CPSC 314
Assignment 2

due: Wednesday, October 2, 2019, 3pm

Answer the questions in the spaces provided on the question sheets.

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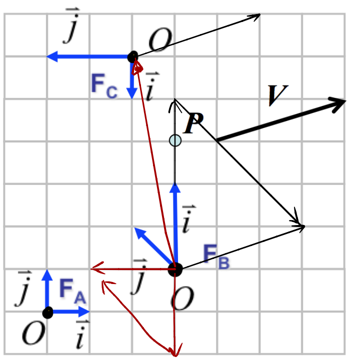
1. Transformations as a change of coordinate frame



$$\begin{bmatrix} x_A \\ y_A \\ 1 \end{bmatrix}_A = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ 1 \end{bmatrix}_C$$

$$\begin{bmatrix} x_B \\ y_B \\ 1 \end{bmatrix}_B = \begin{bmatrix} -0.5 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ 1 \end{bmatrix}_C$$

$A = M_{C \to A}\, C$

$B = M_{C \to B}\, C.$

$A = M_{B \to A}\, B.$

$M_{C \to A} C = M_{B \to A} M_{C \to B} C.$

(a) (3 points) Express the coordinates of point P with respect to coordinate frames A, B, and C.

$P_A (3, 4) \quad P_B (1.5, 0) \quad P_c (2, -0.5)$

(b) (3 points) Express the coordinates of vector V with respect to coordinate frames A, B, and C.

$V_A (3, 1) \quad V_B (2, -3) \quad V_c (-1, -1.5)$

(c) (3 points) Fill in the 2D transformation matrix, $M_{C \to A}$, that takes points from F_C to F_A, as given to the right of the above figure.

(d) (3 points) Fill in the 2D transformation matrix, $M_{C \to B}$, that takes points from F_C to F_B, as given to the right of the above figure.

(e) (2 points) Given an expression for the 2D transformation matrix, $M_{B \to A}$, in terms of the matrices $M_{C \to A}$ and $M_{C \to B}$. Then evaluate the expression using your results above. Use any tool you like to compute the required matrix inverse, i.e., matlab, an online web page, or you can also develop the matrix yourself in the same way you developed the other matrices. Test your solution using point P.

$A = M_{C \to A} C$

$B = M_{C \to B} C.$

$A = M_{B \to A} B. \iff M_{C \to A} C = M_{B \to A} M_{C \to B} C$

$\iff M_{C \to A} = M_{B \to A} M_{C \to B}.$

$\iff M_{C \to A} M_{C \to B}^{-1} = M_{B \to A} I_3$

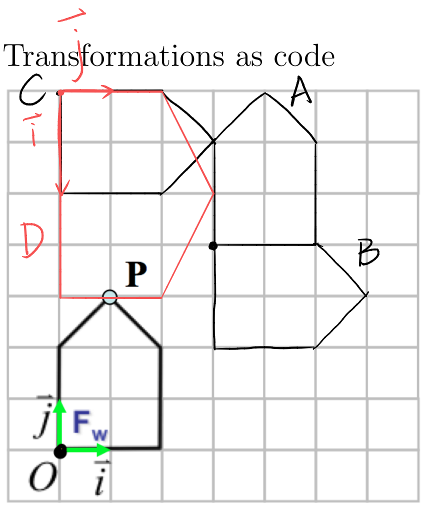
$\therefore M_{B \to A} = M_{C \to A} M_{C \to B}^{-1}$

$$= \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 5 \\ 0 & 0.5 & -0.5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_B = (1.5, 0)$

$$\begin{bmatrix} 0 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = P_A (3, 4)$$

2. Transformations as code



$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_w = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{obj}$$

(a) (2 points) The drawing of the above house represents its untransformed shape. Sketch the transformed version of the house in the above diagram, that results after each of the steps listed below. Label each sketch with its corresponding label, i.e., a,b,c,d. Assume that all transformations applied to M do right-multiplication.

```
M.identity();
M.translate(3,4,0);    // A
M.rotate(Z,-90);       // B
M.translate(-3,-3,0);  // C
M.scale(2,1,1);        // D
```

(b) (1 point) On the diagram, also sketch the origin and basis vectors corresponding to M after the final transformation, i.e., step D.

