Box Dimensions of Fractal Surfaces: Random Tagaki Surfaces and Typical Surfaces

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Abstract

In this paper we study the box dimensions of highly irregular surfaces. The paper consists of two parts. In the first part we investigate random Takagi surfaces, and prove that the box dimension of the graph of such a surface is $3 + \frac{\log a}{\log 2}$ where $a \in [\frac{1}{2}, 1)$ is the common scaling ratio between levels. This result is somewhat surprising as one may have expected the result to hold only almost surely. In the second half of the paper we study the box dimensions of a typical surface (in the sense of Baire). We deduce that a typical surface is as irregular as possible; having upper box dimension 3 and lower box dimension 2. This improves earlier results of Humke and Petruska.

1 Introduction

The Takagi function is a well known example of a continuous but nowhere differentiable function,

$$f(x) = \sum_{n=0}^{\infty} a^n g(2^n x),$$

where $a \in (1/2, 1)$ and g is a sawtooth function. It is natural to investigate the fractal dimension of the graph of such a function; in ***[some ref]*** determine that the box dimension of the graph of a family of Takagi like functions is $2 + \frac{\log a}{\log 2}$. For $a < 1/2$ the function is Lipschitz, which implies that the graph has box dimension equal to 1. We consider a generalised higher dimensional formulation of this graph, for a 'pyramid' function g,

$$f_\omega(\mathbf{x}) = \sum_{n=0}^{\infty} c_{\omega,n}(\mathbf{x}) a^n g(2^n \mathbf{x}),$$

where $c_{\omega,n}$, a simple function defined on the unit square, is used to determine if the pyramid functions point upwards or downwards. In ***[ref Hunt]***, Hunt investigates the Hausdorff dimension of the graph of functions of the form

$$f_\Theta(x) = \sum_{n=0}^{\infty} a_n h(b_n x + \theta_n),$$

where the applitudes a_n and the frequencies b_n need only exhibit an approximate exponential dependence on n. Hunt deduced that for almost all $\Theta \in [0, 1]^{\mathbb{N}}$ the

Hausdorff dimension of such a graph is $2 + \frac{\log a}{\log b}$. The function h need only be Lipschitz, Periodic, and C^1. This final condition rules out h being a saw-tooth function.

2 Statement of Results

Throughout this paper let $C[0,1]^2$ denote the set of all continuous functions $f : [0,1]^2 \longrightarrow \mathbb{R}$, and the graph of such a function $f \in C[0,1]^2$ is denoted by

$$\Gamma(f) = \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^3 \mid \mathbf{x} \in [0,1]^2\}.$$

We define the δ-mesh grid in $\mathbb{R}^3$, $\mathcal{Q}_\delta$, by

$$\mathcal{Q}_\delta = \{[m_1\delta, (m_1+1)\delta] \times [m_2\delta, (m_2+1)\delta] \times [m_3\delta, (m_3+1)\delta] \mid m_1, m_2, m_3 \in \mathbb{Z}\}. \tag{1}$$

This allows us to define, for $\delta > 0$ and $E \subset \mathbb{R}^n$, $N_\delta(E) = |\{Q \in \mathcal{Q}_\delta \mid Q \cap E \neq \varnothing\}|$. Finally, we define the upper and lower box counting dimensions to be

$$\overline{\dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \qquad \underline{\dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta},$$

respectively, and where they coincide, we write $\dim_B(E)$ to denote this common value. We refer the reader to [Fal90] for more material relating to the box dimension.

We now state our main results, beginning with the construction of a random Takagi surface. The surface is defined analogously to the Takagi curve ***[some ref]***; we first define the 'pyramid' function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$g(x,y) = \begin{cases} \{x\} & \text{if } \{y\} > \{x\} \text{ and } \{y\} \leqslant 1 - \{x\}, \\ 1 - \{x\} & \text{if } \{y\} \leqslant \{x\} \text{ and } \{y\} > 1 - \{x\}, \\ \{y\} & \text{if } \{y\} \leqslant \{x\} \text{ and } \{y\} \leqslant 1 - \{x\}, \\ 1 - \{y\} & \text{if } \{y\} > \{x\} \text{ and } \{y\} > 1 - \{x\}, \end{cases}$$

where $\{\cdot\}$ denotes the fractional part of a number. The graph of g is depicted in the top left of Figure 1. Consider the space $\Omega = \prod_{n=0}^{\infty} \{-1,1\}^n$, a sequence of random signs. For each $\omega = (\omega_n(k,l))_{n \geqslant 0,\, 0 \leqslant k,l < 2^n} \in \Omega$ we associate a sequence of random functions $c_{\omega,0}, c_{\omega,1}, \ldots : [0,1]^2 \longrightarrow \mathbb{R}$ by

$$c_{\omega,n}(\mathbf{x}) = \sum_{k,l=0}^{2^n-1} \omega_n(k,l) 1_{R_n(k,l)}(\mathbf{x}),$$

where

$$R_n(k,l) = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \times \left[\frac{l}{2^n}, \frac{l+1}{2^n}\right].$$

We will now explain the geometric interpretation of $c_{\omega,n}$. For each n, we partition $[0,1]^2$ into the unique 4^n non-overlapping squares, and then toss a coin to decide whether the function takes the value 1 or -1 on the interior of each of the squares. Finally we define the random Takagi surface $f_\omega : [0,1]^2 \longrightarrow \mathbb{R}$ for $a \in (1/2, 1)$ by

$$f_\omega(\mathbf{x}) = \sum_{n=0}^{\infty} c_{\omega,n}(\mathbf{x}) a^n g(2^n \mathbf{x}).$$

