On the Padé Table for e^x and the simple continued fractions for e and $e^{L/M}$

By J H McCabe

Mathematical Institute, St Andrews, Fife, KY16 9SS, UK. email: jhm@st-and.ac.uk

---

ABSTRACT. A recently observed connection between some *Padé approximants* for the exponential series and the convergents of the simple continued fraction for e is established, leading to an alternative proof of the latter. Similar results for the simple continued fraction $e^2, e^{1/M}$ and $e^{2/M}$, when M is a natural number greater than one, are derived.

MSC: 30B70, 41A21

Keywords: regular continued fractions, Padé tables

---

1 The Padé Table

The *Padé table* [see figure 1] of the formal power series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \quad c_0 \neq 0, \tag{1}$$

is an infinite two dimensional array of irreducible rational functions

$$P_{n,m}(x) = \frac{A_{n,m}(x)}{B_{n,m}(x)} = \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_m x^m}{\beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_n x^n} \ , \quad m, n > 0, \tag{2}$$

in each of which the coefficients are such that the expansion of $P_{m,n}(x)$ in powers of x matches that of $f(x)$ as far as possible. The power series and its associated Padé table are said to be *normal* if

$$P_{n,m}(x) = \sum_{k=0}^{m+n} c_k x^k \quad + \quad \text{higher order terms,}$$

in which case every element of the table exists and is different from any other element.

$$
\begin{array}{cccccccc}
P_{0,0} & P_{1,0} & P_{2,0} & P_{3,0} & P_{4,0} & \ldots\ldots & P_{n,0} & \ldots\ldots \\
P_{0,1} & P_{1,1} & P_{2,1} & P_{3,1} & P_{4,1} & \ldots\ldots & P_{n,1} & \ldots\ldots \\
P_{0,2} & P_{1,2} & P_{2,2} & P_{3,2} & P_{4,2} & \ldots\ldots & P_{n,2} & \ldots\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
P_{0,m} & P_{1,m} & P_{2,m} & P_{3,m} & P_{4,m} & \ldots\ldots & P_{n,m} & \ldots\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

$Figure 1. The Pad\acute{e} table$

There are several methods for transforming a normal series into its Padé table, including variations of the quotient difference algorithm, techniques which exploit the close connection between the Padé table and various continued fraction expansions that correspond to the series (1). One such algorithm is the following, developed for transforming two series expansions into two point Padé approximations but applicable to the standard Padé table.

Set $d_1^j = -c_j/c_{j-1}$ for $j = 1, 2, \ldots$ and use these ratios as starting values for the *rhombus rules*

$$
\left.
\begin{aligned}
n_{i+1}^j &= n_i^{j+1} + d_i^{j+1} - d_i^j \\
d_{i+1}^{j+1} &= n_{i+1}^{j+1} \times d_i^j \div n_{i+1}^j
\end{aligned}
\right\}, \tag{3}
$$

for $j = 1, 2, 3, \ldots$ and $i = 1, 2, 3, \ldots$ with $n_1^j = 0$ for all j and $d_i^1 = -n_i^1$ for $i = 2, 3, \ldots$.

The elements generated form the $n - d$ array shown in figure 2 and it can be seen that both recurrences connect four elements that form a rhombus.

$$
\begin{array}{cccccccc}
d_1^1 & n_2^1 & d_2^1 & n_3^1 & d_3^1 & n_4^1 & d_4^1 & \ldots \\
d_1^2 & n_2^2 & d_2^2 & n_3^2 & d_3^2 & n_4^2 & d_4^2 & \ldots \\
d_1^3 & n_2^3 & d_2^3 & n_3^3 & d_3^3 & n_4^3 & d_4^3 & \ldots \\
d_1^4 & n_2^4 & d_2^4 & n_3^4 & d_3^4 & n_4^4 & d_4^4 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

