z} is an optimal solution to (P) if and only if \bar{z} and the components α_i of $\bar{\alpha}$ satisfy the following conditions.

- (i) $\alpha_i \geq 0$, $f_i(\bar{z}) \leq 0$, and $\alpha_i f_i(\bar{z}) = 0$ for all $i \in \{1, \dots, r\}$.
- (ii) $f_i(\bar{z}) = 0$ for $i \in \{r+1, \dots, m\}$.
- (iii) $0_n \in [\partial f_0(\bar{z}) + \sum_{\alpha_i \neq 0} \alpha_i \partial f_i(\bar{z})]$.

Proof. [Roc70, Theorem 28.3] □

Takeaways For strictly convex functions we can derive duality in terms of the optimal solutions.

4 Random Matrix Inequalities

4.1 Matrix Analysis

The **trace** of a square matrix, denoted by tr , is the sum of its diagonal entries, i.e. $\text{tr}(\mathbf{B}) = \sum_{j=1}^d b_{jj}$ for $\mathbf{B} \in \mathbb{M}_d$. The trace is unitarily invariant, i.e. $\text{tr}(\mathbf{B}) = \text{tr}(\mathbf{Q}\mathbf{B}\mathbf{Q}^*)$ for all $\mathbf{B} \in \mathbb{M}_d$ for all unitary $\mathbf{Q} \in \mathbb{M}_d$. In particular, the existence of an eigenvalue value decomposition shows that the trace of a Hermitian matrix equals the sum of its eigenvalues. Let $f : I \rightarrow \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Consider a matrix $\mathbf{A} \in \mathbb{H}_d$ whose eigenvalues are contained in I . We define the matrix $f(\mathbf{A}) \in \mathbb{H}_d$ using an eigenvalue decomposition of \mathbf{A} :

$$f(\mathbf{A}) = \mathbf{Q} \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_d) \end{bmatrix} \mathbf{Q}^* \quad \text{where} \quad \mathbf{A} = \mathbf{Q} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} \mathbf{Q}^*. \quad (4.1)$$

The definition of $f(\mathbf{A})$ does not depend on which eigenvalue decomposition we choose. Any matrix function that arises in this fashion is called a **standard matrix function**.

Proposition 4.1. *Let $f, g : I \rightarrow \mathbb{R}$ be real-valued functions on an interval $I \subseteq \mathbb{R}$, and let $\mathbf{A} \in \mathbb{H}_d$ be a Hermitian matrix whose eigenvalues are contained in I .*

- (i) *If λ is an eigenvalue of \mathbf{A} , then $f(\lambda)$ is an eigenvalue of $f(\mathbf{A})$.*
- (ii) *$f(a) \leq g(a)$ for all $a \in I$ implies $f(\mathbf{A}) \preceq g(\mathbf{A})$.*

Proposition 4.2. (Generalized Klein inequality) *Let u_1, \dots, u_n and v_1, \dots, v_n be real-valued functions on an interval I of the real line. Suppose*

$$\sum_{k=1}^n u_k(a)v_k(b) \geq 0 \quad \text{for all } a, b \in I. \quad (4.2)$$

Then

$$\overline{\text{tr}} \left(\sum_{k=1}^n u_k(\mathbf{A})v_k(\mathbf{B}) \right) \geq 0 \quad \text{for all } \mathbf{A}, \mathbf{B} \in \mathbb{H}_d(I). \quad (4.3)$$

Proof. [Pet94, Proposition 3] □

Proposition 4.3. (Hölder inequality for trace) *Let p and q be Hölder conjugate indices. Then*

$$\text{tr}(\mathbf{B}\mathbf{C}) \leq \|\mathbf{B}\|_p \|\mathbf{C}\|_q \quad \text{for all } \mathbf{B}, \mathbf{C} \in \mathbb{M}_d. \quad (4.4)$$

Proof. [Bha97, Corollary IV.2.6] □

4.2 Matrix Concentration Inequalities via the Method of Exchangeable Pairs

Definition 4.1. Let Z and Z' random variables taking values in a Polish space \mathcal{Z} . We say that (Z, Z') is an **exchangeable pair** if it has the same distribution as (Z', Z) . In particular, Z and Z' must share the same distribution.

