lab2

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2021 年 6 月 18 日

1 实验二：解非线性方程

题目： 求方程 $x^3 - \cos(x) - 5x - 1 = 0$ 的根。

记 $f(x) = x^3 - \cos x - 5x - 1$，作 $f(x)$ 的图像。从图可以看出：方程 $f(x) = 0$ 有 3 个根，第一个位于 $[-3, -1]$ 内，第二个位于 $[-1, 1]$ 内，第三个位于 $[1, 3]$ 内。

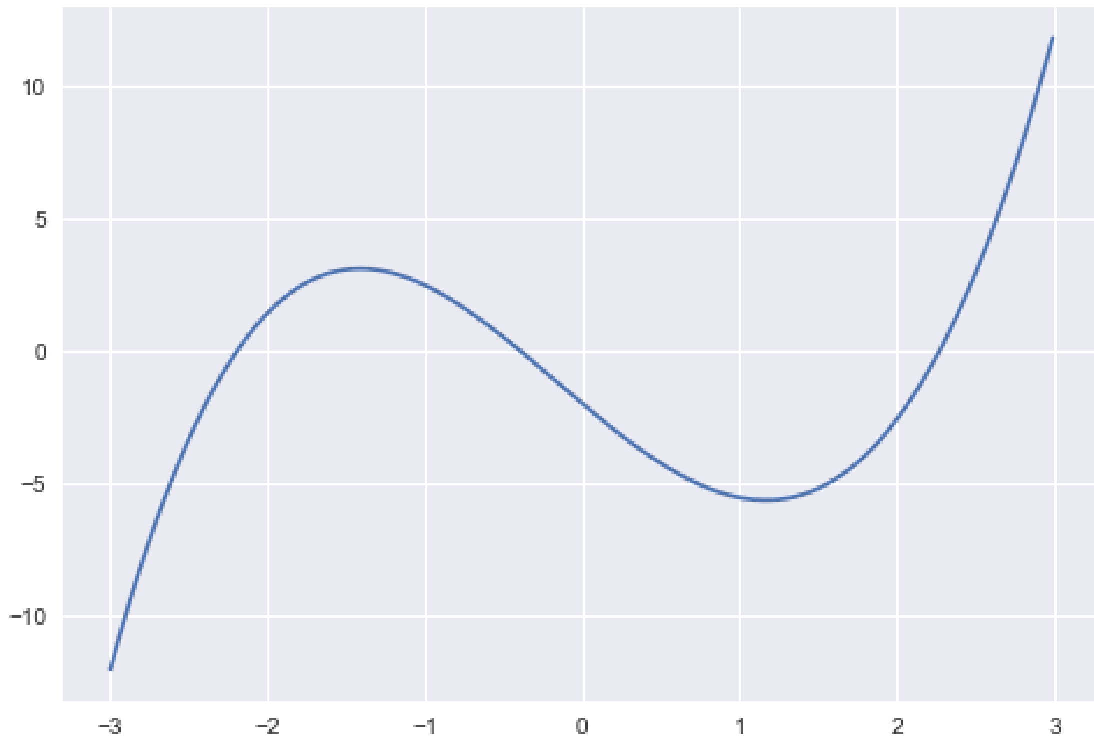
[92]:
```
# Import
import numpy as np
import matplotlib.pyplot as plt
from scipy.misc import derivative
```

[93]:
```
# 定义函数
def f0(x):
    y = x**3-np.cos(x)-5*x-1
    return y
```

[94]:
```
x_space = np.arange(-3,3,0.01)
y_space = f0(x_space)


plt.style.use("seaborn")
plt.plot(x_space, y_space)
```

[94]:
```
[<matplotlib.lines.Line2D at 0x24dfddabf48>]
```



1.1 1. 二分法求根

二分法求根的思路是：对于连续函数 $f(x)$，若其在区间 $[a,b]$ 上满足 $f(a)\cdot f(b)<0$，则其在 (a,b) 内存在零点 $\bar{x}$。且我们可以通过二分法求出 $\bar{x}$，其思路是：

1. 取 $x_0=(a+b)/2$。若 $f(x_0)=0$，则输出 x_0。否则转到下一步。
2. 若 $f(a)\cdot f(x_0)>0$，则 $a_1=x_0, b_1=b$，否则 $a_1=a, b_1=x_0$。
3. 重复第二步，构造 $[a_1,b_1],[a_2,b_2],\ldots$，并计算 $x_k=(a_k+b_k)/2$。直至 $f(x_k)=0$ 或 $|f(x_k)|<\epsilon$ 停止迭代并输出 x_k。