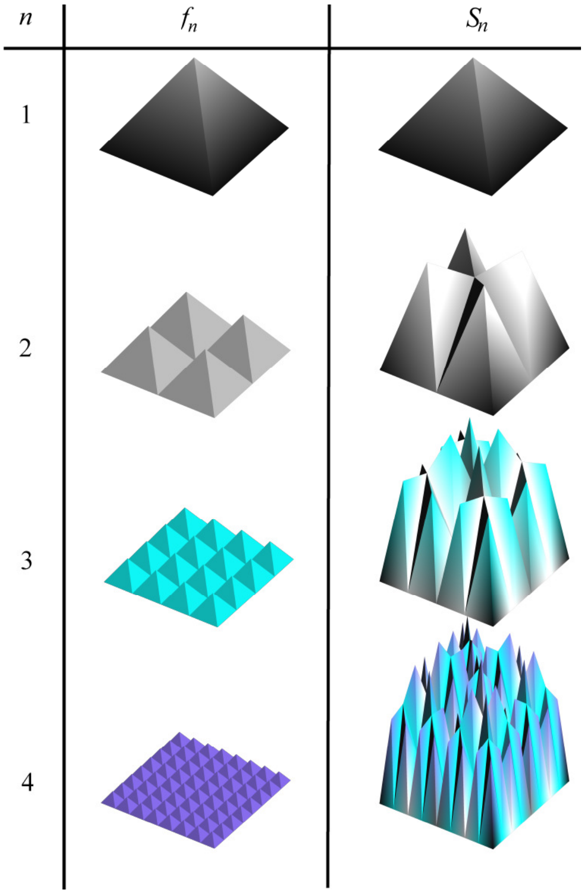


Figure 1. *The construction of Takagi surface with $\omega = (1, 1, \ldots)$ and $a = 0.575$.*

Theorem 2.1 *For a Takagi surface f_ω, with a being the common ratio between levels, we have*

$$\dim_B(\Gamma(f_\omega)) = 3 + \frac{\log a}{\log 2}$$

for all $\omega \in \Omega$.

One would expect the above result to hold almost surely (with respect to a suitable measure on Ω), however it surprisingly holds for all ω without exception.

We now go on to discuss a further generalisation of the Takagi surface, namely with random phase-shifts included. Let $\Theta = \prod_{n=0}^{\infty}([0,1) \times [0,1))^n$. For $\theta = (\boldsymbol{\theta}_n)_n \in \Theta$ and $\omega \in \Omega$ we define the random phase-shifted Takagi function $f_{\omega,\theta}$ by

$$f_{\omega,\theta}(\mathbf{x}) = \sum_{n=0}^{\infty} c_{\omega,n}(\mathbf{x} + \boldsymbol{\theta}_n) a^n g(2^n(\mathbf{x} + \boldsymbol{\theta}_n)).$$

This function is considerably more complicated than the random Takagi function defined previously as the pyramids no longer stack up due to the lack of a common grid pattern. We prove that the upper box dimension of the graph of such a function is bounded above by $3 + \frac{\log a}{\log 2}$.

Proposition 2.2 *For a phased-shifted random Takagi surface $f_{\omega,\theta}$, with a being the common ratio between levels, we have*

$$\overline{\dim}_B(\Gamma(f_{\omega,\theta})) \leqslant 3 + \frac{\log a}{\log 2}$$

for all $\omega \in \Omega$, and for all $\theta \in \Theta$.

We now introduce a measure on Θ. Let μ denote the uniform measure on $[0,1) \times [0,1)$, we let P denote the infinite product measure obtained from μ. We strongly believe that the upper bound in Proposition 2.2 is the correct P almost sure box dimension of $\Gamma(f_{\omega,\theta})$, and we make the following conjecture.

Conjecture 2.3 *For a phased-shifted random Takagi surface $f_{\omega,\theta}$, with a being the common ratio between levels such that for P almost all θ we have*

$$\dim_B(\Gamma(f_{\omega,\theta})) = 3 + \frac{\log a}{\log 2}$$

for all $\omega \in \Omega$.

Next we prove that a the graph associated with a typical continuous function $f \in C([0,1]^2)$ has upper box dimension 3 and lower box dimension 2. We mean typical in the sense of Baire, that is a property is said to hold for a typical surface if it holds for a comeagre subset of $C[0,1]^2$.

Theorem 2.4 *A typical surface has upper box dimension 3 and lower box dimension 2, that is for the set*

$$M = \{f \in C([0,1]^2) \mid \overline{\dim}_B(\Gamma(f)) = 3 \text{ and } \underline{\dim}_B(\Gamma(f)) = 2\}$$

is comeagre.

In [HP88] Humke and Petruska proved that the upper box dimension of the graph of a typical continuous function is 2. Their methods can clearly be adapted to prove that $\overline{\dim}_B(\Gamma(f))$ equals 3 for a typical surface, and our proof of this is only included here for completeness. Humke and Petruska also proved that the Hausdorff dimension of the graph of a typical continuous function is 1. The result concerning lower box dimensions in 2.4 can be adapted to the 1 dimensional case, in which case it is an improvement on [HP88] due to the well known inequality $\dim_H(E) \leqslant \underline{\dim}_B(E)$.

3 Proof of Theorem 2.1

We begin by proving Proposition 2.2, the upper bound to Theorem 2.1 then follows as a corollary. We start with a lemma that gives sufficient conditions for an upper bound for the upper box dimension of the surface.