Definition 4.2. Let (Z, Z') be an exchangeable pair of random variables taking values in a Polish space \mathcal{Z} , and let $\Psi : \mathcal{Z} \rightarrow \mathbb{H}_d$ be a measurable function. Define the random Hermitian matrices

$$\mathbf{X} := \Psi(Z) \quad \text{and} \quad \mathbf{X}' := \Psi(Z'). \quad (4.5)$$

We say that $(\mathbf{X}, \mathbf{X}')$ is a **matrix Stein pair** if there is a constant $\alpha \in (0, 1]$ for which

$$\mathbf{E}[\mathbf{X} - \mathbf{X}' | Z] = \alpha \mathbf{X} \quad \text{almost surely.} \quad (4.6)$$

The constant α is called the **scale factor** of the pair. We always assume $\mathbf{E}[\|\mathbf{X}\|^2] < \infty$.

Lemma 4.1. Suppose that $(\mathbf{X}, \mathbf{X}')$ is a matrix Stein pair with scale factor α . Let $\mathbf{F} : \mathbb{H}_d \rightarrow \mathbb{H}_d$ be a measurable function that satisfies the regularity condition $\mathbf{E}[\|(\mathbf{X} - \mathbf{X}')\mathbf{F}(\mathbf{X})\|] < \infty$. Then

$$\mathbf{E}[\mathbf{X} \cdot \mathbf{F}(\mathbf{X})] = \frac{1}{2\alpha} \mathbf{E}[(\mathbf{X} - \mathbf{X}')(\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}'))]. \quad (4.7)$$

Proof. [MJC⁺14, Lemma 2.4] Suppose that $(\mathbf{X}, \mathbf{X}')$ constructed from an auxiliary exchangeable pair (Z, Z') . The defining property implies

$$\alpha \cdot \mathbf{E}[\mathbf{X} \cdot \mathbf{F}(\mathbf{X})] = \mathbf{E}[\mathbf{E}[\mathbf{X} - \mathbf{X}' | Z] \cdot \mathbf{F}(\mathbf{X})] = \mathbf{E}[(\mathbf{X} - \mathbf{X}')\mathbf{F}(\mathbf{X})] \quad (4.8)$$

□

4.3 Matrix Khintchin Inequality

Theorem 4.1. (Matrix BDG inequality) Let $p = 1$ or $p \geq 3/2$. Suppose that $(\mathbf{X}, \mathbf{X}')$ is a matrix Stein pair where $\mathbf{E}[\|\mathbf{X}\|_{2p}^{2p}] < \infty$. Then

$$\mathbf{E}[\|\mathbf{X}\|_{2p}^{2p}]^{1/(2p)} \leq \sqrt{2p-1} \mathbf{E}[\|\Delta_{\mathbf{X}}\|_p^p]^{1/(2p)}, \quad (4.9)$$

where $\Delta_{\mathbf{X}}$ is the conditional variance.

Proof. [MJC⁺14, §7.3] Apply method of exchangeable pairs, generalized Klein inequality, trace Hölder

□

Theorem 4.2. [MJC⁺14, Corollary 7.3] Suppose that $p = 1$ or $p \geq 3/2$. Consider a finite sequence $(\mathbf{Y}_k)_{k \geq 1}$ of independent, random, Hermitian matrices and a deterministic sequence $(\mathbf{A}_k)_{k \geq 1}$ for which

$$\mathbf{E}[\mathbf{Y}_k] = 0 \quad \text{and} \quad \mathbf{Y}_k^2 \preceq \mathbf{A}_k^2 \quad \text{almost surely for all } k \geq 1. \quad (4.10)$$

Then

$$\mathbf{E} \left[\left\| \sum_{k \geq 1} \mathbf{Y}_k \right\|_{2p}^{2p} \right]^{1/(2p)} \leq \sqrt{p - \frac{1}{2}} \left\| \left(\sum_{k \geq 1} (\mathbf{A}_k^2 + \mathbf{E}[\mathbf{Y}_k^2]) \right)^{1/2} \right\|_{2p}. \quad (4.11)$$