Figure 2. *The n-d array*

The convergents of the continued fraction

$$
c_0 + c_1 x + \ldots + c_{k-1} x^{k-1} + \frac{c_k x^k}{1 + d_1^k x +} \frac{n_2^k x}{1 + d_2^k x +} \frac{n_3^k x}{1 + d_3^k x + \ldots} \tag{4}
$$

for $k = 1, 2, 3, \ldots$ are the Padé approximants $P_{1,k-1},\ P_{2,k-1},\ P_{3,k-1},\ P_{4,k-1}, \ldots$, that is those on the $(k-1)$th row of the Padé table. In fact the $n-d$ array contains what is required to construct the continued fraction whose successive convergents form any chosen sequence of Padé approximants, provided that each member of the sequence is a neighbour of the previous member in the table. These include row sequences, staircase sequences, saw tooth sequences and battlement sequences. See McCabe [3] for details. The starting points for constructing the continued fractions are the three term recurrence relations linking the numerator and denominator polynomials of trios of adjacent approximants

(i) $P_{i-1,j} P_{i,j} P_{i+1,j}$: $\quad P_{i+1,j}(x) = \left(1 + d_{i+1}^{j+1} x\right) P_{i,j}(x) + n_{i+1}^{j+1} x P_{i-1,j}(x)$

(ii) $\begin{array}{cc} P_{i-1,j} & P_{i,j} \\ & P_{i,j+1} \end{array}$ $\quad P_{i,j+1}(x) = P_{i,j}(x) + n_{i+1}^{j+1} x P_{i-1,j}(x)$

(iii) $\begin{array}{cc} P_{i,j-1} & \\ P_{i,j} & P_{i+1,j} \end{array}$ $\quad P_{i+1,j}(x) = P_{i,j}(x) + \left(n_{i+1}^{j+1} + d_{i+1}^{j+1}\right) x P_{i,j-1}(x)$

and, combining (ii) and (iii), we obtain

(iv) $\begin{array}{cc} & P_{i+1,j} \\ P_{i,j+1} & P_{i+1,j+1} \end{array}$ $\quad P_{i+1,j+1} = \frac{-1}{d_{i+1}^{j+1}} \left\{ n_{i+2}^{j+1} P_{i,j+1}(x) - \left(n_{i+1}^{j+1} + d_{i+1}^{j+1}\right) x P_{i+1,j}(x) \right\}$

(v) $\begin{array}{cc} P_{i,j} & P_{i+1,j} \\ P_{i,j+1} & \end{array}$ $\quad P_{i,j+1} = P_{i+1,j} - d_{i+1}^{j+1} x P_{i,j}$

We subsequently make use of (i), (iv) and (v) to derive a particular continued fraction for the exponential function. The relations (ii) and (iii) used in turn repeatedly provides the continued fraction

$$\frac{c_0}{1+}\ \frac{\left(n_1^1 + d_1^1\right) x}{1}\ \underset{+}{}\ \frac{n_2^1 x}{1+}\ \frac{\left(n_2^2 + d_2^2\right) x}{1}\ \underset{+}{}\ \frac{n_3^2 x}{1+}\ \frac{\left(n_3^3 + d_3^3\right) x}{1}\ \underset{+}{}\ \frac{n_4^3 x}{1+} \ldots \tag{5}$$

whose convergents are the staircase sequence $P_{0,0},\ P_{1,0},\ P_{1,1},\ P_{2,1},\ P_{2,2},\ P_{3,2},\ P_{3,3}, \ldots$. Note, these relations must be applied separately to the numerator polynomials and the denominator polynomials of the Padé approximants.

2 Simple (regular) continued fraction expansions for irrationals

Let λ be an irrational number and let the sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ be generated by

$$\lambda_{n+1} = \frac{1}{\lambda_n - a_n}, \quad a_n = [\lambda_n], \quad n = 0, 1, 2, \ldots,$$

with $\lambda_0 = \lambda$ and where $[\lambda_n]$ denotes the integer part of λ_n. It is easily shown that

$$\lambda = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \frac{1}{a_5 +} \frac{1}{a_6 +} \dots$$

is a convergent continued fraction with limit λ. The convergents of the continued fraction form a sequence of rationals, the even members forming a monotonically increasing sequence converging to λ and the odd order members forming a monotonically decreasing sequence converging to λ. The properties of these rationals as approximations to λ, in terms of their *order of approximation* and as *best approximations*, are well known.