Lemma 3.1 *Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function. Assume that there are constants $s \in [2,3]$, $c > 0$, such that*

$$|f(\mathbf{x}) - f(\boldsymbol{y})| \leqslant c|\mathbf{x} - \boldsymbol{y}|^{3-s}$$

for all x, y then

$$\overline{\dim}_B(\Gamma(f)) \leqslant s$$

This lemma is a trivial generalisation of a lemma in [Fal90], the proof is therefore omitted. We will now obtain the upper bound for $\overline{\dim}_B(\Gamma(f_{\omega,\theta}))$ by showing $f_{\omega,\theta}$ satisfies the Holder condition in Lemma 3.1.

Proposition 3.2 *Let $s = 3 + \frac{\log a}{\log 2}$, $\omega \in \Omega$, and $\theta \in \Theta$. For all $\mathbf{x}, \boldsymbol{y} \in [0,1]^2$, we have*

$$|f_{\omega,\theta}(\mathbf{x}) - f_{\omega,\theta}(\boldsymbol{y})| \leqslant c|\mathbf{x} - \boldsymbol{y}|^{3-s}$$

Proof: Let $\mathbf{h} = \mathbf{y} - \mathbf{x}$, and choose $N \in \mathbb{N}$ such that $2^{-(N+1)} < |\mathbf{h}| \leqslant 2^{-N}$. To condense the notation, denote $c_{\omega,n}(\mathbf{x} + \boldsymbol{\theta}_n)g(2^n(\mathbf{x} + \boldsymbol{\theta}_n))$ by $z_n(\mathbf{x})$.

$$\begin{aligned}
|f_{\omega,\theta}(\mathbf{x}+\mathbf{h}) - f_{\omega,\theta}(\mathbf{x})| &\leqslant \sum_{n=0}^{N} |a^n z_n(\mathbf{x}+\mathbf{h}) - a^n z_n(\mathbf{x})| \\
&\quad + \sum_{n=N+1}^{\infty} |a^n z_n(\mathbf{x}+\mathbf{h}) - a^n z_n(\mathbf{x})| \\
&\leqslant \sum_{n=0}^{N} a^n |z_n(\mathbf{x}+\mathbf{h}) - z_n(\mathbf{x})| + \sum_{n=N+1}^{\infty} 2^{(s-3)n} \\
&\leqslant \sum_{n=0}^{N} (2a)^n |\mathbf{h}| + 2^{(s-3)(N+1)} \frac{1}{1 - 2^{(s-3)}} \\
&= |\mathbf{h}| \frac{2^{(s-2)(N+1)} - 1}{2^{(s-2)} - 1} + \frac{1}{1 - 2^{s-3}} \left(\frac{1}{2^{N+1}} \right)^{3-s} \\
&\leqslant |\mathbf{h}| \frac{2^{(s-2)(N+1)}}{2^{(s-2)} - 1} + C_1 |\mathbf{h}|^{(3-s)} \\
&= C_2 |\mathbf{h}| 2^{N(s-2)} + C_1 |\mathbf{h}|^{3-s} \\
&\leqslant (C_1 + C_2)|\mathbf{h}|^{3-s}
\end{aligned}$$

Where $C_1 = (1 - 2^{s-3})^{-1}$ and $C_2 = (1 - 2^{s-2})^{-1}$. $\square$

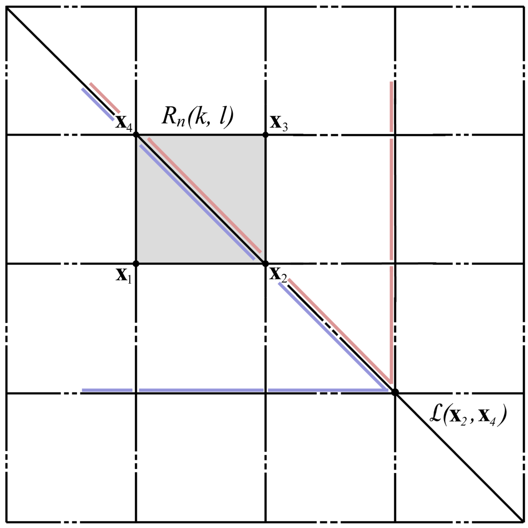
We combine Lemma 3.1 Proposition 3.2 to obtain an upper bound.

Corollary 3.3 *For all $\omega \in \Omega$, $\theta \in \Theta$ we have*

$$\overline{\dim}_B(\Gamma(f_{\omega,\theta})) \leqslant s = 3 + \frac{\log a}{\log 2}$$

The proof of the lower bound is based on the following technical lemma.

Proposition 3.4 *Let $\omega \in \Omega$, $a \in (1/2, 1)$, $n \in \mathbb{N}$ and $0 \leqslant k, l < 2^n$. Let $\mathbf{x}_i$ for $i = 1, 2, 3, 4$ denote the corners of $R_n(k, l)$, and $\boldsymbol{m}$ denote the centre point of $R_n(k, l)$, as shown in the figure below.*



Then

$$\min_{i=1\cdots 4}\{f_\omega(\mathbf{x}_i)\} \leqslant \sum_{i=1}^{n-1} c_{\omega,i}(\boldsymbol{m}) a^i g(2^i \boldsymbol{m}) \leqslant \max_{i=1\cdots 4}\{f_\omega(\mathbf{x}_i)\}$$