(c) (1 point) If the matrices representing each of the four individual steps were available, e.g., $M_D = Scale(2,1,1)$ and so forth, give the linear algebraic expression that combines M_A, M_B, M_C, M_D into the final compound transformation M.

$M = M_A M_B M_C M_D$

(d) (2 points) Give the values of the resulting transformation matrix, M, where $P_W = M P_{obj}$. Complete this in the space given to the right of the diagram.

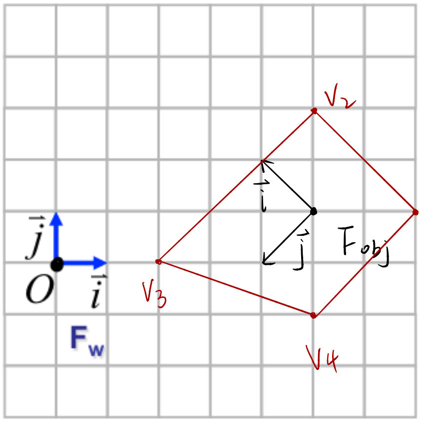
(e) (2 points) Suppose that an alternate sequence of transformations, given by

$M = Rotate(Z, \theta) Scale(a, b, 1) Translate(c, d, 0)$,

is used to implement the same transformation. Provide the suitable values for a, b, c, d and θ that would implement the given transformation.

$\theta = -90$ $a = 2$ $b = 1$ $c = -3.5$ $d = 0$

3. Transformations as matrices



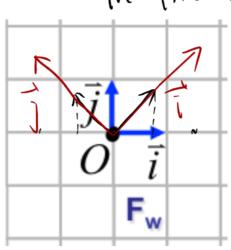
$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_w = \begin{bmatrix} -1 & -1 & 0 & 5 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{obj} \begin{matrix} 1 \\ -1 \\ 0 \\ \end{matrix}$$

(a) (3 points)
(i) On the above diagram, sketch the origin and basis vectors of the coordinate frame F_{obj} that results from the given transformation matrix.
(ii) Draw the object that consists of a single polygon, and has the following 4 vertices, as modeled in in F_{obj}, i.e., the local frame: $V_1(-1,-1), V_2(1,-1), V_3(1,2), V_4(-1,1)$. Give the world coordinates of those same points (computed using the matrix, or taken from the figure).

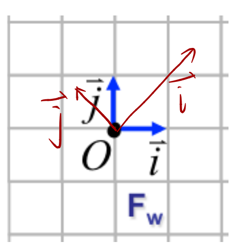
$V_{1w}(7,1)$ $V_{2w}(5,3)$ $V_{3w}(2,0)$ $V_{4w}(5,-1)$

(b) (3 points) In this question, we compare the following affine transformations:
$M_A = Scale(2,1,1)Rotate(Z,45°)$, and $M_B = Rotate(Z,45°)Scale(2,1,1)$.
(i) Sketch the i and j basis vectors of the resulting coordinate frames for M_A and M_B.
(ii) Which one gives a non intuitive result?
(iii) For what values of (a,b,c) would $Rotate(Z,\theta)Scale(a,b,c) = Scale(a,b,c)Rotate(Z,\theta)$?

in fixed coords



M_A



M_B

(ii) M_A gives a non intuitive result.

(iii) We want:

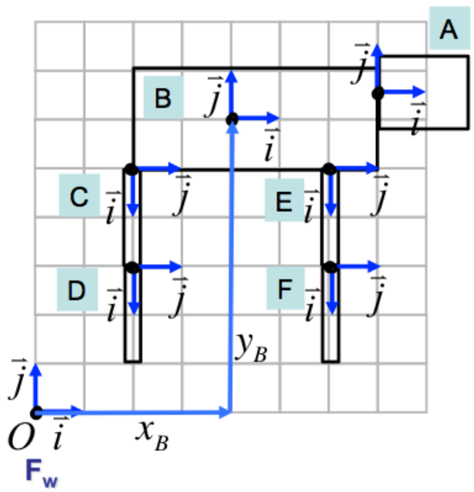
$$\begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

