Box-Counting Dimensions of Takagi-Type Surfaces

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Abstract. We investigate the lower and upper box-counting dimensions of a surface defined analogously to the Takagi function, a function known for its property that it is continuous but nowhere differentiable. In some cases exact values for the dimension can be given, whilst in others upper and lower bounds are known.

Contents

1. Introduction

We investigate here the box dimensions of graphs of Takagi functions. In particular we consider Takagi surfaces with random peak orientation and sequence scalings. From a paper of F. Takeo [**Tak96**] we know that in some cases the box dimension does not exist and we must consider upper and lower box dimensions seperately. We first consider two special cases: when the scaling sequence converges "quickly", and when it converges "slowly". We then find certain bounds for the upper and lower box dimensions.

ST, DR, DS, and NC are grateful to LO for much useful discussion and advice, and also to the Nuffield Foundation and the Carnegie Trust for the Universities of Scotland, for their financial support.

2. Definitions and Preliminary Observations

We adopt the notation

$$I_{n,i,j} := \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right] \times \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right].$$

We extend the graph of a Takagi curve to a surface as follows. Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the function whose graph is a pyramid on each unit square with height $\frac{1}{2}$. Now let $g_k\colon [0,1]\times[0,1] \to \mathbb{R}$ be the function

$$\epsilon_{k,i,j}\frac{1}{2^k} g(2^k x)$$

on $I_{k,i,j}$, with $\epsilon_{k,i,j} \in \{-1, 1\}$ chosen with equal probability for either value. For a sequence $(a_k)_k \in \ell^1$ we define the Takagi surface $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \sum_{k=0}^{\infty} a_k g_k(x)$$

which exists and is continuous by the Weierstrass 'Majorant' theorem. We will be studying the graph of f; that is the set

$$\operatorname{graph} f = \left\{ \big(x, f(x)\big) \mid x \in [0,1]\times[0,1] \right\}.$$

Some approximate visualisations of such sets are provided with this report. We define the δ-grid (of 'cubes') in $\mathbb{R}^3$ by

$$\mathcal{Q}_\delta = \Big\{ [n_1\delta, (n_1+1)\delta] \times [n_2\delta, (n_2+1)\delta] \times [n_3\delta, (n_3+1)\delta] \;\Big|\; n_1, n_2, n_3 \in \mathbb{Z} \Big\},$$

and define $\mathrm{M}_\delta(E)$ to be the number of δ-cubes interesecting a subset E of $\mathbb{R}^3$, thus

$$\mathrm{M}_\delta(E) = \big|\{Q \in \mathcal{Q}_\delta \mid Q \cap E \neq \varnothing\}\big|.$$

This leads to the definition of the upper and lower box dimensions as

$$\overline{\dim}_{\mathrm{B}}(E) = \limsup_{\delta\to 0} \frac{\log \mathrm{M}_\delta(E)}{-\log\delta} \qquad \text{and} \qquad \underline{\dim}_{\mathrm{B}}(E) = \liminf_{\delta\to 0} \frac{\log \mathrm{M}_\delta(E)}{-\log\delta},$$

and to the immediate observation that

$$\underline{\dim}_{\mathrm{B}}(E) \leqslant \overline{\dim}_{\mathrm{B}}(E).$$

It is also useful to note that for a continuous function f on $[0,1]\times[0,1]$ we have

$$\frac{1}{\delta}\sum_{i,j=0}^{[\frac{1}{\delta}]} \operatorname{osc}\big(f, [i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta]\big) \leqslant \mathrm{M}_\delta\big(\operatorname{graph} f\big)$$

$$\leqslant 2\left(\frac{1}{\delta}+1\right)^2 + \frac{1}{\delta}\sum_{i,j=0}^{[\frac{1}{\delta}]} \operatorname{osc}\big(f, [i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta]\big), \quad (1)$$

where $\operatorname{osc}\big(f, [\zeta, \eta]\big) := \sup_{x,y \in [\zeta,\eta]} \big|f(x) - f(y)\big|$.

