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Game forms of the Ramsey type problems

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Содержание

1 Terminology

Hypergraph F is a set of sets. Those sets are called edges, and elements of edges are called vertices. Let's denote $\mathcal{V}(F)$ the set of all vertices. Everywhere below we deal with finite hypergraphs only $(|F| < \infty \wedge \forall S \in F\, |S| < \infty)$. The hypergraph is called r-uniform if every is edge has r vertices.

For vertices v_1, v_2 let's denotes $d_{v_1 v_2}(F) := |\{S \in F \mid v_1 \in S \wedge v_2 \in S\}|$ — number of edges that contain both v_1 and v_2. Let's also denote $d_2(F) = \max\limits_{v1,v2 \in \mathcal{V}(F))} d_{v_1 v_2}(F)$ — maximum number of edges intersecting by at least two vertices.

For the hypergraph F and $n \in N$ let's define a game $G(F, n)$ by following rules: players first player (Maker) and second player (Breaker) alternately pick previously unpicked vertexes of F, until all vertexes are picked. Maker picks one vertex per move, Breaker picks $n-1$ vertices per move. The first player wins if exists edge $S \in F$ where all points are picked by first player. Second player wins otherwise.

For $n = 2$ there exists a winning condition for Maker — if for F is r-uniform and $|F| > 2^{r-3} d_2(F) |\mathcal{V}(F)|$ then Maker has a winning strategy. We shell prove generalisation of this statement.

2 Sufficient conditions for a winning strategy

2.1 Definition of the potential

Suppose on m-th move first player have picked vertexes $\{x_1 \dots x_m\} = X$ and second player picked $\{y_1 \dots y_{(c-1)m}\} = Y$. Let's denote potential of this position. To do it, let's define potential of the edge S as

$$P_{X,Y}(S) = \begin{cases} 0 & \text{if } S \cap Y \neq \emptyset \\ c^{-|\{S \setminus X\}|} & \text{otherwise} \end{cases}$$

Two notable properties of this functions are: $P_{X,Y \cup Y'}(S) \leq P_{X,Y}(S)$ for all $X, Y, Y', S \in F$ and if $S \cap (X' \cup Y') = \emptyset$ then $P_{X,Y}(S) = P_{X \cup X',Y}(S) = P_{X,Y \cup Y'}(S) = P_{X \cup X',Y \cup Y'}(S)$.

The potential of this state of the game is the sum of potentials of all edges of hypergraph F:

$$P(X, Y) = \sum_{S \in F} P_{X,Y}(S)$$

.

let's define $\Delta_M P_{X,Y}(v) = P(X \cup \{v\}, Y) - P(X, Y)$ — how much potential of the position would increase if Maker picks v on its next move.

$$
\begin{aligned}
&\Delta_M P_{X,Y}(v) = \\
=&P(X \cup \{v\}, Y) - P(X,Y) = \\
=&\sum_{s \in F} \left(P_{X\cup\{v\},Y}(S) - P_{X,Y}(S)\right) = \\
=&\sum_{\substack{s\in F\\ v\notin S}} \left(\underbrace{P_{X\cup\{v\},Y}(S)}_{\substack{=P_{X,Y}(S)\\ \text{because v not in S}}} - P_{X,Y}(S) \right) + \sum_{\substack{s\in F\\ v\in S}} \left(\underbrace{P_{X\cup\{v\},Y}(S)}_{\substack{=cP_{X,Y}(S)\\ \text{because v in S}}} - P_{X,Y}(S) \right) = \\
=&0 + \sum_{\substack{s\in F\\ v\in S}} (c-1) P_{X\cup\{v\},Y}(S) = \\
=&(c-1)\sum_{\substack{s\in F\\ v\in S}} P_{X\cup\{v\},Y}(S) \qquad =
\end{aligned}
$$

Similary we define $\Delta_B P_{X,Y}(v) = P(X, Y \cup \{v\}) - P(X,Y)$ how much potential of the position would increase if Breaker picks v on its move.