Proof: We claim that we may choose opposite corner points $\mathbf{x}_a, \mathbf{x}_b$ of $R_n(k, l)$ such that for all $1 \leqslant i < n$ we have that the points $(\mathbf{x}_a, c_{\omega,i}(\mathbf{x}_a) a^i g(2^i \mathbf{x}_a))$, $(\mathbf{x}_b, c_{\omega,i}(\mathbf{x}_b) a^i g(2^i \mathbf{x}_b)) \in \mathbb{R}^3$ lie on the same face of one of pyramids that make up the graph of the function $c_{\omega,i}(\mathbf{x}) a^i g(\mathbf{x})$. Suppose not, then there exists $0 \leqslant i_1, i_2 < n$ such that

(i) $(\mathbf{x}_1, c_{\omega,i_1}(\mathbf{x}_1) a^{i_1} g(2^{i_1} \mathbf{x}_1))$ and $(\mathbf{x}_3, c_{\omega,i_1}(\mathbf{x}_3) a^{i_1} g(2^{i_1} \mathbf{x}_3))$ lie on different faces of one of the pyramids of graph of the function $c_{\omega,i_1}(\mathbf{x}) a^{i_1} g(\mathbf{x})$.

(ii) $(\mathbf{x}_2, c_{\omega,i_2}(\mathbf{x}_2) a^{i_2} g(2^{i_2} \mathbf{x}_2))$ and $(\mathbf{x}_4, c_{\omega,i_2}(\mathbf{x}_4) a^{i_2} g(2^{i_2} \mathbf{x}_4))$ lie on different faces of one of the pyramids of the graph of the function $c_{\omega,i_2}(\mathbf{x}) a^{i_2} g(\mathbf{x})$.

From (i) we deduce that as $\mathbf{x}_1$ and $\mathbf{x}_3$ lie on different faces of one of the pyramids of the graph of the function $c_{\omega,i_1}(\mathbf{x}) a^{i_1} g(2^{i_1} \mathbf{x})$, so the line $\mathcal{L}(\mathbf{x}_2, \mathbf{x}_4)$ in the $x - y$ plane containing $\mathbf{x}_2$ and $\mathbf{x}_4$ contains a further point $\mathbf{x}_0 = (x_0, y_0)$ satisfying $c_{\omega,i_1}(\mathbf{x}_0) a^{i_1} g(2^{i_1} \mathbf{x}_0) = 0$. The equation of $\mathcal{L}(\mathbf{x}_2, \mathbf{x}_4)$ is given by $2^n(x + y) = k + l + 1$ from which we deduce that $k + l \equiv 1 \mod 2$. Applying

the same argument to (ii) we deduce that $k+l \equiv 0 \mod 2$, a contradiction.
We now choose $\mathbf{x}_a, \mathbf{x}_b$ to be the opposite corner points with the desired property, namely the points $(\mathbf{x}_a, c_{\omega,i}(\mathbf{x}_a)a^i g(2^i\mathbf{x}_a))$, $(\mathbf{x}_b, c_{\omega,i}(\mathbf{x}_b)a^i g(2^i\mathbf{x}_b)) \in \mathbb{R}^3$ lie on the same face of the same pyramid of each of the graphs of the functions $c_{\omega,i}(\mathbf{x})a^i g(2^i\mathbf{x})$ for $0 \leqslant i < n$. So does the midpoint of $R_n(k,l)$, $\mathbf{m}$. In addition we have that $c_{\omega,i}(\mathbf{x}_a) = c_{\omega,i}(\mathbf{x}_b) = c_{\omega,i}(\mathbf{m})$ for $i \geqslant n$, hence

$$c_{\omega,i}(\mathbf{x}_a)a^i g(2^i\mathbf{x}_a) - c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) = c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) - c_{\omega,i}(\mathbf{x}_b)a^i g(2^i\mathbf{x}_b),$$

for all $0 \leqslant i < n$. Summing over $0 \leqslant i < n$ yields

$$f_\omega(\mathbf{x}_a) - \sum_{i=0}^{n-1} c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) = \sum_{i=0}^{n-1} c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) - f_\omega(\mathbf{x}_b)$$

This implies

$$\min(f_\omega(\mathbf{x}_a), f_\omega(\mathbf{x}_b)) \leqslant \sum_{i=0}^{n-1} c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) \leqslant \max(f_\omega(\mathbf{x}_a), f_\omega(\mathbf{x}_b))$$

Which is sufficient for the desired result to hold. □

Proposition 3.5 *For all $\omega \in \Omega$ and $a \in (1/2, 1)$ we have*

$$\sup_{x,y \in R_n(k,l)} |f_\omega(\mathbf{x}) - f_\omega(\mathbf{y})| \geqslant \frac{1}{4}a^n$$

Proof: Fix $n \in \mathbb{N}$, $0 \leqslant k,l < 2^n$. As in the previous proposition let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ denote the corner points of $R_n(k,l)$. We divide the proof into two cases, applying Proposition 3.4 in each.
Case 1: $\omega_n(k,l) = 1$. In this case we have

$$\begin{aligned}
\sup_{x,y \in R_n(k,l)} |f_\omega(\mathbf{x}) - f_\omega(\mathbf{y})| &\geqslant \sum_{i=1}^{n} c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) - \min_{i=1\cdots 4}\{f_\omega(\mathbf{x}_i)\} \\
&= \sum_{i=0}^{n-1} c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) + a^n g(2^n\mathbf{m}) - \min_{i=1\cdots 4}\{f_\omega(\mathbf{x}_i)\} \\
&\geqslant a^n g(2^n\mathbf{m}) = \frac{1}{2}(2a)^n \frac{1}{2^{n+1}} = \frac{1}{4}a^n.
\end{aligned}$$

Case 2: $\omega_n(k,l) = -1$. In this case we have

$$\begin{aligned}
\sup_{x,y \in R_n(k,l)} |f_\omega(\mathbf{x}) - f_\omega(\mathbf{y})| &\geqslant \max_{i=1\cdots 4}\{f_\omega(\mathbf{x}_i)\} - \sum_{i=1}^{n} c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) \\
&= \max_{i=1\cdots 4}\{f_\omega(\mathbf{x}_i)\} - \sum_{i=0}^{n-1} c_{\omega,i}(\mathbf{m})a^i g(2^i\mathbf{m}) + a^n g(2^n\mathbf{m}) \\
&\geqslant a^n g(2^n\mathbf{m}) = \frac{1}{2}(2a)^n \frac{1}{2^{n+1}} = \frac{1}{4}a^n.
\end{aligned}$$

This completes the proof. □

We now introduce a lemma that gives us a lower bound for $N_{\frac{1}{n}}$, for a proof of a generalisation of this lemma see [Fal90].