3. The Exact Upper and Lower Dimensions for some Scaling Sequences

If the sequence $(a_k)_k$ scaling the function tends to zero 'fast enough', then we find that the dimension of the surface is exactly 2.

We will need the following result:

♦ **Lemma 3.1.** *Let* $f \in C\big([0,1] \times [0,1]\big)$ *and suppose* $s \in [2,3]$. *If for every* $x, y \in [0,1] \times [0,1]$ *we have*

$$\big|f(x) - f(y)\big| \leqslant c\big|x - y\big|^{3-s},$$

then

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \leqslant s.$$

Proof. Assume the hypothesis of the lemma. Then, using (1), we have

$$\begin{aligned}
\mathrm{M}_\delta\big(\operatorname{graph} f\big) &\leqslant 2\left(\frac{1}{\delta} + 1\right)^2 + \frac{1}{\delta}\sum_{i,j=0}^{[\frac{1}{\delta}]} \operatorname{osc}\big(f, [i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta]\big) \\
&\leqslant 2\left(\frac{1}{\delta} + 1\right)^2 + \frac{1}{\delta}\sum_{i,j=0}^{[\frac{1}{\delta}]} c(\sqrt{2})^{3-s}|\delta|^{3-s} \\
&\leqslant (4 + 2(\sqrt{2})^{3-s}c)\frac{1}{\delta^s}.
\end{aligned}$$

This then gives

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) = \limsup_{\delta\to 0} \frac{\log \mathrm{M}_\delta\big(\operatorname{graph} f\big)}{-\log\delta} \leqslant \limsup_{\delta\to 0} \frac{\log \dfrac{4 + 2(\sqrt{2})^{3-s}c}{\delta^s}}{-\log\delta} = s.$$

■

We now assume that

$$\frac{a_n}{\alpha^n} \to 0 \text{ as } n \to \infty \text{ for all } \alpha \in \left(\frac{1}{2}, 1\right).$$

(This corresponds to a scaling sequence tending to zero 'quickly'.) Thus we have

$$\left(\forall\alpha \in \left(\frac{1}{2}, 1\right)\right)(\forall\epsilon > 0)\,(\exists N \in \mathbb{N})\,(n > N \Rightarrow a_n \leqslant \alpha^n\epsilon)\,.$$

Fix $\alpha \in \big(\frac{1}{2}, 1\big)$. For $\epsilon = 1$, there is an $N \in \mathbb{N}$ such that $n > N \Rightarrow a_n \leqslant \alpha^n$. Furthermore there exists $A \in \mathbb{R}$ such that $a_k \leqslant A\alpha^k$ for $k \leqslant N$. We set $h = y - x$ for convenience. Now,

$$\begin{aligned}\big|f(x)-f(y)\big| &\leqslant \sum_{k=0}^{N} a_k\big|g_k(x+h)-g_k(x)\big| + \sum_{k=N+1}^{\infty} a_k\big|g_k(x+h)-g_k(x)\big| \\ &\leqslant A\sum_{k=0}^{N}\alpha^k 2^k|h| + \sum_{k=N+1}^{\infty}\alpha^k\big|g_k(x+h)-g_k(x)\big| \\ &\leqslant A\frac{(2\alpha)^{N+1}}{1-2\alpha}|h| + \frac{\alpha^{N+1}}{1-\alpha} \\ &\leqslant b|h|^{3-s}+c|h|^{3-s} = k|x-y|^{3-s} \qquad \text{where } s = 3+\frac{\log\alpha}{\log 2}.\end{aligned}$$

By Lemma 3.1 we find

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \leqslant s = 3+\frac{\log\alpha}{\log 2},$$

and since α was arbitrary we get

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \leqslant 2.$$

We have therefore proved the result:

♦ **Theorem 3.2.** *If the sequence scaling the Takagi surface satisfies $a_n/\alpha^n \to 0$ for every $\frac{1}{2} \leqslant \alpha \leqslant 1$, then*

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) = \underline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) = 2. \quad \blacksquare$$

We now turn to the case where the scaling sequence is 'much slower' than any geometric scaling between 1/2 and 1. That is, we assume

$$\frac{\alpha^n}{a_n} \to 0 \text{ as } n\to\infty \text{ for all } \alpha \in \left(\frac{1}{2},1\right).$$