In particular, when $(\xi_k)_{k \geq 1}$ is an independent sequence of Rademacher random variables,

$$\mathbf{E} \left[\left\| \sum_{k \geq 1} \xi_k \mathbf{A}_k \right\|_{2p}^{2p} \right]^{1/(2p)} \leq \sqrt{2p-1} \left\| \left(\sum_{k \geq 1} \mathbf{A}_k^2 \right)^{1/2} \right\|_{2p}. \quad (4.12)$$

4.4 Matrix Moment Inequality

Theorem 4.3. Assume $n \geq 3$

(i) Suppose that $p \geq 1$, and fix $r \geq p \vee 2 \log(n)$. Consider a finite sequence $(\mathbf{S}_k)_{k \geq 1}$ of independent, random, positive-semidefinite matrices with dimension $n \times n$. Then

$$\mathbf{E} \left[\left\| \sum_{k \geq 1} \mathbf{S}_k \right\|^p \right]^{1/p} \leq \left[\left\| \sum_{k \geq 1} \mathbf{E}[\mathbf{S}_k] \right\|^{1/2} + 2\sqrt{er} \mathbf{E}[\max_{k \geq 1} \|\mathbf{S}_k\|^p]^{1/(2p)} \right]^2. \quad (4.13)$$

(ii) Suppose that $p \geq 2$, and fix $r \geq p \vee 2 \log(n)$. Consider a finite sequence $(\mathbf{Y}_k)_{k \geq 1}$ of independent, symmetric, random, self-adjoint matrices with dimension $n \times n$. Then

$$\mathbf{E} \left[\left\| \sum_{k \geq 1} \mathbf{Y}_k \right\|^p \right]^{1/p} \leq \sqrt{er} \left\| \left(\sum_{k \geq 1} \mathbf{E}[\mathbf{Y}_k^2] \right)^{1/2} \right\| + 2er \mathbf{E}[\max_{k \geq 1} \|\mathbf{S}_k\|^p]^{1/p}. \quad (4.14)$$

4.5 Intrinsic Dimension

Definition 4.3. For a positive-semidefinite matrix \mathbf{S} , the **intrinsic dimension** is the quantity

$$\text{intdim}(\mathbf{A}) := \frac{\text{tr} \mathbf{A}}{\|\mathbf{A}\|}.$$

Lemma 4.2. (Intrinsic dimension) Let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a convex function with $\varphi(0) = 0$. For any positive-semidefinite matrix \mathbf{S} it holds that

$$\text{tr}(\varphi(\mathbf{S})) \leq \text{intdim}(\mathbf{S}) \cdot \varphi(\|\mathbf{S}\|).$$

Proof. [Tro15, Lemma 7.5.1] Since φ is convex on any interval $[0, L]$ with $L > 0$ and $\varphi(0) = 0$, it holds

$$\varphi(a) \leq \left(1 - \frac{a}{L}\right) \varphi(0) + \frac{a}{L} \varphi(L) = \frac{a}{L} \varphi(L) \quad \text{for all } a \in [0, L]. \quad (4.15)$$

Since \mathbf{S} is positive-semidefinite, the eigenvalues of \mathbf{S} fall in the interval $[0, L]$, where $L = \|\mathbf{S}\|$.

$$\text{tr}(\varphi(\mathbf{S})) = \sum_{i=1}^d \varphi(\lambda_i) \leq \frac{\sum_{i=1}^d \lambda_i}{\|\mathbf{S}\|} \varphi(\|\mathbf{S}\|) = \frac{\text{tr}(\mathbf{S})}{\|\mathbf{S}\|} \varphi(\|\mathbf{S}\|) = \text{intdim}(\mathbf{S}) \cdot \varphi(\|\mathbf{S}\|). \quad (4.16)$$

□

5 Empirical Processes

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and (\mathcal{X}, Σ) a measurable space. Let $X_j : (\Omega, \mathcal{A}, \mathbf{P}) \rightarrow (\mathcal{X}, \Sigma)$, $j = 1, \dots, n$ be independent and identically-distributed (i.i.d.) random variables with probability distribution \mathbf{P}_X and \mathcal{F} a family of measurable functions $f : (\mathcal{X}, \Sigma) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider the map

$$f \mapsto G_n f := \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n f(X_i) - \mathbf{P}_X f \right), \quad (5.1)$$

where $\mathbf{P}_X f := \int_{\mathcal{X}} f d\mathbf{P}_X$. We call $(G_n f)_{f \in \mathcal{F}}$ the empirical process indexed by \mathcal{F} . Furthermore

$$\|G_n f\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |G_n f|. \quad (5.2)$$