$$
\begin{aligned}
&\Delta_B P_{X,Y}(v) = && (1)\\
=&P(X, Y \cup \{v\}) - P(X,Y) = && (2)\\
=&\sum_{s \in F} \left(P_{X,Y\cup\{v\}}(S) - P_{X,Y}(S)\right) = && (3)\\
=&\sum_{\substack{s\in F\\ v\notin S}} \left(\underbrace{P_{X,Y\cup\{v\}}(S)}_{\substack{=P_{X,Y}(S)\\ \text{because v not in S}}} - P_{X,Y}(S) \right) + \sum_{\substack{s\in F\\ v\in S}} \left(\underbrace{P_{X,Y\cup\{v\}}(S)}_{\substack{=0\\ \text{because v in S}}} - P_{X,Y}(S) \right) = && (4)\\
=&0 + \sum_{\substack{s\in F\\ v\in S}} -P_{X,Y}(S) = && (5)\\
=&-\frac{1}{c-1}\Delta_M P_{X,Y}(v) && (6)
\end{aligned}
$$

One notable property of Δ_B is

$\Delta_B P_{X,Y\cup Y'}(v) \geq \Delta_B P_{X,Y}(v)$ for all $X, Y, Y' \in F$ and $v \in \mathcal{V}(F)\Delta_M P_{X,Y\cup Y'}(v) \leq \Delta_M P_{X,Y}(v)$ for all X, Y, Y

(7)

. It is true because $\Delta_B P_{X,Y\cup Y'}(v) = -\sum_{\substack{s\in F\\ v\in S}} P_{X,Y\cup Y'}(S) \geq -\sum_{\substack{s\in F\\ v\in S}} P_{X,Y}(S) = \Delta_B P_{X,Y}(v)$ and $\Delta_M P_{X,Y} = -(c-1)\Delta_M P_{X,Y}$.

2.2 Sufficient condition for a Maker to have a winning strategy

Theorem 1 *If n-uniform hypergraph F and natural c satisfy $\frac{|F|}{|\mathcal{V}(F))|} > (c-1)^2 c^{n-3} d_2(F)$ then the first player has winning strategy in $G(F, c)$.*

Lets can formulate the strategy for the Maker — to pick the vertex x that increases the potential of position the most.

$$x = \underset{x \in F \setminus (X \cup Y)}{\text{argmax}} \ \Delta_M P_{X,Y}(x)$$

To prove that this strategy is winning, we shall prove, that it guarantees that in the end potential is positive. That would be impossible if all edges has a vertex occupied by Breaker (Breaker wins), since potential of the position would be $\sum\limits_{S \in F} 0 = 0$. To do it let's split sequence moves of players $\{x_1, y_1, \cdots y_c, x_2, y_{c+1} \ldots y_{2c}, \ldots y_m\}$ into pairs (one move of Maker and one move of Breaker) $x_i, \{y_{ic+1} \cdot y_{ic+c}\}$. The last pair may contain less than c moves of breaker.

Lets see, how potential of positions changes along game progresses.

The potential of position in before the first move is $P(\emptyset, \emptyset) = \sum\limits_{S \in F} P_{\emptyset,\emptyset}(S) = \sum\limits_{S \in F} c^{-n} = |F| c^{-n}$

Lemma 1 *After one pair of moves the potential of the position decreases at most by $\left(1 - \frac{1}{c}\right)^2 d_2(F)$, i.e. $P(X \cup \{x\}, Y \cup Y_{m+1}) - P(X, Y) \geq \left(1 - \frac{1}{c}\right)^2 d_2(F)$.*

Proof Lets calculate the change of potential after one pair of moves.

Informally if potential is $P(X, Y)$ on m-th move then after Makers move the potential becomes $P(X \cup \{x\}, Y)$ and after second player picks $c-1$ vertexes $Y' = \{y_1, y_2, \ldots y_{c-1}\}$ the potential of the position becomes $P(X \cup \{x\}, Y \cup \{y_1\})$ then $P(X \cup \{x\}, Y \cup \{y_1, y_2)\}$, ... then $P(X \cup \{x\}, Y \cup \{y_1, y_2, \ldots y_{c-}$

Formally

$$\begin{aligned}
&P(X \cup \{x\}, Y \cup Y_{m+1}) - P(X, Y) = \\
&= [-P(X, Y) + P(X \cup \{x\}, Y)] + \\
&+ [-P(X \cup \{x\}, Y) + P(X \cup \{x\}, Y \cup \{y_1\})] + \\
&+ [-P(X, Y \cup \{y_1\}) + P(X \cup \{x\}, Y \cup \{y_1, y_2\})] + \\
&+ \cdots + \\
&+ [-P(X, Y \cup \{y_1, y_2 \ldots y_{c-2}\}) + P(X \cup \{x\}, Y \cup \{y_1, y_2 \ldots y_{c-1}\})] = \\
&= \Delta_M P_{X,Y}(x) + \Delta_B P_{X \cup \{x\}, Y}(y_1) + \Delta_B P_{X \cup \{x\}, Y \cup \{y_1\}}(y_2) + \Delta_B P_{X \cup \{x\}, Y \cup \{y_1, y_2\}}(y_3) + \ldots \\
&\cdots + \Delta_B P_{X \cup \{x\}, Y \cup \{y_1, y_2, \ldots y_{c-2}\}}(y_{c-1}) \geq \\
&\geq \Delta_M P_{X,Y}(x) + \sum_{y \in Y'} \Delta_B P_{X \cup \{x\}, Y}(y)
\end{aligned}$$

