Improving the Minkowski Constant Using the Exponent of Decay

Rongqing Wang, Zhongyi Li, Tianer Tang

August 15, 2025

Guided by Professor Nikolay Moshchevitin

Contents

1.1 Introduction to Multidimensional Diophantine Approximation

To approximate an irrational vector $\boldsymbol{\alpha} \in \mathbb{R}^d \setminus \mathbb{Q}^d$ with integer vectors under a norm $\|\cdot\|$:

- **Linear Form Approximation (LF)**: $(\mathbf{q}, p) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{\mathbf{0}\}$ which minimize $|\mathbf{q} \cdot \boldsymbol{\alpha} - p| = |q_1\alpha_1 + \cdots + q_d\alpha_d - p|$
- **Simultaneous Approximation (SA)**: $(q, \mathbf{p}) \in \mathbb{Z}^+ \times \mathbb{Z}^d$ which minimize $\|q\boldsymbol{\alpha} - \mathbf{p}\|$

In this presentation we will work with two-dimensional LF approximations under the Euclidean norm.

Require $1, \alpha_1, \alpha_2$ linearly independent over $\mathbb{Q}$.

Definition (Minkowski Constant for two-dimensional LF)

A positive constant c such that for any $\boldsymbol{\alpha}$ there exist infinitely many approximations $(\mathbf{q}, p) \in \mathbb{Z}^3$ which satisfy $|\mathbf{q} \cdot \boldsymbol{\alpha} - p| < \dfrac{1}{c\|\mathbf{q}\|^2}$.

Geometrical Interpretation

Theorem (Minkowski, [2])

Let $A \subset \mathbb{R}^3$ *be a set which is convex and symmetric about the origin. If* $\mathrm{vol}(A) > 2^3$, *then* A *contains a nonzero point in* $\mathbb{Z}^3$*.*

For positive constants c_l, c_s, define the following sets:

$$S_l := \left\{ (x_1, x_2, y) \in \mathbb{R}^3 : \ |\alpha_1 x_1 + \alpha_2 x_2 - y| < \frac{1}{c_l \|(x_1, x_2)\|^2} \right\}$$

Geometrical Interpretation

Theorem (Minkowski, [2])

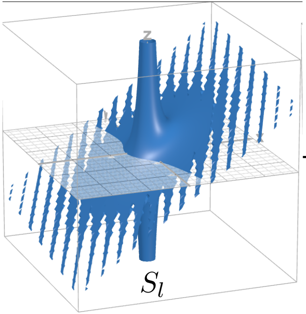
Let $A \subset \mathbb{R}^3$ be a set which is convex and symmetric about the origin. If $\operatorname{vol}(A) > 2^3$, then A contains a nonzero point in $\mathbb{Z}^3$.

For positive constants c_l, c_s, define the following sets:

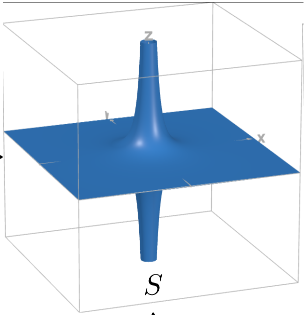
$$S_l := \left\{(x_1, x_2, y) \in \mathbb{R}^3 : \ |\alpha_1 x_1 + \alpha_2 x_2 - y| < \frac{1}{c_l \|(x_1, x_2)\|^2}\right\}$$

$$S_s := \left\{(y_1, y_2, x) \in \mathbb{R}^3 : \ \|(\alpha_1 x - y_1, \alpha_2 x - y_2)\| < \frac{1}{c_s \sqrt{|x|}}\right\}$$

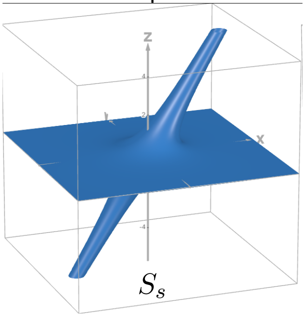
$$S := \left\{(x, y, z) : \ \|(x, y)\|^2 |z| < \frac{1}{k}\right\}$$



$\xrightarrow{\chi^\top}$



$\uparrow \chi$



Choose $c_s^2 = k = c_l$:

$$\chi = \begin{bmatrix} 1 & & -\alpha_1 \\ & 1 & -\alpha_2 \\ & & 1 \end{bmatrix}, \quad \chi^\top = \begin{bmatrix} 1 & & \\ & 1 & \\ -\alpha_1 & -\alpha_2 & 1 \end{bmatrix}$$