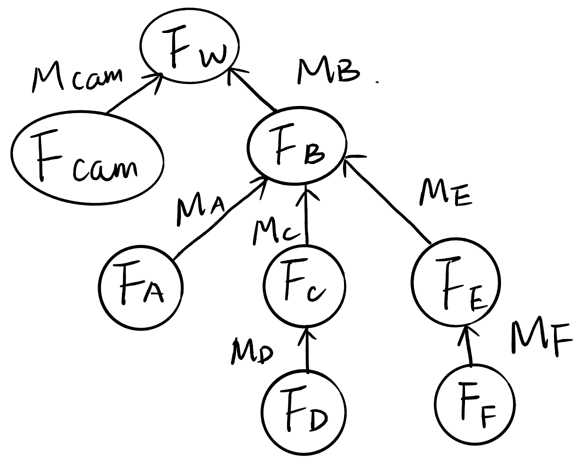
$$\begin{bmatrix} a\cdot\cos\theta & -b\cdot\sin\theta & & \\ a\cdot\sin\theta & b\cdot\cos\theta & & \\ 0 & 0 & c & \\ 0 & 0 & & 1 \end{bmatrix} = \begin{bmatrix} a\cos\theta & -a\sin\theta & & \\ b\sin\theta & b\cos\theta & & \\ 0 & 0 & c & \\ 0 & 0 & & 1 \end{bmatrix}$$

thus: $-b\sin\theta = -a\sin\theta$ $\therefore a = b$
or $\theta = k \times 180°$ (k: constant)

4. Scene Graphs

(a) (2 points) In the space to the right below, sketch a scene graph for the simple 2D dog. The labeled nodes should represent the coordinate frames, e.g., F_A, for each link. Use directed edges, i.e., arrows, to represent transformations that connect the nodes in the graph. Label each edge with a unique name, e.g., M_A, which indicates the transformation matrix that takes points from F_A to its parent frame. Use the world coordinate frame as the root note of the scene graph. Assume that the body, F_B, and the camera, F_{cam}, are children of the world frame, F_W. Assume that all other parts are defined relative to their parent links, i.e., the links closer to the body.





(b) (1 point) Give a set of 4 local vertex coordinates (x, y) that could be used to represent link C.

LU (0,-0.2)   LD (2,-0.2)
RU (0, 0.2)   RD (2, 0.2)

(c) (2 points) Use your scene graph to develop an algebraic expression for the composite transformation to develop $M_{D\to cam}$, where $P_{cam} = MP_D$, i.e., it transforms point P_D to the camera coordinate frame, F_{cam}. Express your answer as a product of the matrices used to label your scene graph.

$P_W = M_B M_C M_D P_D$. $M_{cam} P_{cam} = M_B M_C M_D P_D$.

$P_W = M_{cam} P_{cam}$. $\underbrace{M_{cam}^{-1} M_{cam}}_{= I} P_{cam} = M_{cam}^{-1} M_B M_C M_D P_D$.

$\therefore M_{D\to cam} = M_{cam}^{-1} M_B M_C M_D$.

(d) (2 points) Similarly, give an algebraic expression for the composite transformation that takes point, P_D, as defined in frame F_D, to the head frame, F_A.

$P_B = M_A P_A = M_C M_D P_D$ $P_A = \underbrace{M_A^{-1} M_C M_D}_{= M_{D\to A}} P_D$

(e) (3 points) The full position of the dog is described by $x_B, y_B, \theta_A, \theta_B, \theta_C, \theta_D, \theta_E, \theta_F$, where x_B, y_B locate the body, and the angles define rotations about joints, given by the Z-axis of the local coordinate frames. As drawn, the figure shows the dog in its reference configuration, i.e., with $\theta_A, \theta_B, \theta_C, \theta_D, \theta_E, \theta_F = 0$. Give expressions for each of M_A, M_B, M_C, M_D, M_E and M_F in terms of a product of translation and rotation matrices, e.g., of the form `Trans(a,b,0)` and `Rot(z,θ)`.