If we apply this algorithm to the irrational number e we obtain

$$e = 2 + \frac{1}{1 +} \frac{1}{2 +} \frac{1}{1 +} \frac{1}{1 +} \frac{1}{4 +} \frac{1}{1 +} \frac{1}{1 +} \frac{1}{6 +} \frac{1}{1 +} \frac{1}{1 +} \frac{1}{8 +} \frac{1}{1 +} \frac{1}{1 +} \frac{1}{10 +} \frac{1}{1 +} \frac{1}{1 +} \frac{1}{12 +} \dots \tag{6}$$

which suggests $e = \left[2; \overline{1, 2n, 1}\right]_{n=1}^{\infty}$. Euler, in studying the *Ricatti* equation proved that e had the above expansion and it was alternatively established by Hermite in his proof of the transcendence of e, see for example *Olds* [4]. The proof presented below is a development of an observation made in a recent article, *Cohn* [2], on the relationship between the successive convergents of (6) and a particular sequence of Padé approximants for e^x.

Specifically, the initial convergents of (6) are $\frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}, \frac{1264}{465}, \frac{1457}{536}, \frac{2721}{1001} \dots$ and these rationals appear to be the values of the sequence $P_{0,1}(x), P_{1,1}(x), P_{2,1}(x), P_{1,2}(x), P_{2,2}(x),$ $P_{3,2}(x), P_{2,3}(x), P_{3,3}(x), P_{4,3}(x) \dots$ when we set $x = 1$, that is the sequence $\{P_{r-1,r}(1), P_{r,r}(1), P_{r+1,r}(1)\}_{r=1}^{\infty}$.

This is in fact the case, as is now proved.

3 A continued fraction for e^x

The $n - d$ *array* constructed from the coefficients of the exponential series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^k}{k!} + \dots$$

begins

j	d_1^j	n_2^j	d_2^j	n_3^j	d_3^j	n_4^j
1	-1	$1/2$	$-1/2$	$2/6$	$-1/3$	$3/12$
2	$-1/2$	$1/6$	$-1/3$	$2/12$	$-1/4$	$3/20$
3	$-1/3$	$1/12$	$-1/4$	$2/20$	$-1/5$	$3/30$
4	$-1/4$	$1/20$	$-1/5$	$2/30$	$-1/6$	$3/42$
5	$-1/5$	$1/30$	$-1/6$	$2/42$	$-1/7$	$3/56$
6	$-1/6$	$1/42$	$-1/7$	$2/56$	$-1/8$	$3/72$

and it is easily established that

$$d_i^j = -\frac{1}{i+j-1} \qquad n_i^j = \frac{i-1}{(i+j-2)(i+j-1)} \tag{7}$$

for $i, j = 1, 2, 3, \ldots$. These values provide through (5) the continued fraction for e^x

$$e^x = \frac{1}{1-}\,\frac{x}{1+}\,\frac{x/2}{1\;-}\,\frac{x/6}{1\;+}\,\frac{x/6}{1\;-}\,\frac{x/10}{1\;+}\,\frac{x/10}{1\;-}\,\frac{x/14}{1\;+}\,\frac{x/14}{1\;+}\ldots$$

which is equivalent to the more familiar expansion

$$e^x = \frac{1}{1-}\,\frac{x}{1+}\,\frac{x}{2-}\,\frac{x}{3+}\,\frac{x}{2-}\,\frac{x}{5+}\,\frac{x}{2-}\,\frac{x}{7+}\,\frac{x}{2+}\ldots.$$

Using these n and d values in the three term recurrences (i), (iv) and (v) above we obtain

$$\left.\begin{aligned} P_{r+1,r}(x) &= \left(1-\frac{x}{2r+1}\right)P_{r,r}(x)+\frac{x}{2(2r+1)}P_{r-1,r}(x) \\ P_{r,r+1}(x) &= P_{r+1,r}(x)+\frac{x}{2r+1}P_{r,r}(x) \\ P_{r+1,r+1}(x) &= \frac{1}{2}\left\{P_{r,r+1}(x)+P_{r+1,r}(x)\right\} \end{aligned}\right\}. \tag{8}$$

If we use these in turn for $r = 1, 2, 3, 4, \ldots$, starting with approximants $P_{0,1}(x)$ and $P_{1,1}(x)$, we will generate separately the numerator polynomials and denominator polynomials of $P_{2,1}(x), P_{1,2}(x), P_{2,2}(x)$, $P_{3,2}(x), P_{2,3}(x), P_{3,3}(x), P_{4,3}(x)\ldots$.