Lemma 3.6 *For all $n \in \mathbb{N}$ we have*

$$N_{\frac{1}{n}}(\Gamma(f)) \geqslant n \sum_{k,l=0}^{n-1} \sup_{\mathbf{x},\mathbf{y} \in R_n(k,l)} |f(\mathbf{x}) - f(\mathbf{y})|$$

Corollary 3.7 *For all $\omega \in \Omega$ and $a \in (1/2, 1)$ we have*

$$\underline{\dim}_B(\Gamma(f_\omega)) \geqslant s = 3 + \frac{\log a}{\log 2}$$

Proof: Applying Lemmas 3.6 and Proposition 3.5, we have:

$$\begin{aligned} N_{\frac{1}{n}}(\Gamma(f_\omega)) &\geqslant 2^n \sum_{k,l=0}^{2^n-1} \sup_{x,y \in R_n(k,l)} |f_\omega(\mathbf{x}) - f_\omega(\mathbf{y})| \\ &\geqslant 2^n \sum_{k,l=0}^{2^n-1} \frac{1}{4} a^n = \frac{1}{4}(8a)^n. \end{aligned}$$

Hence

$$\begin{aligned} \underline{\dim}_B(\Gamma(f_\omega)) &= \liminf_{n\to\infty} \frac{\log N_{\frac{1}{n}}(\Gamma(f_\omega))}{-\log 2^{-n}} \\ &\geqslant \liminf_{n\to\infty} \frac{\log \frac{1}{4}(8a)^n}{n \log 2} \\ &\geqslant \frac{\log 8a}{\log 2} = s. \end{aligned}$$

$\square$

4 Proof of Theorem 2.4

In this section we give the proof of Theorem 2.4, the upper bound we include for completeness, see [HP88] for more details. We begin by proving a number of auxiliary lemmas leading to the proof of Theorem 2.1.

Lemma 4.1 *Let $(s_n)_n$ be a sequence such that $\lim_{n\to\infty} s_n = \infty$. For each non-empty open set $U \subseteq C[0,1]^2$ and $k \in \mathbb{N}$, there exists a non-empty open set $V_{U,k} \subseteq C([0,1]^2)$ such that*

(i) $V_{U,k} \subseteq U$

(ii) $V_{U,k} \subseteq \left\{ f \in C([0,1]^2) \mid \exists m > k \ : \ N_{\frac{1}{m}}(\Gamma(f)) m^{-3} s_m > k \right\}$

PROOF: Let $U \subseteq C[0,1]^2$ be open. Since U is non-empty, we can choose $f \in U$ and $\epsilon > 0$ such that $B(f,\epsilon) \subseteq U$. We now introduce some notation. Namely, let

$$\begin{aligned} A &= \{(\mathbf{x},y) \in [0,1]^2 \times \mathbb{R} \mid |f(\mathbf{x}) - y| < \epsilon\}, \\ \mathcal{J}_m &= \{Q \in \mathcal{Q}_{\frac{1}{m}} \mid Q \subseteq A\}. \end{aligned}$$

Recall that $\mathcal{Q}_\delta$ is defined in equation (1). We claim that we can find a positive integer m such that the following conditions hold:

(i) $s_m > \frac{k}{\epsilon}$

(ii) $\Gamma(f) \subseteq \bigcup_{Q \in \mathcal{J}_m} Q$

(iii) $\lambda^3(\bigcup_{Q \in \mathcal{J}_m} Q) \geqslant \epsilon$

note needs counter fixed
As $s_n \to \infty$, we may choose a positive integer, m_1 such that

$$s_n > \frac{k}{\epsilon} \quad \text{for all } n > m_1.$$

Next, we claim that we may choose a further positive integer m_2 such that

$$\Gamma(f) \subseteq \bigcup_{Q \in \mathcal{J}_m} Q \quad \text{for all } m > m_2.$$

Indeed, as f is uniformly continuous, there exists a $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$ we have $|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\epsilon}{3}$. Hence, if $m_2 > \frac{1}{\delta}$, then we have that

$$\Gamma(f) \subseteq \bigcup_{Q \in \mathcal{J}_m} Q \quad \text{for all } m > m_2.$$

Finally, we have $\bigcup_{n \in \mathbb{N}} \bigcup_{Q \in \mathcal{J}_n} Q = A$. In addition, for all $n \in \mathbb{N}$, $\bigcup_{Q \in \mathcal{J}_{2^n}} Q \subseteq \bigcup_{Q \in \mathcal{J}_{2^{n+1}}} Q$. Thus by continuity of the Lebesgue measure we have

$$2\epsilon = \lambda^3(A) = \lambda^3\left(\bigcup_{m \in \mathbb{N}} \bigcup_{Q \in \mathcal{J}_{2^m}} Q\right) = \lim_{n \to \infty} \lambda^3\left(\bigcup_{Q \in \mathcal{J}_{2^m}} Q\right).$$

Thus we may choose m_3 such that

$$\lambda^3\left(\bigcup_{Q \in \mathcal{J}_m} Q\right) \geqslant \epsilon \quad \text{for all } m > m_3$$

Finally we set $m = \max(m_1, m_2, m_3)$.