2.1 Best Approximations

For vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $1, \alpha_1, \alpha_2$ linearly independent over $\mathbb{Q}$, we have a sequence of approximation vectors called 'Best Approximations'

$$(\mathbf{q}_\nu, p_\nu) = (q_{1,\nu}, q_{2,\nu}, p_\nu) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$$

with the property that for the approximation error $\delta_\nu = |\mathbf{q}_\nu \cdot \boldsymbol{\alpha} - p_\nu|$ holds

$$\delta_\nu < |\mathbf{q} \cdot \boldsymbol{\alpha} - p|$$

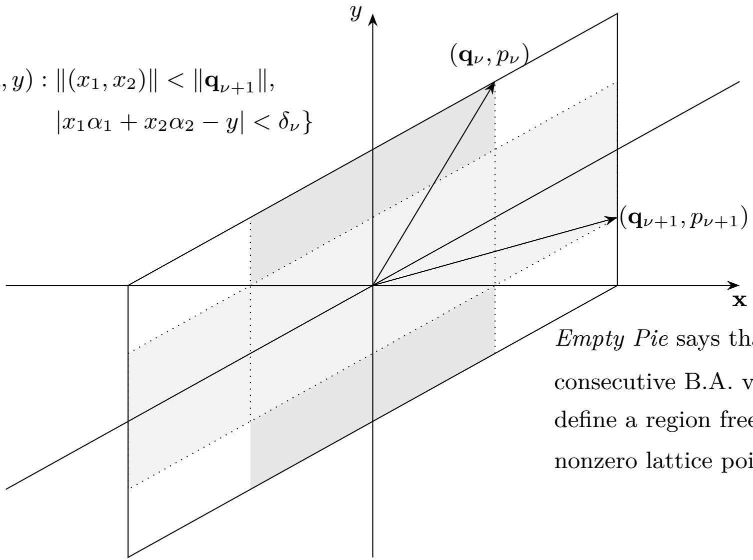
for any vector $\mathbf{q} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ with $\|\mathbf{q}\| < \|\mathbf{q}_\nu\|$.

Lemma (Empty Pie)

Define $\Pi := \{(x_1, x_2, y) \in \mathbb{R}^3 : \ \|(x_1, x_2)\| < \|\boldsymbol{q}_{\nu+1}\|, |x_1\alpha_1 + x_2\alpha_2 - y| < \delta_\nu\}$. *Then* Π_ν *contains no nonzero lattice points.*

2.2 Main Proof

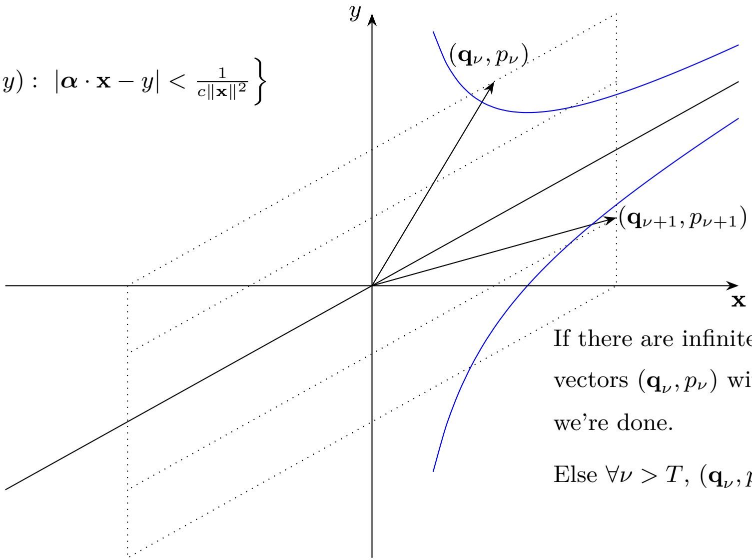
$$\Pi_\nu := \{(\mathbf{x}, y) : \|(x_1, x_2)\| < \|\mathbf{q}_{\nu+1}\|, \\ |x_1\alpha_1 + x_2\alpha_2 - y| < \delta_\nu\}$$



Empty Pie says that two consecutive B.A. vectors define a region free of nonzero lattice points.