Remembering the three term recurrence relations satisfied by the numerators and denominators of continued fractions, the relations (8) allow us to write down the continued fraction for which

$$P_{0,1}(x), P_{1,1}(x), P_{2,1}(x), P_{1,2}(x), P_{2,2}(x), P_{3,2}(x), P_{2,3}(x), P_{3,3}(x), P_{4,3}(x)\ldots$$

are successive convergents.

It is

$$e^x = 1 + x + \frac{x^2/2}{1-x/2+}\,\frac{x/6}{1-x/3+}\,\frac{x/3}{1}\,\frac{1/2}{+1/2+}\,\frac{x/10}{1-x/5+}\,\frac{x/5}{1}\,\frac{1/2}{+1/2+}\,\frac{x/14}{1-x/7+}\,\frac{x/7}{1}\,+\ldots.$$

This is in the form for which the numerators and denominators of the convergents are equal to unity when x is set to zero. We obtain them in monic form from the equivalent continued fraction

$$e^x = 1 + x + \frac{x^2}{2-x+}\,\frac{x}{3-x+}\,\frac{x}{1+}\,\frac{1}{1+}\,\frac{x}{5-x+}\,\frac{x}{1+}\,\frac{1}{1+}\,\frac{x}{7-x+}\,\frac{x}{1+}\,\frac{1}{1+}\,\frac{x}{9-x+}\ldots \tag{9}$$

whose sequence of convergents begins

$$P_{0,1}(x) = \frac{1+x}{1},\ P_{1,1}(x) = \frac{2+x}{2-x},\qquad P_{2,1}(x) = \frac{6+2x}{6-4x+x^2}$$

$$P_{1,2}(x) = \frac{6+4x+x^2}{6-2x},\ P_{2,2}(x) = \frac{12+6x+x^2}{12-6x+x^2},\qquad P_{3,2}(x) = \frac{60+24x+3x^2}{60-36x+9x^2-x^3}$$

$$P_{2,3}(x) = \frac{60+36x+9x^2+x^3}{60-24x+3x^2},\qquad P_{3,3}(x) = \frac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3},$$

Clearly, as expected since $e^{-x} = \frac{1}{e^x}$, the diagonal elements of the Padé table and the off diagonal ones satisfy, respectively,

$$P_{k,k}(x) = \frac{A_{k,k}(x)}{A_{k,k}(-x)} \text{ and } P_{k,k+1}(x) = \frac{A_{k,k+1}(x)}{B_{k,k+1}(x)} = \frac{B_{k,k-1}(-x)}{A_{k,k-1}(-x)} = \frac{1}{P_{k,k-1}(-x)}.$$

Setting $x = 1$ in the continued fraction (9) yields the expansion (6), namely

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{10+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{12+}\ldots \tag{6}$$

and setting $x = 1$ in the approximants above yields the sequence

$$\frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}, \frac{1264}{465}, \frac{1457}{536}, \frac{2721}{1001}\cdots,$$

which are the convergents of (6).

Convergence of the continued fraction (6) follows from that of any sequence of diagonal and off diagonal Padé approximants for the exponential function, see Baker & Graves-Morris [1] for instance.

An alternative continued fraction expansion for e, but not a simple one of course, is the one obtained by setting $x = 1$ in the continued fraction whose convergents form the sequence starting at the first element of the Padé table,

$$P_{0,0}(x), P_{1,0}(x), P_{0,1}(x), P_{1,1}(x), P_{2,1}(x), P_{1,2}(x), P_{2,2}(x), P_{3,2}(x), P_{2,3}(x), P_{3,3}(x), P_{4,3}(x) \ldots .$$

This fraction is

$$\frac{1}{1-}\frac{x}{1+}\frac{x}{1+}\frac{1}{1+}\frac{x}{3-x+}\frac{x}{1+}\frac{1}{1+}\frac{x}{5-x+}\frac{x}{1+}\frac{1}{1+}\frac{x}{7-x+}\frac{x}{1+}\frac{1}{1+}\frac{x}{9-x+}\ldots$$

and setting $x = 1$ yields the irrational number λ,

$$\lambda = \frac{1}{1-}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{8+}\ldots .$$