可以使用以下函数实现二分法：

[95]:
```python
# 1.对区间二分法求根
def halfsearch(f, a, b, uplimit):
    if f(a)*f(b)>=0:
        print("f(a)*f(b) should be less than 0")
        return 0
    if (a>b):
        print("a should be less than b")
```

```python
        return 0
    while(b-a>uplimit):
        x0 = (a+b)/2
        if f(x0)==0:
            return x0
        elif f(x0)*f(a)>0:
            a = x0
        else:
            b = x0
    x0 = (a+b)/2
    return x0
```

分别在 $[-3,-1],[-1,1],[1,3]$ 上使用二分法，所得结果如下所示：

[97]:
```python
# 要求误差为 1e-6
limit = 1e-6

print("The result between -3 and -1:", "%.6f"%halfsearch(f0,-3,-1,limit))
print("The result between -1 and 1:","%.6f"%halfsearch(f0,-1,1,limit))
print("The result between 1 and 3:","%.6f"%halfsearch(f0,1,3,limit))
```

```
The result between -3 and -1: -2.193133
The result between -1 and 1: -0.396959
The result between 1 and 3: 2.270829
```

1.2 2. 不动点迭代法

对于 $f(x)=0$，可以将其变换为 $x=\phi(x)$。对于初值 x_0，构造序列 $x_1=\phi(x_0),x_2=\phi(x_1),...,x_k=\phi(x_{k-1}),...$。如果序列 $\{x_i\}_{i=0}^{\infty}$ 收敛，则其收敛值即为方程 $f(x)=0$ 的根。其思路是：

1. 对于 x_k，计算 $x_{k+1}=\phi(x_k)$。
2. 若 $|x_{k+1}-x_k|<\epsilon$，则输出 x_{k+1}。
3. 若迭代次数超过一定次数，停止算法，输出算法不收敛。
4. 回到第 1 步进行迭代。

可以使用以下函数实现不动点法：

[98]:
```python
def fixpoint(f, x0, uplimit):
    xold = x0
```

```
    xnew = f(xold)
    cnt = 1
    while abs(xnew-xold)>uplimit:
        xold = xnew
        xnew = f(xold)
        cnt = cnt+1
        if(cnt>10000 or abs(xnew)>1e10):
            return 0
    return (xnew,cnt)
```

下面对于两个不同的迭代函数进行不动点迭代法。其中 $\phi_1(x) = \frac{x^3-\cos(x)-1}{5}, \phi_2(x) = \sqrt[3]{\cos(x)+5x+1}$。初值取分别设置为-3,-2,-1,0,1,2,3。

(1) $\phi_1(x) = \frac{x^3-\cos(x)-1}{5}$

对 $\phi_1(x)$ 使用不动点迭代法可得到如下结果:

[102]:
```
def phi0(x):
    y = (x**3-np.cos(x)-1)/5
    return y

print("Fix-point Method for phi1")
for i in [-3,-2,-1,0,1,2,3]:
    if fixpoint(phi0,i,limit) == 0:
        print("初值 = %d 时不收敛"%i)
    else:
        result, itertime = fixpoint(phi0,i,limit)
        print("Start point:", i, "Result point:", "%.6f"%result, "Iterate times:
↪", itertime)
```

```
Fix-point Method for phi1
初值 = -3 时不收敛
Start point: -2 Result point: -0.396958 Iterate times: 8
Start point: -1 Result point: -0.396958 Iterate times: 6
Start point: 0 Result point: -0.396958 Iterate times: 4
Start point: 1 Result point: -0.396958 Iterate times: 5
Start point: 2 Result point: -0.396958 Iterate times: 7
初值 = 3 时不收敛
```