We now construct a function $g_0 \in C[0,1]^2$. Let $0 \leqslant k, l < 2^m$ and consider the square $R_m(k,l)$. Let $\mathbf{k}_1 = (k,l), \mathbf{k}_2 = (k+1,l), \mathbf{k}_3 = (k,l+1)$ and $\mathbf{k}_4 = (k+1,l+1)$ denote its 4 corner points, and $\mathbf{m} = (\frac{2k+1}{2^m}, \frac{2l+1}{2^m})$ its centre point. For each pair (i,j) for which $\mathbf{k}_i$ and $\mathbf{k}_j$ are adjacent corner points, consider the unique plane containing the three points $(\mathbf{k}_i, f(\mathbf{k}_i)$,$(\mathbf{k}_j, f(\mathbf{k}_j)$, and $(\mathbf{m}, f(\mathbf{m})) \in$

$\mathbb{R}^3$. Let $r_{max} = \max\{r | R_m(k,l) \times [\frac{r}{m}, \frac{r+1}{m}] \subseteq \mathcal{J}_m\}$, $r_{min} = \min\{r | R_m(k,l) \times [\frac{r}{m}, \frac{r+1}{m}] \subseteq \mathcal{J}_m\}$. On one of the planes we adjoin a spike extending to within exactly $1/2m$ of the upper face of the cube $R_m(k,l) \times [\frac{r_{max}}{m}, \frac{r_{max}+1}{m}]$. Similarly on another plane we adjoin a spike extending to within exactly $1/2m$ of lower face of the cube $R_m(k,l) \times [\frac{r_{min}}{m}, \frac{r_{min}+1}{m}]$, see Fig 2.

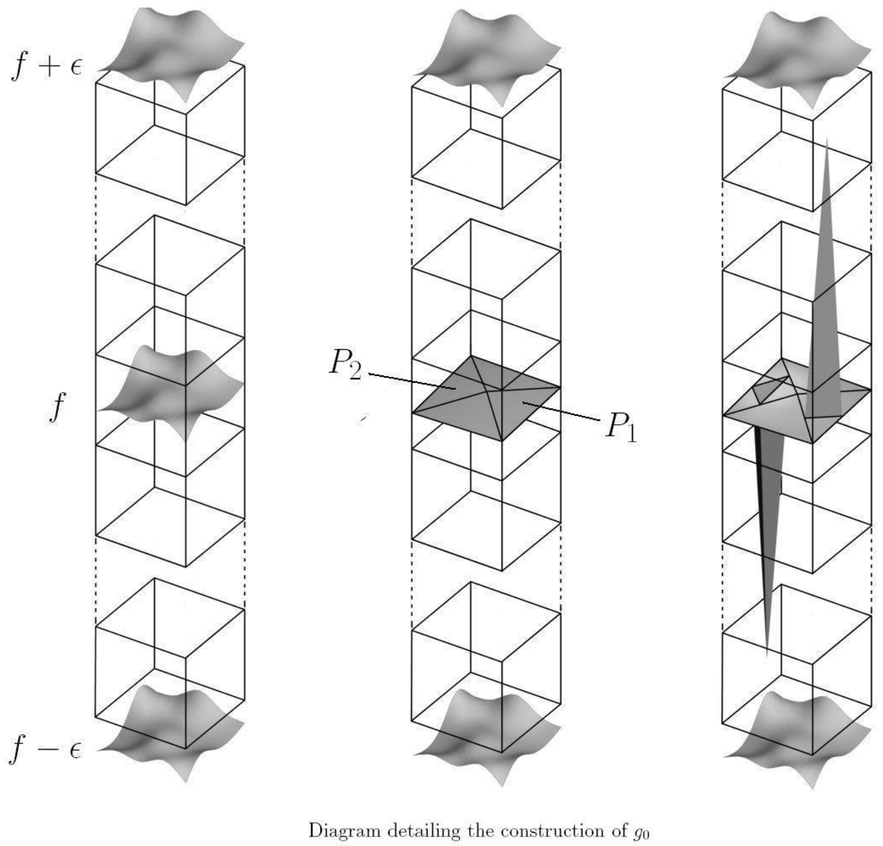


Figure 2. *The construction of g_0 on the square $R_n(k,l)$.*

If we repeat this process for each $0 \leqslant k, l < m$, it is easy to see that we obtain a function $g_0 \in C([0,1]^2)$ such that:

(i) $\Gamma(g_0) \subseteq \bigcup_{Q \in \mathcal{J}_m} Q$

(ii) $\Gamma(g_0) \cap Q \neq \emptyset$ for all $Q \in \mathcal{J}_m$

(iii) If $\|g - g_0\|_\infty < \dfrac{1}{4m}$, then $N_{\frac{1}{m}}(\Gamma(g_0)) = N_{\frac{1}{m}}(\Gamma(g))$

note needs counter fixed
Let

$$V_{U,k} = B\,(g_0, 1/4m).$$

If $g \in V_{U,k}$, then $\Gamma(g) \subseteq \bigcup_{Q \in \mathcal{J}_m} Q \subseteq A$ [by (i) and (iii)] and so $V_{U,k} \subseteq U$. Finally if $g \in V_{U,k}$, then

$$\begin{aligned}\frac{N_{\frac{1}{m}}(\Gamma(g))}{m^3}s_m &= \frac{N_{\frac{1}{m}}(\Gamma(g_0))}{m^3}s_m \quad \text{[by (ii)]} \\ &= \lambda^3(\bigcup_{Q\in\mathcal{J}_m} Q)s_m \\ &> \epsilon s_m \\ &> k.\end{aligned}$$