Clearly from (6) this can be written

$$\lambda = \frac{1}{1-}\frac{1}{1+}\frac{1}{e-1} = \frac{1}{1-}\frac{e-1}{e} = e.$$

4 The continued fraction for $e^{\frac{L}{M}}, M \in \mathbb{N}$

It is not only the simpled continued fraction for e that has a regularity, those of rational powers of e can also be written down. For example

$$\begin{aligned} e^{1/2} &= [1; 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, 1, 1, 17, \ldots] \\ e^{1/3} &= [1; 2, 1, 1, 8, 1, 1, 14, 1, 1, 20, 1, 1, 26, \ldots] \\ e^{1/4} &= [1; 3, 1, 1, 11, 1, 1, 19, 1, 1, 27, 1, 1, 35, \ldots] \\ e^{1/5} &= [1; 4, 1, 1, 14, 1, 1, 24, 1, 1, 34, 1, 1, 44, \ldots] \end{aligned} .$$

These are each a special case of the general expansion, for $M > 1$

$$e^{1/M} = [1; M-1, 1, 1, 3M-1, 1, 1, 5M-1, 1, 1, 7M-1, 1, 1, 9M-1, \ldots] \tag{10}$$

which is established by Osler [5] using certain integrals.

It is easily conjectured that the successive convergents of these continued fractions form the following sequence of Padé approximants evaluated at $x = 1/M$,

$P_{0,0}(1/M), P_{1,0}(1/M), P_{0,1}(1/M), P_{1,1}(1/M), P_{2,1}(1/M), P_{1,2}(1/M), P_{2,2}(1/M), P_{3,2}(1/M) \ldots$, that is $\{P_{r-1,r-1}(1/M), P_{r,r-1}(1/M), P_{r-1,r}(1/M)\}_{r=1}^{\infty}$. This is the same sequence as in the case for e but with the $P_{0,0}(1/M)$ and $P_{1,0}(1/M)$ added on at the beginning.

The conjecture is proved by constructing the continued fraction which has this sequence as convergents by making successive use of the second and third relations in (8), with $r = 0$ and then all three in order with $r = 1, 2, 3, \ldots$. We obtain

$$e^{1/M} = 1 + \frac{1/M}{1-1/M+}\frac{1/M}{1}+\frac{1/2}{1/2+}\frac{1/6M}{1-1/3M+}\frac{1/3M}{1}+\frac{1/3M}{1}+\frac{1/2}{1/2}\frac{1/10M}{1-1/5M+}\frac{1/5M}{1}$$
$$+\frac{1/2}{1/2+}\frac{1/14M}{1-1/7M+}\frac{1/7M}{1}+\ldots$$

which is equivalent to

$$e^{1/M} = 1+\frac{1}{M-1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3M-1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{5M-1++}\frac{1}{1+}\frac{1}{1+}\frac{1}{7M-1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{9M-1+}\frac{1}{1+}\frac{1}{1+}\ldots$$

as in (10) above.

What about the continued fraction expansions of integer powers of $e^{L/M}$, $M \in \mathbb{N}$ and values of L other than 1? There is no known formula for the partial quotients of the simple continued fraction expansion of $e^{L/M}$ unless $L = 1$ or 2 and, in the latter case, with M odd since if M were even it would be covered by (10) above.

Perron [6] gives the expansions

$$e^2 = 7 + \frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{18+}\frac{1}{5+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{30+}\ldots = \left[7; \overline{3n+2, 1, 1, 3n+3, 12n+18}\right]_{n=0}^{\infty} \tag{11}$$

and

$$e^{2/M} = \left[1; \overline{\frac{1}{2}\{6n+1)M-1\}, 6(2n+1)M, \frac{1}{2}\{6n+5)M-1\}, 1, 1}\right]_{n=0}^{\infty}. \tag{12}$$

We now show that these expansions can be established, like (6) and (10), by showing that the convergents of each form a sequence of Padé approximants evaluated, respectively, at $x = 2$ and at $x = 2/M$. The sequences are different from those above, but they still contain only diagonal and off diagonal elements of the Padé table for e^x.