Thus we have

$$\left(\forall\alpha\in\left(\frac{1}{2},1\right)\right)(\forall\epsilon>0)\,(\exists N\in\mathbb{N})\,(n>N \Rightarrow \alpha^n \leqslant a_n).$$

Fix $\alpha \in (\frac{1}{2},1)$. For $\epsilon = 1$ there is an $N\in\mathbb{N}$ such that $n > N \Rightarrow \alpha^n \leqslant a_n$. Thus we have that

$$(\exists B\in\mathbb{R})(B<1)(\forall n\in\mathbb{N})(\alpha^n \leqslant Ba_n)$$

We then have, from the proof of Lemma 5.1 (page 7) that

$$\operatorname{osc}(f, I_{n,i,j}) \geqslant \tfrac{1}{2}a_n \geqslant \tfrac{1}{2}B\alpha^n = B\frac{1}{2}\left(\frac{1}{2^m}\right)^{2-s}$$

This and the inequality (1) leads to:

$$\underline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \geqslant s = 3+\frac{\log\alpha}{\log 2}.$$

Since α was arbitrary, we get $\underline{\dim}_{\mathrm{B}}(\mathrm{graph}\, f) \geqslant s = 3$. Hence we can construct surfaces that are continuous yet so irregular that their box-counting dimension is equal to three. This is formalised in the following theorem:

♦ **Theorem 3.3.** *If the scaling sequence of the random Takagi surface is such that*

$$\alpha^n / a_n \to 0 \text{ for every } \alpha \in \left(\tfrac{1}{2}, 1\right)$$

then

$$\underline{\dim}_{\mathrm{B}}(\mathrm{graph}\, f) = \overline{\dim}_{\mathrm{B}}(\mathrm{graph}\, f) = 3. \quad \blacksquare$$

4. An Upper Bound for the Upper Dimension

In this section we examine the less straightforward case where we cannot approximate the dimension of the surface by that of a geometrically scaled one.

♦ **Lemma 4.1.** *Assume that the sequence $(a_n)_n$ scaling the Takagi surface is such that*

$$\frac{1}{2} < \limsup_{n\to\infty} \left(\log a_n^{1/n}\right) < 1, \tag{2}$$

and let $I_{n,i,j}$ be given. Then for every $\epsilon > 0$ there exists a constant A such that

$$|f(x) - f(y)| \leqslant A|x-y|^{3-(\bar{s}+\epsilon)} \qquad \text{for every } x, y \in I_{n,i,j},$$

where

$$\bar{s} = 3 + \limsup_{n\to\infty} \frac{(1/n)\log a_n}{\log 2}.$$

Proof. Let $\epsilon > 0$, and let $x, y \in I_{n,i,j}$. Furthermore set $h = y - x$. Then for some $M \in \mathbb{N}$ we have

$$\frac{1}{2^{M+1}} \leqslant |h| \leqslant \frac{1}{2^M}, \tag{3}$$

and since $x, y \in I_{n,i,j}$ we have $M > n$. Also,

$$\begin{aligned} \left|f(x+h) - f(x)\right| &= \sum_{k=0}^{M} a_k \left|g_k(x+h) - g_k(x)\right| + \sum_{k=M+1}^{\infty} a_k \left|g_k(x+h) - g_k(x)\right| \\ &\leqslant \sum_{k=0}^{M} 2^k a_k |h| + \sum_{k=M+1}^{\infty} a_k. \end{aligned} \tag{4}$$

Now, we have

$$3 - (\bar{s} + \epsilon) < 3 - \bar{s} = \frac{1}{\log 2} \liminf_{n\to\infty} \left(\frac{-1}{n} \log a_n\right)$$

and since the inequality is strict there exists $K_\epsilon \in \mathbb{N}$ such that

$$n \geqslant K_\epsilon \Rightarrow a_n \leqslant \left(\frac{1}{2^n}\right)^{3-(\bar{s}+\epsilon)} \qquad \text{(by rearranging).}$$