Since

$$\Delta_B P_{X\cup\{x\},Y}(y) = -\sum_{\substack{S\in F\\ y\in S}} P_{X\cup\{x\},Y}(S) = -\sum_{\substack{S\in F\\ y\in S\\ x\notin S}} \underbrace{P_{X\cup\{x\},Y}(S)}_{\substack{=P_{X,Y}(S)\\ \text{because } x\notin S}} - \sum_{\substack{S\in F\\ y\in S\\ x\in S}} \underbrace{P_{X\cup\{x\},Y}(S)}_{\substack{=cP_{X,Y}(S)\\ \text{because } x\in S}} =$$

$$= -\sum_{\substack{S\in F\\ y\in S\\ x\notin S}} P_{X,Y}(S) - \sum_{\substack{S\in F\\ y\in S\\ x\in S}} cP_{X,Y}(S) =$$

$$= \underbrace{-\sum_{\substack{S\in F\\ y\in S}} P_{X\cup\{x\},Y}(S)}_{=\Delta_B P_{X,Y}(y)} - (c-1)\sum_{\substack{S\in F\\ y\in S\\ x\in S}} \underbrace{P_{X,Y}(S)}_{\substack{\leq c^{-2}\\ \text{because } \{x,y\}\subset S\setminus X}} \geq$$

$$\geq \Delta_B P_{X,Y}(y) - (c-1)\left|\{S\in F \mid \{x,y\}\subset S\}\right| c^{-2} \geq$$

$$\geq \Delta_B P_{X,Y}(y) - (c-1)\, c^{-2} d_2(F)$$

The calculation above can be continued as

$$P(X\cup\{x\}, Y\cup Y_{m+1}) - P(X,Y) =$$

$$\geq \Delta_M P_{X,Y}(x) + \sum_{y\in Y'} \Delta_B P_{X\cup\{x\},Y}(y) \geq$$

$$\geq \Delta_M P_{X,Y}(x) + \sum_{y\in Y'} \Delta_B P_{X,Y}(y) - \sum_{y\in Y'} (c-1)\, c^{-2} d_2(F) \geq$$

$$\geq \underbrace{\Delta_M P_{X,Y}(x)}_{\substack{=\max_{x\in F\setminus(X\cup Y)} \Delta_M P_{X,Y}\\ \text{in Makers strategy}\\ \text{move } x \text{ is chosen}\\ \text{to maximize this}}} - \frac{1}{c-1}\sum_{y\in Y'} \Delta_M P_{X,Y}(y) - \sum_{y\in Y'} (c-1)\, c^{-2} d_2(F) \geq$$

$$\geq \underbrace{\max_{x\in F\setminus(X\cup Y)} \Delta_M P_{X,Y} - \frac{1}{|Y'|}\sum_{y\in Y'} \Delta_M P_{X,Y}(y)}_{\geq 0} - (c-1)(c-1)\, c^{-2} d_2(F) \geq$$

$$\geq (c-1)^2 c^{-2} d_2(F)$$

Thus, if Maker plays according to the strategy (of picking the move that has maximize the $\Delta_M P_{X,Y}$), then after one pair of moves the potential can decrease only by $(c-1)^2 c^{-2} d_2(F)$.

Therefore, after all $\frac{|F|}{c}$ pairs of moves are played, the potential is at least

$$P(\emptyset,\emptyset) - (c-1)^2 c^{-2} d_2(F)\frac{|\mathcal{V}(F)|}{c} = |F|\, c^{-n} - (c-1)^2 c^{-3} d_2(F)\, |\mathcal{V}(F)|$$

which is positive under the condition of the theorem.