2.2 Main Proof

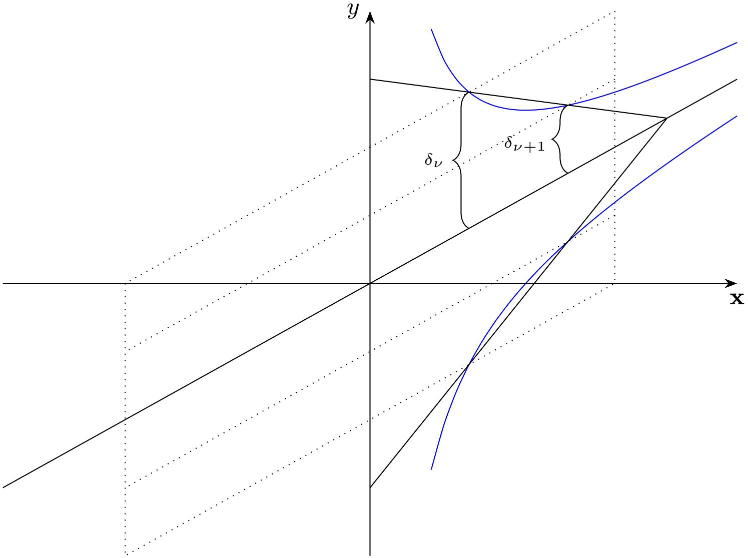
$$S := \left\{ (\mathbf{x}, y) : \ |\boldsymbol{\alpha} \cdot \mathbf{x} - y| < \frac{1}{c\|\mathbf{x}\|^2} \right\}$$



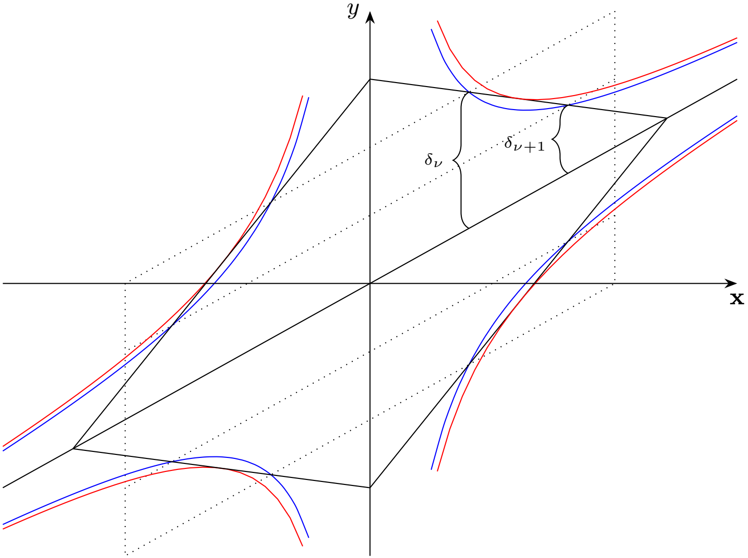
If there are infinitely many vectors $(\mathbf{q}_\nu, p_\nu)$ within S, we're done.

Else $\forall \nu > T$, $(\mathbf{q}_\nu, p_\nu) \notin S$

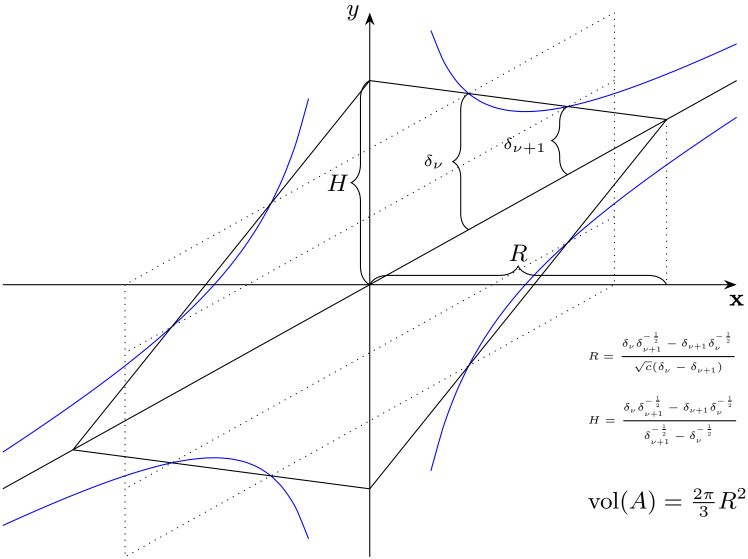
2.2 Main Proof



2.2 Main Proof



2.2 Main Proof



$$R = \frac{\delta_\nu \delta_{\nu+1}^{-\frac{1}{2}} - \delta_{\nu+1}\delta_\nu^{-\frac{1}{2}}}{\sqrt{c}(\delta_\nu - \delta_{\nu+1})}$$