Since there are only finitely many terms before the K_ϵth, we can find $C > 0$ such that

$$a_n \leqslant C\left(\frac{1}{2^n}\right)^{3-(\bar{s}+\epsilon)}.$$

Then, the finite sum becomes

$$\begin{aligned}|h|\sum_{k=0}^{M} 2^k a_k &\leqslant |h| C \sum_{k=0}^{M} 2^k \left(\frac{1}{2^k}\right)^{3-(\bar{s}+\epsilon)}\\ &\leqslant C'|h|\left(\frac{1}{2^{M+1}}\right)^{2-(\bar{s}+\epsilon)}\\ &\leqslant C'|h|^{3-(\bar{s}+\epsilon)}\end{aligned}$$

and the infinite sum becomes

$$\begin{aligned}\sum_{k=M+1}^{\infty} a_k &\leqslant C \sum_{k=M+1}^{\infty} \left(\frac{1}{2^k}\right)^{3-(\bar{s}+\epsilon)}\\ &= C''\left(\frac{1}{2^{M+1}}\right)^{3-(\bar{s}+\epsilon)}\\ &\leqslant C''|h|^{3-(\bar{s}+\epsilon)}.\end{aligned}$$

Finally, on setting $A = C' + C''$ we have $\big|f(x) - f(y)\big| \leqslant A\big|x-y\big|^{3-(\bar{s}+\epsilon)}$. ■

♦ **Theorem 4.2.** *Suppose that* (2) *holds. For every* $\epsilon > 0$ *we have*

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \leqslant \bar{s} + \epsilon.$$

Proof. First, recall that

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) = \limsup_{\delta\to 0} \frac{\log \mathrm{M}_\delta\big(\operatorname{graph} f\big)}{-\log\delta},$$

and second, note that it is a standard result [**Fal90**, p. 41] that

$$\limsup_{\delta\to 0} \frac{\log \mathrm{M}_\delta\big(\operatorname{graph} f\big)}{-\log\delta} = \limsup_{n\to\infty} \frac{\log \mathrm{M}_{2^{-n}}\big(\operatorname{graph} f\big)}{-\log 2^{-n}}. \tag{5}$$

Let $\epsilon > 0$. By (5) it is sufficient to examine $\delta = 2^{-n}$ regarded as a sequence tending to zero. Now, by (1),

$$\mathrm{M}_{2^{-n}}\big(\operatorname{graph} f\big) \leqslant 2\left(\frac{1}{2^{-n}} + 1\right)^2 + 2^n \sum_{i,j=0}^{2^n} \operatorname{osc}(f,\, I_{n,i,j});$$

and in applying Lemma 4.1 to each interval $I_{n,i,j}$ we obtain

$$\mathrm{M}_{2^{-n}}\big(\operatorname{graph} f\big) \leqslant 2(2^n+1)^2 + 2^n A \sum_{i,j=0}^{2^n} \left(\frac{1}{2^n}\right)^{3-(\bar{s}+\epsilon)},$$

where A is the biggest of all the constants given by Lemma 4.1. This expression is then bounded above thus:

$$\begin{aligned}
\mathrm{M}_{2^{-n}}\big(\operatorname{graph} f\big) &\leqslant 2(2^n+1)^2 + 2^n A(2^n+1)^2\left(\frac{1}{2^n}\right)^{3-(\bar{s}+\epsilon)} \\
&\leqslant 2(2^n)^2 + 4(2^n) + 2 + A(2^{2n}+2^{n+1}+1)(2^n)^{(\bar{s}+\epsilon)-2} \\
&\leqslant K\cdot(2^n)^{\bar{s}+\epsilon} \qquad (6)
\end{aligned}$$

where K is a constant, and the first line was obtained by evaluating the sum. Finally,

$$\begin{aligned}
\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) &= \limsup_{n\to\infty}\frac{\log \mathrm{M}_{2^{-n}}\big(\operatorname{graph} f\big)}{-\log 2^{-n}} \\
&\leqslant \limsup_{n\to\infty}\frac{\log K(2^n)^{\bar{s}+\epsilon}}{\log 2^n} \qquad \text{(by (6))} \\
&= \bar{s}+\epsilon.
\end{aligned}$$

■

Thus, to summarize, for the condition (2), we have $\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \leqslant \bar{s}$.