So, at the end of the game, potential is positive, and therefore Maker wins. $\square$

2.3 Sufficient condition for a Breaker to have a winning strategy

Theorem 2 *If $|F| < c^{n-1}$, then Breaker has a winning strategy in the $G(F,c)$.*

Proof Let's define strategy for a Breaker: in each move $c-1$ times select unpicked vertex that decreases the potential the most

$$y = \underset{v \in \mathcal{V}(F) \setminus (X \cup Y)}{\operatorname{argmin}} \Delta_B P_{X,Y}(v) = \left(= \underset{v \in \mathcal{V}(F) \setminus (X \cup Y)}{\operatorname{argmax}} |\Delta_b P_{X,y}(v)| \right)$$

, where X and Y — are sets of points picked by Maker and Breaker respectively up to this point. We shell prove that this strategy guarantees winning for the Breaker.

To do this we shall prove, that in the end the potential of the position is less than one. Indeed, that would be impossible if exists an edge completely occupied by Maker (i.e. Maker wins), since potential of it would be 1 and therefore positional of position would be at least 1.

Let's split moves of players $\{x_1, y_1, \cdots y_{c-1}, x_2, y_c \dots y_{2(c-1)} \dots y_m\}$ into first move of Maker $\{x_1\}$, pairs of moves one by Breaker and one by Maker $\{y_{i(c-1)+1} \cdot y_{(i+1)(c-1)}, x_{i+1}\}$ and the last move of Breaker $\{y_i\}$ if it exists.

Lets see, how potential of the position changes as game progresses. Before players make any moves the potential is $P(\emptyset, \emptyset) = \sum\limits_{S \in F} P_{\emptyset,\emptyset}(S) = \sum\limits_{S \in F} c^{-n} = |F| c^{-n} \le c^{-2}$.

After first move $\{x_1\}$ potential of each edge increase at most in c times. Therefore after this move the potential of position $P(\{x_1\}, \emptyset) < 1$.

Lemma 2 *If Breaker follows the strategy, then after one pair of moves $Y' = \{Y_1, Y_2 \dots Y_{c-1}\}$ and $X' = \{x_{m+1}\}$ potential of the position does not increase.*

Proof Informally we look how the potential changes with each new added Y_i and then X_i.

Formally the change of the potential is

$$P(X \cup X', Y \cup Y') - P(X, Y) = \tag{8}$$
$$= P(X \cup X', Y \cup Y') - P(X, Y \cup Y') + P(X, Y \cup Y') - P(X, Y) = \tag{9}$$
$$= \Delta_M P_{X, Y \cup Y'}(x_{m+1}) + \underbrace{P(X, Y \cup Y') - P(X, Y)}_{\text{term}} \le \tag{10}$$

Since this term can be bounded
$P(X, Y \cup \{Y_1, Y_2, \dots Y_{c-1}\}) - P(X, Y) =$

$$\begin{aligned}
&= \underbrace{(P(X, Y \cup \{Y_1, Y_2, \dots Y_{c-1}\}) - P(X, Y \cup \{Y_1, Y_2, \dots Y_{c-2}\}))}_{= \Delta_B P_{X, Y \cup \{Y_1, Y_2, \dots Y_{c-2}\}}(Y_{c-1})} + \\
&+ \underbrace{(P(X, Y \cup \{Y_1, Y_2, \dots Y_{c-2}\}) - P(X, Y \cup \{Y_1, Y_2, \dots Y_{c-3}\}))}_{= \Delta_B P_{X, Y \cup \{Y_1, Y_2, \dots Y_{c-3}\}}(Y_{c-2})} + \\
&+ \dots + \\
&+ \underbrace{(P(X, Y \cup \{Y_1\}) - P(X, Y))}_{\Delta_B P_{X, Y \cup \emptyset}(Y_1)} =
\end{aligned}$$

$= \Delta_B P_{X,Y\cup\{Y_1,Y_2,\dots Y_{c-2}\}}\left(Y_{c-1}\right) + \Delta_B P_{X,Y\cup\{Y_1,Y_2,\dots Y_{c-3}\}}\left(Y_{c-2}\right) + \dots + \Delta_B P_{X,Y\cup\emptyset}\left(Y_1\right) \leq$
[because Y_i is selected to minimize this expression]
$\leq \Delta_B P_{X,Y\cup\{Y_1,Y_2,\dots Y_{c-2}\}}\left(Y_{c-1}\right) + \Delta_B P_{X,Y\cup\{Y_1,Y_2,\dots Y_{c-3}\}}\left(Y_{c-1}\right) + \dots + \Delta_B P_{X,Y\cup\emptyset}\left(Y_{c-1}\right) \leq$
[because $Y \cup \{Y_1, Y_2, \dots Y_i\} \subset Y \cup Y'$ and property **??**]
$\leq \Delta_B P_{X,Y\cup Y'\setminus\{Y_{c-1}\}}\left(Y_{c-1}\right) + \Delta_B P_{X,Y\cup Y'\setminus\{Y_{c-1}\}}\left(Y_{c-1}\right) + \dots + \Delta_B P_{X,Y\cup Y'\setminus\{Y_{c-1}\}}\left(Y_{c-1}\right) =$
$= (c-1)\,\Delta_B P_{X,Y\cup Y'}\left(Y_{c-1}\right)$