Define the convergents of (11) by $\dfrac{A_k}{B_k}, k = 0, 1, 2, \ldots$. It is easily seen by direct calculation of these rational numbers and by evaluation of certain Padé approximants at $x = 2$ that

$$\begin{array}{llll}
\dfrac{A_0}{B_0} = \dfrac{7}{1} = P_{2,2}(2) & \dfrac{A_1}{B_1} = \dfrac{15}{2} = P_{3,2}(2) & \dfrac{A_2}{B_2} = \dfrac{22}{3} = P_{2,3}(2) & \dfrac{A_3}{B_3}\dfrac{37}{5} = P_{3,3}(2) \\
\dfrac{A_4}{B_4} = \dfrac{133}{18} = P_{4,4}(2) & \dfrac{A_5}{B_5} = \dfrac{2431}{329} = P_{5,5}(2) & \dfrac{A_6}{B_6} = \dfrac{12288}{1663} = P_{6,5}(2) & \dfrac{A_7}{B_7} = \dfrac{14719}{1992} = P_{5,6}(2)
\end{array}$$

and the pattern continues with

$\dfrac{A_8}{B_8} = P_{6,6}(2), \dfrac{A_9}{B_9} = P_{7,7}(2), \dfrac{A_{10}}{B_{10}} = P_{8,8}(2), \dfrac{A_{11}}{B_{11}} = P_{9,8}(2), \dfrac{A_{12}}{B_{12}} = P_{8,9}(2), \dfrac{A_{13}}{B_{13}} = P_{9,9}(2)$,
$\frac{A_{14}}{B_{14}} = P_{10,10}(2)$ and $\dfrac{A_{15}}{B_{15}} = P_{11,11}(2)$ etc..

The obvious conjecture is that the values of the rationals $\dfrac{A_k}{B_k}, k = 0, 1, 2, \ldots$ are the sequence $[P_{3n-1,3n-1}(2), P_{3n,3n-1}(2), P_{3n-1,3n}(2), P_{3n,3n}(2), P_{3n+1,3n+1}(2)]_{n=1}^{\infty}$, which define a route through the Padé table made up of the series of paths shown below,

$$\begin{array}{ccccc}
P_{3k-1,3k-1}(2) & \rightarrow & P_{3k,3k-1}(2) & & \\
& \swarrow & & & \\
P_{3k-1,3k}(2) & \rightarrow & P_{3k,3k}(2) & & \\
& & & \searrow & \\
& & & & P_{3k+1,3k+1}(2)
\end{array}$$

for $k = 1, 2, 3, \ldots$.

In order to construct the continued fraction, starting from the Padé approximants $P_{0,0}(x)$ and $P_{1,1}(x)$ to yield $P_{2,2}(x)$ and then following the above paths to reach the first element in each of the above paths, that is $P_{5,5}(x), P_{8,8}(x), P_{11,11}(x), \ldots$ we use sequentially the following recurrence relations

$$\begin{array}{llll} P_{r-1,r-1}(x) & & & \\ & \searrow & & \\ & P_{r,r}(x) & & \\ & & \searrow & \\ & & P_{r+1,r+1}(x) = P_{r,r}(x) + \frac{x^2}{4(4r^2-1)} P_{r-1,r-1}(x) & \qquad r = 1, 4, 7, \ldots \end{array}$$

$$\begin{array}{lll} P_{r-1,r-1}(x) & & \\ & \searrow & \\ & P_{r,r}(x) \to P_{r+1,r}(x) = \left(1 - \frac{x}{2(2r+1)}\right) P_{r,r}(x) + \frac{x^2}{4(4r^2-1)} P_{r-1,r-1}(x) & \qquad r = 2, 5, 8, \ldots \end{array}$$

$$\begin{array}{ll} P_{r,r}(x) \to P_{r+1,r}(x) & \\ \swarrow & \\ P_{r,r+1}(x) = P_{r+1,r}(x) + \frac{x}{2r+1} P_{r,r}(x) & \qquad r = 2, 5, 8, \ldots \end{array}$$

$$\begin{array}{ll} \qquad P_{r+1,r}(x) & \\ \swarrow & \\ P_{r,r+1}(x) \to P_{r+1,r+1}(x) = \frac{1}{2}\{P_{r,r+1}(x) + P_{r+1,r}(x)\} & \qquad r = 2, 5, 8, \ldots \end{array}$$

$$\begin{array}{lll} P_{r-1,r}(x) \to P_{r,r}(x) & & \\ & \searrow & \\ & P_{r+1,r+1}(x) = \left(1 - \frac{x}{2(2r+1)}\right) P_{r,r}(x) + \frac{x}{2(2r+1)} P_{r-1,r}(x) & \qquad r = 3, 6, 9, \ldots \end{array}$$