5. A Lower Bound for the Lower Dimension

♦ **Lemma 5.1.** *Let m denote the centre of $I_{n,i,j}$. Then there exist two points α, $\beta \in I_{n,i,j}$ such that*

$$f(\alpha) \leqslant \sum_{k=0}^{n-1} a_k g_k(m) \leqslant f(\beta).$$

Proof. Given an interval $I_{n,i,j}$, define α and β by the following algorithm:

(1) If a diagonal of $[0,1]\times[0,1]$ crosses $I_{n,i,j}$ then let α and β be the two corners of $I_{n,i,j}$ which lie on that diagonal.
(2) If no diagonal of $[0,1]\times[0,1]$ crosses $I_{n,i,j}$ then look at the quarter interval I_{1,i_1,j_1} containing $I_{n,i,j}$. If a diagonal of I_{1,i_1,j_1} crosses $I_{n,i,j}$ then let α and β be the two corners of $I_{n,i,j}$ which lie on that diagonal.
(3) If no diagonal of I_{1,i_1,j_1} crosses $I_{n,i,j}$ then look at the quarter interval I_{2,i_2,j_2} of I_{1,i_1,j_1} containing $I_{n,i,j}$ and proceed as above.

Note that the above process will always produce two distinct points α and β since one of the diagonals of the interval $I_{n-1,i_{n-1},j_{n-1}}$ containing $I_{n,i,j}$ crosses $I_{n,i,j}$. It is important, however, that α and β be the two corners which *first* lie on a diagonal, in the above sense.

We now define the function $h\colon \{\alpha+t(\beta-\alpha)\mid 0\leqslant t\leqslant 1\}\to\mathbb{R}$ by

$$h(x) = \sum_{k=0}^{n-1} a_k g_k(x).$$

Note that h agrees with f at α and β, since for $m \geqslant n$ we have $g_m(x) = 0$. Furthermore, because of the way α and β were chosen, h is an affine transformation. This means that

$$f(\alpha) \leqslant h(x) \leqslant f(\beta) \qquad \text{for every } x \in \{\, \alpha + t(\beta - \alpha) \mid 0 \leqslant t \leqslant 1 \,\}\,.$$

In particular this holds for the midpoint m. ■

Proposition 5.2. *For every $\epsilon > 0$ there exists $K_\epsilon \in \mathbb{N}$ such that*

$$n > K_\epsilon \Rightarrow \operatorname{osc}(f,\, I_{n,i,j}) \geqslant \frac{1}{2}\left(\frac{1}{2^n}\right)^{3-(\underline{s}-\epsilon)} \quad \textit{for every } i,j \in \{0,\ldots,2^n\},$$

where

$$\underline{s} = 3 + \liminf_{m\to\infty} \frac{(1/n)\log a_n}{\log 2}.$$

Proof. First note that

$$3 - \underline{s} + \epsilon > 3 - \underline{s} = \frac{1}{\log 2} \limsup_{n\to\infty} \left(\frac{-1}{n}\log a_n\right).$$

Since the inequality is strict there exists $K_\epsilon \in \mathbb{N}$ such that

$$n > K_\epsilon \Rightarrow 3 - \underline{s} + \epsilon > \frac{1}{\log 2}\frac{-1}{n}\log a_n,$$

from which it follows that

$$\left(\frac{1}{2^n}\right)^{3-(\underline{s}-\epsilon)} < a_n. \tag{7}$$

Given $I_{n,i,j}$, let m be the centre of the interval. If $k > n$ then $g_k(m) = 0$; moreover $g_n(m) = \pm\frac{1}{2}$. Hence

$$f(m) = \sum_{k=0}^{n-1} a_k g_k(m) + a_n g_n(m).$$

Using Lemma 5.1 we have for some α and β:

$$f(\alpha) - f(m) \leqslant -a_n g_n(m) \leqslant f(\beta) - f(m).$$

Depending on the sign of $g_n(m)$ we have one of the following:

(1) $\operatorname{osc}(f,\, I_{n,i,j}) \geqslant |f(\beta) - f(m)| \geqslant \frac{1}{2}a_n$;
(2) $\operatorname{osc}(f,\, I_{n,i,j}) \geqslant |f(\alpha) - f(m)| \geqslant \frac{1}{2}a_n$.