We can continue calculation 10

$$\leq \Delta_M P_{X,Y\cup Y'}\left(x_{m+1}\right) + (c-1)\,\Delta_B P_{X,Y\cup Y'(Y_{c-1})} \leq \text{[because of 2.1]} \leq$$
$$\geq \Delta_M P_{X,Y\cup Y'\setminus\{Y_{c-1}\}}\left(x_{m+1}\right) + (c-1)\,\Delta_B P_{X,Y\cup Y'(Y_{c-1})} = 0\,\text{[because of 6]}$$

□

Therefore after all pairs of moves are played the potential of position is still less than 1. After last Breaker's move it can't increase, therefore at the end it is lesser than 1. □

3 Vander Van der Game with for c colors

3.1 Condition for a Maker to have a winning strategy

Let's define Van der Wander Game with k colors as follows. Two players alternately pick colors from the set $\{1, 2, \dots N\}$, the first player (Maker) picks one number per move, and the second player (Breaker) picks n per move. Maker wins if there exists an arithmetic progression of length n where all elements are picked by Maker.

Theorem 3 *If $N > 2\,(n-1)^2\, n\,(c-1)^2\, c^{n-3}$ then Maker has a winning strategy.*

Proof Consider hypergraph F made of all arithmetic progressions in $\{1, 2, \dots N\}$. Let's check the condition for maker to have a winning strategy.

1. By definition each edge contains n elements, so F is n-uniform
2. $|\mathcal{V}(F)| = |\{1, 2 \dots N\}| = N$
3. For every number $b < \frac{N}{2}$ for every step $s < \frac{N}{2(n-1)}$ the arithmetic progression $b, b+s, b+2s, \dots b+(n-1)\,s$ is contained in $\mathcal{V}(F)$ (and two different progressions cannot have both same beginnings and steps). Because of that the number of edges is at least $\frac{N}{2}\frac{N}{2(n-1)} = \frac{N^2}{4(n-1)}$
4. By two numbers a and b and their number in arithmetic progression the can be unambiguously determined. Because of that for given two numbers a and b the number of arithmetic progressions containing both of them is at most the number of pairs of indexes, which is $\binom{n}{2} = \frac{n(n-1)}{2}$

Therefore $\frac{|F|}{|\mathcal{V}(F)|} \geq \frac{N}{4(n-1)} \geq (c-1)^2\, c^{n-3}\frac{n(n-1)}{2} = (c-1)^2\, c^{n-3} d_2(F)$, so the condition for Maker to have a winning strategy is met. □

3.2 Condition for a Breaker to have a winning strategy

Theorem 4 *If $N < \sqrt{(n-1)\,c^{n-1}}$ than Breaker has a winning strategy.*

Proof Consider the same hypergraph of all arithmetic progressions of length n. Let's check that condition for Breaker to have a winning strategy is met.

1. F is n-uniform.
2. $|\mathcal{V}(F)| = N$
3. Since for every first element of progression (N options to choose) and for each step (at most $\frac{N}{n-1}$ options to choose) there can be only one progression with theese start and step (and all progressions are counted that way) $|F| \leq \frac{N^2}{n-1}$.

So, $|F| = \frac{N^2}{n-1} < \frac{(n-1)c^{n-1}}{n-1} = c^{n-1}$ □

4 Games on $\mathbb{R}^n$

Let $A = \{a_0, a_1, \ldots, a_n\} \subset U$ — where U is an open set in $\mathbb{R}^n$, $n \geq 2$. We can define Maker-Breaker game $G(A, U)$. Maker and Breaker alternately pick points on U. In each turn, Maker picks one point, while Breakers picks c. Maker wins if he occupies a figure equal to A (two figures A and B are equal if there exists isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(A) = B$).

Theorem 5