(2) $\phi_2(x) = \phi_2(x) = \sqrt[3]{\cos(x) + 5x + 1}$

对 $\phi_2(x)$ 使用不动点迭代法可得到如下结果：

[103]:
```
def phi1(x):
    y = np.cbrt(np.cos(x)+5*x+1)
    return y

print("Fix-point Method for phi2")
for i in [-3,-2,-1,0,1,2,3]:
    if fixpoint(phi1,i,limit) == 0:
        print("初值 = %d时不收敛"%i)
    else:
        result, itertime = fixpoint(phi1,i,limit)
        print("Start point:", i, "Result point:", "%.6f"%result, "Iterate times:
↪", itertime)
```

```
Fix-point Method for phi2
Start point: -3 Result point: -2.193133 Iterate times: 16
Start point: -2 Result point: -2.193132 Iterate times: 14
Start point: -1 Result point: -2.193132 Iterate times: 17
Start point: 0 Result point: 2.270829 Iterate times: 13
Start point: 1 Result point: 2.270829 Iterate times: 12
Start point: 2 Result point: 2.270829 Iterate times: 11
Start point: 3 Result point: 2.270829 Iterate times: 12
```

1.3 3. 埃特金加速法

对于不动点迭代法中的迭代函数 $\phi(x)$，不动点迭代法使用迭代公式 $x_{k+1} = \phi(x_k)$。埃特金加速法使用如下的迭代公式：$y_k = \phi(x_k), z_k = \phi(y_k), x_{k+1} = x_k - \frac{(y_k - x_k)^2}{z_k - 2y_k + x_k}$。

更一般的，我们将使用埃特金加速法的不动点迭代法称为斯蒂芬森迭代法。其思路如下：

1. 对于 x_k，计算 $x_{k+1} = x_k - \frac{(\phi(x_k) - x_k)^2}{\phi(\phi(x_k)) - 2\phi(x_k) + x_k}$。
2. 若 $|x_{k+1} - x_k| < \epsilon$，则输出 x_{k+1}。
3. 若迭代次数超过一定次数，停止算法，输出算法不收敛。
4. 回到第 1 步进行迭代。

可以使用如下函数实现斯蒂芬森迭代法：

[105]:
```python
def aitken(f, x0, uplimit):
    xnew = x0
    xold = np.inf
    cnt = 0
    while abs(xold-xnew)>uplimit:
        xold = xnew
        xk1 = f(xold)
        xk2 = f(xk1)
        xnew = xold-((xk1-xold)**2/(xk2-2*xk1+xold))
        cnt = cnt+1
        if (cnt > 10000 or abs(xnew)>1e10):
            return 0
    return (xnew,cnt)
```

斯蒂芬森迭代法在迭代函数 $\phi_1(x) = \frac{x^3-\cos(x)-1}{6}, \phi_2(x) = \sqrt[3]{\cos(x)+6x+1}$ 和初值-3,-2,-1,0,1,2,3 下的结果为:

(1) $\phi_1(x) = \frac{x^3-\cos(x)-1}{5}$

对 $\phi_1(x)$ 使用斯蒂芬森迭代法可得到如下结果:

[106]:
```python
print("Aitken Method for phi1")
for i in [-3,-2,-1,0,1,2,3]:
    temp = aitken(phi0,i,limit)
    if temp==0:
        print("初值 = %d时不收敛"%i)
    else:
        result, itertime = temp
        print("Start point:", i, "Result point:", "%.6f"%result, "Iterate times:
↪", itertime)
```

```
Aitken Method for phi1
Start point: -3 Result point: -2.193133 Iterate times: 8
Start point: -2 Result point: -2.193133 Iterate times: 6
Start point: -1 Result point: -0.396958 Iterate times: 4
Start point: 0 Result point: -0.396958 Iterate times: 3
Start point: 1 Result point: -0.396958 Iterate times: 4
Start point: 2 Result point: 2.270829 Iterate times: 6
```

```
Start point: 3 Result point: 2.270829 Iterate times: 7
```

(2) $\phi_2(x) = \phi_2(x) = \sqrt[3]{\cos(x) + 5x + 1}$

对 $\phi_2(x)$ 使用斯蒂芬森迭代法可得到如下结果：

[107]:
```
print("Aitken Method for phi2")
for i in [-3,-2,-1,0,1,2,3]:
    temp = aitken(phi1,i,limit)
    if temp==0:
        print("初值 = %d时不收敛"%i)
    else:
        result, itertime = temp
        print("Start point:", i, "Result point:", "%.6f"%result, "Iterate times:
 ↪", itertime)
```

```
Aitken Method for phi2
Start point: -3 Result point: -2.193133 Iterate times: 4
Start point: -2 Result point: -2.193133 Iterate times: 3
Start point: -1 Result point: -2.193133 Iterate times: 5
Start point: 0 Result point: 2.270829 Iterate times: 4
Start point: 1 Result point: 2.270829 Iterate times: 4
Start point: 2 Result point: 2.270829 Iterate times: 3
Start point: 3 Result point: 2.270829 Iterate times: 3
```