This concludes the proof for Lemma 4.1. □

Lemma 4.2 *Let $(s_n)_n$ be a sequence such that $\lim_{n\to\infty} s_n = \infty$, then there exists $M \subseteq C([0,1]^2)$ such that*

(i) $M \subseteq \left\{ f \in C([0,1]^2) \mid \limsup_{n\to\infty} N_{\frac{1}{n}}(\Gamma(f))n^{-3}s_n = \infty \right\}$

(ii) *M is comeagre*

Proof: First we define the set M. For an open set U and positive integer k, let $V_{U,k}$ denote a set whose existence is guaranteed by Lemma 4.1. For each $k \in \mathbb{N}$ define the set

$$A_k = \bigcup_{U\subseteq C[0,1]^2\,,\ U \text{ open}} V_{U,k}\,,$$

and put

$$M = \bigcap_{k\in\mathbb{N}} A_k.$$

Now we prove (ii), it is clear that A_k is open. We claim that A_k is dense in $C[0,1]^2$. Let $f \in C[0,1]^2$ and $\epsilon > 0$. Then as $B = B(f,\epsilon)$ is open we have that $V_{B,k} \subseteq B$. Moreover $V_{B,k} \subset A_k$. This shows that A_k is dense. As M is the intersection of a countable collection of dense open sets we have that M is comeagre.

Next we prove (i). Let $f \in M$. Since $f \in M$ we conclude that $f \in A_k$ for all $k \in \mathbb{N}$. Thus for each $k \in \mathbb{N}$ there exists $m_k > k$ such that

$$\frac{N_{\frac{1}{m_k}}(\Gamma(f))}{{m_k}^3}s_{m_k} > k\,.$$

Hence

$$\begin{aligned}\limsup_{m\to\infty}\frac{N_{\frac{1}{m}}(\Gamma(f))}{m^3}s_m &\geqslant \limsup_{k\to\infty}\frac{N_{\frac{1}{m_k}}(\Gamma(f))}{{m_k}^3}s_{m_k} \\ &\geqslant \limsup_{k\to\infty} k \\ &= \infty\end{aligned}$$

This proves (i). □

Lemma 4.3 *Let $p : [0,1]^2 \longrightarrow \mathbb{R}$ be a polynomial, then we have for all $m \in \mathbb{N}$,*

$$N_{\frac{1}{m}}(\Gamma(p)) \leqslant \sqrt{2}m^2\|\nabla p\|_\infty + 2m^2$$

PROOF: Let $m \in \mathbb{N}$, for $0 \leqslant k, l < m$ and consider $R_m(k,l)$. Then by applying an appropriate higher dimensional generalisation of the mean value theorem, see for example [Rud76], we have that there exists $\mathbf{c}(\mathbf{x},\mathbf{y}) \in R_m(k,l)$ such that

$$\begin{aligned} N_{\frac{1}{m}}(\Gamma(p) \cap (R_m(k,l) \times \mathbb{R})) &\leqslant m \sup_{\mathbf{x},\mathbf{y} \in R_m(k,l)} |p(\mathbf{x}) - p(\mathbf{y})| + 2 \\ &= m \sup_{\mathbf{x},\mathbf{y} \in R_m(k,l)} |(\mathbf{x} - \mathbf{y}) \cdot (\nabla p)(\mathbf{c}(\mathbf{x},\mathbf{y}))| + 2 \\ &\leqslant \sqrt{2} \sup_{\mathbf{c} \in R_m(k,l)} |(\nabla p)(\mathbf{c})| + 2 \end{aligned}$$

Now summing over $0 \leqslant k, l < m$ yields the desired result. □

Lemma 4.4 *Let $(s_n)_n$ be a sequence such that $\lim_{n\to\infty} s_n = 0$. For each open set, $U \subseteq C([0,1]^2)$ and $k \in \mathbb{N}$, there exists an open set $W_{U,k} \subseteq C[0,1]^2$ such that*

(i) $W_{U,k} \subseteq U$

(ii) $W_{U,k} \subseteq \left\{ f \in C([0,1]^2) \mid \exists m > k \ : \ N_{\frac{1}{m}}(\Gamma(f)) m^{-2} s_m < 1 \right\}$

PROOF: Let $U \subseteq C[0,1]^2$ be open, $k \in \mathbb{N}$. We may assume that U is non empty and we may thus choose $f \in U$ and $\epsilon > 0$ such that

$$B(f,\epsilon) \subseteq U.$$

By the Weierstrass approximation theorem we may find a polynomial, p such that $\|p - f\|_\infty < \dfrac{\epsilon}{2}$. Choose $m > k$ such that $s_m < \dfrac{1}{\sqrt{2}\|\nabla p\|_\infty + 4}$. Let $\delta < \min(\epsilon/2, 1/m)$. Set

$$W_{U,k} = B(p,\delta).$$

For $g \in W_{U,k}$ we have

$$\|g - f\|_\infty \leqslant \|g - p\|_\infty + \|p - f\|_\infty < \delta + \frac{\epsilon}{2} < \epsilon.$$