We obtain the continued fraction

$$\frac{1}{1-}\,\frac{2x}{2-x+}\,\frac{x^2}{6}+\frac{x^2}{10-x+}\,\frac{2x}{1}+\frac{1}{1+}\,\frac{2x}{14-x+}\,\frac{x^2}{18+}\,\frac{x^2}{22-x+}\,\frac{2x}{1}+\frac{1}{1+}\,\frac{2x}{26-x+}\,\frac{x^2}{30+}\,\frac{x^2}{34-x+}\ldots$$

and setting $x = 2$ yields

$$\delta = \frac{1}{1-}\,\frac{4}{4+}\,\frac{4}{6+}\,\frac{4}{8+}\,\frac{4}{1+}\,\frac{1}{1+}\,\frac{4}{12+}\,\frac{4}{18+}\,\frac{4}{20+}\,\frac{4}{1+}\,\frac{1}{1+}\,\frac{4}{24+}\,\frac{4}{30+}\,\frac{4}{32+}\,\frac{4}{1+}\,\frac{1}{1+}\,\frac{4}{36+}\ldots$$

or, equivalently,

$$\delta = \frac{1}{1-}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{18+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{30+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{9+}\ldots.$$

We see from (11) that this can be written as

$$\delta = \frac{1}{1-}\,\frac{1}{1+}\,\frac{1}{e^2-1} = \frac{1}{1-}\,\frac{e^2-1}{e^2} = e^2.$$

The convergents of this continued fraction, after the first two, are thos of the simple continued fraction

$$e^2 = [7; \overline{3n+2, 1, 1, 3n+3, 12n+18}]_{n=0}^{\infty}$$

given above.

Just as the convergents of the simple continued fractions of e and of $e^{1/M}$ follow the same paths in the Padé table, but evaluated at $x = 1$ and $x = 1/M$ respectively, those of the simple continued fraction of $e^{2/M}$ follow the same path as those of e^2, but evaluated at $x = 2/M$ instead of $x = 2$. Thus the convergents of (12), namely

$$e^{2/M} = \left[1; \overline{\frac{1}{2}\{(6n+1)M - 1\}, 6(2n+1)M, \frac{1}{2}\{(6n+5)M - 1\}, 1, 1}\right]_{n=0}^{\infty}$$

are $P_{0,0}(2/M), P_{1,1}(2/M)$ then

$[P_{3n-1,3n-1}(2/M), P_{3n,3n-1}(2/M), P_{3n-1,3n}(2/M), P_{3n,3n}(2/M), P_{3n+1,3n+1}(2/M)]_{n=1}^{\infty}$.

Since there are no known formula for the partial quotients of the simple continued fraction expansion of $e^{L/M}$ unless $L = 1$ or 2 it is unlikely that the convergents form evaluations of an ordered seqence of Padé approximants.

References

[1] Baker, G A and Graves-Morris, P, *Padé approximants Part 1: Basic Thoery*, Encyclopedia of Mathematics and its applications, Volume 13, Addison Wesley, 1981.

[2] H Cohn, "A short proof of the simple continued fraction expansion of e", *American Mathematical Monthly*, Vol. 13, No. 1, January 2006, MAA.

[3] J H McCabe, "The quotient difference algorithm and the Padé table: An alternative form and a general continued fraction", Mathematics of Computation, Vol. 41, No. 6, pp183-197, 1983.

[4] Olds, C, "The simple continued fraction expansion of e", *American Mathematical Monthly*, Vol. 77, pp968-974, 1970.

[5] T J Osler, A proof of the Continued Fraction of $e^{1/M}$, *American Mathematical Monthly*, Vol. 13, No. 1, January 2006, MAA.

[6] O Perron, Die Lehre von den Kettenbrüchen, Band I, B. G. Teubner, Stuttgart, 1957.