1.4 4. 牛顿法

牛顿法对函数 $f(x)$ 在 x_k 处做泰勒展开 $f(x^*) \approx f(x_k) + f'(x_k)(x^* - x_k)$，其中 x^* 是方程 $f(x) = 0$ 的根。因此可以得到递推式 $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$。牛顿法的思路如下：

1. 对于 x_k，计算 $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$。
2. 若 $|x_{k+1} - x_k| < \epsilon$，则输出 x_{k+1}。
3. 若迭代次数超过一定次数，停止算法，输出算法不收敛。
4. 回到第 1 步进行迭代。

牛顿法主要有两个缺点：第一、牛顿法需要计算导数 $f'(x)$，计算量较大。第二、牛顿法不能保证算法的收敛性。针对这两个问题，可以分别使用简化的牛顿法和牛顿下山法对牛顿法进行改进。

简化的牛顿法：使用 $f'(x_0)$ 代替 $f'(x_k)$ 减少计算量。其递推式为 $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_0)}$。

牛顿下山法：引入压缩因子 λ，将递推公式变为 $x_{k+1} = x_k - \lambda\frac{f(x_k)}{f'(x_k)}$，使得 $|f(x_{k+1})| < |f(x_k)|$，从

而保证其收敛性。其中，λ 的取值开始时为 1，在每次递推中不断折半，直至满足递减的条件。牛顿下山法的思路如下：

1. 对于 x_k，设定 $\lambda = 1$。

- 1.1. 计算 $x_{k+1} = x_k - \lambda\frac{f(x_k)}{f'(x_k)}$。
- 1.2. 判断 $|f(x_{k+1})| < |f(x_k)|$。若满足则调到第 2 步。
- 1.3.$\lambda = \lambda/2$，并回到 1.1 步。

2. 若 $|x_{k+1} - x_k| < \epsilon$，则输出 x_{k+1}。
3. 若迭代次数超过一定次数，停止算法，输出算法不收敛。
4. 回到第 1 步进行迭代。

可以使用以下函数实现牛顿法及其改进方法：

普通牛顿法代码：

[108]:
```
def newton_phi(f, x):
    y = x-f(x)/derivative(f, x, dx=1e-6)
    return y

def newton(f, x0, uplimit):
    xold = x0
    xnew = newton_phi(f,xold)
    cnt = 1
    while abs(xnew-xold)>uplimit:
        xold = xnew
        xnew = newton_phi(f,xold)
        cnt = cnt+1
        if cnt > 10000 or abs(xnew)>1e10:
            return 0
    return (xnew,cnt)
```

结果如下所示：

[110]:
```
print("Newton Method for f0")
for i in [-3,-2,-1,0,1,2,3]:
    temp = newton(f0,i,limit)
    if temp == 0:
        print("初值 = %d时不收敛"%i)
```

```
    else:
        result, itertime = temp
        print("Start point:", i, "Result point:", "%.6f"%result, "Iterate times:
↪", itertime)
```

```
Newton Method for f0
Start point: -3 Result point: -2.193133 Iterate times: 6
Start point: -2 Result point: -2.193133 Iterate times: 5
Start point: -1 Result point: -0.396958 Iterate times: 4
Start point: 0 Result point: -0.396958 Iterate times: 4
Start point: 1 Result point: -2.193133 Iterate times: 7
Start point: 2 Result point: 2.270829 Iterate times: 5
Start point: 3 Result point: 2.270829 Iterate times: 5
```

简化牛顿法代码:

[112]:
```
def newton_phi_simplified(f,C,x):
    y = x-f(x)/C
    return y

def newton_simplified(f, x0, uplimit):
    xold = x0
    C = derivative(f, x0, dx=1e-6)
    xnew = newton_phi_simplified(f,C,xold)
    cnt = 1
    while abs(xnew-xold)>uplimit:
        xold = xnew
        xnew = newton_phi_simplified(f,C,xold)
        cnt = cnt+1
        if cnt > 10000 or  abs(xnew)>1e10:
            return 0
    return (xnew,cnt)
```

结果如下所示:

[113]:
```
print("Newton Simplified Method for f0")
for i in [-3,-2,-1,0,1,2,3]:
    temp = newton_simplified(f0,i,limit)
```

```
    if temp == 0:
        print("初值 = %d时不收敛"%i)
    else:
        result, itertime = temp
        print("Start point:", i, "Result point:", "%.6f"%result, "Iterate times:
↪", itertime)
```

```
Newton Simplified Method for f0
Start point: -3 Result point: -2.193134 Iterate times: 25
Start point: -2 Result point: -2.193133 Iterate times: 15
Start point: -1 Result point: -0.396959 Iterate times: 44
Start point: 0 Result point: -0.396958 Iterate times: 4
初值 = 1 时不收敛
Start point: 2 Result point: 2.270829 Iterate times: 16
Start point: 3 Result point: 2.270830 Iterate times: 18
```

牛顿下山法代码:

[116]:
```
def newton_phi_downhill(f, x, lam):
    y = x-lam*f(x)/derivative(f, x, dx=1e-6)
    return y

def newton_downhill(f, x0, uplimit):
    xold = np.inf
    xnew = x0
    cnt = 0
    while abs(xnew-xold)>uplimit:
        xold = xnew
        lam = 1
        x_temp = newton_phi_downhill(f,xold,lam)
        while(abs(f(x_temp))>abs(f(xold))):
            lam = lam/2
            x_temp = newton_phi_downhill(f,xold,lam)
        xnew = x_temp
        cnt = cnt+1
        if cnt > 10000 or abs(xnew)>1e10:
            return 0
```

```
    return (xnew,cnt)
```

结果如下所示：

[117]:
```
print("Newton Downhill Method for f0")
for i in [-3,-2,-1,0,1,2,3]:
    temp = newton_downhill(f0,i,limit)
    if temp == 0:
        print("初值 = %d时不收敛"%i)
    else:
        result, itertime = temp
        print("Start point:", i, "Result point:", "%.6f"%result, "Iterate times:
↪", itertime)
```

```
Newton Downhill Method for f0
Start point: -3 Result point: -2.193133 Iterate times: 6
Start point: -2 Result point: -2.193133 Iterate times: 5
Start point: -1 Result point: -0.396958 Iterate times: 4
Start point: 0 Result point: -0.396958 Iterate times: 4
Start point: 1 Result point: -0.396958 Iterate times: 5
Start point: 2 Result point: 2.270829 Iterate times: 5
Start point: 3 Result point: 2.270829 Iterate times: 5
```

1.5 5. 埃特金加速法的收敛速度

埃特金加速法的收敛速度和不动点法的收敛速度可如下表所示。

对于迭代函数 $\phi_1(x) = \frac{x^3-\cos(x)-1}{5}$，其结果为

初值	不动点法的根	不动点法的迭代次数	斯蒂芬森法的根	斯蒂芬森法的迭代次数
-3	NaN	NaN	-2.193133	8
-2	-0.396958	8	-2.193133	6
-1	-0.396958	6	-2.193133	4
0	-0.396958	4	2.270829	3
1	-0.396958	5	2.270829	4
2	-0.396958	7	2.270829	6
3	NaN	NaN	2.270829	7

其中 NaN 代表不收敛，下同

对于迭代函数 $\phi_2(x) = \sqrt[3]{\cos(x) + 5x + 1}$，其结果为

初值	不动点法的根	不动点法的迭代次数	斯蒂芬森法的根	斯蒂芬森法的迭代次数
-3	-2.193133	16	-2.193133	4
-2	-2.193132	14	-2.193133	3
-1	-2.193132	17	-0.396958	5
0	2.270829	13	-0.396958	4
1	2.270829	12	-0.396958	4
2	2.270829	11	2.270829	3
3	2.270829	12	2.270829	3

1.6 6. 牛顿法的收敛速度

牛顿法的收敛速度可如下表所示。

初值	普通牛顿法 (根/迭代次数)	简化牛顿法 (根/迭代次数)	牛顿下山法 (根/迭代次数)
-3	(-2.193133,6)	(-2.193134,25)	(-2.193133,6)
-2	(-2.193133,5)	(-2.193133,15)	(-2.193133,5)
-1	(-0.396958,4)	(-0.396959,44)	(-0.396958,4)
0	(-0.396958,4)	(-0.396958,4)	(-0.396958,4)
1	(-2.193133,7)	(NaN,NaN)	(-0.396958,5)
2	(2.270829,5)	(2.270829,16)	(2.270829,5)
3	(2.270829,5)	(2.270830,18)	(2.270829,5)

```
[ ]:
```