Thus $W_{U,k} \subseteq U$. Finally we have for $g \in W_{U,k}$

$$\begin{aligned} \frac{N_{\frac{1}{m}}(\Gamma(g))}{m^2} s_m &\leqslant \frac{N_{\frac{1}{m}}(\Gamma(p)) + 2m^2}{m^2} s_m \text{needs explanation} \\ &\leqslant \frac{\sqrt{2}m^2\|\nabla p\|_\infty + 4m^2}{m^2} \qquad \text{[by Lemma 4.3]} \\ &= \left(\sqrt{2}\|p'\|_\infty + 4\right) s_m \\ &< 1. \end{aligned}$$

This completes the proof of Lemma 4.4 □

Lemma 4.5 *Let $(s_n)_n$ be a sequence such that $\lim_{n\to\infty} s_n = 0$, then there exists $M \subseteq C[0,1]^2$ such that*

(i) $M \subseteq \left\{ f \in C([0,1]^2) \mid \liminf_{n\to\infty} \frac{N_{\frac{1}{n}}(\Gamma(f))}{n^2} s_n < \infty \right\}$

(ii) *M is comeagre*

Proof: First we define the set M. For an open set U an integer k, let $W_{U,k}$ denote the set whose existence is guaranteed by Lemma 4.4. For each $k \in \mathbb{N}$ we define the set

$$A_k = \bigcup_{U \subseteq C[0,1]^2,\ U \text{ open}} W_{U,k}\ ,$$

and put

$$M = \bigcap_{k\in\mathbb{N}} A_k\ .$$

Now we prove (ii), it is clear that A_k is open. We claim that A_k is dense in $C[0,1]^2$. Let $f \in C[0,1]^2$ and $\epsilon > 0$. Then as $B = B(f,\epsilon)$ is open we have that $W_{U,k} \subseteq B$. Moreover $W_{U,k} \subseteq A_k$. This shows that A_k is dense. As M is the intersection of a countable collection of dense open sets we have that M is comeagre.
Next we prove (i). Let $f \in M$. Since $f \in M$, we conclude that $f \in A_k$ for all $k \in \mathbb{N}$. Thus for each $k \in \mathbb{N}$ there exists $m_k > k$ such that

$$\frac{N_{\frac{1}{m_k}}(\Gamma(f))}{{m_k}^2} s_{m_k} < 1\ .$$

Hence

$$\begin{aligned}\liminf_{k\to\infty} \frac{N_{\frac{1}{m}}(\Gamma(f))}{m^2} s_m &\leqslant \liminf_{k\to\infty} \frac{N_{\frac{1}{m_k}}(\Gamma(f))}{{m_k}^2} s_{m_k} \\ &\leqslant \liminf_{k\to\infty} 1 \\ &< \infty\ .\end{aligned}$$

This proves (i). □

We now prove our main theorem. We will prove the theorem in two parts.

Proof: It suffices to show that both of the sets

$$A = \{f \in C[0,1]^2 \mid \overline{\dim}_B(\Gamma(f)) = 3\},$$

$$B = \{f \in C[0,1]^2 \mid \underline{\dim}_B(\Gamma(f)) = 2\}.$$

are comeagre in $C[0,1]^2$. We first prove that A is comeagre . Let $\epsilon > 0$ and define the sequence $s_n(\epsilon)$ by $s_n(\epsilon) = n^\epsilon$. Next consider the set

$$M_\epsilon = \left\{ f \in C[0,1]^2 \mid \limsup_{m\to\infty} N_{\frac{1}{m}}(\Gamma(f)) m^{\epsilon-3} = \infty \right\},$$

and put

$$M = \bigcap_{\epsilon\in\mathbb{Q}^+} M_\epsilon$$

It is clear that $M \subseteq A$. Hence is suffices to show that M is comeagre. By Lemma 4.2 we have that $C[0,1]^2 \setminus M_\epsilon$ is meagre for all $\epsilon > 0$, thus

$$\begin{aligned} C[0,1]^2 \setminus M &= C[0,1]^2 \setminus \bigcap_{\epsilon \in \mathbb{Q}^+} M_\epsilon \\ &= \bigcup_{\epsilon \in \mathbb{Q}^+} C[0,1]^2 \setminus M_\epsilon \end{aligned}$$

is a countable union of meagre sets and therefore meagre. Next we show that B is comeagre.

Let $\epsilon > 0$, define the sequence $s_n(\epsilon)$ by $s_n(\epsilon) = n^{-\epsilon}$. Next consider the set

$$L_\epsilon = \left\{ f \in C[0,1]^2 \mid \liminf_{m \to \infty} \frac{N_{\frac{1}{m}}(\Gamma(f))}{m_k^{2+\epsilon}} < \infty \right\},$$

and put

$$L = \bigcap_{\epsilon \in \mathbb{Q}^+} L_\epsilon$$

It is clear that $L \subseteq B$. Hence it suffices to show that L is comeagre. By Lemma 4.5 we have that $C[0,1]^2 \setminus L_\epsilon$ is meagre for all $\epsilon > 0$, thus

$$\begin{aligned} C[0,1]^2 \setminus L &= C[0,1]^2 \setminus \bigcap_{\epsilon \in \mathbb{Q}^+} L_\epsilon \\ &= \bigcup_{\epsilon \in \mathbb{Q}^+} C[0,1]^2 \setminus L_\epsilon \end{aligned}$$

is a countable union of meagre sets and therefore meagre. □

5 ToDo

1. Relabel references of eqns according to section.
2. Lists need renumbering according to section.
3. Some vectors which should be just bold are also italic because they are nested inside a lemma.
4. Prop 3.4 no longer has the midpoint referred to showing in the diagram.

References

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