M_A = Trans (3, 0.5, 0) Rot (z, 0) M_E = Trans (2, -1, 0) Rot (z, -90)

M_B = Trans (4, 6, 0) Rot (z, 0) M_F = Trans (2, 0, 0) Rot (z, 0)

M_C = Trans (-2, -1, 0) Rot (z, -90)

M_D = Trans (2, 0, 0) Rot (z, 0)

5. A requirement of a 3×3 rotation matrix, $R = \begin{bmatrix} \vec{\mathbf{a}} & \vec{\mathbf{b}} & \vec{\mathbf{c}} \end{bmatrix}$, is that the columns, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, have unit magnitude and are mutually orthogonal, i.e., a zero dot product. Furthermore, for a right-handed coordinate system, we require $\mathbf{a} \times \mathbf{b} = \mathbf{c}$.

(a) (1 point) Given a rotation matrix defined by three column vectors, $R = \begin{bmatrix} \vec{\mathbf{a}} & \vec{\mathbf{b}} & \vec{\mathbf{c}} \end{bmatrix}$ compute the resulting matrix product, $M = R^T R$.

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ a_1b_1+a_2b_2+a_3b_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ a_1c_1+a_2c_2+a_3c_3 & b_1c_1+b_2c_2+b_3c_3 & c_1^2+c_2^2+c_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

(b) (2 points) A 4×4 rigid body transformation M is defined by a rotation matrix and a translation, T, as shown below. Develop an expression for the inverse of this transformation matrix. Hint: Find values for a matrix M^{-1} such that $I = M^{-1}M$. Note that your answer to part (a) provides most of the solution.

$$\begin{bmatrix} d_x & e_x & f_x & T_x \\ d_y & e_y & f_y & T_y \\ d_z & e_z & f_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$d \quad e \quad f \quad T$

$$\begin{array}{c} d \\ e \\ f \\ \\ \end{array} \begin{bmatrix} d_x & d_y & d_z & -T \cdot d \\ e_x & e_y & e_z & -T \cdot e \\ f_x & f_y & f_z & -T \cdot f \\ 0 & 0 & 0 & 1 \end{bmatrix} = M^{-1}$$

(c) (3 points) Determine if the matrices below are rotations and explain why or why not. Use the criteria for rotation matrices, as described earlier.

(i) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(iii) $\left\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2+1^2} = \sqrt{2} \neq 1$

$\therefore$ it's not a rotation.

(i) $\left\| \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\| = \sqrt{1+1} = \sqrt{2} \neq 1$

$\therefore$ not a rotation

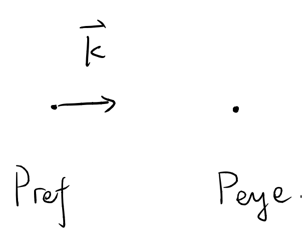
(ii) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = 0 \ (\checkmark)$

$\left\| \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\| = \sqrt{1^2} = 1 \ (\checkmark)$

$\therefore$ it is a rotation

6. (4 points) Viewing Transformation

Determine the viewing transformation, M_{view}, that takes points from WCS (world coordinates) to VCS (viewing or camera coordinates), for the following camera parameters:
$P_{eye} = (5, 10, 10)$, $P_{ref} = (0, 10, 10)$, $V_{up} = (0, 1, 0)$. Show your work.



$$\vec{k} = \frac{P_{eye} - P_{ref}}{\| P_{eye} - P_{ref} \|} = \frac{\langle 5, 10, 10 \rangle - \langle 0, 10, 10 \rangle}{\| \langle 5, 0, 0 \rangle \|} = \frac{\langle 5, 0, 0 \rangle}{\sqrt{25}} = \langle 1, 0, 0 \rangle$$

$$\vec{i} = \frac{\vec{V_{up}} \times \vec{k}}{\| \vec{V_{up}} \times \vec{k} \|} = \frac{\langle 0, 1, 0 \rangle \times \langle 1, 0, 0 \rangle}{\| \langle 0, 0, -1 \rangle \|} = \langle 0, 0, -1 \rangle$$

$$\vec{j} = \vec{k} \times \vec{i} = \langle 1, 0, 0 \rangle \times \langle 0, 0, -1 \rangle = \langle 0, 1, 0 \rangle$$

$$M_{view} = \begin{bmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 10 \\ -1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$