In both cases, applying (7) gives

$$\operatorname{osc}(f,\, I_{n,i,j}) \geqslant \frac{1}{2}\left(\frac{1}{2^n}\right)^{3-(\underline{s}-\epsilon)}.$$ ■

Theorem 5.3. *For every $\epsilon > 0$ we have*

$$\underline{s} - \epsilon \leqslant \underline{\dim}_{\mathrm{B}}\bigl(\operatorname{graph} f\bigr)$$

where $\underline{s}$ is as defined as above.

PROOF. As in the proof of Theorem 4.2 it is sufficient to examine only the sequence $1/2^n$ in the limit inferior calculation. By the observation (1) we obtain, if $\delta = 1/2^n$,

$$\mathrm{M}_{2^{-n}}\big(\operatorname{graph} f\big) \geqslant 2^n \sum_{i,j=0}^{2^n} \operatorname{osc}(f,\, I_{n,i,j}).$$

Now let $\epsilon > 0$. It is permissible to assume that $n > K_\epsilon$ as in Proposition 5.2 since we can discard finitely many terms in calculating the limit inferior. So,

$$\begin{aligned}\mathrm{M}_{2^{-n}}\big(\operatorname{graph} f\big) &\geqslant 2^n \sum_{i,j=0}^{2^n} \frac{1}{2}\left(\frac{1}{2^n}\right)^{3-(\underline{s}-\epsilon)} \geqslant \frac{1}{2}\left(\frac{1}{2^n}\right)^{-(\underline{s}-\epsilon)} \\ &= \tfrac{1}{2}(2^n)^{\underline{s}-\epsilon}.\end{aligned}$$

It is a matter of performing the calculation:

$$\underline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \geqslant \liminf_{n\to\infty} \frac{\log \frac{1}{2}(2^n)^{\underline{s}-\epsilon}}{\log 2^n} = \underline{s} - \epsilon. \qquad \blacksquare$$

6. A Lower Bound for the Upper Dimension for certain scalings

We show now that $\bar{s}$ is also a lower bound for the upper box dimension. Using (1) and the proof of Proposition 5.2, we know that

$$\begin{aligned}\mathrm{M}_{2^{-n}}(\operatorname{graph} f) &\geqslant 2^n \sum_{i,j=0}^{2^n} \operatorname{osc}(f, I_{n,i,j}) \\ &\geqslant 2^n \sum_{i,j=0}^{2^n} \tfrac{1}{2} a_n \\ &= \tfrac{1}{2} 2^n \left(2^n + 1\right)^2 a_n \geqslant \tfrac{1}{2} 2^{3n} a_n.\end{aligned}$$

We now approximate the upper box dimension:

♦ **Theorem 6.1.** *We have, if $\bar{s}$ is as defined previously:*

$$\overline{\dim}_{\mathrm{B}}\big(\operatorname{graph} f\big) \geqslant \bar{s}.$$

PROOF. By using the preceeding arguments, we find that

$$\begin{aligned}\overline{\dim}_{\mathrm{B}} &= \limsup_{n\to\infty} \frac{\log \mathrm{M}_{2^{-n}}(\operatorname{graph} f)}{-\log \frac{1}{2^n}} \\ &\geqslant \limsup_{n\to\infty} \frac{\log \frac{1}{2} + \log 2^{3n} + \log a_n}{\log 2^n} \\ &= 3 + \limsup_{n\to\infty} \left(\frac{\frac{1}{n}\log a_n}{\log 2} - \frac{1}{n} \right) \\ &= \bar{s}. \end{aligned} \qquad \blacksquare$$

7. Further Investigation

F. Takeo is able to find further bounds on the dimensions of generalised Takagi curves. We believe that his approaches may be adaptable to our random Takagi surface. Once this is done it may be possible to unify to some extent the results presented herein.

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