Lemma 5.1. (Bernstein Inequality for Empirical Processes) *For any bounded, measurable function f it holds for all $t > 0$*

$$\mathbf{P}(|G_n f| > t) \leq 2 \exp \left(-\frac{1}{4} \frac{t^2}{\mathbf{P}_X(f^2) + t \|f\|_{\infty} / \sqrt{n}} \right) \quad (5.3)$$

Proof. By the Markov inequality it holds for all $\lambda > 0$

$$\mathbf{P}(G_n f > t) \leq e^{-\lambda t} \mathbf{E} \exp(\lambda G_n f) \quad (5.4)$$

□

Lemma 5.2. *For any finite class \mathcal{F} of bounded, measurable, square-integrable functions, with $|\mathcal{F}|$ elements, it holds*

$$\mathbf{E} \|G_n f\|_{\mathcal{F}} \lesssim \max_{f \in \mathcal{F}} \frac{\|f\|_{\infty}}{\sqrt{n}} \log(1 + |\mathcal{F}|) + \max_{f \in \mathcal{F}} \|f\|_{\mathbf{P}, 2} \sqrt{\log(1 + |\mathcal{F}|)}. \quad (5.5)$$

6 Simple yet useful Calculations

Theorem 6.1. (Multivariate Taylor Theorem) *Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$. Then for all $x, \Delta \in \mathbb{R}^n$ there exists $\xi \in [0, 1]$ such that it holds*

$$\begin{aligned} f(x + \Delta) = f(x) &+ \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} \Delta_i + \sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{\partial^2 f(x + \xi \Delta)}{\partial x_i \partial x_j} \Delta_i \Delta_j \\ &+ \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 f(x + \xi \Delta)}{\partial x_i^2} \Delta_i^2 \end{aligned} \quad (6.1)$$

Corollary 6.1.1. *Let $f \in C^2(\mathbb{R})$. Then for all $a, x, \Delta \in \mathbb{R}^n$ there exist $\xi \in [0, 1]$ such that it holds*

$$f(a^T(x + \Delta)) - f(a^T x) = f'(a^T x) \Delta^T a + \frac{1}{2} f''(a^T(x + \xi \Delta)) \Delta^T A \Delta, \quad (6.2)$$

where $A := aa^T \in \mathbb{R}^{n \times n}$.

Proof. By the chain rule we have for all $a, x, \Delta \in \mathbb{R}^n$ and $\xi \in [0, 1]$

$$\frac{\partial^2 f(a^T(x + \xi \Delta))}{\partial x_i \partial x_j} = f''(a^T(x + \xi \Delta)) a_i a_j. \quad (6.3)$$

Since $A := aa^T$ is symmetric we have

$$\Delta^T A \Delta = 2 \sum_{\substack{i,j=1 \\ i \neq j}}^n a_i a_j \Delta_i \Delta_j + \sum_{i=1}^n a_i^2 \Delta_i^2. \quad (6.4)$$

Plugging (6.3) and (6.4) into (6.1) yields (6.2). □

Proposition 6.1. *For all $x, y \in \mathbb{R}$ it holds*

$$|x + y| - |x| \geq -|y| \quad (6.5)$$

Proof. Checking all 6 combinations of $x + y, x, y$ being nonnegative or negative yields the result. □

Notation Index

$\#A$ cardinality of the set A

$\mathbf{E}[X|Y]$ conditional expectation of the random variable X with respect to $\sigma(Y)$

$\mathbf{E}[X]$ expectation of the random variable X

$\mathbf{Var}[X]$ variance of the random variable X

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ extension of the real numbers

$\xrightarrow{\mathcal{D}}$ convergence of distributions

\mathbf{P} generic probability measure

$\mathbf{P}_X = \mathbf{P} \circ X^{-1}$ distribution of the random variable X

\mathbb{R} set of real numbers

$x \vee y, x \wedge y, x^+, x^-$ maximum, minimum, positive part, negative part of real numbers

$X \sim \mu$ the random variable has distribution μ

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