$$H = \frac{\delta_\nu \delta_{\nu+1}^{-\frac{1}{2}} - \delta_{\nu+1}\delta_\nu^{-\frac{1}{2}}}{\delta_{\nu+1}^{-\frac{1}{2}} - \delta_\nu^{-\frac{1}{2}}}$$

$$\mathrm{vol}(A) = \frac{2\pi}{3} R^2 H > 8$$

2.3 Results

Lemma (Kissing Number Lemma[1])

$$\forall \nu \in \mathbb{Z}^+ : \ \delta_\nu > \delta_{\nu+5} + \delta_{\nu+6}$$

As a corollary, $\exists$ infinitely many ν with $\delta_\nu > t \cdot \delta_{\nu+1}$, where $t^6 - t - 1 = 0 : \ t \approx 1.134^+$

Computations show that $c = \frac{\pi}{12} \cdot \frac{\left(t - t^{-1/2}\right)^3}{(t-1)^2\left(1 - t^{-1/2}\right)}$ guarantees $\mathrm{vol}(A) > 8$ for all such ν.
By *Minkowski*, we have found a lattice point within A and S. Vary ν to find infinitely many such points.

$$c \approx 1.772^+ > 1.767^+ = \frac{9\pi}{16}^{\dagger}$$

† *We call an improvement like this 'too bad to be false'*

2.4 Generalization

- Euclidean norm $\to$ Any norm
- $\boldsymbol{\alpha} \in \mathbb{R}^2 \to \boldsymbol{\alpha} \in \mathbb{R}^d$
- Linear Form Approximation $\to$ Simultaneous Approximation

$v = \mathrm{vol}(B_N^d(1))$ and $t > 1$ is the exponent of decay.

$$c = \frac{v\left(t - t^{-\frac{1}{d}}\right)^{d+1}}{2^d(d+1)(t-1)^d\left(1 - t^{-\frac{1}{d}}\right)} > \frac{v}{2^d}\left(\frac{d+1}{d}\right)^d$$

2.4 Generalization

- Euclidean norm → Any norm
- $\boldsymbol{\alpha} \in \mathbb{R}^2 \to \boldsymbol{\alpha} \in \mathbb{R}^d$
- Linear Form Approximation → Simultaneous Approximation

$v = \mathrm{vol}(B_N^d(1))$ and $t > 1$ is the exponent of decay.

$$c = \frac{v\left(t - t^{-\frac{1}{d}}\right)^{d+1}}{2^d(d+1)(t-1)^d\left(1 - t^{-\frac{1}{d}}\right)} > \frac{v}{2^d}\left(\frac{d+1}{d}\right)^d$$

$$c_s = \sqrt[d]{\frac{v\left(t_s - t_s^{-\frac{1}{d}}\right)^{d+1}}{2^d(d+1)(t_s-1)^d\left(1 - t_s^{-\frac{1}{d}}\right)}} > \sqrt[d]{\frac{v}{2^d}}\frac{d+1}{d}$$

3 Future Plans

- Generalize this method to general approximation case $\|\Theta\mathbf{q} - \mathbf{p}\|$ where $\Theta \in \mathbb{R}^{n\times m}$, and $\mathbf{q} \in \mathbb{Z}^m, \mathbf{p} \in \mathbb{Z}^n$.
- Combine this method with *Minkowski* for non-convex bodies to get better results[3].
- Improve the exponent of decay of LF approximation because that benefits our result.

References

[1] Evgeny V. Ermakov. "Simultaneous two-dimensional best Diophantine approximations in the Euclidean norm". In: *arXiv preprint arXiv:1002.2713* (2010). arXiv: `1002.2713 [math.NT]`. URL: `https://arxiv.org/abs/1002.2713`.

[2] Hermann Minkowski. *Geometrie der Zahlen*. Lecture and subsequent paper. Presented in Halle, Germany. Introduced the Convex Body Theorem. Original publication in German. 1893.

[3] Nikolay G. Moshchevitin. "To the Blichfeldt–Mullender–Spohn theorem on simultaneous approximations". In: *Proceedings of the Steklov Mathematical Institute* 239 (2002), pp. 268–274. DOI: `10.1134/S008154380202014X`.

Thank You!